Computing Tutte Paths

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Abstract

Tutte paths are one of the most successful tools for attacking problems on long cycles in planar graphs. Unfortunately, results based on them are non-constructive, as their proofs inherently use an induction on overlapping subgraphs and these overlaps prevent any attempt to bound the running time by a polynomial.

For special cases however, computational results of Tutte paths are known: For 4-connected planar graphs, Tutte paths are in fact Hamiltonian paths and Chiba and Nishizeki [5] showed how to compute such paths in linear time. For 3-connected planar graphs, Tutte paths have a significantly more complicated structure, and it has only recently been shown that they can be computed in polynomial time [24]. However, Tutte paths are defined for general 2-connected planar graphs and this is what most applications need. In this unrestricted setting, no computational results for Tutte paths are known.

We give the first efficient algorithm that computes a Tutte path (in this unrestricted setting). One of the strongest existence results about such Tutte paths is due to Sanders [23], which allows one to prescribe the end vertices and an intermediate edge of the desired path. Encompassing and strengthening all previous computational results on Tutte paths, we show how to compute such a special Tutte path efficiently. Our method refines both, the existence results of Thomassen [29] and Sanders [23], and avoids that the subgraphs arising in the inductive proof intersect in more than one edge by using a novel iterative decomposition along 2-separators. Finally, we show that our algorithm runs in time $O(n^2)$.

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1 Introduction

The question whether a graph $G = (V, E)$ is *Hamiltonian*, i.e. contains a cycle of length $n := |V|$, is among the most fundamental graph problems. For planar graphs and graphs embeddable on higher surfaces, *Tutte paths* have proven to be one of the most successful tools for attacking Hamiltonicity problems and problems on long cycles. For this reason, there is a wealth of existential results in which Tutte paths serve as main ingredient; in chronological order, these are [31, 29, 26, 4, 22, 23, 27, 33, 16, 28, 11, 13, 18, 21, 20, 17, 24, 7, 2].

As a historical starting point to these results, Whitney [32] proved that every 4-connected maximal planar graph is Hamiltonian. Tutte extended this to arbitrary 4-connected planar graphs by showing that every 2-connected planar graph G contains a Tutte path [30, 31]

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(for a definition of Tutte paths, see Section 2). Thomassen [29] in turn proved the following generalization, which also implies that every 4-connected planar graph is Hamiltonian-connected, i.e. contains a path of length $n-1$ between any two vertices. For a plane graph G, let C_G be its outer face.

▶ **Theorem 1** (Thomassen [29]). *Let G be a 2-connected plane graph, $x \in V(C_G)$, $\alpha \in E(C_G)$ and $y \in V(G) - x$. Then G contains a Tutte path from x to y through α.*

Sanders [23] then generalized Thomassen's result further by allowing to choose the start vertex x of the Tutte path arbitrarily.

▶ **Theorem 2** (Sanders [23]). *Let G be a 2-connected plane graph, $x \in V(G)$, $\alpha \in E(C_G)$ and $y \in V(G) - x$. Then G contains a Tutte path from x to y through α.*

On top of the above series of fundamental results, Tutte paths have been used in two research branches: While the first deals with the existence of Tutte paths on graphs embeddable on higher surfaces [26, 3, 27, 33, 28, 17], the second [15, 9, 3, 10, 16, 11, 19] investigates generalizations or specializations of Hamiltonicity such as *k-walks*, *long cycles* and *Hamiltonian connectedness*.

Unfortunately, in all the results mentioned so far, very little is known about the complexity of finding a Tutte path. This is crucial, as the task of finding Tutte paths is almost always the only reason that hinders the computational tractability of the problem. The main obstruction so far is that Tutte paths are found by decomposing the input graph into overlapping subgraphs, on which induction is applied. Although this is enough to prove existence results, these overlapping subgraphs do not allow to bound the running time polynomially (as argued in [12, 24]). The only known computational results on Tutte paths [12, 1, 5, 21, 24] deal therefore with very restricted settings, such as the case that Tutte paths are just Hamiltonian paths: While it is known how to compute Tutte paths for planar 4-connected graphs [5] efficiently (in which case Tutte paths are just Hamiltonian paths), for planar 3-connected graphs a first polynomial-time algorithm was only recently shown [24].

However, no efficient algorithm is published so far that computes Tutte paths in general 2-connected planar graphs (i.e. the ones of Theorem 1 or 2). In fact, the claimed algorithmic results in [26, 27] require polynomial running times for computing such Tutte paths, without giving proofs that such algorithms exist. Given the subtlety of the arguments inherent to Tutte paths, we feel that giving such a proof is necessary. Indeed, history shows that even for the much easier setting that Tutte paths are Hamiltonian paths, an existence result for Tutte paths has been incorrectly claimed to imply a polynomial-time algorithm [29, 4] (again, due to overlapping subgraphs). For finding Tutte paths in certain restrictions of 2-connected and 3-connected planar graphs, the related results in [22, 17] claim polynomial running times as well.

Our Results

Our motivation is two-fold. First, we want to make Tutte paths accessible to algorithms. We will show that Tutte paths can be computed in time $O(n^2)$. This has impact on almost all the applications using Tutte paths listed above.

For several of them, e.g. [26, 22, 27, 17], we immediately obtain polynomial-time algorithms where no efficient algorithms were published before. In addition, Tutte paths were also used in [7, 8] to show that every essentially 4-connected polyhedral graph contains a cycle of length proportional to n. As the existence proofs in this paper are constructive, our result directly implies a efficient (in fact, an $O(n^2)$-time) algorithm for the computation of these cycles. Furthermore, [2] showed that every 3-connected planar graph having at most three 3-separators is Hamiltonian. If a 3-connected planar graph contains at most one 3-separator, our algorithm shows that a Hamiltonian cycle can be computed in $O(n^2)$ time, by using a suitable choice of the intermediate edge α.

