On the Complexity of Infinite Advice Strings

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Abstract

We investigate in this paper a notion of comparison between infinite strings. In a general way, if $\mathcal{M}$ is a computation model (e.g. Turing machines) and $\mathcal{C}$ a class of objects (e.g. languages), the complexity of an infinite word α can be measured with respect to the amount of objects from $\mathcal{C}$ that are presentable with machines from $\mathcal{M}$ using α as an oracle.

In our case, the model $\mathcal{M}$ is finite automata and the objects $\mathcal{C}$ are either recognized languages or presentable structures, known respectively as advice regular languages and advice automatic structures. This leads to several different classifications of infinite words that are studied in detail; we also derive logical and computational equivalent measures. Our main results explore the connections between classes of advice automatic structures, MSO-transductions and two-way transducers. They suggest a closer study of the resulting hierarchy over infinite words.

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Related Version An extended version with full proofs is available at [9], `https://arxiv.org/abs/1801.04908`.

1 Introduction

Several measures have been defined to describe the (intuitive) complexity of infinite strings; among others we mention subword complexity, Kolmogorov complexity, and Turing degrees. Whereas the two first methods focus on the intrinsic information contained in a string, the other one studies the relation of computability from one word to another, defining a preorder whose properties are now quite well understood. Equivalently, this preorder compares the expressive power of Turing machines that use an infinite word as oracle.

This paper follows a similar idea: we consider finite automata that can access an infinite *advice* string while processing their input. Such automata define classes of *advice regular languages* [17], that generalize standard regularity. This notion enables us to introduce a way to compare infinite words: α is simpler (in the sense of languages) than β if every language recognized by an automaton with advice α can also be recognized with advice β. It corresponds to some intuition that α contains less information than β.

Before going further, we evoke the current motivations around advice regular languages. Standard regular languages can be used to encode finite-signature structures, known as *automatic structures*. This concept, derived from Büchi's early automata-logic connections, has been shown especially relevant since its formalization in the 1990's (see e.g. [8]). The model opened the door to a vast range of decision procedures via automata constructions, but it suffers from a lack of expressiveness, since e.g. $\langle \mathbb{Q}, + \rangle$ is not automatic [20]. However, $\langle \mathbb{Q}, + \rangle$ is an example of *advice automatic structure*: it can be encoded using advice regular

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languages (instead of regular languages) [13]. Such structures share many properties with the former automatic structures, furthermore the use of advices builds a rich framework to discuss algorithmic meta-theorems [2]. We shall not follow a model-theoretic point of view on advice automatic structures, but we use them to define another notion of comparison over infinite words as follows: α is simpler (in the sense of structures) than β if every automatic structure with advice α is also automatic with advice β.

Objectives and outline. This paper is structured as a quest for a relevant way to compare infinite strings through the notion of advice. The informal criteria we use to define a "good" complexity measure are the following: it should have a simple definition, be robust enough, but not too coarse because we want to separate simple classes of sequences. Note that Turing degrees do not match this intuition since they make no distinction between all computable (thus useful in practise) sequences. Our results will establish an interesting correspondence between the expressive power of advices (compared more or less using *languages*) and certain forms of *transductions*, when considering the way they classify infinite strings. This is somehow surprising, since the theory of transformations between words tends to be more fruitful and more difficult than the study of languages, following an early remark of Dana Scott [18]: "the functions computed by the various machines are more important - or at least more basic - than the sets accepted by these devices". The concept of advice helps unifying these frameworks. Furthermore, we shall use this idea to provide slightly new perspectives on (advice) automatic presentations and logic over infinite words.

After recalling preliminary results on formal languages, structures and logic, we present formally in Section 3 the notion of regularity with advice, under several variants. We study the comparisons of words provided by the classes of advice languages, as evoked above. An easy correspondence is drawn with transductions, for instance we show that every regular language with advice α is also regular with β if and only if α is the image of β under a Mealy machine. Nevertheless, we conclude that comparisons via languages are far from being robust. The last part of this section introduces the classes of advice automatic structures and briefly describes some of their properties. We then show that some variants of advice regular languages have no influence on the classes of presentable structures. This first involved result is also a first step to obtain a new robust notion of comparison.

Section 4 intends to understand the comparison over infinite words defined with respect to advice automatic structures (see above); it develops our most interesting contributions. Similar investigations were built in [14] under the formalism of set-interpretations, a very close notion. We particularize their results to show that every automatic structure with advice α is also automatic with β if and only if α is the image of β under an MSO-transduction (some logical transformation between words). We then give a more handy equivalent statement: α is the image of β under a two-way transducer. This result is quite specific and original, since such transducers are however not powerful enough to realize all *functions* of infinite words defined by MSO-transductions [4]. In Section 5 we investigate the structural properties of this relation of comparison (defined in particular by two-way transductions). Even if no previous research was done on the subject, a similar study was carried out in [10] for comparison by one-way finite transducers. In the light of their results, we rough out the structure of a new hierarchy and explain why a more involved questioning may be fruitful.

2 Preliminaries

Greek capitals Σ, Γ and Δ are used to denote alphabets, i.e. finite sets of letters; $\square$ is a padding letter that never belongs to these alphabets. If w is a (possibly infinite) word, let

$|w| \in \mathbb{N} \cup \{\omega\}$ be its length, and for $n \geq 0$ let $w[n]$ be its $(n+1)$-th letter (when defined). For $0 \leq m \leq n$, let $w[m:n] = w[m]w[m+1] \cdots w[n-1]$ (when defined, possibly ε). We write $w[:n]$ for the prefix $w[0:n]$, and $w[n:]$ for the (possibly infinite) suffix $w[n]w[n+1] \cdots$. Denote by Reg (resp. ωReg) the class of regular (resp. ω-regular) languages.

We shall deal with structures over a finite (relational) signature, denoted by fraktur letters $\mathfrak{A}$, $\mathfrak{B}$, etc. Equality implicitly belongs to every signature. If τ is a signature and $\mathcal{L}$ a logic, $\mathcal{L}[\tau]$-formulas are $\mathcal{L}$-formulas over the signature τ. Let $\alpha \in \Gamma^\omega$, its word structure $\mathfrak{W}^\alpha = \langle \mathbb{N}, <, (P_a)_{a \in \Gamma} \rangle$ is defined with $<$ the usual ordering, and $n \in P_a$ if $\alpha[n] = a$. For succinctness reasons, $\alpha \models \phi$ stands for $\mathfrak{W}^\alpha \models \phi$ and $\mathsf{MSO}[<, \Gamma]$ for $\mathsf{MSO}[<, (P_a)_{a \in \Gamma}]$. Recall that MSO-formulas can be interpreted using weak semantics (WMSO), where we allow set quantifications to range only over finite sets.

