Counting Ascents in Generalized Dyck Paths

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Abstract

Non-negative Łukasiewicz paths are special two-dimensional lattice paths never passing below their starting altitude which have only one single special type of down step. They are well-known and -studied combinatorial objects, in particular due to their bijective relation to trees with given node degrees.

We study the asymptotic behavior of the number of ascents (i.e., the number of maximal sequences of consecutive up steps) of given length for classical subfamilies of general non-negative Łukasiewicz paths: those with arbitrary ending altitude, those ending on their starting altitude, and a variation thereof. Our results include precise asymptotic expansions for the expected number of such ascents as well as for the corresponding variance.

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Supplement Material Supplementary SageMath [10] worksheets producing the results of this article can be found at `https://benjamin-hackl.at/publications/lukasiewicz-ascents/`.

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1 Introduction

Two-dimensional lattice paths can be defined as sequences of points in the plane $\mathbb{R}^2$ where for any point, the vector pointing to the succeeding point ("step") is from a predefined finite set, the *step set*. In general, lattice paths are very classical combinatorial objects with a variety of applications in, amongst others, Biology, Physics, and Chemistry.

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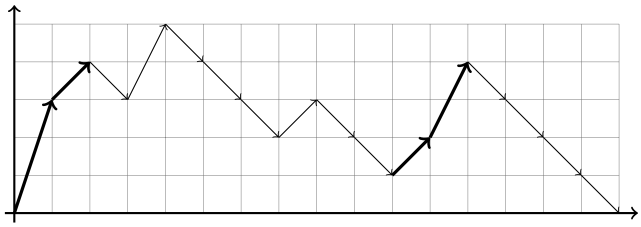

Figure 1 Simple Łukasiewicz excursion of length 16 with emphasized 2-ascents where $\mathcal{S} = \{-1, 1, 2, 3\}$

In this paper, our focus lies on a special class of two-dimensional lattice paths: *non-negative simple Łukasiewicz paths*. A lattice path is said to be *simple* if the horizontal coordinate is the same (e.g. is 1) for all possible steps. In case of a simple path family, we define the *step set* $\mathcal{S}$ as the set of allowed height differences, i.e., the respective y-coordinates between the points of the path. If, additionally, the step set $\mathcal{S} \subseteq \mathbb{Z}$ is integer-valued and contains -1 as the single negative value (meaning that all other values in $\mathcal{S}$ are non-negative), then the corresponding paths are called *simple Łukasiewicz paths*.

If a lattice path starts at the origin and never passes below the horizontal axis, then the path is said to be a *meander* (or non-negative path). And in case such a non-negative path ends on the horizontal axis, it is called an *excursion*.

Lattice path families of this type have been studied intensely, see [1] for a detailed survey on general simple lattice paths, and, for example, [2, 9] for investigations concerning Łukasiewicz paths.

We are interested in analyzing the number of *ascents* in these paths. An *ascent* is an inclusion-wise maximal sequence of up steps (i.e., steps in $\mathcal{S} \setminus \{-1\}$; this might also include the horizontal step corresponding to 0). For an integer $r \geq 1$, if an ascent consists of precisely r steps, then the ascent is said to be an r-ascent. As an example, Figure 1 depicts some non-negative Łukasiewicz excursion with emphasized 2-ascents.

In this paper, we give a precise analysis of the number of r-ascents for non-negative simple Łukasiewicz paths of given length, as well as of variants of this class of lattice paths. Our investigation is motivated by [6], where the number of 1-ascents in a special lattice path class related to the classic Dyck paths was analyzed explicitly by elementary methods.

Main Results

Within this paper, three special classes of non-negative Łukasiewicz paths are of interest:

- *excursions*, i.e., paths that end on the horizontal axis,
- *dispersed excursions*, i.e., excursions where horizontal steps are not allowed except on the horizontal axis,
- *meanders*, i.e., general non-negative Łukasiewicz paths without additional restrictions.

Formally, we conduct our analysis by investigating random variables $E_{n,r}$, $D_{n,r}$, $M_{n,r}$ which model the number of r-ascents in a random excursion, dispersed excursion, and meander of length n, respectively. The underlying probability models are based on equidistribution: within a family, all paths of length n are assumed to be equally likely.

Given $r \in \mathbb{N}$ and considering $n \in \mathbb{N}_0$ with $n \to \infty$, we prove that for excursions we have

$$\mathbb{E}E_{n,r} = \mu n + c_0 + O(n^{-1/2}) \qquad \text{and} \qquad \mathbb{V}E_{n,r} = \sigma^2 n + O(n^{1/2}),$$

for some constants μ, c_0, σ^2 depending on the chosen step set $\mathcal{S}$. The constants are given explicitly in Theorem 7. Additionally, if n is not a multiple of the so-called *period* of the step set, then the random variable degenerates and we have $E_{n,r} = 0$; see Theorem 7 for details.

For dispersed excursions, the corresponding computations get rather messy, which is why we restrict ourselves to the investigation of d_n, the number of dispersed excursions of length n, as well as the expected value $\mathbb{E}D_{n,r}$. In particular, for all step sets $\mathcal{S}$ (except for the special case of dispersed Dyck paths with $\mathcal{S} = \{-1, 1\}$), d_n satisfies

$$d_n = c_0\kappa^n n^{-3/2} + O(\kappa^n n^{-5/2}),$$

with constants c_0 and κ depending on the chosen step set. For the expected number of ascents in this particular lattice path family, we find

$$\mathbb{E}D_{n,r} = \mu n + O(1)$$

for some constant μ depending on $\mathcal{S}$. Explicit values for these constants and more details are given in Theorem 10.

In the context of meanders we are able to show that for all step sets (with two special exceptions: Dyck meanders with $\mathcal{S} = \{-1, 1\}$, and Motzkin meanders with $\mathcal{S} = \{-1, 0, 1\}$) we have

$$\mathbb{E}M_{n,r} = \mu n + c_0 + O(n^{5/2}\kappa^n) \qquad \text{and} \qquad \mathbb{V}M_{n,r} = \sigma^2 n + O(1),$$

for constants μ, c_0, $\kappa \in (0, 1)$, σ^2 depending on $\mathcal{S}$. Also, the random variable $M_{n,r}$ is asymptotically normally distributed; see Theorem 12 for explicit formulas for the constants and more details.

