Polynomial Vector Addition Systems With States

Jérôme Leroux[1]
Univ.Bordeaux, CNRS, Bordeaux-INP, Talence, France
jerome.leroux@labri.fr

Abstract

The reachability problem for vector addition systems is one of the most difficult and central problems in theoretical computer science. The problem is known to be decidable, but despite intense investigation during the last four decades, the exact complexity is still open. For some sub-classes, the complexity of the reachability problem is known. Structurally bounded vector addition systems, the class of vector addition systems with finite reachability sets from any initial configuration, is one of those classes. In fact, the reachability problem was shown to be polynomial-space complete for that class by Praveen and Lodaya in 2008. Surprisingly, extending this property to vector addition systems with states is open. In fact, there exist vector addition systems with states that are structurally bounded but with Ackermannian large sets of reachable configurations. It follows that the reachability problem for that class is between exponential space and Ackermannian. In this paper we introduce the class of polynomial vector addition systems with states, defined as the class of vector addition systems with states with size of reachable configurations bounded polynomially in the size of the initial ones. We prove that the reachability problem for polynomial vector addition systems is exponential-space complete. Additionally, we show that we can decide in polynomial time if a vector addition system with states is polynomial. This characterization introduces the notion of iteration scheme with potential applications to the reachability problem for general vector addition systems.

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1 Introduction

Vector addition systems or equivalently Petri nets are one of the most popular formal methods for the representation and the analysis of parallel processes [4]. The reachability problem is central since many computational problems (even outside the parallel processes) reduce to the reachability problem. In 1981, Mayr [13] provided the first decidability proof of the reachability problem. Later, that proof was first simplified by Kosaraju [7], and then ten years later by Lambert [9]. This last proof still remains difficult and the complexity upper bound of the corresponding algorithm is just known to be non-primitive recursive [11]. Nowadays, up to some details, there are only two different known reachability algorithms for general vector addition systems; one based on the Kosaraju-Lambert-Mayr (KLM) decomposition; and a recent one based on Presburger inductive invariants [10]. Despite intense investigation during the last four decades, it is still an open problem whether an elementary complexity upper bound for the reachability problem exists.

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When the reachability set of a vector addition system is finite, the KLM decomposition degenerates and it just corresponds to the regular language of all possible executions of the vector addition system from an initial configuration to a final one. Even in that case, the complexity of the KLM algorithm is Ackermannian and no better complexity upper bound are known.

In 2008, Praveen and Lodaya proved that the reachability problem for structurally bounded vector addition systems, the class of vector addition systems with finite reachability sets from any initial configuration is polynomial-space complete [17]. Surprisingly, extending this property to vector addition systems with states is open. In fact, there exist vector addition systems with states that are structurally bounded but with Ackermannian large sets of reachable configurations. It follows that the reachability problem for that class is between exponential space and Ackermannian.

Intuitively, for structurally bounded vector addition systems with states, the KLM algorithm fails to avoid enumerating all the possible reachable configurations since it tries to detect cycles of edges that can be iterated to obtain arbitrarily large components (such a cycle cannot exists due to the structurally bounded condition). Characterizing indexes that can be very large but not necessarily arbitrarily large should provide new insights on how to overcome the Ackermannian complexity of the KLM algorithm.

Our contributions

In this paper we introduce the class of *polynomial vector addition systems with states* defined as the vector addition systems with states such that reachable configurations have sizes polynomially bounded with respect to sizes of initial configurations. We prove that a vector addition system with states is not polynomial if, and only if, it contains a so-called *iteration scheme* that can increase some components. We prove that we can decide in polynomial time if a vector addition system with states is polynomial, and we show that the reachability problem for polynomial vector addition systems with states is exponential-space complete. Up to our knowledge, our notion of iteration scheme is new and provide a potential application to patch the KLM algorithm.

Outline

In Section 2 we introduce vector addition systems with states (VASS for short), and the subclass of polynomial VASS. Iteration schemes are defined in Section 3. Intuitively iteration schemes are sequences of cycles that can be iterated many times (at least an exponential number of times). Indexes that can be increased by an iteration scheme are called iterable indexes, and edges that occur in iteration schemes are called iterable edges. We show that reachable configurations cannot be polynomially bounded with respect to the size of the initial configurations on any iterable index. It follows that VASS with iterable indexes cannot be polynomial. In Section 4, we recall some general properties about the Kirchoff's functions and the Euler's lemma. Those definitions are used in Section 5 to prove the correctness of a polynomial-time algorithm inspired by the Kosaraju-Sullivan algorithm for computing the set of iterable indexes and the set of iterable edges. In Section 6 we show that reachable configurations are polynomially bounded on the non-iterable indexes with respect to the size of the initial configurations. Finally in Section 7 we show that we can decide in polynomial time if a VASS is polynomial and we prove that the reachability problem for polynomial VASS is exponential-space complete.

2 Polynomial Vector Addition Systems With States

In this section we first introduce the vector addition systems, and the structurally bounded ones. Then, we recall how the reachability problem for that subclass can be solved in polynomial space[17]. Next, we introduce the vector addition systems with states (VASS) and we show that the previous approach for VAS no longer apply to VASS. Finally, we introduce the class of polynomial VASS, the main class of VASS studied in this paper.

