A Diagrammatic Axiomatisation of Fermionic Quantum Circuits

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Abstract

We introduce the fermionic ZW calculus, a string-diagrammatic language for fermionic quantum computing (FQC). After defining a fermionic circuit model, we present the basic components of the calculus, together with their interpretation, and show how the main physical gates of interest in FQC can be represented in the language. We then list our axioms, and derive some additional equations. We prove that the axioms provide a complete equational axiomatisation of the monoidal category whose objects are quantum systems of finitely many local fermionic modes, with operations that preserve or reverse the parity (number of particles mod 2) of states, and the tensor product, corresponding to the composition of two systems, as monoidal product. We achieve this through a procedure that rewrites any diagram in a normal form. We conclude by showing, as an example, how the statistics of a fermionic Mach-Zehnder interferometer can be calculated in the diagrammatic language.

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1 Introduction

The ZW calculus is a string-diagrammatic language for qubit quantum computing, introduced by the first author in [16]. Developing ideas of Coecke and Kissinger [6], it refined and extended the earlier ZX calculus [4, 1], while keeping some of its most convenient properties, such as the ability to handle diagrams as undirected labelled multigraphs. In the version of [17, Chapter 5], it provided the first complete equational axiomatisation of the monoidal category of qubits and linear maps, with the tensor product as monoidal product. Soon after its publication, the third author and Q. Wang derived from it a universal completion of the ZX calculus [28, 18].

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Since its early versions, the ZX calculus has had the advantage of including familiar gates from the circuit model of quantum computing [29, Chapter 4], such as the Hadamard gate and the CNOT gate, either as basic components of the language, or as simple composite diagrams. This facilitates the transition between formalisms and the application to known algorithms and protocols, and is related to the presence of a simple, well-behaved "core" of the ZX calculus, modelling the interaction of two strongly complementary observables [5], in the guise of special commutative Frobenius algebras [9]. Access to complementary observables is fundamental in quantum computing schemes such as the one-way quantum computer, to which the ZX calculus was applied in [11].

The ZW calculus only includes one special commutative Frobenius algebra, corresponding to the computational basis, as a basic component. On the other hand, as noted already in [16], the ZW calculus has a fundamentally different "core", which is obtained by removing a single component that does not interact as naturally with the rest. This core has the property of only representing maps that have a definite parity with respect to the computational basis: the subspaces spanned by basis states with an even or odd number of 1s are either preserved, or interchanged by a map. This happens to be compatible with an interpretation of the basis states of a single qubit as the empty and occupied states of a *local fermionic mode*, the unit of information of the *fermionic quantum computing* (FQC) model.

Fermionic quantum computing is computationally equivalent to qubit computing [3]. The connection with the ZW calculus suggested that an independent fermionic version of the calculus could be developed, combining the best of both worlds with respect to FQC rather than qubit computing: the superior structural properties of the ZW calculus, including an intuitive normalisation procedure for diagrams, together with the superior hands-on features of the ZX calculus.

In this paper, we present such an axiomatisation, to which we refer as the *fermionic ZW calculus*. We start by defining our model in Section 2: the monoidal category **LFM** of local fermionic modes and maps that either preserve or reverse the parity of a state, with the tensor product of $\mathbb{Z}_2$-graded Hilbert spaces as the monoidal product. We introduce a number of physical gates from which one may build fermionic quantum circuits: the beam splitter, the phase gates, the fermionic swap gate, and the empty and occupied state preparations. Finally, we describe our diagrammatic language with its interpretation in **LFM**, and show that all the physical gates have simple diagrammatic representations.

In Section 3, we list the axioms of the fermionic ZW calculus, and state several derived equations, whose proofs are appended at the end of the paper. We introduce short-hand notation for certain composite diagrams (sometimes called the "spider" notation in categorical quantum mechanics [7, Section 8.2]), and prove inductive generalisations of the axioms. Then, in Section 4, we prove our main theorem, that the fermionic ZW calculus is an axiomatisation of **LFM**. We achieve this by defining a normal form for diagrams, from which one can easily read the interpretation in **LFM**, and showing that any diagram can be rewritten in normal form using the axioms.