In theory, our approach can be used to obtain arbitrarily precise asymptotic expansions for all the quantities above. For the sake of readability we have chosen to only give the main term as well as one additional term, wherever possible.

On a more technical note, in order to deal with general Łukasiewicz step sets in our setting, we make use of a generating function approach (see [3, Chapter I]). In particular, we heavily rely on the technique of *singular inversion* (see [3, Chapter VI.7], [8]), which deals with finding an asymptotic expansion for the growth of the coefficients of generating functions $y(z)$ satisfying a functional equation of the type

$$y = z\,\phi(y)$$

with a suitable function ϕ.

Notation and Special Cases

Throughout this paper, the step set is denoted as $\mathcal{S} = \{-1, b_1, \ldots, b_{m-1}\}$ with integers $b_j \geq 0$ for all j and $m \geq 1$. The b_j are referred to as up steps – even if the step is a horizontal one.

The so-called characteristic polynomial of the lattice path class, i.e., the generating function corresponding to the set $\mathcal{S}$, is denoted by $S(u) := \sum_{s\in\mathcal{S}} u^s$. The strongly related generating function of the non-negative steps is denoted by $S_+(u) := \sum_{s\in\mathcal{S},\, s\geq 0} u^s$.

In this context, observe that the particular step set $\mathcal{S} = \{-1, 0\}$ corresponds to a, in some sense, pathological family of Łukasiewicz paths. In this case, there is only precisely one non-negative Łukasiewicz path of any given length. The family of meanders and excursions coincides, and also the random variables degenerate in the sense that we have[1]

[1] We make use of the Iversonian notation popularized in [4, Chapter 2]: $[\![expr]\!]$ takes value 1 if *expr* is true, and 0 otherwise.

$E_{n,r} = M_{n,r} = [\![n = r]\!]$. Thus, further investigation of this case is not required – which is why we exclude the case $\mathcal{S} = \{-1, 0\}$ from now on.

While in the case of a general step set $\mathcal{S}$ we are forced to deal with implicitly given quantities, for special cases like $\mathcal{S} = \{-1, 1\}$ (Dyck paths), everything can be made completely explicit as we will demonstrate in the course of our investigations.

Finally, we make use of the following well-established notation: For a generating function $f(z) = \sum_{n\geq 0} f_n z^n$, the coefficient belonging to z^n is denoted as $f_n = [z^n]f(z)$.

Structure of this Abstract

In Section 2, we determine suitable generating functions required to analyze the number of ascents. The approach is based on the inherent relation between Łukasiewicz paths and plane trees with given vertex degrees. Formulas for the respective generating functions are given in Proposition 4. Note that in the full version of this extended abstract, [5], we demonstrate another approach (following the kernel method and the "Adding a new slice approach") to determine the suitable generating functions.

Section 3 contains the actual analysis of ascents for the different lattice path families mentioned above. In particular, in Section 3.1 we investigate excursions; the main result is stated in Theorem 7. Section 3.2 deals with the analysis of ascents in dispersed excursions. In this case, the expected number of r-ascents for all but one given step sets is analyzed within Theorem 10, and the analysis for the remaining one is conducted in Proposition 11. Finally, Section 3.3 contains our results for ascents in meanders. Similarly to the previous section, the analysis for most step sets is given in Theorem 12, and the remaining cases are investigated in Propositions 13 and 14.

Proofs and additional details can be found in the full version of this extended abstract, see [5]. Appendix A contains several important tools necessary for a detailed analysis of the inverse function in the center of this abstract. In particular, with Propositions 15 and 16 we prove useful extensions of [3, Theorem VI.6; Remark VI.17].

2 Generating Functions: A Combinatorial Approach

In this section we will introduce and discuss the preliminaries required in order to carry out the asymptotic analysis of ascents in the different path classes. We begin by taking a closer look at the structure of Łukasiewicz paths.

Of course, the number of excursions of given length n strongly depends on the structure of the step set $\mathcal{S}$. For example, in the case of Dyck paths, i.e., $\mathcal{S} = \{-1, 1\}$, there cannot be any excursions of odd length – Dyck paths are said to be periodic lattice paths.

▶ **Definition 1** (Periodicity of lattice paths)**.** Let $\mathcal{S}$ be a Łukasiewicz step set with corresponding characteristic polynomial $S(u) = \sum_{s\in\mathcal{S}} u^s$. Then the period of $\mathcal{S}$ (and the associated lattice path family) is the largest integer p for which a polynomial Q satisfying

$$u\,S(u) = Q(u^p)$$

exists. If $p = 1$, then $\mathcal{S}$ is said to be aperiodic, otherwise $\mathcal{S}$ is said to be p-periodic.

▶ Remark. Observe that if a step set $\mathcal{S}$ has period p, then there are only excursions of length n where $n \equiv 0 \pmod p$. This can be seen by considering the generating function enumerating unrestricted paths of length n with respect to their height, i.e., $S(u)^n$. Obviously, the number of excursions of length n is at most the number of unrestricted paths ending at altitude 0, and the latter one can be written as

$$[u^0]\, S(u)^n = [u^n](u\, S(u))^n = [u^n]Q(u^p)^n.$$

Hence, if $n \not\equiv 0 \pmod p$, there are no unrestricted paths ending on the horizontal axis – and thus also no excursions.

The following proposition describes an integral relation which allows us to construct a suitable generating function later on.

▶ **Proposition 2.** *The excursions of Łukasiewicz paths of length n with respect to some step set $\mathcal{S}$ correspond to rooted plane trees with $n+1$ nodes and node degrees contained in the set $1+\mathcal{S}$.*

An r-ascent in a Łukasiewicz excursion with respect to the step set $\mathcal{S}$ corresponds to a rooted subtree such that the leftmost leaf in this subtree has height r, and additionally the root node of the subtree is not a leftmost child itself (in the original tree).

Proof. As pointed out in e.g. [1, Example 3], this bijection between rooted plane trees with given node degrees and Łukasiewicz excursions is well known. See [7, Section 11.3] for an approach using words. However, as this bijection and its consequences makes up an integral part of the argumentation within this paper, we present a short proof ourselves. Furthermore, proving the bijection allows us to find the substructure in the tree corresponding to an r-ascent.