Second, we aim for computing the strongest possible known variant of Tutte paths, encompassing the many incremental improvements on Tutte paths made over the years. We will therefore develop an algorithm for Sander's existence result [23], which was proven to be best possible in many aspects. Sanders result has also an immediate extension to connected planar graphs [20], which can be computed simply and efficiently from our result by using block-cut trees.

We will first give a decomposition that refines the original ones used for Theorems 1 and 2, and allows to decompose G into graphs that pairwise intersect in at most one edge. We then show that this small overlap does not prevent us from achieving a polynomial running time. All graphs will be simple. We proceed by showing how this decomposition can be computed efficiently in order to find the Tutte paths of Theorem 2. Our main result is hence the following, giving the first polynomial-time algorithm for computing Tutte paths.

▶ **Theorem 3.** *Let G be a 2-connected plane graph, $x \in V(G)$, $\alpha \in E(C_G)$ and $y \in V(G) - x$. Then a Tutte path of G from x to y through α can be computed in time $O(n^2)$.*

Section 3 presents the decomposition with small overlap that proves the existence of Tutte paths. On the way to our main result, we give full algorithmic counterparts of the approaches of Thomassen and Sanders; for example, we describe small overlap variants of Theorem 1 and of the *Three Edge Lemma* [26, 22], which was used in the purely existential result of Sanders [23] as a black box.

Our Techniques

We broadly follow the idea of [5] and construct a Tutte path that is based on certain 2-separators of the graphs constructed during our decomposition. This depends on many structural properties of the given graph. In [5], the necessary properties follow from the restriction to the class of internally 4-connected planar graphs, the restriction on the endpoints of the desired Tutte path, and the fact that the Tutte paths computed recursively are actually Hamiltonian. In contrast, here we give new insights into the much wilder structure of Tutte paths of 2-connected planar graphs, allow $x, y \notin C_G$, and hence extend this technique. We show that based on the prescribed vertices and edge, there are always unique non-interlacing 2-separators that are contained in every possible Tutte path of the given graph. We then use this set of 2-separators to iteratively construct a preliminary Tutte path and use this iterative procedure to avoid overlaps of more than one edge in the decomposition of the input graph.

2 Preliminaries

We assume familiarity with standard graph theoretic notations as in [6]. Let $deg(v)$ be the degree of a vertex v. We denote the subtraction of a graph H from a graph G by $G - H$, and the subtraction of a vertex or edge x from G by $G - x$.

A *k-separator* of a graph $G = (V, E)$ is a subset $S \subseteq V$ of size k such that $G - S$ is disconnected. A graph G is *k-connected* if $|V| > k$ and G contains no $(k-1)$-separator. For a path P and two vertices $x, y \in P$, let xPy be the smallest subpath of P that contains x and y. For a path P from x to y, let $inner(P) := V(P) - \{x, y\}$ be the set of its inner vertices. Paths that intersect pairwise at most at their endvertices are called *independent*.

A connected graph without a 1-separator is called a *block*. A *block of a graph* G is an inclusion-wise maximal subgraph that is a block. Every block of a graph is thus either 2-connected or has at most two vertices. It is well-known that the blocks of a graph partition its edge-set. A graph G is called a *chain of blocks* if it consists of blocks $B_1, B_2, \dots, B_k$ such that $V(B_i) \cap V(B_{i+1})$, $1 \leq i < k$, are pairwise distinct 1-separators of G and G contains no other 1-separator. In other words, a chain of blocks is a graph, whose block-cut tree [14] is a path.

A *plane* graph is a planar embedding of a graph. Let C be a cycle of a plane graph G. For two vertices x, y of C, let xCy be the clockwise path from x to y in C. For a vertex x and an edge e of C, let xCe be the clockwise path in C from x to the endvertex of e such that $e \notin xCe$ (define eCx analogously). Let the subgraph of G *inside* C consist of $E(C)$ and all edges that intersect the open set inside C into which C divides the plane. For a plane graph G, let C_G be its outer face.

A central concept for Tutte paths is the notion of H-bridges (see [31] for some of their properties): For a subgraph H of a 2-connected plane graph G, an H*-bridge* of G is either an edge that has both endvertices in H but is not itself in H or a component K of $G - H$ together with all edges (and the endvertices of these edges) that join vertices of K with vertices of H. An H-bridge is called *trivial* if it is just one edge. A vertex of an H-bridge L is an *attachment* of L if it is in H, and an *internal* vertex of L otherwise. An *outer* H-bridge of G is an H-bridge that contains an edge of C_G.

A *Tutte path* (*Tutte cycle*) of a plane graph G is a path (a *cycle*) P of G such that every outer P-bridge of G has at most two attachments and every P-bridge at most three attachments. In most of the cases we consider, G will be 2-connected, so that every P-bridge has at least two attachments. For vertices x, y and an edge $\alpha \in C_G$, let an x-α-y*-path* be a Tutte path from x to y that contains α. An x-y*-path* is an x-α-y*-path* for an arbitrarily chosen edge $\alpha \in C_G$.

3 Decomposition with Small Overlap

After excluding several easy cases of the decomposition, we prove Thomassen's Theorem 1 constructively and then show how to use this for a proof of the Three Edge Lemma. The Three Edge Lemma, in turn, allows us to give a constructive proof of Sander's Theorem 2 in which only small overlaps occur in the induction. Due to space constraints, we have to omit this proof, it however derived from [23] in a similar way as Theorem 1 from [29].

We will use induction on the number of vertices. In all proofs about Tutte paths of this section, the induction base is a triangle, in which the desired Tutte path can be found trivially; thus, we will assume in these proofs by the induction hypothesis that graphs with fewer vertices contain Tutte paths. All graphs in the induction will be simple.

The following sections cover different cases of the induction steps of the three statements to prove, starting with some easy cases for which a decomposition into edge disjoint subgraphs was already given [29]. For the remainder of the article, let G be a simple plane 2-connected graph with outer face C_G and let $x \in V(G)$, $\alpha \in E(C_G)$ and $y \in V(G) - x$. If $\alpha = xy$, the desired path is simply xy; thus, assume $\alpha \neq xy$. Since G is 2-connected, C_G is a cycle.

