How to Navigate Through Obstacles?

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Abstract

Given a set of obstacles and two points in the plane, is there a path between the two points that does not cross more than k different obstacles? This is a fundamental problem that has undergone a tremendous amount of work by researchers in various areas, including computational geometry, graph theory, wireless computing, and motion planning. It is known to be NP-hard, even when the obstacles are very simple geometric shapes (*e.g.*, unit-length line segments). The problem can be formulated and generalized into the following graph problem: Given a planar graph G whose vertices are colored by color sets, two designated vertices $s, t \in V(G)$, and $k \in \mathbb{N}$, is there an s-t path in G that uses at most k colors? If each obstacle is connected, the resulting graph satisfies the color-connectivity property, namely that each color induces a connected subgraph.

We study the complexity and design algorithms for the above graph problem with an eye on its geometric applications. We prove a set of hardness results, among which a result showing that the color-connectivity property is crucial for any hope for fixed-parameter tractable (FPT) algorithms, as without it, the problem is W[SAT]-hard parameterized by k. Previous results only implied that the problem is W[2]-hard. A corollary of this result is that, unless W[2] = FPT, the problem cannot be approximated in FPT time to within a factor that is a function of k. By describing a generic plane embedding of the graph instances, we show that our hardness results translate to the geometric instances of the problem.

We then focus on graphs satisfying the color-connectivity property. By exploiting the planarity of the graph and the connectivity of the colors, we develop topological results that allow us to prove that, for any vertex v, there exists a set of paths whose cardinality is upper bounded by a function of k, that "represents" the valid s-t paths containing subsets of colors from v. We employ these structural results to design an FPT algorithm for the problem parameterized by both k and the treewidth of the graph, and extend this result further to obtain an FPT algorithm for the parameterization by both k and the length of the path. The latter result generalizes and explains previous FPT results for various obstacle shapes, such as unit disks and fat regions.

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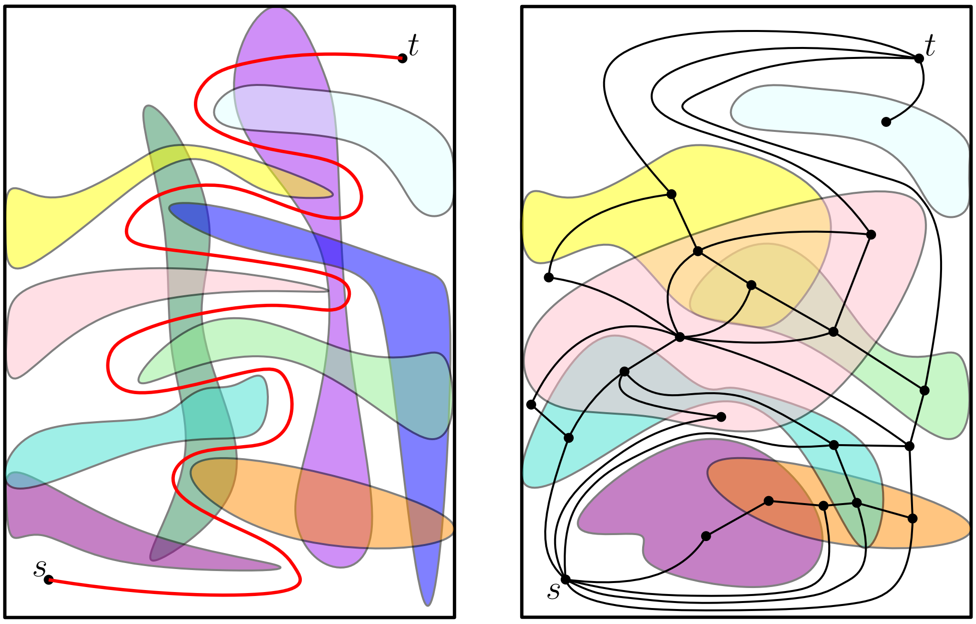

Figure 1 Illustration of instances of the problem under consideration drawn within a bounding box. The left figure shows an instance in which the optimal path crosses two obstacles, zigzagging between the other obstacles. The right figure shows an instance and its auxiliary plane graph.

1 Introduction

We consider the following problem: Given a set of obstacles and two designated points in the plane, is there a path between the two points that does not cross more than k obstacles? Equivalently, can we remove k obstacles so that there is an obstacle-free path between the two designated points? We refer to this problem as Obstacle Removal, and to its restriction to instances in which each obstacle is connected as Connected Obstacle Removal.

By considering the auxiliary plane graph that is the dual of the plane subdivision determined by the obstacles, Obstacle Removal was formulated and generalized into the following graph problem, referred to as Colored Path (see Figure 1 for illustrations):

Colored Path
Given: A planar graph G; a set of colors C; $\chi : V \longrightarrow 2^C$; two designated vertices $s, t \in V(G)$; and $k \in \mathbb{N}$
Question: Does there exist an s-t path in G that uses at most k colors?

Denote by Colored Path-Con the restriction of Colored Path to instances in which each color induces a connected subgraph of G.

As we discuss next, Connected Obstacle Removal and Colored Path are fundamental problems that have undergone a tremendous amount of work, albeit under different names and contexts, by researchers in various areas, including computational geometry, graph theory, wireless computing, and motion planning.

1.1 Related Work

In motion planning, the goal is generally to move a robot from a starting position to a final position, while avoiding collision with a set of obstacles. This is usually referred to as the *piano-mover*'s problem. Obstacle Removal is a variant of the piano-mover's problem, in which the obstacles are in the plane and the robot is represented as a point. Since determining if there is an obstacle-free path for the robot in this case is solvable in polynomial time, if no such path exists, it is natural to seek a path that intersects as few obstacles as possible. Motivated by planning applications, CONNECTED OBSTACLE REMOVAL and COLORED PATH were studied under the name MINIMUM CONSTRAINT REMOVAL [7, 9, 10, 11]. CONNECTED OBSTACLE REMOVAL has also been studied extensively, motivated by applications in wireless computing, under the name BARRIER COVERAGE or BARRIER RESILIENCE [1, 2, 12, 13, 15, 16]. In such applications, we are given a field covered by sensors (usually simple shapes such as unit disks), and the goal is to compute a minimum set of sensors that need to fail before an entity can move undetected between two given sites.