Given a rooted plane tree T consisting of n nodes whose outdegrees are contained in $1+\mathcal{S}$, we construct a lattice path as follows: when traversing the tree in preorder[2], if passing a node with outdegree d, take a step of height $d-1$. The resulting lattice path thus consists of n steps, and always ends on altitude -1, which follows from

$$\sum_{v\in T}(\deg(v) - 1) = \sum_{v\in T}\deg(v) - n = (n-1) - n = -1,$$

where $\deg(v)$ denotes the outdegree (i.e., the number of children) of a node v in the tree T. In particular, observe that by taking the first $n-1$ steps of the lattice path, we actually end up with a Łukasiewicz excursion using the steps from $\mathcal{S}$. To see this, first observe that as the last node traversed in preorder certainly is a leaf, meaning that the nth step in the corresponding lattice path is a down step. As the path ends on altitude -1 after n steps, we have to arrive at the starting altitude after $n-1$ steps.

Furthermore, as illustrated in Figure 2, adding one to the current height of the constructed lattice path gives the size of the stack remembering the children that still have to be visited while traversing the tree in preorder. Combining the two previous arguments proves that the first $n-1$ steps in the constructed lattice path form a Łukasiewicz excursion.

Similarly, by simply reversing the lattice path construction, a rooted plane tree of size $n+1$ with node degrees in $1+\mathcal{S}$ can be constructed from any Łukasiewicz excursion of length n with respect to $\mathcal{S}$. This establishes the bijection between the two combinatorial families.

Finally, Figure 3 illustrates what r-ascents in Łukasiewicz paths are mapped to by means of the bijection above. ◀

[2] Traversing a tree in preorder corresponds to the order in which the nodes are visited when carrying out a depth-first search on it.

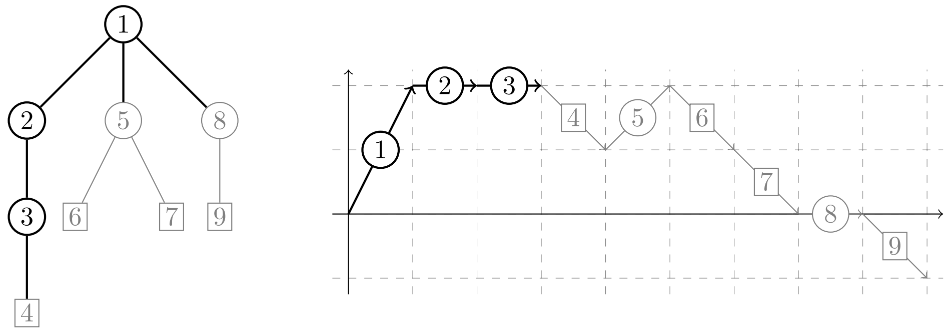

Figure 2 Bijection between Łukasiewicz paths and trees with given node degrees. The emphasized nodes and edges indicate the construction of the tree after the first three steps, which illustrates that the height of the Łukasiewicz path is one less than the number of available node positions in the tree.

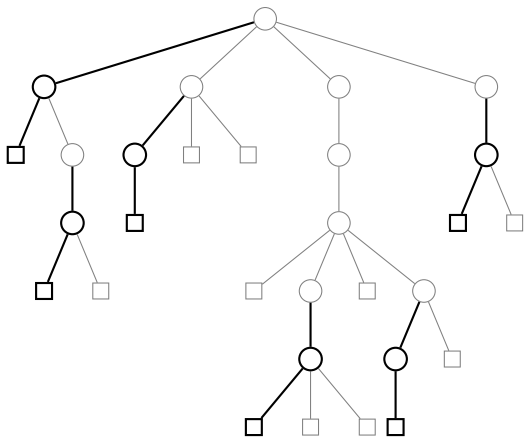

Figure 3 Plane tree with 30 nodes bijective to some Łukasiewicz excursion with respect to the step set $\mathcal{S} = \{-1, 0, 1, 2, 3\}$ whose number of 2-ascents is 6. The edges and nodes corresponding to the 2-ascents are emphasized.

In some sense, the bijection from Proposition 2 can be seen as a generalization of the well-known bijection between Dyck paths and binary trees where the tree is traversed in preorder, internal nodes correspond to up steps and leaves to down steps.

The fact that there is this bijection between Łukasiewicz excursions and these special trees with given node degrees allows us to draw an immediate conclusion regarding the corresponding generating functions.

▶ **Corollary 3.** *Let $V(z,t)$ be the generating function enumerating rooted plane trees with node degrees in $1+\mathcal{S}$ where z marks the number of nodes and t marks the number of r-ascents in the corresponding Łukasiewicz excursion. Then $V(z,t)/z$ enumerates Łukasiewicz excursions with respect to $\mathcal{S}$ based on their length (marked by z) and the number of r-ascents (marked by t).*

Additionally, $V(z,t)$ satisfies the equations

$$V(0,t) = 0, \qquad V(z,t) = zL(z,t,V(z,t)), \tag{1}$$

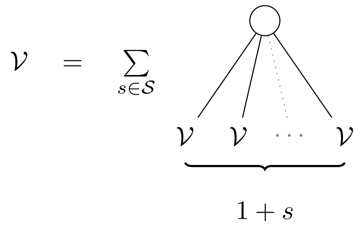

Figure 4 Symbolic equation for the family of plane trees $\mathcal{V}$ with outdegrees in $1+\mathcal{S}$. The generating function for $\mathcal{V}$ is $V(z)$, and the root node is enumerated by z.

where

$$L(z,t,v) = \frac{1}{1 - z\,S_+(v)} + (t-1)(z\,S_+(v))^r$$

enumerates sequences of up steps. The power series representation of $V(z,t)$ *is given by*

$$V(z,t) = \sum_{j\geq 0} g_j(t) z^{jp+1} \tag{2}$$

where p *denotes the period of* $\mathcal{S}$*. In particular,* $V(z) \coloneqq V(z,1)$*, the ordinary generating function enumerating plane trees with node degrees in* $1+\mathcal{S}$ *with respect to their size, satisfies*

$$V(0) = 0, \qquad V(z) = zV(z)\,S(V(z)). \tag{3}$$

Proof. The first part of this statement is an immediate consequence of the bijection from Proposition 2. In order to prove (1), we observe that $\mathcal{V}$, the combinatorial class of plane trees with vertex outdegrees in $1+\mathcal{S}$, can be constructed combinatorially by means of the symbolic equation

$$\mathcal{V} = \bigcirc \times \operatorname{SEQ}\Bigl(\bigcirc \times \sum_{\substack{s\in\mathcal{S}\\ s\geq 0}} \mathcal{V}^s\Bigr).$$