3.1 The Easy Cases

We say that G is *decomposable into* G_L *and* G_R if it contains subgraphs G_L and G_R such that $G_L \cup G_R = G$, $V(G_L) \cap V(G_R) = \{c, d\}$, $x \in V(G_L)$, $\alpha \in E(G_R)$, $V(G_L) \neq \{x, c, d\}$ and $V(G_R) \neq \{c, d\}$ (or the analogous setting with y taking the role of x) (see Figure 1). In

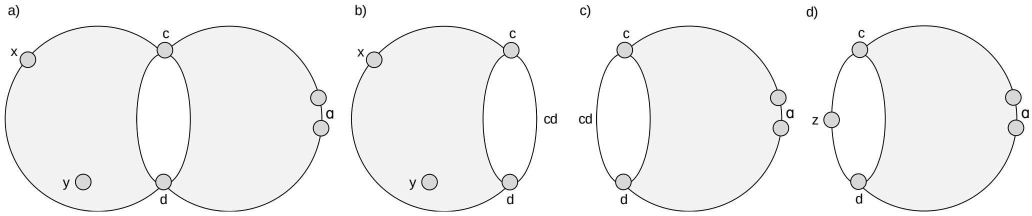

Figure 1 a) shows a graph G that is decomposable into G_L and G_R. The figures b) to d) show the graphs G'_L, G'_R and G^*_R (in this order) that are constructed to process G in [29] .

particular, $G_L \neq \{c, d\}$, even if $x \in \{c, d\}$. Hence $\{c, d\}$ is a 2-separator of G. There might exist multiple pairs (G_L, G_R) into which G is decomposable; we will always choose a pair that minimizes $|V(G_R)|$. Note that G_R intersects C_G (for example, in α), but G_L does not have to intersect C_G. In [29], it was shown that every decomposable graph G contains a Tutte path, without using recursion on overlapping subgraphs.

▶ **Lemma 4** ([29]). *If G is decomposable into G_L and G_R, then G contains an x-α-y-path.*

Even if G is not decomposable into G_L and G_R, G may contain other 2-separators $\{c, d\}$ that allow for a similar reduction as in Lemma 4 (for example, when modifying its prerequisites to satisfy $\{x, \alpha, y\} \subseteq G_R - \{c, d\}$).

▶ **Lemma 5** ([29]). *Let $\{c, d\}$ be a 2-separator of G and let J be a $\{c, d\}$-bridge of G having an internal vertex in C_G such that x, y and α are not in J. Then G contains an x-α-y-path.*

3.2 Proof of Theorem 1

We now prove that G contains a Tutte path from $x \in V(C_G)$ to $y \in V(G) - x$ through $\alpha \in E(C_G)$. For simplicity, if y is not in $V(C_G)$ but has degree two and both of its neighbors are in $V(C_G)$, then we change the embedding of G (and therefore C_G) such that y belongs to the outer face. If Lemma 4 or 5 can be applied, we obtain such a Tutte path directly, so assume their prerequisites are not met. Let l_α be the endvertex of α that appears first when we traverse C_G in clockwise order starting from x, and let r_α be the other endvertex of α. If $y \in xC_Gl_\alpha$, we interchange x and y (this does not change l_α); hence, we have $y \notin xC_Gl_\alpha$. If $y = r_\alpha$, we mirror the embedding such that y becomes l_α and proceed as in the previous case; hence, $y \notin xC_Gr_\alpha$.

We define two paths P and Q in G, whose union will, step by step, be modified into a Tutte path of G. Let $Q := xC_Gl_\alpha$ and let $H := G - V(Q)$; in particular, $y \notin Q$ and, if x is an endvertex of α, $Q = \{x\}$. Since G is not decomposable, we have $deg(r_\alpha) \geq 3$, as otherwise the neighborhood of r_α would be the 2-separator of such a decomposition. Since $deg(r_\alpha) \geq 3$, r_α is incident to an edge $e \notin C_G$ that shares a face with α. Let B_1 be the block of H that contains e. It is straight-forward to prove the following about B_1 (see Thomassen [29]), which shows that every vertex of C_G is either in Q or in B_1.

▶ **Lemma 6** ([29]). *B_1 contains $C_G - V(Q)$ and is the only block of H containing r_α.*

Consider a component A of H that does not contain B_1. Then the neighborhood of A in G is in Q and must contain a 2-separator of G due to planarity. Hence, either $y \in A$ and we can apply Lemma 4 or $y \notin A$ and we can apply Lemma 5. Since both contradicts our assumptions, H is connected and contains B_1 and y. Let K be the minimal plane chain of blocks $B_1, \ldots, B_l$ of H that contains B_1 and y (hence, $y \in B_l$). Let v_i be the intersection of B_i and B_{i+1} for $1 \leq i \leq l - 1$; in addition, we set $v_0 := r_\alpha$ and $v_l := y$.

Consider any $(K \cup C_G)$-bridge J. Since Lemma 5 cannot be applied, J has an attachment $v_J \in K$. Further, J cannot have two attachments in K, as this would contradict the maximality of the blocks in K. Let $C(J)$ be the shortest path in C_G that contains all vertices in $J \cap C_G$ and does not contain r_α as inner vertex (here, r_α serves as a *reference vertex* of C_G that ensures that the paths $C(J)$ are chosen consistently on C_G). Let l_J be the endvertex of $C(J)$ whose counterclockwise incident edge in C_G is not in $C(J)$ and let r_J be the other endvertex of $C(J)$.

3.2.1 Decomposing along Maximal 2-Separators

At this point we will deviate from the original proof of Theorem 1 in [29], which continues with an induction on every block of K that leads to overlapping subgraphs in a later step of the proof. Instead, we will show that a v_0-v_l-path P of K can be found iteratively such that the graphs in the induction have only small overlap.