Kumar *et al.* [13] were the first to study CONNECTED OBSTACLE REMOVAL. They showed that for unit-disk obstacles in some restricted setting, the problem can be solved in polynomial time. The complexity of the general case for unit-disk obstacles remains open. Several works showed the NP-hardness of the problem, even when the obstacles are very simple geometric shapes such as line segments (*e.g.*, see [1, 15, 16]). The complexity of the problem when each obstacle intersects a constant number of other obstacles is open [9, 11].

Bereg and Kirkpatrick [2] designed approximation algorithms when the obstacles are unit disks by showing that the length, referred to as the *thickness* [2] (*i.e.*, number of regions visited), of a shortest path that crosses k disks is at most $3k$; this follows from the fact that a shortest path does not cross a disk more than a constant number of times.

Korman *et al.* [12] showed that CONNECTED OBSTACLE REMOVAL is FPT parameterized by k for unit-disk obstacles, and extended this result to similar-size fat-region obstacles with a constant *overlapping number*, which is the maximum number of obstacles having nonempty intersection. Their result draws the observation, which was also used in [2], that for unit-disk (and fat-region) obstacles, the length of an optimal path can be upper bounded by a linear function of the number of obstacles crossed (*i.e.*, the parameter). This observation was then exploited by a branching phase that decomposes the path into subpaths in (simpler) restricted regions, enabling a similar approach to that of Kumar *et al.* [13].

Motivated by its applications to networking, among other areas, the problem of computing a minimum-colored path in a graph received considerable attention (*e.g.*, see [3, 17]). The problem was shown to be NP-hard in several works [3, 4, 11, 17]. Most of the NP-hardness reductions start from SET COVER, and result in instances of COLORED PATH (*i.e.*, planar graphs), as was also observed by [2]. These reductions are FPT-reductions, implying the W[2]-hardness of COLORED PATH. Moreover, these reductions imply that, unless P = NP, the minimization version of COLORED PATH cannot be approximated to within a factor of $c \lg n$, for any constant $c < 1$. Hauser [11], and Gorbenko and Popov [10], implemented exact and heuristic algorithms for the problem on general graphs. Very recently, Eiben *et al.* [7] designed exact and heuristic algorithms for COLORED PATH and OBSTACLE REMOVAL, and proved computational lower bounds on their subexponential-time complexity, assuming the Exponential Time Hypothesis.

1.2 Our Results and Techniques

We study the complexity and parameterized complexity of COLORED PATH and COLORED PATH-CON, eyeing the implications on their geometric counterparts OBSTACLE REMOVAL and CONNECTED OBSTACLE REMOVAL, respectively. The proofs of the hardness results we obtain are too long and technical to be included, and for those we refer to the full paper [8].

Clearly, COLORED PATH is in the parameterized class XP. We show that the color-connectivity property is crucial for any hope for an FPT-algorithm, since even very restricted instances and combined parameterizations of COLORED PATH are W[1]-complete:

▶ **Theorem 1.1.** Colored Path, *restricted to instances of pathwidth at most 4, and in which each vertex contains at most one color and each color appears on at most 2 vertices, is* W[1]-complete parameterized by k.

▶ **Theorem 1.2.** Colored Path, *parameterized by both k and the length of the sought path ℓ, is* W[1]-complete.

Without restrictions, the problem sits high in the parameterized complexity hierarchy:

▶ **Theorem 1.3.** Colored Path, *parameterized by k, is* W*[SAT]-hard and is in* W[P].

A corollary of Theorem 1.3 is that, unless W[2] = FPT, Colored Path cannot be approximated in FPT time to within a factor that is a function of k.

We can produce a generic construction [8] that can be used to realize any graph instance of Colored Path as a geometric instance of Obstacle Removal. Using this generic construction, the hardness results in Theorems 1.1–1.3, and the inapproximability result discussed above, translate to Obstacle Removal. Previously, Colored Path was only known to be W[2]-hard, via the standard reduction from Set Cover [3, 11, 17]. Our results refine the parameterized complexity and approximability of Colored Path and Obstacle Removal.

As it turns out, the color-connectivity property without planarity is hopeless: We can tradeoff planarity for color-connectivity by adding a single vertex that serves as a color-connector, thus establishing the W[SAT]-hardness of the problem on apex graphs.

The above hardness results show that we can focus our attention on Colored Path-Con. We show the following algorithmic result:

▶ **Theorem 1.4** (Theorem 4.7). Colored Path-Con, *parameterized by both k and the treewidth ω of the input graph, is* FPT.

We remark that bounding the treewidth does not make Colored Path-Con much easier, as we can show that Colored Path-Con is NP-hard even for 2-outerplanar graphs of pathwidth at most 3 [8].

The folklore dynamic programming approach based on tree decomposition, used for the Hamiltonian Path/Cycle problems, does not work for Colored Path-Con to prove the result in Theorem 1.4 for the following reasons. As opposed to the Hamiltonian Path/Cycle problems, where it is sufficient to keep track of how the path/cycle interacts with each bag in the tree decomposition, this is not sufficient in the case of Colored Path-Con because we also need to keep track of which color sets are used on both sides of the bag. Although (by color connectivity) any subset of colors appearing on both sides of a bag must appear on vertices in the bag as well, there can be too many such subsets (up to $|C|^k$, where C is the set of colors), and certainly we cannot afford to enumerate all of them if we seek an FPT algorithm. To overcome this issue, we develop in Section 3 topological structural results that exploit the planarity of the graph and the connectivity of the colors to show the following. For any vertex $w \in V(G)$, and for any pair of vertices $u, v \in V(G)$, the set of (valid) u-v paths in $G - w$ that use colors appearing on vertices in the face of $G - w$ containing w can be "represented" by a minimal set of paths whose cardinality is a function of k.