Concerning notations used in this paper, we denote by $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ the set of *integers*, *natural numbers*, and *rational numbers*. The absolute value of a *rational* $\lambda \in \mathbb{Q}$ is denoted by $|\lambda|$. Let $d \in \mathbb{N}$ be a natural number. *Vectors* in $\mathbb{Q}^d$ are denoted in bold face, and we denote by $\mathbf{v}[1], \ldots, \mathbf{v}[d]$ the components of $\mathbf{v}$, i.e. $\mathbf{v} = (\mathbf{v}[1], \ldots, \mathbf{v}[d])$. Every operations are performed component-wise on the vectors; for instance the sum $\mathbf{x} + \mathbf{y}$ of two vectors in $\mathbb{Q}^d$ is the vector $\mathbf{z}$ in $\mathbb{Q}^d$ satisfying $\mathbf{z}[i] = \mathbf{x}[i] + \mathbf{y}[i]$ for every $i \in \{1, \ldots, d\}$. We write $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x}[i] \leq \mathbf{y}[i]$ for every $1 \leq i \leq d$, and we write $\mathbf{x} < \mathbf{y}$ if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. We denote by $\mathbf{u}_i$ the ith *unit vector* of $\mathbb{Q}^d$ defined by $\mathbf{u}_i[j] = 0$ if $j \neq i$ and $\mathbf{u}_i[i] = 1$. Notice that $\mathbf{x} = \sum_{i=1}^{d} \mathbf{x}[i]\mathbf{u}_i$ with our notations. The *norm* of a vector $\mathbf{v} \in \mathbb{Q}^d$ is the rational number $||\mathbf{v}|| = \max_i |\mathbf{v}[i]|$. The *norm* of a finite set $\mathbf{V} \subseteq \mathbb{Q}^d$ is defined as $||\mathbf{V}|| = \max_{\mathbf{v} \in \mathbf{V}} ||\mathbf{v}||$. We introduce the sets $|\mathbf{v}|^+ = \{i \mid \mathbf{v}[i] > 0\}$ and $|\mathbf{v}|^- = \{i \mid \mathbf{v}[i] < 0\}$. The *dot product* of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$ is the rational number $\sum_i \mathbf{x}[i] \cdot \mathbf{y}[i]$ denoted as $\mathbf{x} \cdot \mathbf{y}$.

2.1 Vector Addition Systems

A *vector addition system* (*VAS* for short) is a non-empty finite set $\mathbf{A} \subseteq \mathbb{Z}^d$ of *actions*. A vector in $\mathbb{N}^d$ is called a *configuration* of the VAS $\mathbf{A}$. The *semantics* is defined thanks to the binary relation $\rightarrow$ over the configurations by $\mathbf{x} \rightarrow \mathbf{y}$ if $\mathbf{y} = \mathbf{x} + \mathbf{a}$ for some action $\mathbf{a} \in \mathbf{A}$. The reflexive and transitive closure of $\rightarrow$ is denoted as $\xrightarrow{*}$ and it is called the *reachability relation*. If $\mathbf{x} \xrightarrow{*} \mathbf{y}$, we say that $\mathbf{y}$ is *reachable* from $\mathbf{x}$.

The reachability problem consists in deciding for a triple $(\mathbf{x}, \mathbf{A}, \mathbf{y})$ where $\mathbf{x}, \mathbf{y}$ are two configurations of a VAS $\mathbf{A}$, if $\mathbf{x} \xrightarrow{*} \mathbf{y}$. The problem is decidable [14] but its complexity remains elusive; the problem is known to be exponential-space hard [12], and the best known upper bound is non-primitive recursive [11].

A VAS $\mathbf{A}$ is said to be *bounded* from an initial configuration $\mathbf{x}$ if the set of configurations reachable from $\mathbf{x}$ is finite. The boundedness problem is known to be exponential-space complete [18]. Since the size of reachable configurations are at most Ackermannian in that case [15, 5], it follows that the reachability problem can be decided in Ackermannian complexity (space and time are equivalent for that class of complexity). This is the best known upper bound, far from the exponential-space lower bound [12].

When enforcing the VAS to be bounded for any initial configuration, we obtain the so-called structurally bounded VAS. More formally, a VAS is said to be *structurally bounded* if it is bounded from any initial configuration. In polynomial time, one can decide if a VAS is structurally bounded. In fact, a VAS $\mathbf{A}$ is not structurally bounded if, and only if, the following linear system is satisfiable over the non negative rational numbers: $(\lambda_{\mathbf{a}})_{\mathbf{a} \in \mathbf{A}}$:

$$\sum_{\mathbf{a} \in \mathbf{A}} \lambda_{\mathbf{a}} \mathbf{a} > \mathbf{0}$$

The previous observation combined with the Farkas Lemma [19] shows that a VAS is structurally bounded if, and only if, there exists a vector $\mathbf{v}$ in $\mathbb{N}^d$, called a *place invariant* such that $\mathbf{v}[i] > 0$ for every i, and such that $\mathbf{v} \cdot \mathbf{a} \leq 0$ for every action $\mathbf{a}$ in $\mathbf{A}$.

▶ **Example 1.** The VAS $\mathbf{A} = \{2\mathbf{u}_{i+1} - \mathbf{u}_i \mid 1 \leq i < d\}$ admits the place invariant $\mathbf{v} \in \mathbb{N}^d$ defined by $\mathbf{v}[i] = 2^{d-i}$ for every i.

Notice that if $\mathbf{x} \xrightarrow{*} \mathbf{y}$ then $\mathbf{v} \cdot \mathbf{y} \leq \mathbf{v} \cdot \mathbf{x}$ for any place invariant $\mathbf{v}$. We deduce that $||\mathbf{y}|| \leq (\sum_{i=1}^{d} \mathbf{v}[i])||\mathbf{x}||$. The norm of reachable configurations is therefore bounded linearly in the norm of the initial one. As observed by Praveen and Lodaya [17], the norm of the vector $\mathbf{v}$ can be bounded thanks to a small solution theorem of Borosh and Treybig [1] in such a way the reachability problem for structurally bounded VAS is decidable in polynomial space. Based on the reduction of QBF to the reachability problem of structurally bounded VAS, Praveen and Lodaya deduced the following result.