Finally, in Section 5, as a first practical example, we calculate in the diagrammatic language the statistics of a simple circuit, the fermionic Mach-Zehnder interferometer.

2 The model and the components

The basic systems in FQC are *local fermionic modes* (LFMs), physical sites that are either empty or occupied by a single spinless fermionic particle [3]. We indicate the empty and occupied states of a LFM as $|\,0\,\rangle$ and $|\,1\,\rangle$, respectively, in bra-ket notation.

Much like the computational basis states of a qubit, we can see these as an orthonormal basis for the two-dimensional complex Hilbert space B. We note that the "naive" translation from LFMs to qubits does not preserve entanglement and locality properties [15, 10]; see however [13].

States of a composite system of n LFMs correspond to states of the n-fold tensor product $B^{\otimes n}$. However, not all physical states or operations on qubits are accessible as physical states or operations on LFMs. The Hilbert space of a system of n LFMs splits as $H_0 \oplus H_1$, where H_0 is spanned by states where an even number of LFMs is occupied, and H_1 by states where an odd number of LFMs is occupied. Then, any physical operation $f : H_0 \oplus H_1 \to K_0 \oplus K_1$ must either preserve, or invert the parity, that is, either map H_0 to K_0 and H_1 to K_1, or map H_0 to K_1 and H_1 to K_0. This is called the *parity superselection rule*; see [2, 10] for a discussion.

These operations assemble into a category, as follows.

▶ **Definition 1.** A $\mathbb{Z}_2$*-graded Hilbert space* is a complex Hilbert space H decomposed as a direct sum $H_0 \oplus H_1$. A *pure map* $f : H \to K$ of $\mathbb{Z}_2$-graded Hilbert spaces is a bounded linear map $f : H \to K$ such that $f(H_0) \subseteq K_0$ and $f(H_1) \subseteq K_1$ (*even* map), or $f(H_0) \subseteq K_1$ and $f(H_1) \subseteq K_0$ (*odd* map).

Given two $\mathbb{Z}_2$-graded Hilbert spaces H, K, the tensor product $H \otimes K$ can be decomposed as $(H \otimes K)_0 := (H_0 \otimes K_0) \oplus (H_1 \otimes K_1)$, and $(H \otimes K)_1 := (H_0 \otimes K_1) \oplus (H_1 \otimes K_0)$. Then, the tensor product (as maps of Hilbert spaces) of a pair of pure maps $f : H \to K$, $f' : H' \to K'$ is a pure map $f \otimes f' : H \otimes H' \to K \otimes K'$ of $\mathbb{Z}_2$-graded Hilbert spaces. The $\mathbb{Z}_2$-graded Hilbert space $\mathbb{C} \oplus 0$ acts as a unit for the tensor product.

We write $\mathbf{Hilb}^{\mathbb{Z}_2}$ for the symmetric monoidal category of $\mathbb{Z}_2$-graded Hilbert spaces and pure maps, with the tensor product as monoidal product.

▶ Remark 2. The zero maps $0 : H \to K$ are the only pure maps between two $\mathbb{Z}_2$-graded Hilbert spaces H, K that are both even and odd.

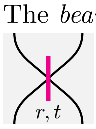
▶ **Definition 3.** We write **LFM** for the full monoidal subcategory of $\mathbf{Hilb}^{\mathbb{Z}_2}$ whose objects are n-fold tensor products of $B := \mathbb{C} \oplus \mathbb{C}$, for all $n \in \mathbb{N}$.

Here, B_0 is the span of $|\,0\,\rangle$, and B_1 is the span of $|\,1\,\rangle$. As customary, we write $|\,b_1 \ldots b_n\,\rangle$ for the basis state $|\,b_1\,\rangle \otimes \ldots \otimes |\,b_n\,\rangle$ of $B^{\otimes n}$, where $b_i \in \{0, 1\}$, for $i = 1, \ldots, n$.