In Section 4, we extend the notion of a minimal set of paths w.r.t. a single vertex to a "representative set" of paths w.r.t. a specific bag, and a specific enumerated configuration for the bag, in a tree decomposition of the graph. This enables us to use the upper bound on the size of a minimal set of paths, derived in Section 3, to upper bound the size of a representative set of paths w.r.t. a bag and a configuration. This, in turn, yields an upper bound on the size of the table stored at a bag, in the dynamic programming algorithm, by a function of both k and the treewidth of the graph, thus yielding the desired result.

In Section 5, we extend the FPT result for COLORED PATH-CON in Theorem 1.4 w.r.t. the parameters k and ω, to the parameterization by both k and the length ℓ of the path:

▶ **Theorem 1.5** (Theorem 5.1). COLORED PATH-CON, *and hence* CONNECTED OBSTACLE REMOVAL, *parameterized by both k and ℓ is* FPT.

The dependency on both ℓ and k is *essential* for the result in Theorem 1.5, as we can show that if we parameterize only by k, or only by ℓ, then the problem becomes W[1]-hard [8].

The result in Theorem 1.5 generalizes and explains Korman *et al.*'s results [12] showing that CONNECTED OBSTACLE REMOVAL is FPT parameterized by k for unit-disk obstacles, which they also generalized to similar-size fat-region obstacles with bounded overlapping number. Their results exploit the obstacle shape to upper bound the length of the path by a linear function of k, and then use branching to reduce the problems to a simpler setting. Our result directly implies that, regardless of the (connected) obstacle shapes, as long as the path length is upper bounded by some function of k (Corollary 5.2), the problem is FPT. The FPT result in Theorem 1.5 also implies that:

▶ **Corollary 1.6** (Corollary 5.3). *For any computable function h,* COLORED PATH-CON *restricted to instances in which each color appears on at most $h(k)$ vertices, is* FPT *parameterized by k.*

The result in Corollary 1.6 has applications to CONNECTED OBSTACLE REMOVAL, in particular, to the interesting case when the obstacles are convex polygons, each intersecting a constant number of other polygons. The question about the complexity of this problem was posed in [9, 11], and remains open. The result in Corollary 1.6 implies that this problem is FPT.

We finally mention that it remains open whether COLORED PATH-CON and CONNECTED OBSTACLE REMOVAL are FPT parameterized by k only.

2 Preliminaries

We assume familiarity with graph theory and parameterized complexity. We refer the reader to the standard books [5, 6].

For a set S, we denote by 2^S the power set of S. Let $G = (V, E)$ be a graph, let $C \subset \mathbb{N}$ be a finite set of colors, and let $\chi : V \longrightarrow 2^C$. A vertex v in V is *empty* if $\chi(v) = \emptyset$. A color c *appears on*, or is *contained in*, a subset S of vertices if $c \in \bigcup_{v \in S} \chi(v)$. For two vertices $u, v \in V(G)$, $\ell \in \mathbb{N}$, a u-v path $P = (u = v_0, \dots, v_r = v)$ in G is *ℓ-valid* if $|\bigcup_{i=0}^{r} \chi(v_i)| \leq \ell$; that is, if the total number of colors appearing on the vertices of P is at most ℓ. A color $c \in C$ is *connected* in G, or simply *connected*, if $\bigcup_{c \in \chi(v)} \{v\}$ induces a connected subgraph of G. The graph G is *color-connected*, if for every $c \in C$, c is connected in G.

For an instance (G, C, χ, s, t, k) of COLORED PATH or COLORED PATH-CON, if s and t are nonempty vertices, we can remove their colors and decrement k by $|\chi(s) \cup \chi(t)|$ because their colors appear on every s-t path. If afterwards k becomes negative, then there is no k-valid s-t path in G. Moreover, if s and t are adjacent, then the path (s, t) is a path with the minimum number of colors among all s-t paths in G. Therefore, we will assume:

▶ **Assumption 2.1.** *For an instance (G, C, χ, s, t, k) of* COLORED PATH *or* COLORED PATH-CON*, we can assume that s and t are nonadjacent empty vertices.*

3 Structural Results

Let G be a color-connected plane graph, C a set of colors, and $\chi : V \longrightarrow 2^C$. In this section, we present structural results that are the cornerstone of the FPT-algorithm for Colored Path-Con presented in the next section. We start by giving an intuitive description of the plan for this section.

As mentioned in Section 1, the main issue facing a dynamic programming algorithm based on tree decomposition, is how to upper bound, by a function of k and the treewidth, the number of k-valid paths between (any) two vertices u and v that use color sets contained in a certain bag. As it turns out, this number cannot be upper bounded as desired. Instead, we "represent" those paths using a minimal set $\mathcal{P}$ of k-valid u-v paths, in the sense that any k-valid u-v path can be replaced by a path from $\mathcal{P}$ that is not "worse" than it. To do so, it suffices to represent the k-valid u-v paths that use color sets contained in a third vertex w, by a set whose cardinality is a function of k. This will enable us to extend the notion of a minimal set of k-valid u-v paths w.r.t. a single vertex to a representative set for the whole bag, which is the key ingredient of the dynamic programming FPT-algorithm – based on tree decomposition – in the next section.

As it turns out, the paths that matter are those that use "external" colors w.r.t. w (defined below), since those colors have the potential of appearing on both sides of a bag containing w. Therefore, the ultimate goal of this section is to define a notion of a minimal set $\mathcal{P}$ of k-valid u-v paths with respect to w (Definition 3.5), and to upper bound $|\mathcal{P}|$ by a function of k. Upper bounding $|\mathcal{P}|$ by a function of k turns out to be quite challenging, and requires ideas and topological results that will be discussed later in this section.