For every block $B_i \neq B_1$ of K, we choose an arbitrary edge $\alpha_i = l_{\alpha_i} r_{\alpha_i}$ in C_{B_i}. In B_1 we choose α_1 such that α_1 is incident to the endvertex of $C_{B_1} \cap C_G$ that is not r_α. As done for G, we may assume for every B_i that l_{α_i} is the endvertex of α_i that is contained in $v_{i-1} C_{B_i} \alpha_i$ and that $v_i \notin v_{i-1} C_{B_i} r_{\alpha_i}$ and (by mirroring the planar embedding and interchanging v_i and v_{i-1} if necessary). However, unlike G, every B_i may satisfy the prerequisites of Lemmas 4 and 5. By the induction hypothesis of Theorem 1, B_i contains a v_{i-1}-α_i-v_i-path P_i. In [29], the outer P_i-bridges of B_i are not only being processed during this induction step, but also in a later induction step when modifying Q. We avoid such overlapping subgraphs by using a new iterative structural decomposition of B_i along certain 2-separators on C_{B_i}. This decomposition allows us to construct P_i iteratively such that the outer P_i-bridges of B_i are not part of the induction applied on B_i. Eventually, $P := \bigcup_{1 \leq i \leq l} P_i$ will be the desired v_0-v_l-path of K.

The outline is as follows. After explaining the basic split operation that is used by our decomposition, we give new insights into the structure of the Tutte paths P_i of the blocks B_i. These are used in Section 3.2.2 to define the iterative decomposition of every block B_i into a modified block $\eta(B_i)$, which will in turn allow to compute every P_i step-by-step. This gives the first part P of the desired Tutte path x-α-y of G. Subsequently, we will show how the remaining path Q can be modified to obtain the second part.

For a 2-separator $\{c, d\} \subseteq C_B$ of a block B, let B^+_{cd} be the $\{c, d\}$-bridge of B that contains cC_Bd and let B^-_{cd} be the union of all other $\{c, d\}$-bridges of B (note that B^+_{cd} contains the edge cd if and only if B^+_{cd} is trivial). For a 2-separator $\{c, d\} \subseteq C_B$, let *splitting off* B^+_{cd} (from B) be the operation that deletes all internal vertices of B^+_{cd} from B and adds the edge cd if cd does not already exist in B. Our decomposition proceeds by iteratively splitting off bridges B^+_{cd} from the blocks B_i of K for suitable 2-separators $\{c, d\} \subseteq C_{B_i}$ (we omit the subscript i in such bridges B^+_{cd}, as it is determined by c and d). The following lemma restricts these 2-separators to be contained in specific parts of the outer face.

▶ **Lemma 7.** *Let P' be a Tutte path of a block B such that P' contains an edge α' and two vertices $a, b \in C_B$. Then every outer P'-bridge J of B has both attachments in aC_Bb or both in bC_Ba. If additionally J is non-trivial and $P' \neq \alpha'$, the attachments of J form a 2-separator of B.*

Proof. Let e be an edge in $J \cap C_B$ and assume without loss of generalization that $e \in aC_Bb$. Let c and d be the last and first vertices of the paths aC_Be and eC_Bb, respectively, that are contained in P' (these exist, as a and b are in P'). Then J has attachments c and d and no further attachment, as P' is a Tutte path. This gives the first claim. For the second claim, let z be an internal vertex of J. Since $P' \neq \alpha'$, P' contains a third vertex $c \notin \{a, b\}$. As c is not contained in J, $\{c, d\}$ separates z and c and is thus a 2-separator of B. ◀

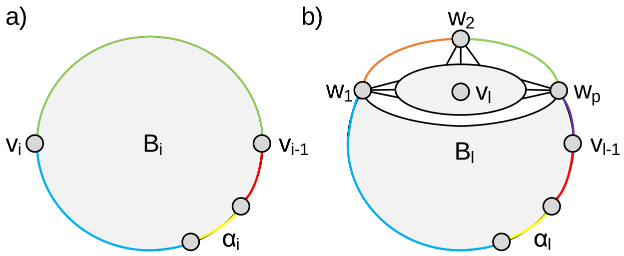

Figure 2 a) The boundary points and -parts of a block $B_i \neq B_l$. b) An instance in which the block B_l contains a 2-separator $\{w_1, w_p\}$ that splits off v_l.

For every block $B_i \neq B_l$ of K, let the *boundary points* of B_i be the vertices $v_{i-1}, l_{\alpha_i}, r_{\alpha_i}, v_i$ and let the *boundary parts* of B_i be the inclusion-wise maximal paths of C_{B_i} that do not contain any boundary point as inner vertex (see Figure 2a; note that boundary parts may be single vertices). Hence, every boundary point will be contained in any possible v_{i-1}-α_i-v_i-path P_i, and there are exactly four boundary parts, one of which is α_i. Now, if $P_i \neq \alpha_i$, applying Lemma 7 for all boundary points $a, b \in \{v_{i-1}, l_{\alpha_i}, r_{\alpha_i}, v_i\}$ and $\alpha' := \alpha_i$ implies that the two attachments of every outer non-trivial P_i-bridge of B_i form a 2-separator that is contained in one boundary part of B_i. For this reason, our decomposition will split off only 2-separators that are contained in boundary parts.