In a nutshell, this constructs trees in $\mathcal{V}$ by explicitly building the path to the leftmost leaf (the first factor in the equation above) in the tree as a sequence of nodes. Apart from a leftmost child, these nodes also have an additional $s \in \mathcal{S}$ branches, $s \geq 0$, where again a tree from $\mathcal{V}$ is attached. Considering that we obtain an r-ascent when using this construction with a sequence of length r, this is precisely what is enumerated by $L(z,t,V(z,t))$. Thus, the symbolic equation directly translates into the functional equation in (1). The condition $V(0,t) = 0$ is a consequence of the fact that there are no rooted trees without nodes.

The power series representation in (2) follows immediately from the considerations on periodicity at the beginning of this section.

Setting $t = 1$ in (1) leads to (3). We also want to give a combinatorial proof of (3):

The implicit equation follows from the observation that a tree with node degrees from $1+\mathcal{S}$ can be seen as a root node (enumerated by z) where $1+s$ for $s \in \mathcal{S}$ such trees are attached. Translating this into the language of generating functions via the symbolic equation illustrated in Figure 4, yields

$$V(z) = z\sum_{s\in\mathcal{S}} V(z)^{1+s} = zV(z)\,S(V(z)). \qquad ◀$$

The shape of the functional equation (3), which is an immediate consequence of the recursive structure of the underlying trees, is rather special. While it is tempting to cancel $V(z)$ on both sides of this equation, it is better to leave it in the present form: on the one hand, $S(u)$ starts with the summand $1/u$ – and on the other hand, we require (3) to be in this special form $y = z\,\phi(y)$ such that we can use singular inversion to obtain the asymptotic behavior of the coefficients of the generating function $V(z)$.

▶ **Proposition 4.** *Let $F(z,t,v)$ be the trivariate ordinary generating function counting non-negative Łukasiewicz paths with step set $\mathcal{S}$ starting at 0, where z marks the length of the path, t marks the number of r-ascents, and v marks the final altitude of the path. Then $F(z,t,v)$ can be expressed as*

$$F(z,t,v) = \frac{v - V(z,t)}{v - zL(z,t,v)} L(z,t,v), \tag{4}$$

where $V(z,t)$ and $L(z,t,v)$ are defined as in Corollary 3.

Proof. It is not hard to see that by considering a sequence of paths enumerated by $L(z,t,v)$ followed by a single down step (the corresponding generating function for this class is $\frac{1}{1-L(z,t,v)\,z/v}$), any unrestricted Łukasiewicz path with respect to $\mathcal{S}$ ending on a down step can be constructed.

We want to subtract all paths that pass below the starting altitude in order to obtain the trivariate generating function $\Phi(z,t,v)$ enumerating just the non-negative Łukasiewicz paths. The paths passing below the axis can be decomposed into an excursion enumerated by $V(z,t)/z$ (see Corollary 3), followed by an (illegal) down step enumerated by z/v, and ending with an unrestricted path again. Thus, the paths to be subtracted are enumerated by

$$\frac{V(z,t)}{z} \frac{z}{v} \frac{1}{1 - L(z,t,v)\frac{z}{v}}.$$

Therefore, we find

$$\Phi(z,t,v) = \frac{v - V(z,t)}{v - zL(z,t,v)}.$$

Keeping in mind that $\Phi(z,t,v)$ only enumerates those non-negative Łukasiewicz paths ending on a down step $\searrow$, the generating function $F(z,t,v)$ enumerating all such paths can be obtained by appending another sequence of upsteps, i.e.,

$$F(z,t,v) = \Phi(z,t,v)L(z,t,v).$$

This proves the statement. ◀

Now, with an appropriate generating function at hand let us discuss our approach for the asymptotic analysis of the number of ascents in a nutshell.

Basically, we set $v = 0$ to obtain a bivariate generating function enumerating Łukasiewicz excursions, and we set $v = 1$ to obtain a generating function enumerating Łukasiewicz meanders. The appropriate generating functions for the factorial moments of $E_{n,r}$ and $M_{n,r}$ (from which expected value and variance can be computed) are then obtained by first differentiating the corresponding generating function with respect to t (possibly more often than once) and then setting $t = 1$ in this partial derivative. The growth of the coefficients of this function can then be extracted by means of singularity analysis.

In particular, this means that in order to compute the asymptotic expansions for the quantities we are interested in, we only need more information on $V(z,1)$.

▶ Notation. For the sake of simplicity, and because we will deal with these expressions throughout the entire paper, we omit the second argument in $V(z,t)$ in case $t = 1$, i.e., we set $V(z) := V(z,1)$, $V_t(z) := V_t(z,1) = \frac{\partial}{\partial t}V(z,t)|_{t=1}$, $V_z(z) := V_z(z,1) = \frac{\partial}{\partial z}V(z,t)|_{t=1}$, and so on.

▶ **Example 5** (Explicit $F(z,t,v)$). In the case of $\mathcal{S} = \{-1,1\}$ and $r = 1$ the generating function $F(z,t,v)$ can be computed explicitly and we find

$$F(z,t,v) = \frac{(1+(t-1)vz(1-vz))((1-2vz)(1-(t-1)z^2) - \sqrt{(1-(t+3)z^2)(1-(t-1)z^2)}}{2z(1-(t-1)z^2)(z-v+v^2z+vz^2(t-1)(1-z))}. \tag{5}$$

Now, as we have derived a suitable generating function, we are interested in extracting information like, for example, asymptotic growth rates from $F(z,t,v)$. To this end, we need more information on the function $V(z,t)$.