▶ **Definition 1** (convolution)**.** If u and v are (possibly infinite) words, their *convolution* $u \otimes v$ is the word of length $\max(|u|, |v|)$ such that:
- $(u \otimes v)[n] = (u[n], v[n])$ if $n < \min(|u|, |v|)$;
- $(u \otimes v)[n] = (u[n], \square)$ if $|v| \leq n < |u|$;
- $(u \otimes v)[n] = (\square, v[n])$ if $|u| \leq n < |v|$.

▶ **Definition 2** (presentation)**.** Let $\mathfrak{A} := \langle A, R_1 \dots R_n \rangle$ be a relational structure and $\mathcal{C}$ a class of languages (possibly over infinite words). A $\mathcal{C}$*-presentation* of $\mathfrak{A}$ is a tuple $(L, L_=, L_1 \dots L_n)$ of languages from $\mathcal{C}$ such that there exists a surjective function $\nu : L \to A$ with:
- $L_= = \{w \otimes w' \mid w, w' \in L \text{ and } \nu(w) = \nu(w')\}$;
- for R_i (arity r_i), $L_i = \{w_1 \otimes \cdots \otimes w_{r_i} \mid \forall 1 \leq j \leq r_i, w_j \in L \text{ and } (\nu(w_1), \dots, \nu(w_{r_i})) \in R_i\}$.

The function ν describes how A is encoded in L. Since we never consider the elements of A directly, it does not belong explicitly to the presentation and can be considered as a notation. The alphabet of L is called *encoding alphabet* and often denoted Σ. The presentation is said *injective* if $L_= = \{w \otimes w \mid w \in L\}$. Fairly recently, the class of (ω)Reg-presentable structures generated much attention, under the name of *(ω-)automatic structures* [8]. Such structures can be effectively represented using a tuple of automata recognizing the previous languages. We denote by (ω)AutStr the class of (ω-)automatic structures.

▶ **Example 3.** $\langle \mathbb{N}, +, 0, 1 \rangle \in$ AutStr.

▶ **Proposition 4** (folklore, [8])**.** *Every (ω-)automatic structure has a decidable FO-theory.*

Automatic structures enjoy several other useful properties, but the presentation fails for simple structures with decidable theory, as shown in the next theorem.

▶ **Theorem 5** ([20])**.** $\langle \mathbb{Q}, + \rangle$ *is not an (ω-)automatic structure.*

A model-theoretic notion closely related to presentations is concept of *interpretation*, where we describe a structure in another (host) structure via a tuple of logical formulas.

▶ **Definition 6** (interpretation)**.** Let $\mathfrak{A}$ be a structure over a signature τ, $\mathcal{L}$ be a logic and $\mathcal{I} := (\phi_\delta(\overline{x}), \phi_=(\overline{x}, \overline{y}), \phi_1(\overline{x_1} \dots \overline{x_{r_1}}) \dots \phi_p(\overline{x_1} \dots \overline{x_{r_p}}))$ a tuple of $\mathcal{L}[\tau]$-formulas where $\overline{x}, \overline{y}$ and the $\overline{x_i}$ are k-tuples of free variables. Let
- $A_\delta := \{\overline{a} = (a_1 \dots a_k) \mid \mathfrak{A} \models \phi_\delta(\overline{a})\}$;
- $\sim$ is a binary relation on A_δ with $\overline{a} \sim \overline{b}$ if $\mathfrak{A} \models \phi_=(\overline{a}, \overline{b})$;
- for $1 \leq i \leq p$, R_i is a relation on A_δ defined as $(\overline{a_1} \dots \overline{a_{r_i}}) \in R_i$ if $\mathfrak{A} \models \phi_i(\overline{a_1} \dots \overline{a_{r_i}})$;

we say that $\mathcal{I}$ is a *k-dimensional $\mathcal{L}$-interpretation* of a structure $\mathfrak{B}$ in the structure $\mathfrak{A}$ if:
- $\sim$ defines an congruence relation on A_δ with respect to $R_1 \dots R_p$;
- $\langle A_\delta, R_1 \dots R_p \rangle / \sim$ is isomorphic to $\mathfrak{B}$.

The interpretation is said *injective* if $\sim$ is the equality relation of A_δ. In the literature, interpretations are often directly assumed to be *1-dimensional injective interpretations*. The choice of the logic $\mathcal{L}$ provides several kinds of interpretation, detailed in Definition 7.

▶ **Definition 7.**

1. An *FO-interpretation* is a tuple of FO-formulas. The elements of $\mathfrak{A}$ are encoded as tuples of elements in the host structure $\mathfrak{B}$.
2. An *MSO-interpretation* is a tuple of MSO-formulas with free first-order variables. If we use the weak semantics, we speak of WMSO-interpretation. Once more, the elements of $\mathfrak{A}$ are encoded as tuples of elements of $\mathfrak{B}$.
3. An *S-interpretation* (set) is a tuple of MSO-formulas with free set variables. If we use weak semantic, we speak of *FS-interpretation* (finite set). The elements of $\mathfrak{A}$ are encoded as tuples of (finite) sets of elements in the host structure.

▶ **Fact 8** (closure under composition)**.**

1. *If $\mathfrak{A}$ is k-dimensionally FO-interpretable in $\mathfrak{B}$ which is l-dimensionally FO-interpretable in $\mathfrak{C}$, then $\mathfrak{A}$ is directly kl-dimensionally FO-interpretable in $\mathfrak{C}$.*
2. *If $\mathfrak{A}$ is k-dimensionally MSO-interpretable in $\mathfrak{B}$ which is 1-dimensionally MSO-interpretable in $\mathfrak{C}$, then $\mathfrak{A}$ is directly k-dimensionally MSO-interpretable in $\mathfrak{C}$*

▶ Remark. The presence of sets and the use of several dimensions force to be careful in the statements of Fact 8. Indeed, there is no reason why the composition of two S-interpretations should be a S-interpretation, since we obtain sets of sets in the whole transformation. A similar argument works for MSO-interpretations without restrictions on the dimension.