The category **LFM** admits, in fact, the structure of a dagger compact closed category in the sense of [31]: each object $B^{\otimes n}$ is self-dual, and the dagger of a pure map $f : B^{\otimes n} \to B^{\otimes k}$ is its adjoint $f^\dagger : B^{\otimes k} \to B^{\otimes n}$.

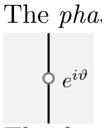
Operationally, we are interested in representing circuits built from the following *logical* components, shown here in diagrammatic form (read from bottom to top), next to their interpretation as maps in **LFM**.

1. The *beam splitter* with parameters $r, t \in \mathbb{C}$, such that $|r|^2 + |t|^2 = 1$:



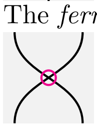
$$|\,00\,\rangle \mapsto |\,00\,\rangle, \qquad |\,10\,\rangle \mapsto r|\,10\,\rangle + t|\,01\,\rangle,$$
$$|\,01\,\rangle \mapsto -\bar{t}|\,10\,\rangle + \bar{r}|\,01\,\rangle, \qquad |\,11\,\rangle \mapsto |\,11\,\rangle.$$

2. The *phase* gate with parameter $\vartheta \in [0, 2\pi)$:



$$|\,0\,\rangle \mapsto |\,0\,\rangle, \quad |\,1\,\rangle \mapsto e^{i\vartheta}|\,1\,\rangle.$$

3. The *fermionic swap* gate:



$$|\,00\,\rangle \mapsto |\,00\,\rangle, \quad |\,10\,\rangle \mapsto |\,01\,\rangle,$$
$$|\,01\,\rangle \mapsto |\,10\,\rangle, \quad |\,11\,\rangle \mapsto -|\,11\,\rangle.$$

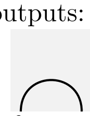
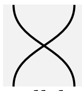
4. *Empty state* and *occupied state* preparation:

 $1 \mapsto |\,0\,\rangle$, $1 \mapsto |\,1\,\rangle$.

All of these are isometries, which makes them, at least in principle, physically implementable gates; see for example [19] for the description of an electron beam splitter.

Apart from the fermionic swap gate, which exploits the antisymmetry of fermionic particles under exchange, these operations are structurally the same as those used in implementations of linear optical quantum computing (LOQC), such as the Knill-Laflamme-Milburn scheme [25], which employ photons, that is, bosonic particles as resources. The two models seem closely related; given the way that the fermionic swap ties the other components together in our axiomatisation, and that the impossibility for two particles to occupy the same mode – a constraint for the bosons in LOQC – is simply a consequence of Pauli exclusion for fermions, it seems possible to us that the logical features of the optical model are a consequence of the features of the fermionic model.

In addition to the logical components, we need the following structural components – the *dualities* and the *swap* – which allow us to treat all our diagrams as components of a circuit diagram, which can be connected together in an undirected fashion, permuting and transposing their inputs and outputs:

 $1 \mapsto |\,00\,\rangle + |\,11\,\rangle$,  $|\,b_1 b_2\,\rangle \mapsto \begin{cases} 1, & b_1 = b_2, \\ 0, & b_1 \neq b_2, \end{cases}$  $|\,b_1 b_2\,\rangle \mapsto |\,b_2 b_1\,\rangle$.

While the swap, dualities, fermionic swap, and phase gates will be basic components of our diagrammatic calculus, we are going to further decompose beam splitters and state preparations. The components so obtained may not correspond to physical operations by themselves, but they have the property that the result of transposing or swapping any of their inputs or outputs only depends on the final number of inputs and outputs. This allows us to treat their diagrammatic representations as vertices of an undirected vertex-labelled multigraph: only the overall arity matters. In addition to making calculations simpler, this enables one to implement the calculus in graph rewriting software, such as Quantomatic [24].

The additional components, given here in "all-output form" together with their interpretation in **LFM**, are the following.