Throughout this section, we shall assume that G is color-connected. We start with the following simple observation that holds because of this assumption:

▶ **Observation 3.1.** *Let $x, y \in V(G)$ be such that there exists a color $c \in \mathcal{C}$ that appears on both x and y. Then any x-y vertex-separator in G contains a vertex on which c appears.*

Let G' be a plane graph, let $w \in V(G')$, and let f be the face in $G' - w$ such that w is interior to f; we call f the *external face* w.r.t. w in $G' - w$, and the vertices incident to f *external vertices* w.r.t. w in $G' - w$. A color $c \in C$ is an *external color* w.r.t. w in $G' - w$, or simply *external* to w in $G' - w$, if c appears on an external vertex w.r.t. w in $G' - w$; otherwise, c is *internal* to w in $G' - w$. The following observation is easy to see:

▶ **Observation 3.2.** *Let G be a color-connected graph, and let $w \in V(G)$. Let H be any subgraph of $G - w$. If c is an external color to w in $G - w$ and c appears on some vertex in H, then c is an external color to w in H. This also implies that the set of internal colors to w in H is a subset of the set of internal colors to w in $G - w$.*

▶ **Definition 3.3.** Let s, t be two designated vertices in G, and let x, y be two adjacent vertices in G such that $\chi(x) = \chi(y)$. Define the following operation to x and y, referred to as a *color contraction* operation, that results in a graph G', a color function χ', and two designated vertices s', t' in G', obtained as follows:

- G' is obtained from G by contracting the edge xy, which results in a new vertex z;
- $s' = s$ (resp. $t' = t$) if $s \notin \{x, y\}$ (resp. $t \notin \{x, y\}$), and $s' = z$ (resp. $t' = z$) otherwise; and
- $\chi' : V(G') \longrightarrow 2^C$ is defined as $\chi'(w) = \chi(w)$ if $w \neq z$, and $\chi'(z) = \chi(x) = \chi(y)$.

G is *irreducible* if there does not exist two vertices in G to which the color contraction operation is applicable.

▶ **Lemma 3.4.** *Let G be a color-connected plane graph, C a color set, $\chi : V \longrightarrow 2^C$, $s, t \in V(G)$, and $k \in \mathbb{N}$. Suppose that the color contraction operation is applied to two vertices in G to obtain G', χ', s', t', as described in Definition 3.3. Then G' is a color-connected plane graph, and there is a k-valid s-t path in G if and only if there is a k-valid s'-t' path in G'.*

▶ **Definition 3.5.** Let $u, v, w \in V(G)$. A set $\mathcal{P}$ of k-valid u-v paths in $G - w$ is said to be *minimal* w.r.t. w if:

(i) There does not exist two paths $P_1, P_2 \in \mathcal{P}$ such that $\chi(P_1) \cap \chi(w) = \chi(P_2) \cap \chi(w)$;
(ii) there does not exist two paths $P_1, P_2 \in \mathcal{P}$ such that $\chi(P_1) \subseteq \chi(P_2)$; and
(iii) for any $P \in \mathcal{P}$, there does not exist a u-v path P' in $G - w$ such that $\chi(P') \subsetneq \chi(P)$.

Clearly, for any $u, v, w \in V(G)$, a minimal set of k-valid u-v paths in $G - w$ exists.

▶ **Observation 3.6.** *Let $u, v, w \in V(G)$. Any set of u-v paths that is minimal w.r.t. w contains at most one path whose vertices contain* only *internal colors w.r.t. w in $G - w$.*

To derive an upper bound on the cardinality of a minimal set $\mathcal{P}$ of k-valid u-v paths w.r.t. a vertex w, we select a maximal set $\mathcal{M}$ of color-disjoint paths in $\mathcal{P}$. We first upper bound $|\mathcal{M}|$ by a function of k, which requires developing several results of topological nature. The key ingredient for upper bounding $|\mathcal{M}|$ is showing that the subgraph induced by the paths in $\mathcal{M}$ has a u-v vertex-separator of cardinality $O(k)$ (Lemma 3.10). We then upper bound $|\mathcal{M}|$ (Lemma 3.12) by upper bounding the number of different traces of the paths of $\mathcal{M}$ on this small separator, and inducting on both sides of the separator. Finally, we show (Theorem 3.13) that $|\mathcal{P}|$ is upper bounded by a function of $|\mathcal{M}|$, which proves the desired upper bound on $|\mathcal{P}|$. We proceed to the details.

For the rest of this section, let $u, v, w \in V(G)$, and let $\mathcal{P}$ be a set of minimal k-valid u-v paths in $G - w$. Let $\mathcal{M}$ be a set of minimal k-valid color-disjoint u-v paths in $G - w$, and let M be the subgraph of $G - w$ induced by the edges of the paths in $\mathcal{M}$.

▶ **Observation 3.7.** *If $P \in \mathcal{M}$ contains a color c that is external to w in M, then c appears on a vertex in P that is incident to the external face to w in M.*

▶ **Lemma 3.8.** *Let G' be a plane graph with a face f, let $u, v \in V(G')$, and let $u_1, \ldots, u_r$, $r \geq 3$, be the neighbors of u. Suppose that, for each $i \in [r]$, there exists a u-v path P_i containing u_i and a vertex incident to f different from v, and such that P_i does not contain any u_j, $j \in [r]$, $j \neq i$. Then there exist two paths P_i, P_j, $i, j \in [r]$, $i \neq j$, such that $V(P_i) \cup V(P_j) - \{v\}$ is a vertex-separator separating $\{u_1, \ldots, u_r\} \setminus \{u_i, u_j\}$ from v.*

▶ **Lemma 3.9.** *Let x, y be two vertices in an irreducible subgraph G' of G, and let f be a face in G'. Then there are at most two color-disjoint x-y paths in G' that contain only colors that appear on f.*

▶ **Lemma 3.10.** *Suppose that M is irreducible, then there exist paths $P_1, P_2, P_3 \in \mathcal{M}$ such that $M - P_1 - P_2 - P_3$ has a u-v vertex-separator of cardinality at most $2k + 3$.*

Proof. By Observation 3.6 and Observation 3.2, $\mathcal{M}$ contains at most one path that contains only internal colors w.r.t. w in M. Therefore, it suffices to show that $\mathcal{M}$ contains two paths P_1, P_2 such that $M - P_1 - P_2$ has a u-v vertex-separator of cardinality at most $2k + 3$, assuming that every path in $\mathcal{M}$ contains an external color w.r.t. w in M.