In principle, we will do the same for the block B_l. If $v_l \in C_{B_l}$, we define the boundary points of B_l just as before for $i < l$. However, B_l is special in the sense that v_l may not be in C_{B_l}. Then we have to ensure that we do not loose v_l when splitting off a 2-separator, as v_l is supposed to be contained in P_l (see Figure 2b). To this end, consider for $v_l \notin C_{B_l}$ the 2-separator $\{w_1, w_p\} \subseteq C_{B_l}$ of B_l such that $B^+_{w_1,w_p}$ contains v_l, the path $w_1 C_{B_l} w_p$ is contained in one of the paths in $\{v_{l-1} C_{B_l} \alpha_l, \alpha_l, \alpha_l C_{B_l} v_{l-1}\}$ and $w_1 C_{B_l} w_p$ is of minimal length if such a 2-separator exists. The restriction to these three parts of the boundary is again motivated by Lemma 7: If $P_l \neq \alpha_l$ and there is an outer non-trivial P_l-bridge of B_l, its two attachments are in P_l and thus we only have to split off 2-separators that are in one of these three paths to avoid these P_l-bridges in the induction. If the 2-separator $\{w_1, w_p\}$ exists, let $w_1, \dots, w_p$ be the $p \geq 2$ attachments of the $w_1 C_{B_l} w_p$-bridge of B_l that contains v_l, in the order of appearance in $w_1 C_{B_i} w_p$; otherwise, let for notational convenience $w_1 := \dots := w_p := l_{\alpha_i}$. In the case $v_l \notin C_{B_l}$, let the *boundary points* of B_l be $v_{l-1}, l_{\alpha_l}, r_{\alpha_l}, w_1, \dots, w_p$ and let the *boundary parts* of B_l be the inclusion-wise maximal paths of C_{B_l} that do not contain any boundary point as inner vertex.

▶ **Lemma 8.** *If the 2-separator $\{w_1, w_p\}$ exists, it is unique and every v_{l-1}-α_l-v_l-path P_l of B_l contains the vertices $w_1, \dots, w_p$.*

Proof. Let $J \subset B^+_{w_1,w_p}$ be the $w_1 C_{B_l} w_p$-bridge of B_l that contains v_l and has attachments $w_1, \dots, w_p$. For the first claim, assume to the contrary that there is a 2-separator $\{w'_1, w'_{p'}\} \neq \{w_1, w_p\}$ of B_l having the same properties as $\{w_1, w_p\}$. By the connectivity of J and the property that restricts $\{w'_1, w'_{p'}\}$ to the three parts of the boundary of B_l, $\{w'_1, w'_{p'}\}$ may only split off a subgraph containing v_l if $w_1 C_{B_l} w_p \subset w'_1 C_{B_l} w'_{p'}$. This however contradicts the minimality of the length of $w'_1 C_{B_l} w'_{p'}$.

For the second claim, let P_l be any v_{l-1}-α_l-v_l-path of B_l. Assume to the contrary that $w_j \notin P_l$ for some $j \in \{1, \dots, p\}$. Then w_j is an internal vertex of an outer P_l-bridge J' of B_l. By Lemma 7, both attachments of J' are in C_{B_l}. However, since J contains a path from $w_j \notin P_l$ to $v_l \in P_l$ in which only w_j is in C_{B_l}, at least one attachment of J' is not in C_{B_l}, which gives a contradiction. ◀

Lemma 8 ensures that the boundary points of any B_i are contained in every Tutte path P_i of B_i. Every block $B_i \neq B_l$ has exactly four boundary parts and B_l has at least three boundary parts (three if $v_l \notin C_{B_l}$ and $\{w_1, w_p\}$ does not exist), some of which may have length zero. For every $1 \leq i \leq l$, the boundary parts of B_i partition C_{B_i}, and one of them consists of α_i. This implies in particular that B_i has at least two boundary parts of length at least one unless $B_i = \alpha_i$. We need some notation to break symmetries on boundary parts. For a boundary part Z of a block B, let $\{c, d\}^* \subseteq Z$ denote two elements c and d (vertices or edges) such that cC_Bd is contained in Z (this notation orders c and d consistently to the clockwise orientation of C_B); if cC_Bd is contained in some boundary part of B that is not specified, we just write $\{c, d\}^* \subseteq C_B$.

We now define which 2-separators are split off in our decomposition. Let a 2-separator $\{c, d\}^* \subseteq C_B$ of B be *maximal in a boundary part Z of B* if $\{c, d\} \subseteq Z$ and Z does not contain a 2-separator $\{c', d'\}$ of B such that $cC_Bd \subset c'C_Bd'$. Let a 2-separator $\{c, d\}^* \subseteq C_B$ of B be *maximal* if $\{c, d\}^*$ is maximal with respect to at least one boundary part of B. Hence, every maximal 2-separator is contained in a boundary part, and 2-separators that are contained in a boundary part are maximal if they are not properly "enclosed" by other 2-separators on the same boundary part.

Let two maximal 2-separators $\{c, d\}^*$ and $\{c', d'\}^*$ of B *interlace* if $\{c, d\} \cap \{c', d'\} = \emptyset$ and their vertices appear in the order c, c', d, d' or c', c, d', d on C_B (in particular, both 2-separators are contained in the same boundary part of B). In general, maximal 2-separators of a block B_i of K may interlace; for example, consider the two maximal 2-separators when B_i is a cycle on four vertices in which v_{i-1} and v_i are adjacent. However, the following lemma shows that such interlacing is only possible for very specific configurations.

▶ **Lemma 9.** *Let $\{c, d\}^*$ and $\{c', d'\}^*$ be interlacing 2-separators of B_i in a boundary part Z such that $c' \in cC_{B_i}d$ and at least one of them is maximal. Then $d'C_{B_i}c = v_{i-1}v_i = \alpha_i$.*

Proof. Since $\{c, d\}$ is a 2-separator, $B_i - \{c, d\}$ has at least two components. We argue that there are exactly two. Otherwise, $B_i - \{c, d\}$ has a component that contains the inner vertices of a path P' from c to d in $B_i - (C_{B_i} - \{c, d\})$. Then $B_i - \{c', d'\}$ has a component containing $(P' \cup C_{B_i}) - \{c', d'\}$ and no second component, as this would contain the inner vertices of a path from c' to d' in $B_i - ((P' \cup C_{B_i}) - \{c', d'\})$, which does not exist due to planarity. Since this contradicts that $\{c', d'\}$ is a 2-separator, we conclude that $B_i - \{c, d\}$, and by symmetry $B_i - \{c', d'\}$, have exactly two components.