1. The binary and ternary black vertex:

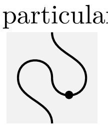
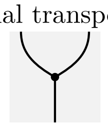
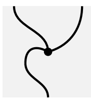
 $1 \mapsto |\,10\,\rangle + |\,01\,\rangle$,  $1 \mapsto |\,100\,\rangle + |\,010\,\rangle + |\,001\,\rangle$.

2. The binary white vertex with parameter $z \in \mathbb{C}$:

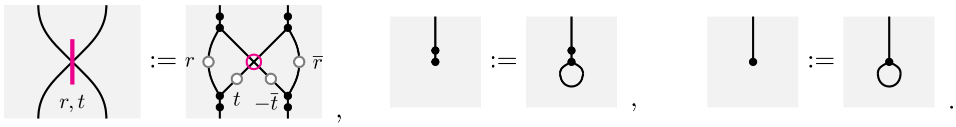
$1 \mapsto |\,00\,\rangle + z|\,11\,\rangle$.

▶ Remark 4. Up to a normalising factor, the interpretations of the binary and ternary vertex are known as EPR state and W state, respectively, in qubit theory [12].

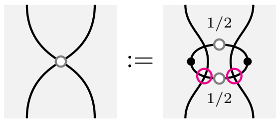
When we draw black and white vertices with a different partition of inputs and outputs, we assume that a particular partial transposition has been fixed, for example to the left:

 :=  ,  :=  .

Now, a phase gate with parameter ϑ is simply a binary white vertex with parameter $e^{i\vartheta}$. The beam splitter with parameters r, t, and the state preparations can be decomposed as follows:



We also introduce a simplified notation for a composite diagram that plays an important role in our calculus, whose interpretation is the *projector* on the even subspace of two LFMs:



$|\,00\,\rangle \mapsto |\,00\,\rangle, \qquad |\,11\,\rangle \mapsto |\,11\,\rangle,$
$|\,01\,\rangle, |\,10\,\rangle \mapsto 0.$

As the notation suggests, this corresponds to the quaternary white vertex of the original ZW calculus. Similarly to the black and white vertices already introduced, its interpretation is symmetric under transposition and swapping of inputs and outputs, so we can freely draw quaternary white vertices with a different partition of inputs and outputs.

▶ Remark 5. Our calculus does not include measurements, probabilistic mixing, or any kind of classical control as internal operations. In future work, we hope to extend our axiomatisation to a mixed quantum-classical calculus, in the style of [8] (see [7, Chapter 8] for a more recent version), incorporating all these elements.

For now, we can calculate the probability of detecting particles at the output ends of a circuit by closing the circuit with occupied and empty state diagrams; a closed circuit is then interpreted as a map $\mathbb{C} \to \mathbb{C}$, that is, a scalar. This will be the probability amplitude of detecting a particle where we have closed with an occupied state, and not detecting it where we have closed with an empty state.

To reason rigorously about our diagrammatic calculus, we rely on the theory of PROs (PROduct categories) [26], strict monoidal categories that have $\mathbb{N}$ as set of objects, and monoidal product given, on objects, by the sum of natural numbers. Morphisms $n \to m$ in a PRO represent operations with n inputs and m outputs. Given a *monoidal signature*, that is, a set of operations with arities $T := \{f_i : n_i \to m_i\}_{i \in I}$, one can generate the free PRO $F[T]$ on T, whose operations are free sequential and parallel compositions of the f_i, modulo the axioms of monoidal categories. By a classic result of Street and Joyal [20, Theorem 1.2], this is equivalent to the PRO whose morphisms are obtained by horizontally juxtaposing and vertically plugging string diagrams with the correct arity, one for each generator, then quotienting by planar isotopy of diagrams. Thus, in the remainder, we will not distinguish between the two, identifying diagrams and operations.