By Observation 3.7, every path in $\mathcal{M}$ passes through an external vertex w.r.t. w in M that contains a color external to w in M. Because the paths in $\mathcal{M}$ are pairwise color-disjoint and u and v are empty vertices, every path in $\mathcal{M}$ passes through a vertex on the external face of w in M that is different from u and v. Let $u_1, \ldots, u_q$ be the neighbors of u in M, and note that since u is empty and M is irreducible, each u_i, $i \in [q]$, contains a color. Let $P_1, \ldots, P_q$ be the paths in $\mathcal{M}$ containing $u_1, \ldots, u_q$, respectively, and note that since the paths in $\mathcal{M}$ are color-disjoint, no P_i passes through u_j, for $j \neq i$. By Lemma 3.8, there are two paths in $P_1, \ldots, P_q$, say P_1, P_2 without loss of generality, such that $V_{12} = V(P_1) \cup V(P_2) - \{v\}$ is a vertex-separator that separates $\{u_3, \ldots, u_q\}$ from v.

We proceed by contradiction and assume that $M^- = M - P_1 - P_2$ does not have a u-v vertex-separator of cardinality $2k+3$. By Menger's theorem [5], there exists a set $\mathcal{D}$ of $r' \geq 2k+3$ vertex-disjoint u-v paths in M^-. Since V_{12} separates $\{u_3, \ldots, u_q\}$ from v in M, every u-v path in M^- intersects at least one of P_1, P_2 at a vertex other than v. It follows that there exists a path in $\{P_1, P_2\}$, say P_1, that intersects at least $k+2$ paths in $\mathcal{D}$ at vertices other than v. Since the paths in $\mathcal{D}$ are vertex-disjoint and incident to u, we can order the paths in $\mathcal{D}$ that intersect P_1 around u (in counterclockwise order) as $\langle Q_1, \ldots, Q_r \rangle$, where $r \geq k+2$, and Q_{i+1} is counterclockwise from Q_i, for $i \in [r-1]$. P_1 intersects each path Q_i, $i \in [r]$, possibly multiple times. Moreover, since the paths in $\mathcal{M}$ are pairwise color-disjoint, each intersection between P_1 and a path Q_i, $i \in [r]$, must occur at an empty vertex. We choose $r-1$ subpaths, $P_1^1, \ldots, P_1^{r-1}$, of P_1 satisfying the property that the endpoints of P_1^i are on Q_i and Q_{i+1}, for $i = 1, \ldots, r-1$, and the endpoints of P_1^i are the only vertices on P_1^i that appear on a path Q_j, for $j \in [r]$. It is easy to verify that the subpaths $P_1^1, \ldots, P_1^{r-1}$ of P_1 can be formed by following the intersection of P_1 with the sequence of (ordered) paths $Q_1, \ldots, Q_r$.

Recall that the endpoints of $P_1^1, \ldots, P_1^{r-1}$ are empty vertices. Since M is irreducible, no two empty vertices are adjacent, and hence, each subpath P_1^i must contain an internal vertex v_i that contains at least one color. We claim that no two vertices v_i, v_j, $1 \leq i < j \leq r-1$, contain the same color. Suppose not, and let v_i, v_j, $i < j$, be two vertices containing a color c. Since v_i, v_j are internal to P_1^i and P_1^j, respectively, $Q_1, \ldots, Q_r$ are vertex-disjoint u-v paths, and by the choice of the subpaths $P_1^1, \ldots, P_1^{r-1}$, the paths Q_i and Q_{i+1} form a Jordan curve, and hence a vertex-separator in G, separating v_i from v_j.

By Observation 3.1, color c must appear on a vertex in Q_p, $p \in \{i, i+1\}$, and this vertex is clearly not in P_1 since P_1 intersects Q_p at empty vertices. Since every vertex in M appears on a path in $\mathcal{M}$, and c appears on $P_1 \in \mathcal{M}$ and on a vertex not in P_1, this contradicts that the paths in $\mathcal{M}$ are pairwise color-disjoint, and proves the claim.

Since no two vertices v_i, v_j, $1 \leq i < j \leq r$, contain the same color, the number $r-1$ of subpaths $P_1^1, \ldots, P_1^{r-1}$ is upper bounded by the number of distinct colors that appear on P_1, which is at most k. It follows that r is at most $k+1$, contradicting our assumption above and proving the lemma. ◀

▶ **Lemma 3.11.** *Let S be a minimal u-v vertex-separator in M. Let M_u, M_v be a partition of $M - S$ containing u and v, respectively, and such that there is no edge between M_u and M_v. For any vertex $x \in S$, M_u is contained in a single face of $M_v + x$.*

▶ **Lemma 3.12.** *$|\mathcal{M}| \leq g(k)$, where $g(k) = \mathcal{O}(c^k k^{2k})$, for some constant $c > 1$.*

Proof (sketch). By Observation 3.6, there is at most one path in $\mathcal{M}$ that contains only internal colors w.r.t. w in $G - w$. Therefore, it suffices to upper bound the number of paths in $\mathcal{M}$ that contain at least one external color to w in $G - w$. By Observation 3.2, every such path in $\mathcal{M}$ contains a color that is external to w in M.