▶ **Theorem 2** (Theorem 3.10 and Theorem 3.11 of [17])**.** *The reachability problem for structurally bounded VAS is polynomial-space complete.*

2.2 Vector Addition Systems With States

The previous approach no longer apply for structurally bounded vector addition systems with states. Formally, a *vector addition systems with states* (*VASS* for short) is a graph $G = (Q, \mathbf{A}, E)$ where Q is a non empty finite set of states, $\mathbf{A}$ is a VAS, and $E \subseteq Q \times \mathbf{A} \times Q$ is a finite set of *edges*. A *configuration* is a pair $(q, \mathbf{x})$ in $Q \times \mathbb{N}^d$ denoted as $q(\mathbf{x})$ in the sequel. The semantics of an edge e is defined thanks to the binary relation $\xrightarrow{e}$ over the configurations by $p(\mathbf{x}) \xrightarrow{e} q(\mathbf{y})$ if $e = (p, \mathbf{y} - \mathbf{x}, q)$. We associate to a word $\pi = e_1 \dots e_k$ of edges the binary relation $\xrightarrow{\pi}$ over the configurations defined as the following composition:

$$\xrightarrow{e_1} \cdots \xrightarrow{e_k}$$

Notice that $\xrightarrow{\varepsilon}$ is the identity binary relation. The *reachability relation* of a VASS G is the binary relation $\xrightarrow{*}$ defined as the union $\bigcup_{\pi \in E^*} \xrightarrow{\pi}$. A configuration $q(\mathbf{y})$ is said to be *reachable* from a configuration $p(\mathbf{x})$ if $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$.

The following lemma states the so-called *VASS monotony property.* We refer to that lemma when we mention a monotony property in the sequel.

▶ **Lemma 3.** *We have* $p(\mathbf{x} + \mathbf{c}) \xrightarrow{\pi} q(\mathbf{y} + \mathbf{c})$ *for every* $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ *and for every* $\mathbf{c} \in \mathbb{N}^d$.

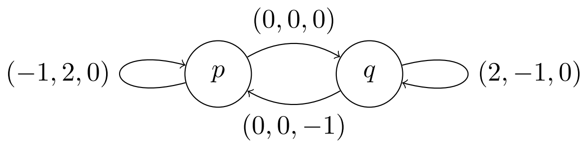
Proof. By induction on the length of π. ◀

We associate to a VASS $G = (Q, \mathbf{A}, E)$ the functions $\text{src}, \text{tgt} : E \to Q$ and $\Delta : E \to \mathbf{A}$ satisfying $e = (\text{src}(e), \Delta(e), \text{tgt}(e))$ for every $e \in E$. The states $\text{src}(e)$ and $\text{tgt}(e)$ are respectively called the *source* and *target* states. The vector $\Delta(e)$ is called the *displacement* of e. We extend the displacement function to words $\pi = e_1 \dots e_k$ of edges by $\Delta(\pi) = \Delta(e_1) + \cdots + \Delta(e_k)$. Given a sequence $\pi_1, \dots, \pi_k$ of words of edges, the vector $\Delta(\pi_1) + \cdots + \Delta(\pi_k)$ is also called the *displacement* of the sequence. A word $\pi = e_1 \dots e_k$ of edges is called a *path* of G from a state p to a state q, if there exists a sequence $q_0, \dots, q_k$ of states with $q_0 = p$ and $q_k = q$ such that $(\text{src}(e_j), \text{tgt}(e_j)) = (q_{j-1}, q_j)$ for every $1 \leq j \leq k$. A path is said to be *simple* if $q_i = q_j$ implies $i = j$. A *cycle* on a state q is a path from q to q. A cycle is said to be *elementary* if $q_i = q_j$ and $i < j$ implies $i = 0$ and $j = k$.

Let $T \subseteq E$ be a subset of edges. An edge e of T is said to be *recurrent* for T if there exists a path from $\text{tgt}(e)$ to $\text{src}(e)$ in T, otherwise, it is said to be *transient* for T. We denote by $\text{rec}(T)$ the set of edges of T that are recurrent for T. The set T is said to be *reccurent* if every edge in T is recurrent, i.e. $\text{rec}(T) = T$. We observe $\text{rec}(T)$ is recurrent for any set

T. We associate to a subset T the equivalence relation $\sim_T$ over $\text{rec}(T)$ defined by $e \sim_T e'$ if there exists a path from $\text{tgt}(e)$ to $\text{src}(e')$ and a path from $\text{tgt}(e')$ to $\text{src}(e)$. The equivalence classes of $\sim_T$ are called the strongly connected components of T, and they are denoted as $\mathcal{SCC}(T)$. The set T is said to be *strongly connected* if $\mathcal{SCC}(T) = \{T\}$. We also denote by $\mathcal{SCC}(G)$ the set $\mathcal{SCC}(E)$, and we say that G is strongly connected if E is strongly connected.

▶ **Example 4.** We adapt the VASS introduced in [6] by introducing the following VASS:



Notice that $p(1, 0, n) \xrightarrow{*} p(4^n, 0, 0)$ for every natural number n since for every $n, m \in \mathbb{N}$ with $n \geq 1$, we have:

$$p(m, 0, n) \xrightarrow{(p,(-1,2,0),p)^m (p,(0,0,0),q)(q,(2,-1,0),q)^{2m}(q,(0,0,-1),p)} p(4m, 0, n-1)$$

A VASS G is said to be *bounded* from an initial configuration $p(\mathbf{x})$ if the reachability set from that configuration is finite. Let us recall that the boundedness problem is decidable in exponential space since the boundedness problem for VASS is logspace reducible to the boundedness for VAS using for instance the encoding of control states as additional counters [6]. A VASS is said to be *structurally bounded* if it is bounded from any initial configuration. Let us recall that a VASS G is not structurally bounded if, and only if, there exists a *cycle* σ such that $\Delta(\sigma) > \mathbf{0}$. Moreover, this property is decidable in polynomial time using the Kosaraju-Sullivan algorithm [8].