▶ **Proposition 6.** *Let $V(z,t)$ be the bivariate generating function from Corollary 3. Let $\tau > 0$ be the uniquely determined positive constant satisfying $S'(\tau) = 0$. Then $V(z)$ has radius of convergence $\rho := 1/S(\tau)$ with a square-root singularity for $z \to \rho$. If $\mathcal{S}$ has period p, then the dominant singularities (i.e., singularities with modulus ρ) are located at $\zeta\rho$ with $\zeta \in G(p)$. The corresponding expansions are given by*

$$V(z) \overset{z\to\zeta\rho}{=} \zeta\tau - \zeta\sqrt{\frac{2\,S(\tau)}{S''(\tau)}}\Big(1-\frac{z}{\zeta\rho}\Big)^{1/2} - \zeta\frac{S(\tau)\,S'''(\tau)}{3\,S''(\tau)^2}\Big(1-\frac{z}{\zeta\rho}\Big) + O\Big(\Big(1-\frac{z}{\zeta\rho}\Big)^{3/2}\Big). \tag{6}$$

3 Analysis of Ascents

3.1 Analysis of Excursions

In this section we focus on the analysis of *excursions*, i.e., paths that start and end on the horizontal axis. On the generating function level, this corresponds to setting $v = 0$ in $F(z,t,v)$ from (4). Also note that from this point on it is quite useful to replace $S_+(v) = S(v) - 1/v$ in $F(z,t,v)$.

Recall that $E_{n,r}$ is the random variable modeling the number of r-ascents in a random non-negative Łukasiewicz excursion of length n with respect to some given step set $\mathcal{S}$.

▶ **Theorem 7.** *Let $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $p \ge 1$ be the period of the step set $\mathcal{S}$. Let τ be the structural constant, i.e., the unique positive solution of $S'(\tau) = 0$. Set $c := \tau\,S(\tau)$.*

Then, the expected number of r-ascents in Łukasiewicz paths of length n for $n \equiv 0 \pmod p$ as well as the corresponding variance grow with $n \to \infty$ according to the asymptotic expansions

$$\begin{aligned}\mathbb{E}E_{n,r} = \frac{(c-1)^r}{c^{r+2}}n + \frac{(c-1)^{r-2}}{2\tau^2c^{r+2}\,S''(\tau)^2}\Big(&S''(\tau)^2\tau^2\big(4c^2-(r+8)c+r+4\big)\\ &- S''(\tau)\,S(\tau)\big(6c^2-6(r+2)c+r^2+5r+6\big)\\ &- S'''(\tau)c(2c^2-(r+4)c+r+2)\Big)\\ &+ O(n^{-1/2})\end{aligned} \tag{7}$$

and

$$\mathbb{V}E_{n,r} = \left(\frac{(c-1)^r}{c^{r+2}} + \frac{(2c-2r-3)(c-1)^{2r}}{c^{2r+4}} - \frac{(c-1)^{2r-2}(2c-r-2)^2}{c^{2r+3}\tau^3\, S''(\tau)}\right)n + O(n^{1/2}). \quad (8)$$

Additionally, for $n \not\equiv 0 \mod p$, we have $E_{n,r} = 0$. All O-constants depend implicitly on r.

By means of Theorem 7 we are immediately able to determine the asymptotic behavior of interesting special cases. We are particularly interested in the most basic setting: $\mathcal{S} = \{-1, 1\}$, i.e., Dyck paths.

▶ **Example 8** (r-Ascents in Dyck paths)**.** In the case of Dyck paths, we have $u\,S(u) = 1 + u^2$. From there, it is easy to see that $\tau = 1$ and $\rho = 1/2$, and that the family of paths is 2-periodic. By the same approach as in the proof of Theorem 7, we can determine the expected number and variance of r-ascents in Dyck paths of length $2n$ with higher precision than stated in Theorem 7, namely as

$$\mathbb{E}D_{2n,r} = \frac{n}{2^{r+1}} - \frac{(r+1)(r-4)}{2^{r+3}} + \frac{(r^2-11r+22)(r+1)r}{2^{r+6}} n^{-1} + O(n^{-2})$$

and

$$\mathbb{V}D_{2n,r} = \left(\frac{1}{2^{r+1}} - \frac{r^2-2r+3}{2^{2r+3}}\right)n - \left(\frac{r^2-3r-4}{2^{r+3}} - \frac{3r^4-20r^3+29r^2-10r-14}{2^{2r+5}}\right) + O(n^{-1/2}).$$

However, as we have a closed expression for $V(z)$, we can do even better. Because of

$$\frac{V(z)}{z} = \frac{1-\sqrt{1-4z^2}}{2z^2},$$

we can also write down the generating function $V_t(z)/z$ for the expected number of r-ascents explicitly.

Ultimately, after extracting the corresponding coefficients of $V_t(z)/z$ we find

$$\mathbb{E}D_{2n,r} = \frac{1}{C_n}\binom{2n-r-1}{n-1}.$$

3.2 Analysis of Dispersed Excursions

Let $\mathcal{S}$ be a Łukasiewicz step set where $0 \notin \mathcal{S}$. In this setting, we define a *dispersed Łukasiewicz excursion* to be an $\mathcal{S}$-excursion where, additionally, horizontal steps can be taken whenever the path is on its starting altitude. Observe that, by our definition of r-ascents, these horizontal steps do not contribute towards ascents, as only the non-negative steps from $\mathcal{S}$ are relevant.

The motivation to study this specific family of Łukasiewicz paths originates from [6], where the authors investigate the total number of 1-ascents in dispersed Dyck paths using elementary methods. Our goal in this section is to find asymptotic expansions for the number of dispersed Łukasiewicz excursions of given length as well as for the expected number of r-ascents in these paths.

We begin our analysis by constructing a suitable bivariate generating function enumerating dispersed Łukasiewicz excursions with respect to their length and the number of r-ascents.

▶ **Proposition 9.** *Let $r \in \mathbb{N}$ and $V(z,t)$ as in Corollary 3. Then the generating function $D(z,t)$ enumerating dispersed $\mathcal{S}$-excursions where z marks the length of the excursion and t marks the number of r-ascents is given by*

$$D(z,t) = \frac{1}{z}\frac{V(z,t)}{1-V(z,t)}. \tag{9}$$

Proof. Let $\mathcal{E}$ denote the combinatorial class of $\mathcal{S}$-excursions. The corresponding bivariate generating function is given by $V(z,t)/z$, as proved in Corollary 3.

