Brief Announcement: Characterizing Demand Graphs for (Fixed-Parameter) Shallow-Light Steiner Network

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Abstract

We consider the Shallow-Light Steiner Network problem from a fixed-parameter perspective. Given a graph G, a distance bound L, and p pairs of vertices $\{(s_i, t_i)\}_{i\in[p]}$, the objective is to find a minimum-cost subgraph G' such that s_i and t_i have distance at most L in G' (for every $i \in [p]$). Our main result is on the fixed-parameter tractability of this problem for parameter p. We exactly characterize the demand structures that make the problem "easy", and give FPT algorithms for those cases. In all other cases, we show that the problem is W[1]-hard. We also extend our results to handle general edge lengths and costs, precisely characterizing which demands allow for good FPT approximation algorithms and which demands remain W[1]-hard even to approximate.

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1 Introduction

We study *length-bounded* variants of Steiner Tree and Steiner Forest, which are related to (but still quite different from) *directed* variants. The direct setting assumes that the edges in the graph are directed. While in the length-bounded setting, we typically assume that the input graph and demands are undirected but each demand has a distance bound, and a solution is only feasible if every demand is connected within distance at most the given bound (rather than just being connected). One of the most basic problems of this form is the Shallow-Light Steiner Tree problem (SLST), where the demands $\{(s_i, t_i)\}_{i\in[p]}$ form a star with root $r = s_1 = s_2 = \cdots = s_p$ and there is a global length bound L (so in any feasible solution the distance from r to t_i is at most L for all $i \in [p]$). This problem has been studied extensively [8, 9, 6, 5], but if we generalize this problem to arbitrary demand pairs then we get the Shallow-Light Steiner Network problem, which to the best of our knowledge has received essentially no study.

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▶ **Definition 1** (Shallow-Light Steiner Network). Given a graph $G = (V, E)$, a cost function $c : E \to \mathbb{R}^+$, a length function $l : E \to \mathbb{R}^+$, a distance bound L, and p pairs of vertices $\{(s_i, t_i)\}_{i\in[p]}$. The objective of SLSN is to find a minimum cost subgraph $G' = (V, S)$, such that for every $i \in [p]$, there is a path between s_i and t_i in G' with length at most L.

Let H be the graph with vertex set $\{s_1, \ldots, s_p, t_1, \ldots, t_p\}$ and edge set $\{(s_i, t_i)\}_{i\in[p]}$. We call H the *demand graph* of the problem.

Both the directed and the length-bounded settings share a dichotomy between considering either star demands (Directed Steiner Tree (DST)/SLST) or totally general demands (Directed Steiner Network (DSN)/SLSN). But this gives an obvious set of questions: what demand graphs make the problem "easy" (in FPT) and what demand graphs make the problem "hard" (W[1]-hard)? Recently Feldmann and Marx [4] gave a complete characterization for this for DSN, proving that if the demand graph is transitively equivalent to an "almost-caterpillar" then the problem is in FPT, and otherwise the problem is W[1]-hard.

While *a priori* there might not seem to be much of a relationship between the directed and the length-bounded settings, there are multiple folklore results that relate them, usually by means of some sort of layered graph. For example, any FPT algorithm for the DST problem can be turned into an FPT algorithm for SLST (with unit edge lengths) and vice versa. However, such a relationship is not known for more general demand graphs.

In light of these relationships between the directed and the length-bounded settings and the recent results of [4], it is natural to attempt to characterize the demand graphs that make SLSN easy or hard. We solve this problem, giving a complete characterization of easy and hard demand graphs. Informally, we show that SLSN is significantly harder than DSN: the only "easy" demand graphs are stars (in which case the problem is just SLST) and constant-size graphs. Even tiny modifications, like a star with a single independent edge, become W[1]-hard (despite being in FPT for DSN).

Connection to Overlay Routing: SLST and SLSN are particularly interesting due to their connection to overlay routing protocols that use so-called *dissemination graphs* for routing rather than traditional paths. Routing on dissemination graphs allows these systems to be significantly more robust, and a length bound corresponds to a latency bound, which is critical for many applications. Recently, Babay et al. [1] designed and built such a system, and demonstrated that it can achieve significantly greater reliability and timeliness than traditional routing with only a slight increase in cost. Finding good solutions to $2\binom{n}{2}$ different SLST instances and $\binom{n}{2}$ different SLSN instances is a crucial piece of this system (as these are the graphs on which routing happens). The search for fast algorithms for these instances was the main motivation behind this work. We refer the interested reader to [1] for a further discussion of this routing system and how it related to SLSN and SLST.

2 Our Results

We first define SLSN with respect to a class (set) of demand graphs.

▶ **Definition 2.** The Shallow-Light Steiner Network problem with restricted demand graph class $\mathcal{C}$ ($\text{SLSN}_\mathcal{C}$) is the SLSN problem with the additional restriction that the demand graph H of the problem must be isomorphic to some graph in $\mathcal{C}$.