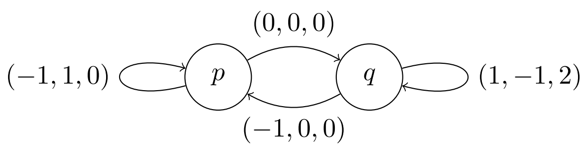
Example 4 shows that reachable configurations of structurally bounded VASS can be exponentially larger than the initial configuration. Unfortunately, it can even be larger by observing that the Ackermannian VASS introduced in [15] are structurally bounded. It follows that the best complexity upper bound for the reachability problem for structurally bounded VASS is Ackermannian. Concerning the lower bound, by observing that Lipton's construction [12, 3] also produces structurally bounded VASS, it follows that the reachability problem is exponential-space hard.

2.3 Polynomial VASS

In this paper we consider a subclass of the structurally bounded VASS, called the polynomial VASS.

▶ **Definition 5.** A VASS is said to be *polynomial* if there exists a polynomial function f such that $||\mathbf{y}|| \leq f(||\mathbf{x}||)$ for every $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$.

▶ **Example 6.** We introduce the following VASS:



We have $p(n,0,0) \xrightarrow{*} p(0,0,n^2+n))$ for every $n \in \mathbb{N}$ since for every $n, m \in \mathbb{N}$ with $m \geq 1$, we have:

$$p(m,0,n) \xrightarrow{(p,(-1,1,0),p)^m (p,(0,0,0),q)(q,(1,-1,2),q)^m (q,(-1,0,0),p)} p(m-1,0,n+2m)$$

We will prove in Example 13 that the VASS is polynomial.

▶ Remark. The VASS given in Example 4 is not polynomial.

We notice that since the Lipton's construction [12, 3] produces polynomial VASS, it follows that the reachability problem is exponential-space hard for polynomial VASS. In this paper we show that (1) we can decide in polynomial time if a VASS is polynomial, (2) the reachability problem is exponential-space complete for polynomial VASS.

3 Iteration Schemes

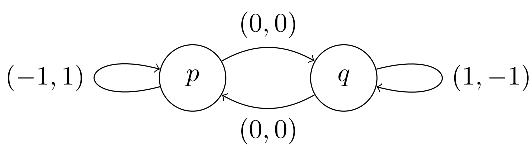
An *iteration scheme* of a VASS G is a finite sequence $\sigma_1, \dots, \sigma_k$ of cycles such that:

$$\bigwedge_{j=1}^{k} \left[\, |\Delta(\sigma_j)|^- \subseteq |\Delta(\sigma_1) + \cdots + \Delta(\sigma_k)|^+ \, \right]$$

Observe that the displacement of an iteration scheme is necessarily a vector in $\mathbb{N}^d$. An index $i \in \{1, \dots, d\}$ such that there exists an iteration scheme with a displacement strictly positive on i is called an *iterable index*. An edge t that occurs in an iteration scheme is called an *iterable edge*. By concatenating iteration schemes, notice that there exists an iteration scheme with a displacement strictly positive on every iterable index, and such that every iterable edge occurs in the scheme.

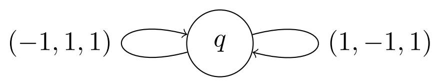
▶ **Example 7.** Let us come back to the VASS introduced in Example 4. Notice that the cycles $(p,(-1,2,0),p)$ and $(q,(2,-1,0),q)$ forms an iteration scheme with a displacement equal to $(1,1,0)$. It follows that the two first indexes are iterable.

▶ **Example 8.** We introduce the following VASS:



Notice that the cycles $(p,(-1,1),p)$ and $(q,(1,-1),q)$ do not form an iteration scheme.

▶ **Example 9.** We cannot restrict iteration schemes to sequences of *elementary* cycles. In fact, let us introduce the following VASS:



Notice that the cycle $(q,(-1,1,1),q)(q,(1,-1,1),q)$ is an iteration scheme proving that the third index is iterable. However notice that any non-empty sequence of elementary cycles is not an iteration scheme.

The following lemma shows that if a strongly connected VASS admits an iteration scheme with a non-zero displacement, then the VASS is not polynomial.

▶ **Lemma 10.** *For every strongly-connected VASS, there exists a rational number $\lambda > 1$ such that for every $n \in \mathbb{N}$ there exists an execution $p_n(\mathbf{x}_n) \xrightarrow{\pi_n} q_n(\mathbf{y}_n)$ such that:*

- $||\mathbf{x}_n|| \geq n$.
- $\mathbf{y}_n[i] \geq \lambda^{||\mathbf{x}_n||}$ *for every iterable index* i.
- *Every iterable edge occurs in* π_n *at least* $\lambda^{||\mathbf{x}_n||}$ *times.*

Proof. The proof is given in a technical report. Intuitively, iteration schemes can be iterated an exponential number of times. ◀

In section 6, we prove that conversely, indexes that are not iterable can be bounded with a polynomial, as well as non iterable edges occur at most a bounded polynomial number of times.

4 Kirchoff's Functions and Euler's Lemma

We recall some classical results about Kirchoff's functions and Euler's lemma. We assume that $G = (Q, \mathbf{A}, E)$ is a VASS.

A *multiset of edges* is a function $\phi : E \to \mathbb{N}$. The set $\{e \in E \mid \phi(e) \neq 0\}$ is called the *domain* of ϕ and it is denoted by $\mathrm{dom}(\phi)$. Given a subset $T \subseteq E$ and a multiset of edges ϕ, we denote by $\phi \cap T$ the multiset of edges defined as follows:

$$\phi \cap T(t) = \begin{cases} \phi(t) & \text{if } t \in T \\ 0 & \text{otherwise} \end{cases}$$

The *Parikh image* of a word π of edges is the multiset of edges ϕ such that $\phi(e)$ is the number of occurrences in π of e for every edge $e \in E$.