By the symbolic method (see [3, Chapter I]), the combinatorial class $\mathcal{D}$ of dispersed excursions can be constructed as $\mathcal{D} = (\mathcal{E} \rightarrow)^{*}\mathcal{E}$, where $\rightarrow^{*}$ represents a (possibly empty) sequence of horizontal steps. Translating this combinatorial construction in the language of (bivariate) generating functions yields (9). ◀

In preparation for the analysis of the generating function $D(z,t)$, we have to investigate the structure of the dominant singularities. In particular, it can be shown that the radius of convergence of $D(z,1)$ (as well as for the corresponding partial derivatives with respect to t) is given by $\rho = 1/S(\tau)$ where $\tau > 0$ is the structural constant with respect to $\mathcal{S}$.

Thus, in the general case of $\tau \neq 1$, the singularities of $D(z,1)$ are of the same type as the singularities of $V(z)$. Therefore, the precise description of the singular structure of $V(z)$ given in Proposition 6 allows us to carry out the asymptotic analysis.

Recall that $D_{n,r}$ is the random variable modeling the number of r-ascents in a random dispersed Łukasiewicz excursion of length n with respect to some step set $\mathcal{S}$.

▶ **Theorem 10.** *Let $p \geq 1$ be the period of the step set $\mathcal{S}$. Assume additionally that for the structural constant τ we have $\tau \neq 1$.*

Then d_n, the number of dispersed Łukasiewicz excursions of length n, satisfies

$$d_n = \frac{1}{\sqrt{2\pi}}\frac{p\tau^k(\tau^p(p-k-1)+k+1)}{(1-\tau^p)^2}\sqrt{\frac{S(\tau)^3}{S''(\tau)}}\,S(\tau)^n n^{-3/2} + O(S(\tau)^n n^{-5/2}) \tag{10}$$

for $n \equiv k \mod p$ and $0 \leq k \leq p-1$. Furthermore, the expected number of r-ascents grows with $n \to \infty$ according to the asymptotic expansion

$$\mathbb{E}D_{n,r} = \frac{(\tau S(\tau)-1)^r}{(\tau S(\tau))^{r+2}}n + O(1). \tag{11}$$

The O-constants depend implicitly on both r as well as on the residue class of n modulo p.

In a nutshell, the proof of this theorem involves a rigorous analysis of the generating functions $D(z,1)$ (for the overall number of dispersed excursions), as well as of $D_t(z,1) = \frac{1}{z}\frac{V_t(z)}{(1-V(z))^2}$ (for the expected number of ascents in these paths). Furthermore, while our results as stated in (10) and (11) only list the asymptotic main term, expansions with higher precision are available in the worksheet as well (they just become rather messy very quickly).

It can be shown that the only family of Łukasiewicz paths that is not covered by Theorem 10 is $\mathcal{S} = \{-1, 1\}$, the case of dispersed Dyck paths. However, as everything is explicitly given, the analysis is quite straightforward.

▶ **Proposition 11.** *Let d_n denote the total number of dispersed Dyck paths of length n, and let $D_{n,r}$ denote the random variable modeling the number of r-ascents in a random dispersed Dyck path of length n.*

Then, d_n is given by

$$d_n = \binom{n}{\lfloor n/2 \rfloor} = \sqrt{\frac{2}{\pi}} 2^n n^{-1/2} - \frac{2 - (-1)^n}{2\sqrt{2\pi}} 2^n n^{-3/2} + O(2^n n^{-5/2}), \tag{12}$$

and the expected number of r-ascents satisfies

$$\mathbb{E} D_{n,r} = \frac{n}{2^{r+2}} - \sqrt{\frac{\pi}{2}} \frac{r-2}{2^{r+2}} n^{1/2} + \frac{(r-1)(r-4)}{2^{r+3}} - \sqrt{\frac{\pi}{2}} \frac{(r-2)(2-(-1)^n)}{2^{r+4}} n^{-1/2} + O(n^{-1}). \tag{13}$$

This completes our analysis of r-ascents in dispersed Łukasiewicz excursions.

3.3 Analysis of Meanders

In this section we study ascents in meanders, i.e., non-negative Łukasiewicz paths without further restriction. The corresponding generating function can be obtained from (4) by setting $v = 1$, which allows arbitrary ending altitude of the path.

In accordance to the results from [1, Theorem 4], the behavior of meanders depends on the sign of the drift (i.e., the quantity $S'(1)$). The following theorem handles the case of positive drift (which, in our setting, is equivalent to $\tau \neq 1$).

Recall that $M_{n,r}$ is the random variable modeling the number of r-ascents in a random non-negative Łukasiewicz path of length n with respect to some given step set $\mathcal{S}$.

▶ **Theorem 12.** *Let $\tau > 0$ be the structural constant, i.e., the unique positive solution of $S'(\tau) = 0$, and assume that $\tau \neq 1$.*

Then, with $\xi = 1/S(1)$, the expected number of r-ascents in Łukasiewicz meanders of length n as well as the corresponding variance grow with $n \to \infty$ according to the asymptotic expansions

$$\mathbb{E} M_{n,r} = \mu n + \frac{(S(1)-1)^r (2S(1)-1-r)}{S(1)^{r+2}} + \frac{(S(1)-1)^r V_z(\xi)}{S(1)^{r+1}(1-V(\xi))} - \frac{V_t(\xi)}{1-V(\xi)} + O\Big(n^{5/2} \Big(\frac{S(\tau)}{S(1)}\Big)^n\Big), \tag{14}$$

and

$$\mathbb{V} M_{n,r} = \sigma^2 n + O(1), \tag{15}$$

where μ and σ^2 are given by

$$\mu = \frac{(S(1)-1)^r}{S(1)^{r+2}} \quad \text{and} \quad \sigma^2 = \frac{(S(1)-1)^r}{S(1)^{r+2}} + \frac{(S(1)-1)^{2r}(2S(1)-3-2r)}{S(1)^{2r+4}}.$$

Moreover, for $n \to \infty$, $M_{n,r}$ is asymptotically normally distributed. All O-constants depend implicitly on r.