▶ Remark. The above composition properties allow - in specific cases - to transfer the decidability of the logical theory from the host structure to the other one.

Interpretations are a key concept to extend standard automata-logic equivalences from regular languages to automatic structures.

▶ **Proposition 9** ([12])**.** *A structure $\mathfrak{A}$ is automatic (resp. ω-automatic) if and only if $\mathfrak{A}$ is FS-interpretable (resp. S-interpretable) in* $(\mathbb{N}, <)$.

3 From advice regular languages to advice automatic structures

We introduce in this section an extension of regular languages known as *regular languages with advice*. This concept enables us to study some preorders over infinite words; we discuss their relevance and establish a first link with transductions. In the last subsection, we describe the structures that can be presented with these classes of languages.

3.1 Terminating languages

The idea of advice regularity is to consider languages accepted by automata that read an infinite advice string while processing its input [5]. We provide an equivalent definition which does not directly deal with automata but only languages.

▶ **Definition 10.** $L \subseteq \Sigma^*$ is *terminating regular* with advice $\alpha \in \Gamma^\omega$ if there exists a regular language $L' \subseteq (\Sigma \times \Gamma)^*$ such that $L = \{w \mid w \otimes \alpha[: |w|] \in L'\}$.

▶ **Example 11.**

1. If $L \subseteq \Sigma^*$ is regular, so is $\{w \otimes w' \mid w \in L, w' \in \Gamma^*, |w| = |w'|\}$, and considering this language shows that L is regular with any advice of Γ^ω;
2. the set $\text{Pref}(\alpha) := \{\alpha[: n] \mid n \geq 0\}$ is regular with advice α.

We denote by $\mathrm{Reg}[\alpha]$ the class of regular languages with advice α. As evoked in the introduction, our goal is to measure the complexity of infinite words, through the expressiveness of their advice classes. We write $\alpha \preccurlyeq_{\mathrm{Reg}} \beta$ whenever $\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$, this relation is clearly a preorder over infinite words. Let the $\preccurlyeq_{\mathrm{Reg}}$-*degrees* be the equivalence classes of the relation $\preccurlyeq_{\mathrm{Reg}} \cap \succcurlyeq_{\mathrm{Reg}}$, they describe the sets of equally complex advices. We remark that ultimately periodic words (i.e. infinite words of the form uv^ω) form the least $\preccurlyeq_{\mathrm{Reg}}$-degree; indeed the inclusion $\mathrm{Reg} \subseteq \mathrm{Reg}[\alpha]$ is strict if and only if α is not ultimately periodic [5, 16]. We now provide a first equivalence with transductions.

▶ **Definition 12.** A *Mealy machine* is a 6-tuple $(Q, q_0, \Delta, \Gamma, \delta, \theta)$ where Q is the finite set of states, $q_0 \in Q$ initial state, Δ is the input alphabet, Γ is the output alphabet, $\delta : Q \times \Delta \to Q$ is the (partial) transition function, and $\theta : Q \times \Delta \to \Gamma$ is the (partial) output function.

A run of a Mealy machine is a run of the underlying deterministic automaton. On input β, the machine outputs α the concatenation of the outputs along the run on β.

▶ **Proposition 13.** *The following conditions are equivalent:*
1. $\mathrm{Reg}[\alpha] \subseteq \mathrm{Reg}[\beta]$*;*
2. α *is the image of* β *under some Mealy machine.*

Comparison via $\preccurlyeq_{\mathrm{Reg}}$ thus corresponds to computability via Mealy machines. The properties of this preorder were studied under this form in [7]. However, tiny changes in the words completely modify their $\preccurlyeq_{\mathrm{Reg}}$-degree: those classes are far from being robust.

▶ **Fact 14** ([7])**.** *Whenever* α *is not ultimately periodic, we have a strictly increasing chain* $\alpha \prec_{\mathrm{Reg}} \alpha[1:] \prec_{\mathrm{Reg}} \cdots \prec_{\mathrm{Reg}} \alpha[n:] \prec_{\mathrm{Reg}} \cdots$*. A strictly decreasing chain can be obtained similarly with* $\alpha \succ_{\mathrm{Reg}} \Box\alpha \succ_{\mathrm{Reg}} \cdots \succ_{\mathrm{Reg}} \Box^n\alpha \succ_{\mathrm{Reg}} \cdots$*.*

An interesting point is the closure properties of these classes.

▶ **Proposition 15** ([5])**.** $\mathrm{Reg}[\alpha]$ *is closed under boolean operations.*

However, when α is not ultimately periodic, $\mathrm{Reg}[\alpha]$ is not closed under projection (with respect to $\otimes$) [16]. This is a serious issue if one intends to encode logical theories, what may explain why automata with advice have remained unused for many years. A possible solution, detailed in the next paragraph, is to use ω-regularity instead of finite regularity.

3.2 Non-terminating languages and ω-regularity

Once more, we shall provide a definition in terms of languages, but it could equivalently be stated with ω-automata that read an advice string.

▶ **Definition 16** ([13])**.** $L \subseteq \Sigma^\omega$ is ω-*regular* with advice $\alpha \in \Gamma^\omega$ if there is an ω-regular language $L' \subseteq (\Sigma \times \Gamma)^\omega$ such that $L = \{w \mid w \otimes \alpha \in L'\}$.

▶ **Example 17.**
1. Every ω-regular language is also ω-regular with any advice;
2. $\{\alpha\}$ is ω-regular with advice α.

We denote by $\omega\mathrm{Reg}[\alpha]$ the class of ω-regular languages with advice α. The next definition generalizes ω-regularity with advice to finite-words languages.

▶ **Definition 18** ([13])**.** A language $L \subseteq \Sigma^*$ is *non-terminating regular* with advice $\alpha \in \Gamma^\omega$ if there is an ω-regular language $L' \subseteq ((\Sigma \uplus \Box) \times \Gamma)^\omega$ such that $L = \{w \mid w \otimes \alpha \in L'\}$.

▶ **Example 19.** $\forall n \geq 0$, $\text{Pref}(\alpha[n :])$ is non-terminating regular with advice α.