The displacement function Δ is extended over the multiset of edges ϕ by $\Delta(\phi) = \sum_{e \in E} \phi(e)\Delta(e)$. The vector $\Delta(\phi)$ are called the *displacement* of ϕ. Notice that if ϕ is the Parikh image of a word π of edges then $\Delta(\phi) = \Delta(\pi)$.

A *Kirchoff* function ϕ is a multiset of edges such that for every $q \in Q$:

$$\sum_{e \in E \mid \mathrm{tgt}(e) = q} \phi(e) = \sum_{e \in E \mid \mathrm{src}(e) = q} \phi(e)$$

Let us recall that a finite sum of Parikh images of cycles is a Kirchoff function and every Kirchoff function is a finite sum of Parikh images of elementary cycles. It follows that the domain of a Kirchoff function is recurrent. The *Euler's Lemma* claims that a Kirchoff function is the Parikh image of a cycle if, and only if, its domain is strongly connected.

5 Computing the Set of Iterable Indexes and Edges

In this section, we show that the set of iterable indexes and the set of iterable edges of a VASS $G = (Q, \mathbf{A}, E)$ are computable in polynomial time. Given a pair (I, T) where I is a subset of $\{1, \ldots, d\}$ and T is a subset of edges, a sequence $\sigma_1, \ldots, \sigma_k$ of cycles of T is called an (I, T)-constrained iteration scheme if

$$\bigwedge_{j=1}^{k} |\Delta(\sigma_j)|^- \subseteq |\Delta(\sigma_1) + \cdots + \Delta(\sigma_k)|^+ \subseteq I$$

We denote by $\Gamma(I,T)$ the pair (I',T') where I' is the set of indexes $i \in I$ such that there exists an (I,T)-constrained iteration scheme with a displacement strictly positive on i, and where T' is the set of edges that occurs in an (I,T)-constrained iteration scheme. Observe that $\Gamma(\{1,\dots,d\},E)$ is the pair (I',T') where I' is the set of iterable indexes and T' is the set of iterable edges.

We are going to compute $\Gamma(I,T)$ inductively by reducing the pair (I,T) into a pair (I',T') such that $\Gamma(I,T)=\Gamma(I',T')$ and such that $\Gamma(I,T)=(I,T)$ if it is not possible to reduce (I,T) anymore. Such an approach is similar to the one used by the Kosaraju-Sullivan algorithm [8, 2] for computing from a VASS the set of edges occurring in cycles with zero displacements. The pair (I',T') is obtained by computing $\Omega(I,T)$ defined as follows.

▶ **Definition 11.** Let (I,T) be a pair such that $I \subseteq \{1,\dots,d\}$ and $T \subseteq E$, and let us consider the following linear system over the variables $(\mu_i)_{i\in I}$ and $(\lambda_t)_{t\in T}$ ranging over the non-negative rational numbers:

$$\bigwedge_{q\in Q} \sum_{t\in T|\mathrm{tgt}(t)=q} \lambda_t = \sum_{t\in T|\mathrm{src}(t)=q} \lambda_t$$

$$\bigwedge_{S\in\mathcal{SCC}(T)} \bigwedge_{i\notin I} \sum_{t\in S} \lambda_t \Delta(t)[i] = 0$$

$$\bigwedge_{i\in I} \sum_{t\in T} \lambda_t \Delta(t)[i] = \mu_i$$

Then $\Omega(I,T)$ is defined as the pair (I',T') where:

- I' is the set of indexes $i\in I$ satisfying the previous linear system and $\mu_i>0$, and
- T' is the set of edges $t\in T$ satisfying the previous linear system and $\lambda_t>0$.

▶ **Example 12.** Let us come back to Example 4. We observe that $\Omega^n(\{1,2,3\},E)$ is equal to $(\{1,2\},E)$ if $n=1$, and it is equal to $(\{1,2\},\{(p,(-1,2,0),p),(q,(2,-1,0),q)\})$ if $n\geq 2$.

▶ **Example 13.** Let us come back to Example 6. We observe that $\Omega^n(\{1,2,3\},E)$ is equal to $(\{3\},E)$ if $n=1$, $(\{3\},\{(p,(-1,1,0),p),(q,(1,-1,2),q)\})$ if $n=2$, and $(\emptyset,\emptyset)$ if $n\geq 3$.

▶ **Example 14.** Let us come back to Example 8. We observe that $\Omega^n(\{1,2\},E)$ is equal to $(\emptyset,E)$ for every $n\geq 1$.

Notice that $\Omega(I,T)$ is computable in polynomial time and $\Omega(I,T)$ reduces the pair (I,T). The following lemma shows that this reduction let unchanged the value of $\Gamma(I,T)$.

▶ **Lemma 15.** *We have* $\Gamma(I,T)=\Gamma(\Omega(I,T))$.

Proof. Let $(I',T')=\Omega(I,T)$. Since $(I',T')\subseteq(I,T)$ we get $\Gamma(I',T')\subseteq\Gamma(I,T)$. For the converse inclusion, we just have to prove that any (I,T)-constrained iteration scheme $\sigma_1,\dots,\sigma_k$ is an (I',T')-constrained iteration scheme. Let ϕ_j be the Parikh image of σ_j, and let $\phi=\sum_{j=1}^k \phi_j$. Notice that $(\mu_i)_{i\in I}$ and $(\lambda_t)_{t\in T}$ defined as $\mu_i=\Delta(\phi)[i]$ and $\lambda_t=\phi(t)$ is a solution of the linear system introduced in Definition 11. It follows that $\mu_i>0$ implies $i\in I'$ and $\lambda_t>0$ implies $t\in T'$. Hence $\sigma_1,\dots,\sigma_k$ is an (I',T')-constrained iteration scheme. ◀

Finally, the following lemma shows that $\Gamma(I,T)$ is equal to (I,T) if (I,T) cannot be reduced anymore.