The proof is by induction on k, over every color-connected plane graph G, every triplet of vertices u, v, w in G, and every minimal set $\mathcal{M}$ w.r.t. w of k-valid pairwise color-disjoint

u-v paths in $G - w$. If $k = 1$, any path in $\mathcal{M}$ contains exactly one external color w.r.t. w in M. By Lemma 3.9, at most two paths in $\mathcal{M}$ contain only external colors. Assume by the inductive hypothesis that, for each $1 \leq i < k$, we have $|\mathcal{M}| \leq g(i)$. We can assume that M is irreducible; otherwise, we can apply the color contraction operation and replace $\mathcal{M}$ with a set of paths satisfying the same properties.

By Lemma 3.10, there are (at most) 3 paths in $\mathcal{M}$, such that the subgraph of M induced by the remaining paths of $\mathcal{M}$ has a u-v vertex-separator S satisfying $|S| \leq 2k + 3$. Remove these 3 paths from $\mathcal{M}$, and now we can assume that M has a u-v vertex-separator S satisfying $|S| \leq 2k + 3$; we will add 3 to the upper bound on $|\mathcal{M}|$ at the end. We can assume that S is minimal. S separates M into two subgraphs M_u and M_v such that $u \in V(M_u)$, $v \in V(M_v)$, and there is no edge between M_u and M_v. We partition $\mathcal{M}$ into the following groups, where each group excludes the paths satisfying the properties of the groups defined before it: (1) The set of paths in $\mathcal{M}$ that contain a nonempty vertex in S; (2) the set of paths $\mathcal{M}_u^k$ consisting of each path P in $\mathcal{M}$ such that all colors on P appear on vertices in M_u (these colors could still appear on vertices in M_v as well); (3) the set of paths $\mathcal{M}_v^k$ consisting of each path P in $\mathcal{M}$ such that all colors on P appear on vertices in M_v; and (4) the set $\mathcal{M}^{<k}$ of remaining paths in $\mathcal{M}$, satisfying that each path contains a nonempty external vertex to w in M and contains less than k colors from each of M_u and M_v.

Since the paths in $\mathcal{M}$ are pairwise color-disjoint, no nonempty vertex in S appears on two distinct paths from group (1). Therefore, the number of paths in group (1) is at most $|S| \leq 2k + 3$. Observe that the vertices in S contained in any path from groups (2)-(4) are empty vertices. To upper bound the number of paths in group (2), for each path P, there is a last vertex x_P (*i.e.*, farthest from u) in P that is in S. Fix a vertex $x \in S$, and let us upper bound the number of paths P in group (2) for which $x = x_P$. Let P_v be the subpath of P from x to v. Note that since v is empty and all the vertices in S that are contained in paths in group (2) are empty, and since M is irreducible, P_v must contain at least one color. Since all colors appearing on P appear on vertices in M_u, all colors appearing on P_v appear in M_u. By Lemma 3.11, M_u is contained in a single face f of $M_v + x$. Since f is a vertex-separator that separates $V(M_u)$ from $V(P_v)$ in G, by Observation 3.1, every color that appears on P_v appears on f. By Lemma 3.9, there are at most two x-v paths that contain only colors that appear on f. This shows that there are at most two paths in group (2) for which x is the last vertex in S. Since $|S| \leq 2k + 3$, this upper bounds the number of paths in group (2) by $2(2k + 3) = 4k + 6$. By symmetry, the number of paths in group (3) is upper bounded by $4k + 6$.

To upper bound the number of paths in group (4), let $S = \{s_2, \ldots, s_{r-1}\}$, and extend S by adding the two vertices $s_1 = u$ and $s_r = v$ to form the set $A = \{s_1, s_2, \ldots, s_r\}$. For every two (distinct) vertices $s_j, s_{j'} \in A$, we define a set of paths $\mathcal{P}_{jj'}$ in $G - w$ whose endpoints are s_j and $s_{j'}$ as follows. For each path P in group (4), partition (the edges in) P into subpaths $P_1, \ldots, P_q$ satisfying the property that the endpoints of each P_i, $i \in [q]$, are in A, and no internal vertex to P_i is in A. For each P_i, $i \in [q]$, such that P_i contains a vertex that contains an external color to w in $G - w$, let P_i' (possibly P_i) be a subpath in $G - w$ between the endpoints of P_i satisfying that $\chi(P_i') \subseteq \chi(P_i)$ and $\chi(P_i')$ is minimal w.r.t. containment. Since P contains a vertex that contains an external color to w in $G - w$, it is easy to see that there exists an $i \in [q]$ such that P_i' contains a vertex containing an external color to w in $G - w$. Pick any $i \in [q]$ satisfying that P_i' contains a vertex containing an external color to w in $G - w$, associate P with P_i', and assign P_i' to the set of paths $\mathcal{P}_{jj'}$ such that s_j and $s_{j'}$ are the endpoints of P_i'. The map that takes each P to its P_i' is clearly a bijection.

Therefore, it suffices to upper bound the number of paths assigned to the sets $\mathcal{P}_{jj'}$. Fix a set $\mathcal{P}_{jj'}$. The paths in $\mathcal{P}_{jj'}$ have $s_j, s_{j'}$ as endpoints, and are pairwise color-disjoint. It is not difficult to show that $\mathcal{P}_{jj'}$ is a minimal set of $(k-1)$-valid s_j-$s_{j'}$ paths in $G - w$ w.r.t. w. By the inductive hypothesis, we have $|\mathcal{P}_{jj'}| \leq g(k-1)$. Since the number of sets $\mathcal{P}_{jj'}$ is at most $\binom{2k+5}{2}$, the number of paths in group (4) is $\mathcal{O}(k^2) \cdot g(k-1)$.