Let $\text{Reg}^{\infty}[\alpha]$ be the class of non-terminating regular languages with advice α. It follows from the definitions that $L \in \text{Reg}^{\infty}[\alpha]$ if and only if $\{w\Box^{\omega} \mid w \in L\} \in \omega\text{Reg}[\alpha]$. These new definitions increase the expressiveness of advice languages, since $\text{Reg}[\alpha] \subseteq \text{Reg}^{\infty}[\alpha]$ and the inclusion is strict when α is not ultimately periodic [13]. Furthermore, they solves the lack of closure properties evoked in the end of Subsection 3.1.

▶ **Proposition 20** ([13]). $\text{Reg}^{\infty}[\alpha]$ *and* $\omega\text{Reg}[\alpha]$ *are closed under boolean operations, cylindrification, and projection (with respect to* $\otimes$*).*

Let us compare infinite words with respect to this ω-regular use of advice. We define the preorders $\preccurlyeq_{\text{Reg}^{\infty}}$ (resp. $\preccurlyeq_{\omega\text{Reg}}$) based on the inclusion of the Reg^{∞} (resp. ωReg) classes, and the corresponding notions of degrees. It is not hard to see that ultimately periodic words are again the least $\preccurlyeq_{\text{Reg}^{\infty}}$- and $\preccurlyeq_{\omega\text{Reg}}$-degree. We now make a non-trivial step towards a generic correspondence between advices, machine transductions, and logic.

▶ **Definition 21.** An ω-regular function f is a (partial) mapping $\Gamma^{\omega} \to \Delta^{\omega}$ whose graph $\{w \otimes f(w) \mid w \in \text{dom}(f)\}$ is an ω-regular language.

▶ **Definition 22** (MSO-relabelling). We say that $\alpha \in \Gamma^{\omega}$ is the image of $\beta \in \Delta^{\omega}$ under an *MSO-relabelling* if there is a tuple $\mathsf{MSO}[<, \Delta]$-formulas $(\phi_a(x))_{a\in\Gamma}$ such that $\forall n \geq 0$, $\alpha[n] = a$ if and only if $\beta \models \phi_a(n)$.

▶ **Proposition 23.** *The following conditions are equivalent:*
1. $\text{Reg}^{\infty}[\alpha] \subseteq \text{Reg}^{\infty}[\beta]$*;*
2. $\omega\text{Reg}[\alpha] \subseteq \omega\text{Reg}[\beta]$*;*
3. α *is the image of* β *under some* ω*-regular function;*
4. α *is the image of* β *under some MSO-relabelling.*

▶ Remark. A word α is the image of β under some Mealy machine if and only if α is the image of β under a *relativized MSO-relabelling*, defined as a relabelling where in the formulas $\phi_a(x)$ every quantification is relativized under x, i.e. of the form $Qy/Y \leq x$.

We obtain in particular $\preccurlyeq_{\omega\text{Reg}} = \preccurlyeq_{\text{Reg}^{\infty}}$ and $\preccurlyeq_{\text{Reg}} \subsetneq \preccurlyeq_{\text{Reg}^{\infty}}$ (see Fact 14 and Example 19). To understand its structure, we briefly give a simple necessary condition for $\alpha \preccurlyeq_{\text{Reg}^{\infty}} \beta$.

▶ **Proposition 24.** *Let* p_{γ} *be the* subword complexity *function of* γ *[3]. If* $\alpha \in \Gamma^{\omega}$ *is the image of* $\beta \in \Delta^{\omega}$ *under some* ω*-regular function, then* $p_{\alpha} \leq K \times p_{\beta}$ *for some constant* K*.*

For all $n \geq 1$, there exists a (computable) string α_n such that $p_{\alpha_n} : k \mapsto n^k$. Necessarily $\text{Reg}^{\infty}[\alpha_n]$ is not contained in any $\text{Reg}^{\infty}[\beta]$ for $\beta \in \{1, \ldots, n-1\}^{\omega}$ because $p_{\beta}(k) \leq (n-1)^k$. This observation shows that the size of the alphabet is an unavoidable parameter for $\preccurlyeq_{\text{Reg}^{\infty}}$, which is not good news when looking for a robust notion of complexity. The rest of this paper will no longer deal with the preorders defined by languages, but it move towards presentable structures in order to describe a more relevant notion of comparison.

3.3 Advice automatic structures

We now turn to classes of structures that are presentable by advice languages. Following Definition 2 and the notations of [2], we denote by $\text{AutStr}[\alpha]$ the class of $\text{Reg}[\alpha]$-presentable structures, $\text{AutStr}^{\infty}[\alpha]$ for $\text{Reg}^{\infty}[\alpha]$-presentable, and $\omega\text{AutStr}[\alpha]$ for $\omega\text{Reg}[\alpha]$-presentable. Such structures are said to be *(*ω*-)automatic with advice* α. Their study is located a level of abstraction higher than what was done above, since the languages have no longer importance in theirselves, but are only used to encode other objects.

An advice automatic structure can be described “effectively” via a tuple of automata (as for standard automatic structures), and a certain advice α. In fact, the decidability feature of automatic structures is preserved as soon as α is decidable enough.

▶ **Proposition 25** ([2]). *If $\mathfrak{W}^\alpha$ has a decidable MSO-theory, every structure in $\omega\mathrm{AutStr}[\alpha]$, $\mathrm{AutStr}^\infty[\alpha]$ or $\mathrm{AutStr}[\alpha]$ has a decidable FO-theory.*

Large classes of infinite words with decidable MSO-theory have been described, see e.g. [6] or [19]. We briefly show why the generalization from automatic structures to advice automatic structures can be fruitful (compare the next result to Theorem 5).

▶ **Fact 26** ([13]). *$\langle \mathbb{Q}, + \rangle \in \mathrm{AutStr}[\alpha]$ for some advice α with decidable MSO-theory.*

We now briefly describe basic properties of presentations with advice.

▶ **Fact 27.** *Inclusion of language classes give $\mathrm{AutStr} \subseteq \mathrm{AutStr}[\alpha] \subseteq \mathrm{AutStr}^\infty[\alpha]$ and $\omega\mathrm{AutStr} \subseteq \omega\mathrm{AutStr}[\alpha]$. Inclusions are equalities if α is ultimately periodic.*

▶ Remark. There is however no immediate argument to deduce $\mathrm{AutStr} \subsetneq \mathrm{AutStr}[\alpha]$ when α is not ultimately periodic. We shall see in Section 5 that this statement is true.