▶ **Lemma 16.** *We have* $\Gamma(I,T)=(I,T)$ *if, and only if,* $\Omega(I,T)=(I,T)$.

Proof. Assume that $\Gamma(I,T)=(I,T)$. Lemma 15 shows that $\Omega(I,T)=(I,T)$. Conversely, assume that $\Omega(I,T)=(I,T)$. By adding many solutions of the linear system introduced in Definition 11, we obtain a solution $(\mu_i)_{i\in I}$ and $(\lambda_t)_{t\in T}$ such that $\mu_i>0$ for every $i\in I$ and $\lambda_t>0$ for every $t\in T$. By multiplying that solution by a large natural number, we can assume that the solution is over the natural numbers. Now, just notice that the multiset of edges ϕ defined by $\phi(t)=\lambda_t$ if $t\in T$ and $\phi(t)=0$ otherwise is a Kirchoff function satisfying $\mathrm{dom}(\phi)=T$, $|\Delta(\phi)|^+=I$, and such that $\Delta(\phi\cap S)[i]=0$ for every $S\in\mathcal{SCC}(T)$ and for every $i\not\in I$, and such that $\Delta(\phi)[i]\geq 0$ for every $i\in I$. Euler's Lemma shows that for every $S\in\mathcal{SCC}(T)$, there exists a cycle σ_S with a Parikh image equal to $\phi\cap S$. Notice that $|\Delta(\sigma_S)|^-=|\Delta(\phi\cap S)|^-\subseteq I$. Moreover $|\sum_S\Delta(\sigma_S)|^+=I$. It follows that $(\sigma_S)_{S\in\mathcal{SCC}(T)}$ is an (I,T)-constrained iteration scheme. This scheme shows that $(I,T)\subseteq\Gamma(I,T)$. Therefore $\Gamma(I,T)=(I,T)$. ◀

Now, let us introduce $\Omega^\infty(I,T)=\bigcap_{n\in\mathbb{N}}\Omega^n(I,T)$. Since $(\Omega^n(I,T))_{n\in\mathbb{N}}$ is a non-increasing sequence, there exists $n\leq|I|.|T|$ such that $\Omega^{n+1}(I,T)=\Omega^n(I,T)$ and for such an n, we have $\Omega^\infty(I,T)=\Omega^n(I,T)$. We deduce that $\Omega^\infty(I,T)$ is computable in polynomial time. Moreover, from Lemma 15 and Lemma 16, we deduce that $\Gamma(I,T)=\Omega^\infty(I,T)$. We have proved the following theorem.

▶ **Theorem 17.** *Iterable indexes and iterable edges are computable in polynomial time.*

6 Non-iterable case

In this section we prove the following theorem.

▶ **Theorem 18.** *Let $G=(Q,\mathbf{A},E)$ be a strongly connected VASS. For every $p(\mathbf{x})\xrightarrow{\pi}q(\mathbf{y})$, the values $\mathbf{y}[i]$ where i is a non iterable index, and the number of occurrences of non iterable edges in π are bounded by:*

$$[(1+||\mathbf{x}||)^2d^2(3||\mathbf{A}||\,|Q|)^{15d^4}]^{4^{d|E|}}$$

We are now ready for proving the following lemma.

▶ **Lemma 19.** *Let $G=(Q,\mathbf{A},E)$ be a VASS such that $||\mathbf{A}||\geq 1$, $I\subseteq\{1,\ldots,d\}$, and T be a recurrent set of edges. We consider a path π such that $p(\mathbf{x})\xrightarrow{\pi}q(\mathbf{y})$. Let $m\geq 1$ satisfying:*

- *The number of occurrences in π of edges not in T is bounded by m, and*
- *$\mathbf{x}[i]+\Delta(\pi')[i]\leq m$ for every every prefix π' of π and for every $i\not\in I$.*

Then m',I',T' defined as follows:

$$m'=m^4(1+||\mathbf{x}||)^2d^2(3\mu)^{15d^4}$$
$$(I',T')=\Omega(I,T)$$

Where $\mu=||\mathbf{A}||\,|Q|$ satisfies:

- *The number of occurrences in π of edges not in T' is bounded by m', and*
- *$\mathbf{x}[i]+\Delta(\pi')[i]\leq m'$ for every every prefix π' of π and for every $i\not\in I'$.*

Proof. Notice that it is sufficient to prove that the number of occurrences in π of edges not in T' is bounded by m', and $\mathbf{y}[i]\leq m'$ for every $i\not\in I'$. In fact, the more general bound $\mathbf{x}[i]+\Delta(\pi')[i]\leq m'$ for every prefix π' of π and for every $i\not\in I'$ can be obtained as a corollary.

Let k be the number of occurrences in π of edges not in T. It follows that π can be decomposed into: $\pi=\pi_0t_1\pi_1\ldots t_k\pi_k$ where $\pi_0,\ldots,\pi_k$ are paths in T and $t_1,\ldots,t_k$ are not in T. We introduce the sequences $(p_j(\mathbf{x}_j))_{1\leq j\leq k}$ and $(q_j(\mathbf{y}_j))_{0\leq j\leq k}$ of configurations such that $p_j(\mathbf{x}_j) \xrightarrow{\pi_j} q_j(\mathbf{y}_j)$ for every $0 \leq j \leq k$, $q_{j-1}(\mathbf{y}_{j-1}) \xrightarrow{t_j} p_j(\mathbf{x}_j)$ for every $1 \leq j \leq k$, and such that $p_0(\mathbf{x}_0) = p(\mathbf{x})$ and $q_k(\mathbf{y}_k) = q(\mathbf{y})$. Notice that $\mathbf{x}_j[i], \mathbf{y}_j[i] \leq m$ for every $i \not\in I$ and for every $0 \leq j \leq k$ since $\mathbf{x}_j$ and $\mathbf{y}_j$ can be obtained trivially as vectors of the form $\mathbf{x} + \Delta(\pi')$ for some prefixes π' of π.