It follows that $|\mathcal{M}| \leq g(k)$, where $g(k)$ satisfies:

$$g(k) \leq 3 + (2k+3) + 2(4k+6) + \mathcal{O}(k^2) \cdot g(k-1) = \mathcal{O}(k^2) \cdot g(k-1),$$

where 3 accounts for the 3 paths removed from $\mathcal{M}$. Solving the aforementioned recurrence relation gives $g(k) = \mathcal{O}(c^k k^{2k})$, where $c > 1$ is a constant. ◀

Applying Lemma 3.12 to a maximal set $\mathcal{M}$ of color-disjoint paths in $\mathcal{P}$, and using an inductive proof, we can show the following theorem:

▶ **Theorem 3.13.** *Let G be a plane color-connected graph, and let $w \in V(G)$. Let G' be a subgraph of $G - w$, and let $u, v \in V(G')$. Every set $\mathcal{P}$ of minimal k-valid u-v paths in G' w.r.t. w satisfies $|\mathcal{P}| \leq h(k)$, where $h(k) = \mathcal{O}(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$.*

4 The Algorithm

In this section, we highlight how the FPT algorithm for Colored Path-Con, parameterized by both k and the treewidth of the input graph works. As pointed out in Section 3, there can be too many (*i.e.*, more than FPT-many) subsets of colors that appear in a bag, and hence, that the algorithm may need to store/remember. To overcome this issue, we extend the notion of a minimal set of k-valid u-v paths w.r.t. a vertex – from the previous section – to a "representative set" of paths w.r.t. a specific bag and a specific enumerated configuration for the bag. This allows us to upper bound the size of the table, in the dynamic programming algorithm, stored at a bag by a function of both k and the treewidth of the graph.

Let (G, C, χ, s, t, k) be an instance of Colored Path-Con. Let $(\mathcal{V}, \mathcal{T})$ be a nice tree decomposition of G. By Assumption 2.1, we can assume that s and t are nonadjacent empty vertices. We add s and t to every bag in $\mathcal{T}$, and now we have $\{s, t\} \subseteq X_i$, for every bag $X_i \in \mathcal{T}$. For a bag X_i, we say that $v \in X_i$ is *useful* if $|\chi(v)| \leq k$. Let U_i be the set of all useful vertices in X_i and let $\overline{U_i} = X_i \setminus U_i$. We denote by V_i the set of vertices in the bags of the subtree of $\mathcal{T}$ rooted at X_i. For any two vertices $u, v \in X_i$, let $G^i_{uv} = G[(V_i \setminus X_i) \cup \{u, v\}]$. We extend the notion of a minimal set of k-valid u-v paths w.r.t. a vertex, developed in the previous section, to the set of vertices in a bag of $\mathcal{T}$.

▶ **Definition 4.1.** A set of k-valid u-v paths $\mathcal{P}_{uv}$ in G^i_{uv} is *minimal* w.r.t. X_i if it satisfies the following properties:

(i) There does not exist two paths $P_1, P_2 \in \mathcal{P}_{uv}$ such that $\chi(P_1) \cap \chi(X_i) = \chi(P_2) \cap \chi(X_i)$;
(ii) there does not exist two paths $P_1, P_2 \in \mathcal{P}_{uv}$ such that $\chi(P_1) \subseteq \chi(P_2)$; and
(iii) for any $P \in \mathcal{P}_{uv}$ there does not exist a u-v path P' in G^i_{uv} such that $\chi(P') \subsetneq \chi(P)$.

The following lemma uses the upper bound on the cardinality of a minimal set of k-valid u-v paths w.r.t. a vertex, derived in Theorem 3.13 in the previous section, to obtain an upper bound on the cardinality of a minimal set of k-valid u-v paths w.r.t. a bag of $\mathcal{T}$:

▶ **Lemma 4.2.** *Let X_i be bag, $u, v \in X_i$, and $\mathcal{P}_{uv}$ a set of k-valid u-v paths in G^i_{uv} that is minimal w.r.t. X_i. Then the number of paths in $\mathcal{P}_{uv}$ is at most $h(k)^{|X_i|}$, where $h(k) = \mathcal{O}(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$.*

▶ **Definition 4.3.** Let X_i be a bag in $\mathcal{T}$. A *pattern* π for X_i is a sequence $(v_1 = s, \sigma_1, v_2, \sigma_2, \ldots, \sigma_{r-1}, v_r = t)$, where $\sigma_i \in \{0,1\}$ and $v_i \in U_i$. For a bag X_i, and a pattern $(v_1 = s, \sigma_1, v_2, \sigma_2, \ldots, \sigma_{r-1}, v_r = t)$ for X_i, we say that a sequence of paths $\mathcal{S} = (P_1, \ldots, P_{r-1})$ *conforms* to (X_i, π) if:

- For each $j \in [r-1]$, $\sigma_j = 1$ implies that P_j is an induced path from v_j to v_{j+1} whose internal vertices are contained in $V_i \setminus X_i$ and P_j is empty otherwise; and
- $|\chi(\mathcal{S})| = |\bigcup_{j\in[r-1]} \chi(P_j)| \leq k$.

▶ **Definition 4.4.** Let X_i be a bag, π a pattern for X_i, and $\mathcal{S}_1, \mathcal{S}_2$ two sequences of paths that conform to (X_i, π). We write $\mathcal{S}_1 \preceq_i \mathcal{S}_2$ if $|\chi(\mathcal{S}_1) \cup (\chi(\mathcal{S}_2) \cap \chi(X_i))| \leq |\chi(\mathcal{S}_2)|$.