As an immediate consequence of the definitions, $\mathrm{AutStr}^\infty[\alpha] \subseteq \omega\mathrm{AutStr}[\alpha]$ and $\omega\mathrm{AutStr}[\alpha]$ contains uncountable structures, whereas $\mathrm{AutStr}^\infty[\alpha]$ does not. This idea can be refined.

▶ **Theorem 28** ([2]). *$\mathrm{AutStr}^\infty[\alpha]$ is exactly the subclass of countable structures of $\omega\mathrm{AutStr}[\alpha]$.*

The next result shows to what extent the advice contains the seeds of every presentation, and how we generalized the case of automatic structures.

▶ **Proposition 29** ([1]).

1. *$\mathfrak{A} \in \omega\mathrm{AutStr}[\alpha]$ if and only if $\mathfrak{A}$ is S-interpretable in $\mathfrak{W}^\alpha$;*
2. *$\mathfrak{A} \in \mathrm{AutStr}^\infty[\alpha]$ if and only if $\mathfrak{A}$ is FS-interpretable in $\mathfrak{W}^\alpha$.*

▶ Remark ([1]). If the presentation is injective and the encoding is alphabet binary, the resulting interpretation can be done 1-dimensional and injective.

Dealing directly with $\mathrm{Reg}[\alpha]$-presentations seems more difficult, since basic properties lack to this class of languages. We now show $\mathrm{AutStr}^\infty[\alpha] = \mathrm{AutStr}[\alpha]$, hence the expression “advice automatic structure” is not ambiguous. To give an intuition of the proof, we note that an ω-automaton performs an infinite run on $w \otimes \alpha$ (for w finite) in two steps: first, it follows a finite run on $w \otimes \alpha[: |w|]$, then it checks some ω-regularity on $\Box^\omega \otimes \alpha[|w| :] \simeq \alpha[|w| :]$. Basically, the ω-regularity feature is only used on suffixes of the advice. On the other hand, a automaton for $\mathrm{Reg}[\alpha]$ is blind to the ω-future. We show that it can nevertheless look at some “finite amount of future” and deduce corresponding ω-regularity on the suffixes. A key idea is that since the advice is *fixed*, so are several properties of its suffixes.

▶ **Theorem 30.** *Let L be an ω-regular language and $\alpha \in \Gamma^\omega$ a fixed word. There is a (finite words) regular language L' and $N \geq 0$ such that for all $n \geq N$, $\alpha[n :] \in L$ if and only if $\alpha[n :]$ has a finite prefix in L'. Furthermore, if L can be described by an FO$[<, \Gamma]$-sentence, L' can be described by an FO$[<, \Gamma]$-sentence as well.*

Proof sketch. The case of FO is treated via equivalence with LTL, known [15] as Kamp's Theorem. For MSO in general, we deduce the result from the work of A.L. Semenov [19]. ◀

Corollary 31 will formalize our intuition that terminating automata can check ω-regular properties on suffixes. It thus enables us to explicit the relationships between $\mathrm{Reg}[\alpha]$ and $\mathrm{Reg}^\infty[\alpha]$, and between $\mathrm{AutStr}[\alpha]$ and $\mathrm{AutStr}^\infty[\alpha]$.

▶ **Corollary 31.** *Let $L \subseteq \Gamma^\omega$ be an ω-regular language and $\alpha \in \Gamma^\omega$. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\{0^n \square^{f(n)} \mid \alpha[n:] \in L\} \in \mathrm{Reg}[\alpha]$.*

▶ **Corollary 32.** *Let $\alpha \in \Gamma^\omega$. For every language $L \in \mathrm{Reg}^\infty[\alpha]$, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\{w \square^{f(|w|)} \mid w \in L\} \in \mathrm{Reg}[\alpha]$.*

▶ **Corollary 33.** *For every advice α, $\mathrm{AutStr}[\alpha] = \mathrm{AutStr}^\infty[\alpha]$.*

▶ Remark. Thanks to these results, we also managed to build a normal form for MSO-formulas (with free variables) when interpreted in a fixed word model, see [9].

4 Complexity of advices when describing structures

After the first results of the previous section on advice automatic structures, we are now able to understand which preorder they describe over infinite words. Corollary 33 implies in particular that $\mathrm{AutStr}[\alpha] \subseteq \mathrm{AutStr}[\beta]$ if and only if $\mathrm{AutStr}^\infty[\alpha] \subseteq \mathrm{AutStr}^\infty[\beta]$. The objective of this section is to show equivalence with $\omega\mathrm{AutStr}[\alpha] \subseteq \omega\mathrm{AutStr}[\beta]$ and give several other characterizations. The climax lies in Theorem 39 and Theorem 45, where we relate our notions to well-known logical transformations and finite transducers.

▶ **Definition 34.** A (k-copying) *MSO-transduction (MSOT)* from Δ^ω to Γ^ω is a tuple of MSO$[<, \Delta]$-formulas with free first-order variables.

$$(\phi_1^a(x))_{a\in\Gamma} \dots (\phi_k^a(x))_{a\in\Gamma}, (\phi_{i,j}^<(x,y))_{1\le i,j\le k})$$

The semantics of an MSOT τ is defined as that of an MSO-interpretation in k disjoint copies of a host word structure. More precisely, the structure $I_\tau(\mathfrak{W}^\beta)$ (not necessarily a word) has signature $\{<, (P_a)_{a\in\Gamma}\}$ and is defined as follows:

- $\mathrm{dom}(I_\tau(\mathfrak{W}^\beta)) = \bigcup_{1\le i\le k}\{(n,i) \mid$ there is $a \in A$ such that $\beta \models \phi_i^a(n)\}$;
- if $(n,i) \in \mathrm{dom}(I_\tau(\mathfrak{W}^\beta))$, then $(n,i) \in P_a$ if and only if $\beta \models \phi_i^a(n)$;
- if $(m,j) \in \mathrm{dom}(I_\tau(\mathfrak{W}^\beta))$, then $(n,i) < (m,j)$ if and only if $\mathcal{U} \models \phi_{i,j}^<(n,m)$.