Let us observe that since π_j is a path in T and since T is recurrent, it follows that there exists $S_j \in \mathcal{SCC}(T)$ such that π_j is a path in S_j. We decompose the Parikh image α_j of π_j into $\alpha_j = \phi_j + \sum_{\ell \in L_j} \psi_{j,\ell}$ where ϕ_j is the Parikh image of a simple path in S_j, and $\psi_{j,\ell}$ is a Parikh image of an elementary cycle in S_j for every ℓ in a finite set L_j.

We introduce the set $\mathbf{Z}$ of displacements of elementary cycles. Notice that $||\mathbf{Z}|| \leq \mu$. We associate to each $S \in \mathcal{SCC}(T)$ and each $\mathbf{z} \in \mathbf{Z}$ the cardinal $n_{S,\mathbf{Z}}$ of the set $\bigcup_{j | S_j = S} \{(j, \ell) \mid \ell \in L_j \wedge \Delta(\psi_{j,\ell}) = \mathbf{z}\}$. Observe that for every S, we have:

$$\begin{aligned}
\sum_{\mathbf{z} \in \mathbf{Z}} n_{S,\mathbf{z}} \mathbf{z} &= \sum_{j | S_j = S} \sum_{\ell \in L_j} \Delta(\psi_{j,\ell}) \\
&= \sum_{j | S_j = S} \Delta(\alpha_j) - \Delta(\phi_j) \\
&= \sum_{j | S_j = S} (\mathbf{y}_j - \mathbf{x}_j - \Delta(\phi_j))
\end{aligned}$$

It follows that for every $i \not\in I$, we have:

$$|\sum_{\mathbf{z} \in \mathbf{Z}} n_{S,\mathbf{z}} \mathbf{z}[i]| \leq (k+1)(m + |Q| ||\mathbf{A}||) \leq 2m(2m\mu) \leq 4m^2\mu(1 + ||\mathbf{x}||)$$

Moreover, we have:

$$\begin{aligned}
\sum_{S \in \mathcal{SCC}(T)} \sum_{\mathbf{z} \in \mathbf{Z}} n_{S,\mathbf{z}} \mathbf{z} &= \sum_{j=0}^{k} (\mathbf{y}_j - \mathbf{x}_j - \Delta(\phi_j)) \\
&= \sum_{j=0}^{k} (\mathbf{y}_j - \mathbf{x}_j) + \sum_{j=1}^{k} \Delta(t_j) - \sum_{j=0}^{k} \Delta(\phi_j) - \sum_{j=1}^{k} \Delta(t_j) \\
&= \mathbf{y} - \mathbf{x} - \sum_{j=0}^{k} \Delta(\phi_j) - \sum_{j=1}^{k} \Delta(t_j)
\end{aligned}$$

If follows that for every i, we have:

$$\begin{aligned}
\sum_{S \in \mathcal{SCC}(T)} \sum_{\mathbf{z} \in \mathbf{Z}} n_{S,\mathbf{z}} \mathbf{z}[i] &\geq -(||\mathbf{x}|| + (k+1)(|Q| - 1)||\mathbf{A}|| + k||\mathbf{A}||) \\
&\geq -3m\mu(1 + ||\mathbf{x}||) \geq -4m^2\mu(1 + ||\mathbf{x}||)
\end{aligned}$$

Let us introduce $\delta = |\mathcal{SCC}(T)|(4m^2\mu(1 + ||\mathbf{x}||)d)^2(3\mu)^{9d^4}$. Notice that $\delta \leq m^4(1 + ||\mathbf{x}||)^2 d^2 (3\mu)^{12d^4}$. Lemma 20 shows that there exists a sequence of natural numbers $(m_{S,\mathbf{z}})_{S,\mathbf{z}}$ such that:

- $\sum_{\mathbf{z} \in \mathbf{Z}} m_{S,\mathbf{z}} \mathbf{z}[i] = \mathbf{0}$ for every S, and every $i \not\in I$.
- $\sum_S \sum_{\mathbf{z} \in \mathbf{Z}} m_{S,\mathbf{z}} \mathbf{z} \geq \mathbf{0}$
- $n_{S,\mathbf{z}} > \delta$ implies $m_{S,\mathbf{z}} > 0$,
- $\sum_S \sum_{\mathbf{z} \in \mathbf{Z}} n_{S,\mathbf{z}} \mathbf{z}[i] > \delta$ implies $\sum_S \sum_{\mathbf{z} \in \mathbf{Z}} m_{S,\mathbf{z}} \mathbf{z}[i] > 0$.

It follows that:

- if $n_{S,\mathbf{z}} > \delta$ then every simple cycle of S with a displacement equal to $\mathbf{z}$ is a simple cycle of T', and

- if $\mathbf{y}[i] > ||\mathbf{x}|| + \delta + m + (k+1)(|Q|-1)||\mathbf{A}|| + k||\mathbf{A}||$ then $i \in I'$.

It follows that $\mathbf{y}[i] \leq ||\mathbf{x}|| + \delta + 3m\mu \leq 3\delta \leq m'$ for every $i \not\in I'$. Moreover, it follows that for every S, the number of occurrences of cycles $\psi_{j,\ell}$ with j such that $S_j = S$, and $\ell \in L_j$ such that $\mathrm{dom}(\psi_{j,\ell}) \not\subseteq T'$ is bounded by $|\mathbf{Z}|\delta \leq m^4(1+||\mathbf{x}||)^2 d^2 (3\mu)^{13d^4}$. It follows that the number of occurrences of edges not in T' in π is bounded by $k + |\mathcal{SCC}(T)|.|Q|.m^4(1+||\mathbf{x}||)^2 d^2 (3\mu)^{13d^4} \leq m^4(1+||\mathbf{x}||)^2 d^2 (3\mu)^{15d^4} = m'$. ◀

The proof of the previous lemma was based on the following one, a kind of "small solution" theorem [16].