It can be shown that Theorem 12 covers all step sets except for $\mathcal{S} = \{-1, 1\}$ and $\mathcal{S} = \{-1, 0, 1\}$. In these cases, we have a similar situation to what we had in Section 3.2: the square root singularity coming from $V(z)$ combines with the zero in the denominator.

The following propositions close this gap.

▶ **Proposition 13.** *The expected number of r-ascents in the Łukasiewicz meanders of length n associated to $\mathcal{S} = \{-1, 1\}$ as well as the corresponding variance grow with $n \to \infty$ according to the asymptotic expansions*

$$\mathbb{E}M_{n,r} = \frac{n}{2^{r+2}} + \frac{\sqrt{2\pi}(r-2)}{2^{r+3}}n^{1/2} - \frac{r^2 - r - 8}{2^{r+3}} + \frac{\sqrt{2\pi}((2-(-1)^n)(r-2)}{2^{r+5}}n^{-1/2} + O(n^{-1}), \quad (16)$$

and

$$\mathbb{V}M_{n,r} = \frac{2^{r+3} - r^2(\pi-2) + 4r(\pi-3) - 4\pi + 10}{2^{2r+5}}n + O(n^{1/2}). \quad (17)$$

▶ **Proposition 14.** *The expected number of r-ascents in the Łukasiewicz meanders of length n associated to $\mathcal{S} = \{-1, 0, 1\}$ as well as the corresponding variance grow with $n \to \infty$ according to the asymptotic expansions*

$$\mathbb{E}M_{n,r} = \frac{2^r}{3^{r+2}}n + \frac{\sqrt{3\pi}(r-4)2^{r-2}}{3^{r+2}}n^{1/2} - (3r^2 - r - 96)\frac{2^{r-4}}{3^{r+2}} + \frac{\sqrt{3\pi}(r-4)2^{r-6}}{3^r}n^{-1/2} + O(n^{-1}) \quad (18)$$

and

$$\mathbb{V}M_{n,r} = \frac{3^{r+2}2^{r+4} - 2^{2r}(3r^2(\pi-2) - 8r(3\pi-10) + 48\pi - 144)}{16 \cdot 3^{2r+4}}n + O(n^{1/2}). \quad (19)$$

References

1 Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.*, 281(1–2):37–80, 2002. doi:10.1016/S0304-3975(02)00007-5.

2 David Bevan. Permutations avoiding 1324 and patterns in Łukasiewicz paths. *J. Lond. Math. Soc. (2)*, 92(1):105–122, 2015. doi:10.1112/jlms/jdv020.

3 Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009. doi:10.1017/CBO9780511801655.

4 Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics. A foundation for computer science*. Addison-Wesley, second edition, 1994.

5 Benjamin Hackl, Clemens Heuberger, and Helmut Prodinger. Ascents in non-negative lattice paths. arXiv:1801.02996 [math.CO], 2018. URL: https://arxiv.org/abs/1801.02996.

6 Kairi Kangro, Mozhgan Pourmoradnasseri, and Dirk Oliver Theis. Short note on the number of 1-ascents in dispersed Dyck paths. arXiv:1603.01422 [math.CO], 2016. URL: https://arxiv.org/abs/1603.01422.

7 M. Lothaire. *Combinatorics on words*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1997. With a foreword by Roger Lyndon and a preface by Dominique Perrin, Corrected reprint of the 1983 original, with a new preface by Perrin. doi:10.1017/CBO9780511566097.

8 Amram Meir and John W. Moon. On an asymptotic method in enumeration. *J. Combin. Theory Ser. A*, 51(1):77–89, 1989. doi:10.1016/0097-3165(89)90078-2.

9 Helmut Prodinger. Returns, hills, and t-ary trees. *J. Integer Seq.*, 19(7):Article 16.7.2, 8, 2016. URL: http://emis.ams.org/journals/JIS/VOL19/Prodinger/prod41.html.

10 The SageMath Developers. *SageMath Mathematics Software (Version 8.1)*, 2017. URL: http://www.sagemath.org.

A Singularity Analysis of Inverse Functions

The aim of this appendix is, on the one hand, to state and prove an extension of [3, Remark VI.17]. In fact, we simply confirm what is announced in the footnote in [3, p. 405] and give more details. Then, we use these results in order to derive relevant information on the generating function $V(z,t)$ from before.

For the following two propositions, we borrow the notation used in [3, Chapter VI.7].

▶ **Proposition 15.** *Let $\phi(u)$ be analytic with radius of convergence $0 < R \le \infty$, $\phi(0) \neq 0$, $[u^n]\,\phi(u) \ge 0$ for all $n \ge 0$ and $\phi(u)$ not affine linear. Assume that there is a positive $\tau \in (0,R)$ such that $\tau\,\phi'(\tau) = \phi(\tau)$. Finally assume that $\phi(u)$ is a p-periodic power series for some maximal p. Denote the set of all pth roots of unity by $G(p)$.*

Then there is a unique function $y(z)$ satisfying $y(z) = z\,\phi(y(z))$ which is analytic in a neighborhood of 0 with $y(0) = 0$. It has radius of convergence $\rho = \tau/\,\phi(\tau)$ around the origin. For $|z| \le \rho$, it has exactly singularities at $z = \rho\zeta$ for $\zeta \in G(p)$. For $z \to \rho$, we have the singular expansion

$$y(z) \overset{z\to\rho}{\sim} \sum_{j\ge 0} (-1)^j d_j \left(1 - \frac{z}{\rho}\right)^{j/2}$$

for some computable constants d_j, $j \ge 0$. We have $d_0 = \tau$ and $d_1 = \sqrt{2\,\phi(\tau)/\,\phi''(\tau)}$. Additionally, we have $[z^n]y(z) = 0$ for $n \not\equiv 1 \pmod p$.

Proof. Existence, uniqueness, radius of convergence as well as singular expansion around $z \to \rho$ of $y(z)$ are shown in [3, Theorem VI.6].