Since we are interested in transformations between words, we only consider the case when $I_\tau(\mathfrak{W}^\beta)$ is a word structure (what is syntactically definable by adding an MSO$[<, \Delta]$-sentence for the domain). Each MSO-transduction τ then realizes a (partial) function $\tau : \Delta^\omega \to \Gamma^\omega$ whose domain is $\{\beta \in \Delta^\omega \mid I_\tau(\mathfrak{W}^\beta)$ is (isomorphic to) a word structure$\}$, the image $\tau(\beta)$ of β being the unique α such that $I_\tau(\mathfrak{W}^\beta) \simeq \mathfrak{W}^\alpha$.

The reader is asked to keep in mind that MSOT define a certain class of functions on infinite strings, even if our main concern is only the existence of a transduction between two fixed words. We write $\alpha \preccurlyeq_{\mathsf{MSOT}} \beta$ if there is a MSO-transduction τ such that $\tau(\beta) = \alpha$.

▶ Remark. MSO-relabelings (see Definition 22), relativized MSO-relabelings, and 1- dimensional MSO-interpretations can all seen as syntactical fragments of 1-copying MSOT.

▶ Remark. Even if MSO-interpretations in general are not closed under composition, it is the case of MSOT [4]. Thus $\preccurlyeq_{\mathsf{MSOT}}$ is transitive, and is even a preorder over infinite words.

▶ **Example 35.**

1. If $\alpha \preccurlyeq_{\mathrm{Reg}^\infty} \beta$ then $\alpha \preccurlyeq_{\mathsf{MSOT}} \beta$ (thus $\preccurlyeq_{\mathsf{MSOT}}$ is a more generic notion of comparison than the preorders of Section 3, we shall see that the increase of power is strict);
2. modifying a finite part of α does not change its MSOT-degree;
3. if w is a finite word, we denote by $\widetilde{w}$ its mirror image; if $\alpha := w_1\#w_2\#\dots \in (\Gamma^*\#)^\omega$, let $\widetilde{\alpha} := \widetilde{w_1}\#\widetilde{w_2}\#\cdots$; then $\widetilde{\alpha} \preccurlyeq_{\mathsf{MSOT}} \alpha$.

4.1 From automatic structures to MSO-transductions

When searching a complete structure of an advice, a naive idea is that $\mathfrak{W}^\alpha \in \text{AutStr}^\infty[\beta]$ if and only if $\text{AutStr}^\infty[\alpha] \subseteq \text{AutStr}^\infty[\beta]$. However, this statement will turn out to be false. We need a stronger object that is presented in Definition 36.

▶ **Definition 36** ([14]). Let $\mathfrak{A} = \langle A, R_1 \dots R_n \rangle$ be a structure, we define its *weak powerset structure* $\mathcal{P}^f(\mathfrak{A})$ as the structure $\langle \mathcal{P}^f(A), R'_1 \dots R'_n, \subseteq \rangle$ where:
- $\mathcal{P}^f(A)$ is the weak powerset (set of finite subsets) of A;
- $\subseteq$ is the inclusion relation on $\mathcal{P}^f(A)$;
- $R'_i(A_1, \dots, A_{r_i})$ holds in $\mathcal{P}^f(\mathfrak{A})$ if and only if $A_1, \dots A_{r_i}$ are singletons $\{a_1\}, \dots, \{a_{r_i}\}$ and $R_i(a_1, \dots a_{r_i})$ holds in $\mathfrak{A}$.

▶ Remark. $\mathfrak{A}$ is FS-interpretable in $\mathfrak{B}$ if and only if $\mathfrak{A}$ is FO-interpretable in $\mathcal{P}^f(\mathfrak{B})$.

▶ **Fact 37.** $\text{AutStr}^\infty[\alpha]$ *is the class of structures FO-interpretable in* $\mathcal{P}^f(\mathfrak{W}^\alpha)$ *(see Proposition 29). We have* $\text{AutStr}^\infty[\alpha] \subseteq \text{AutStr}^\infty[\beta]$ *if and only if* $\mathcal{P}^f(\mathfrak{W}^\alpha) \in \text{AutStr}^\infty[\beta]$.

This result provides a characterization which is abstract and, in some respects, trivial. Nevertheless, we get the intuition that powerset structures are a key notion to understand advice automaticity. In the sequel, a (Δ-labelled) tree structure has the form $\langle A, <, (P_a)_{a \in \Delta} \rangle$ where the domain A is a prefix-closed subset of $\{0,1\}^*$, $w < w'$ holds whenever w is a prefix of w' and the P_a label the nodes of A with $a \in \Delta$. Word structures are particular trees.

▶ **Theorem 38** ([14], Corollary 4.4). *Let* $\mathfrak{A}$ *a structure and* $\mathfrak{T}$ *a tree structure. If* $\mathcal{P}^f(\mathfrak{A})$ *is* 1*-dimensionally injectively FS-interpretable in* $\mathfrak{T}$*, then* $\mathfrak{A}$ *is* 1*-dimensionally injectively WMSO-interpretable in* $\mathfrak{T}$*.*

In the case of advice automatic structures, Theorem 38 is at the same time too generic and too restrictive. On the one hand, we only use interpretations in word structures $\mathfrak{W}^\alpha$. On the other hand, we need arbitrarily dimensional FS-interpretations, and they are not supposed to be injective. We will manage to meet this conditions, up to a slight modification of the advice, and the WMSO-interpretation will be transformed into a more generic MSOT.