Using the relation $\preceq_i$ on the set of sequences that conform to (X_i, π), we can define the key notion of representative sets that makes the dynamic programming approach work:

▶ **Definition 4.5.** Let X_i be a bag and $\pi = (v_1, \sigma_1, v_2 \ldots, \sigma_{r-1}, v_r)$ a pattern for X_i. A set $\mathcal{R}_\pi$ of sequences that conform to (X_i, π) is a *representative* set for (X_i, π) if:

(i) For every sequence $\mathcal{S}_1 \in \mathcal{R}_\pi$, and for every sequence $\mathcal{S}_2 \neq \mathcal{S}_1$ that conforms to (X_i, π), if $\mathcal{S}_1 \preceq_i \mathcal{S}_2$ then $\mathcal{S}_2 \notin \mathcal{R}_\pi$;

(ii) for every sequence $\mathcal{S} \in \mathcal{R}_\pi$, and for every path $P \in \mathcal{S}$ between v_j and v_{j+1}, $j \in [r-1]$, there does not exist a v_j-v_{j+1} path P' in $G^i_{v_j v_{j+1}}$ such that $\chi(P') \subsetneq \chi(P)$; and

(iii) for every sequence $\mathcal{S} \notin \mathcal{R}_\pi$ that conforms to (X_i, π) and satisfies that no two paths in $\mathcal{S}$ share a vertex that is not in X_i, there is a sequence $\mathcal{W} \in \mathcal{R}_\pi$ such that $\mathcal{W} \preceq_i \mathcal{S}$.

The following lemma uses the upper bound on the cardinality of a minimal set of k-valid u-v paths w.r.t. a bag X_i, derived in Lemma 4.2, to obtain an upper bound on the cardinality of a representative set w.r.t. a bag and a fixed pattern (X_i, π):

▶ **Lemma 4.6.** *Let X_i be bag, π a pattern for X_i, and $\mathcal{R}_\pi$ be a representative set for (X_i, π). Then the number of sequences in $\mathcal{R}_\pi$ is at most $h(k)^{|X_i|^2}$, where $h(k) = \mathcal{O}(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$.*

For each bag X_i, we maintain a table Γ_i that contains, for each pattern for X_i, a representative set of sequences $\mathcal{R}_\pi$ for (X_i, π). The rest is a technical dynamic programming algorithm over $(\mathcal{V}, \mathcal{T})$ that computes the table Γ_i at a bag X_i for each bag type (introduce, forget, join) in the nice tree decomposition. We conclude with the following theorem:

▶ **Theorem 4.7.** *There is an algorithm that on input (G, C, χ, s, t, k) of* Colored Path-Con*, either outputs a k-valid s-t path in G or decides that no such path exists, in time $\mathcal{O}^\star(f(k)^{6\omega^2})$, where ω is the treewidth of G and $f(k) = \mathcal{O}(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$. Therefore,* Colored Path-Con *parameterized by both k and the treewidth ω of the input graph is* FPT.

5 Extensions and Applications

In this section, we explain how to extend the FPT result for Colored Path-Con w.r.t. the parameterization by both k and the treewidth of the graph, to the parameterization by both k and the length ℓ of the sought path, and discuss important applications of this extended result. We formally define the problem w.r.t. the parameterization by k and ℓ:

Bounded-length Colored Path-Con
Given: A planar graph G; a set of colors C; $\chi : V \longrightarrow 2^C$; two designated vertices $s, t \in V(G)$; and $k, \ell \in \mathbb{N}$
Question: Does there exist a k-valid s-t path of length at most ℓ in G?

To extend Theorem 4.7, we repeatedly contract every edge uv incident to a vertex v whose distance to s is more than $\ell+1$; we assign the resulting vertex the color set $\chi(u) \cup \chi(v)$. (We do not delete such vertices in order to preserve the color-connectivity property.) Afterwards, the radius of G is at most $\ell+1$, and hence G has treewidth at most $3 \cdot (\ell+1)+1 = 3\ell+4$ [14]. Although the treewidth of G is bounded by a function of ℓ, we cannot use the FPT algorithm for Colored Path-Con, parameterized by k and the treewidth of G, to solve Bounded-length Colored Path-Con because a k-valid path returned by the algorithm for Colored Path-Con may have length more than ℓ. We can extend the FPT results for Colored Path-Con to Bounded-length Colored Path-Con to show the following:

▶ **Theorem 5.1.** Bounded-length Colored Path-Con *parameterized by both k and the length of the path is* FPT*.*

We now describe applications of Theorem 5.1. The first application is a direct consequence of this theorem.

▶ **Corollary 5.2.** *For any computable function h, the restriction of* Colored Path-Con *to instances in which the length of the path is at most $h(k)$ is* FPT *parameterized by k.*

We note that the above restriction of Colored Path-Con can be shown to be NP-hard.

Corollary 5.2 directly implies Korman *et al.*'s results [12], showing that Obstacle Removal is FPT for unit-disk obstacles and for similar-size fat-region obstacles with constant overlapping number. Using Bereg and Kirkpatrick's result [2], the length of a shortest k-valid path for unit-disk obstacles is at most $3k$ (see also Lemma 3 in Korman *et al.* [12]). By Corollary 2 in [12], the length of a shortest k-valid path for similar-size fat-region obstacles with constant overlapping number is linear in k. Corollary 5.2 generalizes these FPT results, which required quite some effort, and provides an explanation to why the problem is FPT for such restrictions, namely because the path length is upper bounded by a function of k.

The second application is related to an open question posed in [9, 11]. For an instance $I = (G, C, \chi, s, t, k)$ of Colored Path-Con, and a color $c \in C$, define the *intersection number* of c, denoted $\iota(c)$, to be the number of vertices in G on which c appears. Define the *intersection number* of G, $\iota(G)$, as $\max\{\iota(c) \mid c \in C\}$.

▶ **Corollary 5.3.** *For any computable function h,* Colored Path-Con *restricted to instances (G, C, χ, s, t, k) satisfying $\iota(G) \leq h(k)$ is* FPT *parameterized by k.*

Corollary 5.3 has applications pertaining to instances of Connected Obstacle Removal whose auxiliary graphs have intersection number bounded by a function of k. An interesting case that was studied is when the obstacles are convex polygons, each intersecting at most a constant number of other polygons. The complexity of this problem was posed as an open question in [9, 11], and remains unresolved. Corollary 5.3 implies that the problem is FPT, even for the more general setting in which the obstacles are arbitrary convex regions satisfying that the number of regions intersected by any region is a constant. (Note that convexity is important here.)

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