As ϕ is a p-periodic power series and $\phi(0) \neq 0$, there exists an aperiodic function χ such that $\phi(u) = \chi(u^p)$. From the non-negativity of the coefficients of $\phi(u)$, it is clear that $\chi(u)$ has non-negative coefficients and is analytic for $|u| < R^p$. We consider $\psi(u) := \chi(u)^p$. Then ψ is again analytic for $|u| < R^p$, it has clearly non-negative coefficients, $\psi(0) \neq 0$ and $\psi(u)$ is not an affine linear function. If $[u^m]\chi(u) > 0$ and $[u^n]\chi(u) > 0$ for some $m < n$, then $[u^{pm}]\psi(u) > 0$ as well as $[u^{pm+(n-m)}]\psi(u) > 0$, which implies that ψ is aperiodic.

Finally, we have

$$\tau^p \psi'(\tau^p) = p\tau^p \chi(\tau^p)^{p-1}\chi'(\tau^p) = \tau\,\phi(\tau)^{p-1}\,\phi'(\tau) = \phi(\tau)^p = \chi(\tau^p)^p = \psi(\tau^p).$$

Considering the functional equation $Y(Z) = Z\psi(Y(Z))$, we see that all assumptions of [3, Theorem VI.6] are satisfied; thus it has a unique solution $Y(Z)$ with $Y(0) = 0$ which is analytic around the origin. By the same result, $Y(Z)$ has radius of convergence

$$\frac{\tau^p}{\psi(\tau^p)} = \frac{\tau^p}{\chi(\tau^p)^p} = \left(\frac{\tau}{\phi(\tau)}\right)^p = \rho^p$$

and, as ψ is aperiodic, the only singularity of $Y(Z)$ with $|Z| \le \rho^p$ is $Z = \rho^p$.

We consider the function $\widetilde{y}(z) := z\chi(Y(z^p))$. By definition, it is analytic for $|z| < \rho$ and its only singularities with $|z| \le \rho$ are those z with $z^p = \rho^p$, i.e., $z = \rho\zeta$ for $\zeta \in G(p)$. It is also clear by definition that $[z^n]\widetilde{y}(z) = 0$ for $n \not\equiv 1 \pmod p$. We have $\widetilde{y}(0) = 0$ and

$$z\,\phi(\widetilde{y}(z)) = z\chi((\widetilde{y}(z))^p) = z\chi(z^p\chi(Y(z^p))^p) = z\chi(z^p\psi(Y(z^p))) = z\chi(Y(z^p)) = \widetilde{y}(z)$$

because $z^p\psi(Y(z^p)) = Y(z^p)$ by definition of Y. This implies that $y = \widetilde{y}$. ◀

While the following proposition is particularly useful in the context of the previous one, it also holds in a slightly more general setting. It gives a detailed description of the singular expansions for p-periodic power series like above.

▶ **Proposition 16.** *Let p be a positive integer and let y be analytic with radius of convergence $0 < \rho \le \infty$, where $[z^n]y(z) = 0$ for $n \not\equiv 1 \pmod p$. Assume that $y(z)$ has p dominant singularites located at $\zeta\rho$ for $\zeta \in G(p)$, and that for some $L \ge 0$ and $z \to \rho$, we have the singular expansion*

$$y(z) \overset{z\to\rho}{=} \sum_{j=0}^{L-1} d_j \Big(1 - \frac{z}{\rho}\Big)^{-\alpha_j} + O\Big(\Big(1 - \frac{z}{\rho}\Big)^{-\alpha_L}\Big),$$

where α_0, α_1, …, α_L are complex numbers such that $\operatorname{Re}(\alpha_j) \ge \operatorname{Re}(\alpha_{j+1})$ for all $0 \le j < L$.

Then, for $\zeta \in G(p)$, the singular expansion of $y(z)$ for $z \to \zeta\rho$ is given by

$$y(z) \overset{z\to\zeta\rho}{=} \sum_{j=0}^{L-1} \zeta d_j \Big(1 - \frac{z}{\zeta\rho}\Big)^{-\alpha_j} + O\Big(\Big(1 - \frac{z}{\zeta\rho}\Big)^{-\alpha_L}\Big),$$

i.e., the expansion for $z \to \zeta\rho$ can be obtained by multiplying the expansion for $z \to \rho$ with ζ and substituting $z \mapsto \zeta/\rho$. Finally, for the coefficients of $y(z)$ we find

$$[z^n]y(z) = [\![p \mid 1-n]\!] [z^n]\Big(p \sum_{j=0}^{L-1} d_j \Big(1 - \frac{z}{\rho}\Big)^{-\alpha_j} + O\Big(\Big(1 - \frac{z}{\rho}\Big)^{-\alpha_L}\Big)\Big), \tag{20}$$

which can be made explicit easily by means of singularity analysis (cf. [3, Chapter VI.4]). In particular,

$$[z^n]y(z) = [\![p \mid 1-n]\!] \Big(\sum_{j=0}^{L-1} \frac{p d_j}{\Gamma(\alpha_j)} n^{\alpha_j - 1}\rho^{-n} + O\big(n^{\operatorname{Re}(\alpha_0)-2}\rho^{-n}\big) + O\big(n^{\operatorname{Re}(\alpha_L)-1}\rho^{-n}\big)\Big).$$

Proof. As $[z^n]y(z) = 0$ for $n \not\equiv 1 \pmod p$ there is a function χ, analytic around the origin, such that $y(z) = z\chi(z^p)$. Thus, for every $\zeta \in G(p)$, we have

$$y(\zeta z) = \zeta z \chi((\zeta z)^p) = \zeta z \chi(z^p) = \zeta y(z)$$

or, equivalently,

$$y(z) = \zeta y\Big(\frac{z}{\zeta}\Big).$$

Thus the singular expansion for $z \to \zeta\rho$ follows from that for $z \to \rho$ by replacing z with z/ζ and multiplication by ζ.

With the singular expansions at all the dominant singularities located at $\zeta\rho$ for $\zeta \in G(p)$ at hand, we are able to extract the overall growth of the coefficients of $y(z)$ by first applying singularity analysis to every expansion separately, and then summing up all these contributions. When doing so, we use the well-known property of roots of unity that

$$\sum_{\zeta \in G(p)} \zeta^m = p[\![p \mid m]\!] \tag{21}$$

for $m \in \mathbb{Z}$ in order to rewrite the occurring sums as $\sum_{\zeta\in G(p)} \zeta^{1-n} = p[\![p \mid 1-n]\!]$. Comparing the resulting asymptotic expansion with (20) proves the statement. ◀