▶ **Theorem 39.** *The following conditions are equivalent:*
1. $\omega\text{AutStr}[\alpha] \subseteq \omega\text{AutStr}[\beta]$*;*
2. $\text{AutStr}^\infty[\alpha] \subseteq \text{AutStr}^\infty[\beta]$*;*
3. $\alpha \preccurlyeq_{\mathsf{MSOT}} \beta$*.*

Proof sktech. We use Proposition 29 several times. The way from 1. to 2. is a consequence of Theorem 28. If 2. holds, we show that $\mathcal{P}^f(\mathfrak{W}^\alpha)$ has an injective binary $\text{Reg}^\infty[\beta']$-presentation for some infinite word β' so that $\beta' \preccurlyeq_{\mathsf{MSOT}} \beta$. As remarked above, $\mathcal{P}^f(\mathfrak{W}^\alpha)$ is thus 1-dimensionally injectively FS-interpretable in the tree $\mathfrak{W}^{\beta'}$, hence Theorem 38 provides a 1-dimensionally WMSO-interpretation of $\mathfrak{W}^\alpha$ in $\mathfrak{W}^{\beta'}$, what implies $\alpha \preccurlyeq_{\mathsf{MSOT}} \beta'$. Composing MSOT concludes that $\alpha \preccurlyeq_{\mathsf{MSOT}} \beta$. If 3. is true and $\mathfrak{A}$ is S-interpretable in $\mathfrak{W}^\alpha$, then $\mathfrak{A}$ is S-interpretable in $\mathfrak{W}^\beta$ by some composition argument. ◀

As a consequence, all the preorders defined by advice-presentable structures converge towards the same comparison via MSO-transductions. This point gives a deep theoretical meaning to their study. Another virtue of Theorem 39 is the ability to translate immediately the results of Example 35 in terms of advice automatic structures.

▶ **Example 40.**
1. If $\alpha \preccurlyeq_{\text{Reg}^\infty} \beta$ then $\text{AutStr}[\alpha] \subseteq \text{AutStr}[\beta]$;
2. modifying a finite part of α does not modify $\text{AutStr}[\alpha]$;
3. if $\alpha \in (\Gamma^* \#)^\omega$, then $\text{AutStr}[\alpha] = \text{AutStr}[\widetilde{\alpha}]$.

4.2 An equivalent computational model: two-way transducers

We will complete our parallel with transductions via an equivalent simple machine model. Furthermore, it will be very useful to describe the structural properties of the preorder.

▶ **Definition 41.** A *two-way finite transducer* (2WFT) is a 6-tuple $(Q, q_0, \Delta \uplus \{\vdash\}, \Gamma, \delta, \theta)$ where Q is the finite set of states, $q_0 \in Q$ is initial, Δ is the input alphabet, Γ is the output alphabet, $\delta : Q \times (\Delta \uplus \{\vdash\}) \to Q \times \{\triangleleft, \triangleright\}$ is the (partial) transition function, and $\theta : Q \times (\Delta \uplus \{\vdash\}) \to \Gamma^*$ is the (partial) output function.

A 2WFT has a two-way read-only input tape and a one-way output mechanism. The component $\{\triangleleft, \triangleright\}$ determines the left or right move of the head on the input tape. When the 2WFT is given $\beta \in \Delta^\omega$ as an input word, this tape contains $\vdash \beta$ (adding a symbol $\vdash$ helps the transducer to notice the beginning of its input when going left). The definition of the (partial) function $\Delta^\omega \to \Gamma^\omega$ realized the 2WFT follows like for Mealy machines.

▶ Remark. The transducer is said to be *one-way* (1WFT, or just finite transducer) if all its transitions are of the form $(q, \triangleright)$. Mealy machines are a particular case of 1WFT.

▶ **Example 42.** There is a three-state 2WFT outputting $\widetilde{\alpha}$ on every $\alpha \in (\Gamma\#)^\omega$. Its behavior is the following: scan a maximal #-free block, read it in a reversed way while outputting, then output # and move to the next block.

When considering definable *functions* between *finite* strings, a well-known equivalence holds between MSOT and 2WFT (Theorem 43). The definitions of MSOT and 2WFT have to be slightly sharpened to get the exact correspondence, see details in [11].

▶ **Theorem 43** ([11]). *(Partial) functions over finite words $\Delta^* \to \Gamma^*$ definable by MSOT are the (partial) functions realized by 2WFT.*

Fairly recently, this result was extended to functions between infinite strings, but some complications quickly appear: deciding the validity of MSO-sentences is not always possible without reading the (variable) input entirely. Thus 2WFT alone are not powerful enough and they need extra features like ω-regular lookahead, i.e. ability to check instantly ω-regular properties of the suffixes of the input starting in the position of the reading head.

▶ **Theorem 44** ([4]). *(Partial) functions over infinite words $\Delta^\omega \to \Gamma^\omega$ definable by MSOT are the (partial) functions realized by 2WFT with ω-regular lookahead whose runs always visit the whole input string.*

When looking closely at Theorem 43 and Theorem 44 in the light of our previous results, a question arises naturally: it is possible to get rid of the lookaheads when fixing the input infinite word? Indeed, we have always considered transformations from a *fixed* word and we noticed in Subsection 3.3 that this restriction simplified certain notions. Theorem 45 gives a positive answer. This involved result is not a direct consequence of Theorem 44, since we are not aware of a simple manner to remove the ω-lookaheads when fixing the input.

▶ **Theorem 45.** *$\alpha \preccurlyeq_{MSOT} \beta$ if and only if $\alpha \preccurlyeq_{2WFT} \beta$.*

Proof sketch. If $\alpha \preccurlyeq_{2WFT} \beta$, the result follows from Theorem 44. Indeed the transformation can be computed by some 2WFT (with a trivial ω-lookahead) whose run visits the whole input. Assume now that $\alpha \preccurlyeq_{MSOT} \beta$. It follows from [4] that α can be computed from β by an *ω-streaming string transducer (SST)*. We provide a rather long argument to show that an SST can be transformed into a 2WFT with a *lookbehind* feature, when the input word is fixed. Lastly, the lookbehind can be removed by some standard techniques (a lookbehind only deals with a *finite* part of the input, which is not the case of an ω-lookahead). ◀

5 The two-way transductions hierarchy

We initiate in this section a study of the previous two-way transductions between infinite words. It can equivalently be seen as the preorder defined by MSOT, or classes $\mathrm{AutStr}[\alpha]$, $\mathrm{AutStr}^{\infty}[\alpha]$ and $\omega\mathrm{AutStr}[\alpha]$; but the 2WFT formulation is - as predicted above - the easiest way to deduce interesting statements. We shall use the term *2WFT hierarchy* to describe the ordered set of 2WFT-degrees (i.e. equivalence classes of $\preccurlyeq_{\mathsf{2WFT}} \cap \succcurlyeq_{\mathsf{2WFT}}$).

A more or less similar work has been done in [10], with the relation $\preccurlyeq_{\mathsf{1WFT}}$ defined by computability via 1WFT. This definition clearly describes a preorder. Even if no previous research exists on the 2WFT hierarchy, we shall see that several results on the 1WFT can be adapted in our context, after a variable amount of work. Note that $\preccurlyeq_{\mathsf{1WFT}} \subseteq \preccurlyeq_{\mathsf{2WFT}}$.