By the same argument, $inner(cC_{B_i}d)$ and $inner(dC_{B_i}c)$ are contained in different components of $B_i - \{c, d\}$ and the same holds for $inner(c'C_{B_i}d')$ and $inner(d'C_{B_i}c')$ in $B_i - \{c', d'\}$. Hence, the component of $B_i - \{c, d'\}$ that contains $inner(cC_{B_i}d') \neq \emptyset$ does not intersect $inner(d'C_{B_i}c)$. If $inner(d'C_{B_i}c) \neq \emptyset$, this implies that $\{c, d'\} \subseteq Z$ is a 2-separator of B_i, which contradicts the maximality of $\{c, d\}$ or of $\{c', d'\}$. Hence, $inner(d'C_{B_i}c) = \emptyset$, which implies that $d'C_{B_i}c$ is an edge. As Z is not an edge, $d'C_{B_i}c = \alpha_i$. Since c and d' are the only boundary points of B_i, either $\{c, d'\} = \{v_{i-1}, v_i\}$ or $B_i = B_l$, $v_l \notin C_{B_l}$, $\{c, d'\} = \{v_{i-1}, w_2\}$, $v_{i-1} = w_1$ and $w_2 = w_p$. However, the latter case is impossible, as then $\{c, d'\}$ would be a 2-separator that separates $inner(cC_{B_i}d') \neq \emptyset$ and v_l, which contradicts the maximality of $\{c, d\}$ or of $\{c', d'\}$. This gives the claim. ◀

If two maximal 2-separators interlace, Lemma 9 thus ensures that these two are the only maximal 2-separators that may contain v_{i-1} and v_i, respectively. This gives the following direct corollary.

▶ **Corollary 10.** *Every block of K has at most two maximal 2-separators that interlace.*

Note that any boundary part may nevertheless contain arbitrarily many (pairwise non-interlacing) maximal 2-separators. The next lemma strengthens Lemma 7.

▶ **Lemma 11.** *Let P_i be a v_{i-1}-α_i-v_i-path of B_i. Let J be a non-trivial outer P_i-bridge of B_i and let e be an edge in $J \cap C_{B_i}$. Then the attachments of J are contained in the boundary part of B_i that contains e.*

Proof. Let c and d be the attachments of J such that $e \in cC_{B_i}d$ and let Z be the boundary part of B_i that contains e. If $P_i = \alpha_i$, $v_{i-1} = l_{\alpha_i}$ and $v_i = r_{\alpha_i}$ are the only boundary points of B_i. Then c and d are the endvertices of $Z = v_iC_{B_i}v_{i-1} \ni e$, which gives the claim.

Otherwise, let $P_i \neq \alpha_i$. By applying Lemma 7 with $a = l_{\alpha_i}$ and $b = r_{\alpha_i}$, $\{c, d\}$ is a 2-separator of B_i that is contained in C_{B_i}. By definition of $w_1, \dots, w_p$, there are at least three independent paths between every two of these vertices in B_i; thus, $\{c, d\}$ does not separate two vertices of $\{w_1, \dots, w_p\}$. Since all other possible boundary points $(v_{i-1}, l_{\alpha_i}, r_{\alpha_i}, v_i)$ are contained in P_i, applying Lemma 7 on these implies that $\{c, d\}$ does not separate two vertices of these remaining boundary points. Hence, if $\{c, d\} \not\subseteq Z$, we have $B_i = B_l$ and $v_l \notin C_{B_l}$ such that $\{c, d\}$ separates $\{w_1, \dots, w_p\}$ from the remaining boundary points. Since the P_i-bridge J does not contain $\alpha_l \in P_i$, $cC_{B_l}d \subseteq J$ contains $\{w_1, \dots, w_p\}$, but $inner(cC_{B_l}d)$ does not contain any other boundary point. As $v_l \in P_i$, at least one of $\{w_1, w_p\}$ must be in P_i, say w_p by symmetry. Then $d = w_p$, as $w_p \in P_i$ cannot be an internal vertex of J. Now, in both cases $p = 2$ (which implies $c \neq w_1$, as $\{c, d\} \not\subseteq Z = w_1C_{B_l}w_2$) and $p \geq 3$, J contains the edge of P_i that is incident to v_l. As this contradicts that J is a P_i-bridge, we conclude $\{c, d\} \subseteq Z$. ◀

Now we relate non-trivial outer P_i-bridges of B_i to maximal 2-separators of B_i. In the next section, we will use this lemma as a fundamental tool for a decomposition into subgraphs having only small overlaps, which will eventually construct P.

▶ **Lemma 12.** *Let P_i be a v_{i-1}-α_i-v_i-path of B_i such that $P_i \neq \alpha_i$. Then the maximal 2-separators of B_i are contained in P_i and do not interlace pairwise. If J is a non-trivial outer P_i-bridge of B_i, there is a maximal 2-separator $\{c, d\}^*$ of B_i such that $J \subseteq B^+_{cd}$.*

3.2.2 Construction of P

We do not know P_i in advance. However, Lemma 12 ensures under the condition $P_i \neq \alpha_i$ that we can split off every non-trivial outer bridge J of P_i by a maximal 2-separator, no matter how P_i looks like. This allows us to construct P_i iteratively by decomposing B_i along its maximal 2-separators. Since maximal 2-separators only depend on the graph B_i (in contrast to the paths P_i, which depend for example on the $K \cup C_G$-bridges), we can access them without knowing P_i itself. We now give the details of such a decomposition.

▶ **Definition 13.** For every $1 \leq i \leq l$, let $\eta(B_i)$ be α_i if $\alpha_i = v_{i-1}v_i$ and otherwise the graph obtained from B_i as follows: For every maximal 2-separator $\{c, d\}^*$ of B_i, split off B^+_{cd}. Moreover, let $\eta(K) := \eta(B_1) \cup \dots \cup \eta(B_l)$.