▶ **Lemma 20.** *Let $(n_{s,\mathbf{z}})_{s,\mathbf{z}}$ be a sequence of natural numbers indexes by s in a non-empty finite set S, and by $\mathbf{z}$ in a finite subset $\mathbf{Z} \subseteq \{-\mu, \dots, \mu\}^d$ for some $\mu \geq 1$. Let $I \subseteq \{1, \dots, d\}$ and $m \geq 1$ such that:*

- *$|\sum_{\mathbf{z}\in\mathbf{Z}} n_{s,\mathbf{z}}\mathbf{z}[i]| \leq m$ for every $s \in S$ and for every $i \not\in I$, and*
- *The vector $\mathbf{v}$ defined as $\sum_{s\in S}\sum_{\mathbf{z}\in\mathbf{Z}} n_{s,\mathbf{z}}\mathbf{z}$ satisfies $\mathbf{v}[i] \geq -m$ for every $i \in \{1, \dots, d\}$.*

There exists a sequence $(m_{s,\mathbf{z}})_{\mathbf{z}\in\mathbf{Z},s\in S}$ of natural numbers such that:

- *$\sum_{\mathbf{z}\in\mathbf{Z}} m_{s,\mathbf{z}}\mathbf{z}[i] = 0$ for every $s \in S$ and for every $i \not\in I$, and*
- *The vector $\mathbf{w}$ defined as $\sum_{s\in S}\sum_{\mathbf{z}\in\mathbf{Z}} m_{s,\mathbf{z}}\mathbf{z}$ satisfies $\mathbf{w} \geq \mathbf{0}$*

and such that $\delta = |S|(md)^2(3\mu)^{9d^4}$ satisfies:

- *If $n_{s,\mathbf{z}} > \delta$ then $m_{s,\mathbf{z}} > 0$, and*
- *If $\mathbf{v}[i] > \delta$ then $\mathbf{w}[i] > 0$.*

Proof. The proof is based on an application of a "small solution" theorem of Pottier [16] on each $s \in S$, and then, on the resulting solutions we apply again the "small solution" theorem for extracting solutions satisfying $\mathbf{w} \geq \mathbf{0}$. ◀

Let us consider $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ and let $(I_n, T_n) = \Omega^n(\{1, \dots, d\}, E)$. We introduce for every n the minimal number $m_n \geq 1$ such that the number of occurrences of edges in π that are not in T_n is bounded by m_n, and $\mathbf{x}[i] + \Delta(\pi')[i] \leq m_n$ for every $i \not\in I_n$ and for every prefix π' of π. Notice that $m_0 = 1$, and Lemma 19 shows that for every $n \geq 0$:

$$m_{n+1} \leq m_n^4(1+||\mathbf{x}||)^2 d^2 (3\mu)^{15d^4}$$

Let us introduce the sequence $(s_n)_{n\geq 0}$ defined by $s_0 = 1$, and the induction for every $n \in \mathbb{N}$:

$$s_{n+1} = s_n^4.(1+||\mathbf{x}||)^2 d^2 (3\mu)^{15d^4}$$

Observe that $m_n \leq s_n$ for every n. Moreover, we have:

$$s_n = [(1+||\mathbf{x}||)^2.d^2(3\mu)^{15d^4}]^{4^n-1}$$

Since $\Gamma(\{1, \dots, d\}, E) = \Omega^\infty(\{1, \dots, d\}, E) = \Omega^{d|E|}(\{1, \dots, d\}, E)$, we have proved Theorem 18.

7 Applications

Theorem 18 shows that a strongly connected VASS without iterable indexe is polynomial. Combined with Lemma 10, we deduce the following characterization.

▶ **Theorem 21.** *A strongly connected VASS is polynomial if, and only if, its set of iterable indexes is empty.*

As a direct consequence of Theorems 21 and 17 and the following Lemma 22, we get the following Theorem 23. Note that the restriction of a VASS $G = (Q, \mathbf{A}, E)$ to a subset $T \subseteq E$ of edges is defined as the VASS $G|_T = (Q, \mathbf{A}, T)$.

▶ **Lemma 22.** *A VASS is polynomial if, and only if, its restriction to every SCC is polynomial.*

Proof. The proof is given in a technical report. ◀

▶ **Theorem 23.** *We can decide in polynomial-time if a VASS is polynomial.*

Moreover, since Theorem 18 shows that reachable configurations are bounded exponentially in space, we derive the following result.

▶ **Theorem 24.** *The reachability problem for polynomial VASS is exponential-space complete.*

Proof. Theorem 18 shows that reachable configurations are bounded exponentially in space. It follows that the reachability problem is decidable in exponential space. We have already observed the lower bound in Section 2.2. ◀

8 Conclusion

In this paper we introduced the class of polynomial VASS and showed that the membership problem of a VASS in that class is decidable in polynomial time. Moreover, we proved that the reachability problem for polynomial VASS is exponential-space complete. Our characterization of polynomial VASS is based on the notion of iteration schemes. Intuitively, whereas a cycle of a VASS with a non-negative displacement can be iterated an arbitrarily number of times to obtain arbitrarily large values on indexes that are strictly increased by the cycle, iteration schemes can be iterated an exponential number of times and provide a way to increase by an exponential number every index that is increased by the iteration scheme. As a future work, we are interested in using iteration schemes rather than iterable cycles in the KLM algorithm to hopefully obtain better complexity upper bound for the reachability problem for general vector addition systems.

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