▶ **Proposition 46.**
1. *There are uncountably many distinct 2WFT-degrees;*
2. *a set of 2WFT-degree has an upper bound if and only if it is countable;*
3. *the 2WFT hierarchy has no greatest degree;*
4. *every 2WFT-degree contains a binary string.*

▶ Remark. Considering binary strings is thus sufficient to describe all the degrees. Comparing this result with Proposition 24 shows that the preorder $\preccurlyeq_{\mathrm{Reg}^{\infty}}$ defined by MSO-relabelings is strictly weaker than $\preccurlyeq_{\mathsf{2WFT}} = \preccurlyeq_{\mathsf{MSOT}}$.

As a consequence of Proposition 46, the 2WFT hierarchy is not trivial. We now show that it is fine-grained enough to distinguish ultimately periodic words.

▶ **Proposition 47.** *Ultimately periodic words are the least 2WFT-degree.*

This result shows, through the equivalences of Section 4, that non-trivial advices *strictly* increase the class of presentable structures (what had no reason to be obvious).

▶ **Corollary 48.** *$\mathcal{P}^f(\mathfrak{W}^{\alpha})$ is automatic if and only if α is ultimately periodic.*

▶ Remark. There are non-ultimately periodic sequences α such that $\mathfrak{W}^{\alpha}$ is automatic [6]. However, no sufficient and necessary condition is known to describe such sequences.

We now turn to a more involved statement. A sequence β is said to be *prime* if it is a minimal but non-trivial word. Formally, β non-ultimately periodic is prime in the 2WFT hierarchy if for all $\alpha \preccurlyeq_{\mathsf{2WFT}} \beta$, either $\beta \preccurlyeq_{\mathsf{2WFT}} \alpha$ or α is ultimately periodic. The existence of prime sequences shows in particular that the 2WFT hierarchy is not dense.

▶ **Theorem 49.** *The sequence $\pi := \prod_{n=0}^{\infty} 0^n 1$ is prime in the 2WFT hierarchy.*

Proof sketch. Our work is to show that if $\alpha \preccurlyeq_{\mathsf{2WFT}} \pi$, then $\alpha \preccurlyeq_{\mathsf{1WFT}} \pi$. Now, since π is prime in the 1WFT-hierarchy [10], either $\pi \preccurlyeq_{\mathsf{1WFT}} \alpha$ or α is in the least 1WFT-degree, which is also the set of ultimately periodic words. ◀

Classifying all infinite strings may neither be relevant nor useful in practice. We now look at two particular classes of infinite words closed under 2WFT transformations.

▶ **Proposition 50** (subhierarchies).
1. *If $\alpha \preccurlyeq_{\mathsf{2WFT}} \beta$ and if β is computable, then α is computable;*
2. *if $\alpha \preccurlyeq_{\mathsf{2WFT}} \beta$ and if $\mathfrak{W}^{\beta}$ has a decidable MSO-theory, so has $\mathfrak{W}^{\alpha}$.*

▶ **Fact 51.** *The string π has a decidable MSO-theory (see e.g. [6]).*

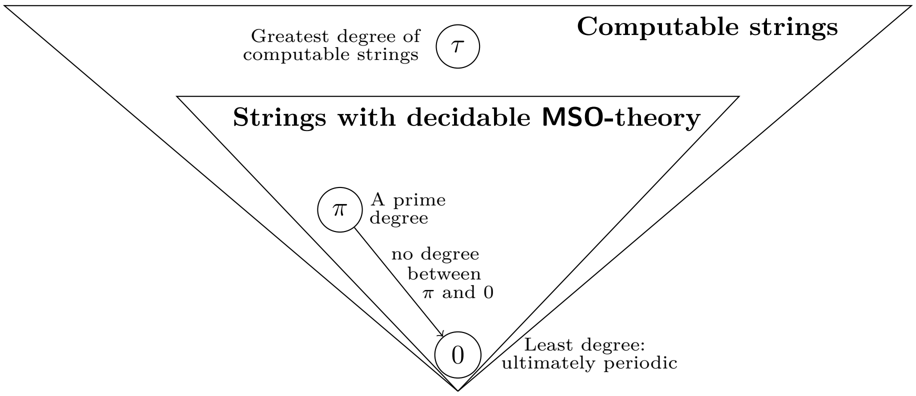

Figure 1 An partial look on the 2WFT hierarchy

Table 1 Equivalent definitions for preorders over ω-words

Advice	Reg	Reg^∞ ωReg	AutStr AutStr^∞ ωAutStr
Logic	rel. MSO-relabelings	MSO-relabelings	MSOT
Machine	Mealy machines	ω-regular functions	2WFT

▶ **Proposition 52** (adapted from [10] for 1WFT)**.** *There exists a greatest degree τ of computable strings in the 2WFT hierarchy.*

▶ **Fact 53.** *The MSO theory of τ is not decidable.*

Figure 1 summarizes the previous results. Note that the 2WFT-degree of ultimately periodic sequences, the 2WFT-degree of π and the 2WFT-degree of τ have to be distinct. Several challenging issues naturally arise about the structure of the 2WFT hierarchy and its subhierarchies. Among others, an interesting question is to describe the degrees of well-known sequences with decidable MSO-theory, for instance morphic words [6].

6 Conclusion and outlook

Preorders of advices, logic and transducers. Our first concern in this paper was the study of various preorders over infinite words, related to the notion of advice strings. The results draw a generic correspondance between definability with advice, logical transductions and machine transductions. Table 1 summarizes this philosophy in an elegant way, note that the notion of (relativized) MSO-relabelings is less standard than MSOT. The gap between MSO-relabelings and MSOT shows that having basic knowledge on the languages is far from being sufficient to understand the richness of presentable structures.

A meaningful hierarchy of infinite words. Two-way transductions appear here to be more basic than relations defined by one-way machines, since they are clearly motivated by logical issues. Furthermore, it fits our informal conditions to be a "good" complexity measure over infinite words. A more involved study of the 2WFT hierarchy may help classifying certain hierarchies of structures, or even understand standard automatic presentations. We recall that such transductions over infinite words are (rather) unexplored.

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