If $\alpha_i \neq v_{i-1}v_i$, α_i cannot be a v_{i-1}-α_i-v_i-path of B_i; hence, the maximal 2-separators of K that were split in this definition do not interlace due to Lemma 12. This implies that the order of the performed splits is irrelevant. In any case, we have $V(C_{\eta(B_i)}) \subseteq V(C_{B_i})$ and the only 2-separators of $\eta(B_i)$ must be contained in some boundary part of B_i, as there would have been another split otherwise. See Figure 3 for an illustration of $\eta(B_l)$. The following lemma highlights two important properties of every $\eta(B_i)$.

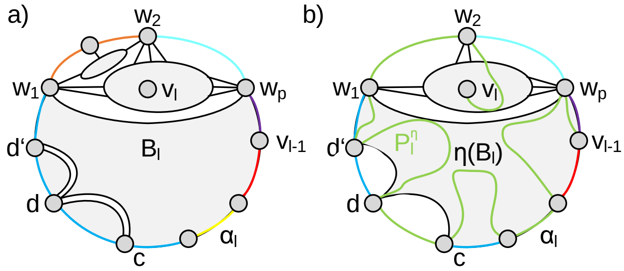

Figure 3 a) A block B_l with boundary points $v_{l-1}, l_{\alpha_l}, r_{\alpha_l}, w_1, \ldots, w_3$ that has two maximal 2-separators on the same boundary part. b) The graph $\eta(B_l)$.

▶ **Lemma 14.** *Every $\eta(B_i)$ is a block. Let P_i^η be a v_{i-1}-α_i-v_i-path of some $\eta(B_i)$ such that $P_i^\eta \neq \alpha_i$. Then every outer P_i^η-bridge of $\eta(B_i)$ is trivial.*

The next lemma shows how we can construct a Tutte path P of K iteratively using maximal 2-separators. We will provide the details of an efficient implementation in Section 4.

▶ **Lemma 15** (Construction of P)**.** *For every $1 \leq i \leq l$, a v_{i-1}-α_i-v_i-path P_i of B_i can be constructed such that no non-trivial outer P_i-bridge of B_i is part of an inductive call of Theorem 1.*

Proof. The proof proceeds by induction on the number of vertices in B_i. If B_i is just an edge or a triangle, the claim follows directly. For the induction step, we therefore assume that B_i contains at least four vertices. If $\alpha_i = v_{i-1}v_i$, we set $P_i := \alpha_i$, so assume $\alpha_i \neq v_{i-1}v_i$. In particular, $\eta(B_i) \neq \alpha_i$ and α_i is no v_{i-1}-α_i-v_i-path of $\eta(B_i)$. As $|V(\eta(B_i))| < n$, we may apply an inductive call of Theorem 1 to $\eta(B_i)$, which returns a v_{i-1}-α_i-v_i-path $P_i^\eta \neq \alpha_i$ of $\eta(B_i)$. This does not violate the claim, since $\eta(B_i)$ does not contain any non-trivial outer P_i^η-bridge by Lemma 14.

Now we extend P_i^η iteratively to the desired v_{i-1}-α_i-v_i-path P_i of B_i by restoring the subgraphs that were split off along maximal 2-separators one by one. For every edge $cd \in C_{\eta(B_i)}$ such that $\{c,d\}^*$ is a maximal 2-separator of B_i (in arbitrary order), we distinguish the following two cases: If $cd \notin P_i^\eta$, we do not modify P_i^η, as in B_i the subgraph B_{cd}^+ will be a valid outer bridge. If otherwise $cd \in P_i^\eta$, we consider the subgraph B_{cd}^+ of B_i. Clearly, $B := B_{cd}^+ \cup \{cd\}$ is a block. Define that the *boundary points* of B are c, d and the two endpoints of some arbitrary edge $\alpha_B \neq cd$ in C_B. This introduces the boundary parts of B in the standard way, and hence defines $\eta(B)$. Note that B may contain several maximal 2-separators in cC_Bd that in B_i were suppressed by $\{c,d\}^*$, as $\{c,d\}^*$ is not a 2-separator of B. In consistency with Lemma 12, which ensures that no two maximal 2-separators of B_i interlace, we have to ensure that no two maximal 2-separators of B interlace in our case $\alpha_i \neq v_{i-1}v_i$, as otherwise $\eta(B)$ would be ill-defined. This is however implied by Lemma 9, as $\alpha_B \neq cd$. Since $|V(\eta(B))| < |V(B_i)|$, a c-α_B-d-path P_B of B can be constructed such that no non-trivial outer P_B-bridge of B is part of an inductive call of Theorem 1. Since $\alpha_B \neq cd$, P_B does not contain cd. We now replace the edge cd in P_i^η by P_B. This gives the desired path P_i after having restored all subgraphs B_{cd}^+. ◀

Applying Lemma 15 on all blocks of K and taking the union of the resulting paths gives P. In the next step, we will modify Q such that $P \cup \{\alpha\} \cup Q$ becomes the desired Tutte path of G. By Lemma 15, no non-trivial outer P-bridge of K was part of any inductive call of Theorem 1 so far, which allows us to use these bridges inductively for the following modification of Q (the existence proof in [29] used these arbitrarily large bridges in inductive calls for both constructing P and modifying Q).

3.2.3 Modification of Q

We show how to modify Q such that $P \cup \{\alpha\} \cup Q$ is an x-α-y-path of G. To this end, consider a $(P \cup \{\alpha\} \cup Q)$-bridge J of G. Since Lemma 5 cannot be applied, J does not have all of its attachments in Q. On the other hand, if J has all of its attachments in $P \subseteq K$, $J \subseteq K$ follows from the maximality of blocks and therefore J satisfies all conditions for a Tutte path of G. Hence, it suffices to consider $(P \cup \{\alpha\} \cup Q)$-bridges that have attachments in both P and Q. The following lemma showcases some of their properties.