We define $\mathcal{C}_\lambda$ as the class of all demand graphs with at most λ edges, and $\mathcal{C}^*$ as the class of all star demand graphs (there is a central vertex called the root, and every other vertex in the demand graph is adjacent to the root and only the root). Our main result is that these are *precisely* the easy classes: We first prove that SLSN is in XP for parameter p (i.e. solvable in $n^{f(p)}$ time for some function f), which immediately implies that $\mathrm{SLSN}_{\mathcal{C}_\lambda}$ can be solved in polynomial time if λ is a constant. Note that $\mathrm{SLSN}_{\mathcal{C}^*}$ is precisely the SLST problem, for which a folklore FPT algorithm exists, thus $\mathrm{SLSN}_{\mathcal{C}^*}$ (while NP-hard) is in FPT for parameter p. We also show that, for any other class $\mathcal{C}$ (i.e., any class which is not just a subset of $\mathcal{C}^* \cup \mathcal{C}_\lambda$ for some constant λ), the problem $\mathrm{SLSN}_{\mathcal{C}}$ is W[1]-hard for parameter p. In other words, if the class of demand graphs includes arbitrarily large non-stars, then the problem is W[1]-hard parameterized by the number of demands. More formally, we prove the following theorems.

▶ **Theorem 3.** *The unit-length arbitrary-cost* SLSN *problem with parameter p is in* XP*, and it can be solved in $n^{O(p^4)}$ time.*

To prove this theorem, we first prove a structural lemma which shows that the optimal solution must be the union of several lowest cost paths with restricted length (these paths may be between steiner nodes, but we show that there cannot be too many). Then we just need to guess all the endpoints of these paths, as well as all the lengths of these paths. We prove that there are only $n^{O(p^4)}$ possibilities, and the running time is also $n^{O(p^4)}$.

▶ **Theorem 4.** *The unit-length arbitrary-cost $\mathrm{SLSN}_{\mathcal{C}^*}$ problem is* FPT *for parameter p.*

As mentioned, $\mathrm{SLSN}_{\mathcal{C}^*}$ is exactly the same as SLST, so we use a folklore reduction between SLST and DST to prove this theorem.

▶ **Theorem 5.** *If $\mathcal{C}$ is a recursively enumerable class, and $\mathcal{C} \nsubseteq \mathcal{C}_\lambda \cup \mathcal{C}^*$ for any constant λ, then $\mathrm{SLSN}_{\mathcal{C}}$ is* W[1]*-hard for parameter p, even in the unit-length and unit-cost case.*

All of the above results are in the unit-length setting. We extend both our upper bounds and hardness results to handle arbitrary lengths, but with some extra complications. Even if $p = 1$, with arbitrary lengths and arbitrary costs the SLSN problem is equivalent to the Restricted Shortest Path problem, which is known to be NP-hard [7]. Therefore we can no longer hope for an FPT algorithm (with parameter p). So we change our notion of "easy" from "solvable in FPT" to "arbitrarily approximable in FPT": we show $(1+\varepsilon)$-approximation algorithms for the easy cases, and prove that there is no $\left(\frac{5}{4} - \varepsilon\right)$-approximation algorithm for the hard cases in $f(p) \cdot poly(n)$ time for any function f.

▶ **Theorem 6.** *For any constant $\lambda > 0$, there is a fully polynomial time approximation scheme (*FPTAS*) for the arbitrary-length arbitrary-cost $\mathrm{SLSN}_{\mathcal{C}_\lambda}$ problem.*

▶ **Theorem 7.** *There is a $(1 + \epsilon)$-approximation algorithm in $O(4^p \cdot poly(\frac{n}{\varepsilon}))$ time for the arbitrary-length arbitrary-cost $\mathrm{SLSN}_{\mathcal{C}^*}$ problem.*

Our next theorem is analogous to Theorem 5, but since costs are allowed to be arbitrary we can prove stronger hardness of approximation (under stronger assumptions).

▶ **Theorem 8.** *Assume that the (randomized) Gap-Exponential Time Hypothesis [2] (Gap-ETH) holds. Let $\varepsilon > 0$ be a small constant, and $\mathcal{C}$ be a recursively enumerable class where $\mathcal{C} \nsubseteq \mathcal{C}_\lambda \cup \mathcal{C}^*$ for any constant λ. Then there is no $\left(\frac{5}{4} - \varepsilon\right)$-approximation algorithm in $f(p) \cdot n^{O(1)}$ time for $\mathrm{SLSN}_{\mathcal{C}}$ for any function f, even with unit-lengths and polynomial-costs.*

Note that this theorem uses a much stronger assumption (Gap-ETH rather than W[1] $\neq$ FPT), which assumes that there is no (possibly randomized) algorithm running in $2^{o(n)}$ time that can distinguish whether a 3SAT formula is satisfiable or at most a $(1 - \varepsilon)$-fraction of its clauses can be satisfied. This enables us to utilize the hardness result for a generalized version of the MCC problem from [3], which will allow us to modify our reduction from Theorem 5 to get hardness of approximation.

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