▶ **Lemma 16.** *Let J be a $(P \cup \{\alpha\} \cup Q)$-bridge of G that has an attachment in P. Then $J \cap K$ is either exactly one vertex in P or exactly one non-trivial outer P-bridge L of K. In particular, J has at most two attachments in P.*

Let J be a $(P \cup \{\alpha\} \cup Q)$-bridge of G that has attachments in both P and Q and recall that $C(J) = l_J C_G r_J$. Because Lemma 5 is not applicable to G, there is no other $(P \cup \{\alpha\} \cup Q)$-bridge than J that intersects $(J \cup C(J)) - P - \{l_J, r_J\}$; in other words, $J \cup C(J)$ is everything that is enclosed by the attachments of J in G. In order to obtain the Tutte path of Theorem 1, we will thus replace the subpath $C(J)$ with a path $Q_J \subseteq (J \cup C(J)) - P$ from l_J to r_J such that any $(Q_J \cup P)$-bridge of G that intersects $(J \cup C(J)) - P - \{l_J, r_J\}$ has at most three attachments and at most two if it contains an edge of C_G. Since l_J and r_J are contained in Q, no other $(P \cup \{\alpha\} \cup Q)$-bridge of G than J is affected by this "local" replacement, which proves its sufficiency for obtaining the desired Tutte path.

We next show how to obtain Q_J. If $C(J)$ is a single vertex, we do not need to modify Q at all (hence, $Q_J := C(J)$), as then $J \cup C(J)$ does not contain an edge of C_G and has at most three attachments in total (one in Q and at most two in P by Lemma 16). If $C(J)$ is not a single vertex, we have the following lemma.

▶ **Lemma 17** ([29, 4])**.** *Let J be a $(P \cup \{\alpha\} \cup Q)$-bridge of G that has an attachment in P and at least two in Q. Then $(J \cup C(J)) - P$ contains a path Q_J from l_J to r_J such that any $(Q_J \cup P)$-bridge of G that intersects $(J \cup C(J)) - P - \{l_J, r_J\}$ has at most three attachments and at most two if it contains an edge of C_G.*

By Lemma 16, any $(P \cup \{\alpha\} \cup Q)$-bridge J of G intersects K in at most one non-trivial P-bridge of K having attachments c and d. By Lemma 15, this non-trivial P-bridge was never part of an inductive call of Theorem 1 before (in fact, at most its edge cd was). Replacing $C(J)$ with Q_J for every such J, as described in Lemma 17 and before, therefore concludes the constructive proof of Theorem 1.

4 A Quadratic Time Algorithm

In this section, we give an algorithm based on the decomposition shown in Section 3 (see Algorithm 1). It is well known that there are algorithms that compute the blocks of a graph and the block-cut tree of G in linear time, see [25] for a very simple one. Using this on $G - Q$, we can compute the blocks $B_1, \dots, B_l$ of K in time $O(n)$.

We now check if Lemma 4 or 5 is applicable at least once to G; if so, we stop and apply the construction of either Lemma 4 or 5. Checking applicability involves the computation of special 2-separators $\{c, d\}$ of G that are in C_G (e.g., we did assume minimality of $|V(G_R)|$ in Lemma 4). In order to find such a $\{c, d\}$ in time $O(n)$, we first compute the *weak dual* G^* of G, which is obtained from the dual of G by deleting its outer face vertex, and note that such pairs $\{c, d\}$ are exactly contained in the faces that correspond to 1-separators of G^*. Once more, these faces can be found by the block-cut tree of G^* in time $O(n)$ using the above

Algorithm 1 TPATH(G, x, α, y) ▷ method, running time without induction

if G is a triangle or $\alpha = xy$ **then return** the trivial x-α-y path of G ▷ $O(1)$
if Lemma 4 or 5 is applicable at least once to G **then** ▷ weak dual block-cut tree, $O(n)$
  apply TPATH on G_L and G_R as described and **return** the resulting path ▷ $O(1)$
if there is a 2-separator $\{c, d\} \in C_G$ of G **then**
  do simple case 2
Compute the minimal plane chain K of blocks of G ▷ block-cut tree of $G - Q$, $O(n)$
Compute $\eta(K)$ ▷ dyn. progr. on weak dual block-cut tree, $O(n)$
Compute P by the induction of Lemma 15 ▷ dyn. prog. precomputes all possible B^+_{cd}, $O(n)$
Modify Q by the induction of Lemma 17 ▷ traversing outer faces of bridges, $O(n)$
return $P \cup \{\alpha\} \cup Q$

algorithm. Since the block-cut tree is a tree, we can perform dynamic programming on all these 1-separators bottom-up the tree in linear total time, in order to find one desired $\{c, d\}$ that satisfies the respective constraints (e.g. minimizing $|V(G_R)|$, or separating x and α).

Now we compute $\eta(K)$. Since the boundary points of every B_i are known from K, all *maximal* 2-separators can be computed in time $O(n)$ by dynamic programming as described above. We compute in fact the nested tree structure of all 2-separators on boundary parts due to Lemma 12, on which we then apply the induction described in Lemma 15. Hence, no non-trivial outer P-bridge of K is touched in the induction, which allows to modify Q along the induction of Lemma 17.

In our decomposition, every inductive call is invoked on a graph having less vertices than the current graph. The key insight is now to show a good bound on the total number of inductive calls to Theorem 2. In order to obtain good upper bounds, we will restrict the choice of α_i for every block B_i of K such that α_i is an edge of $C_{B_i} - v_{i-1}v_i$. This prevents several situations in which the recursion stops because of the case $\alpha = xy$, which would unease the following arguments. The next lemma shows that only $O(n)$ inductive calls are performed. Its argument is, similarly to one in [5], based on a subtle summation of the Tutte path differences that occur in the recursion tree.

▶ **Lemma 18.** *The number of inductive calls for TPATH(G, x, α, y) is at most* $2n - 3$.

Hence, Algorithm 1 has overall running time $O(n^2)$, which proves our main Theorem 3.

▶ **Corollary 19.** *Let G be a 2-connected plane graph and let α, β, γ be edges of C_G. Then a Tutte cycle of G that contains α, β and γ can be computed in time $O(n^2)$.*

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