▶ **Definition 6.** Let T be the monoidal signature with operations $\mathsf{swap} : 2 \to 2$, $\mathsf{dual} : 0 \to 2$, $\mathsf{dual}^\dagger : 2 \to 0$, $\mathsf{fswap} : 2 \to 2$, $\mathsf{black}_2 : 0 \to 2$, $\mathsf{black}_3 : 0 \to 3$, and $\mathsf{white}_z : 0 \to 2$, for all $z \in \mathbb{C}$. The *language* of the fermionic ZW calculus is the free PRO $F[T]$.

The correspondence with the diagrammatic components we listed earlier should be self-explanatory, and their interpretation induces a monoidal functor $f \colon F[T] \to \mathbf{LFM}$.

Given a set E of equations between diagrams with the same arity in $F[T]$, we can quotient $F[T]$ by the smallest equivalence relation including E and compatible with composition and monoidal product, to obtain a PRO $F[T/E]$, together with a quotient functor $p_E : F[T] \to F[T/E]$.

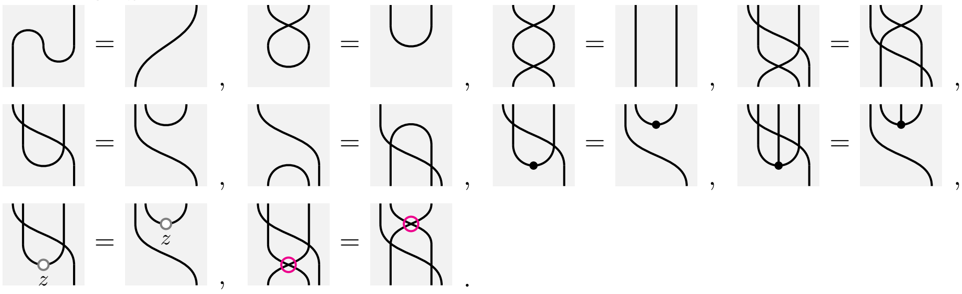
▶ **Definition 7.** The interpretation $f : F[T] \to \mathbf{LFM}$ is *universal* if it is a full functor. A set E of equations is *sound* for $f : F[T] \to \mathbf{LFM}$ if f factors as $f_E \circ p_E$ for a functor $f_E : F[T/E] \to \mathbf{LFM}$. A sound set of equations is *complete* for $\mathbf{LFM}$ if f_E is an equivalence of monoidal categories.

In the next section, we introduce the axioms of the fermionic ZW calculus, in the form of equations between diagrams of $F[T]$; it can be checked that they are all sound for the interpretation. We will later show that they are also complete. This means that whenever two diagrams of the fermionic ZW calculus are "extensionally equal", that is, they have the same interpretation in **LFM**, one can be rewritten into the other by applying a finite sequence of equations.

3 Axioms and derived equations

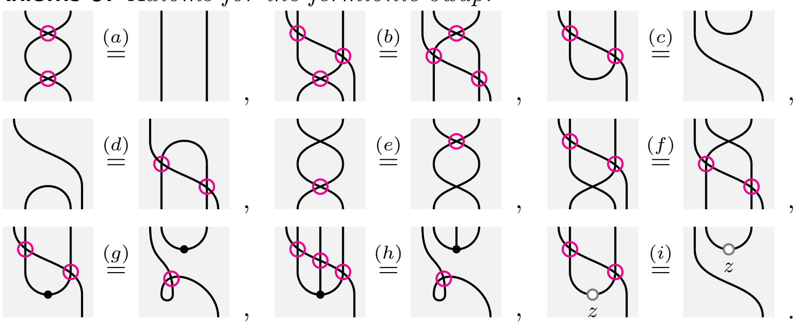
We divide the set E of axioms into four groups, based on the generators to which they mainly pertain.

▶ **Axioms 8.** *Structural axioms.*



Together, these axioms imply that the swap and dualities make $F[T/E]$ a compact closed category on a self-dual object. The Kelly-Laplaza coherence theorem [22, Theorem 8.2] then allows us to extend our topological reasoning to the swapping and transposition of wires.

▶ **Axioms 9.** *Axioms for the fermionic swap.*

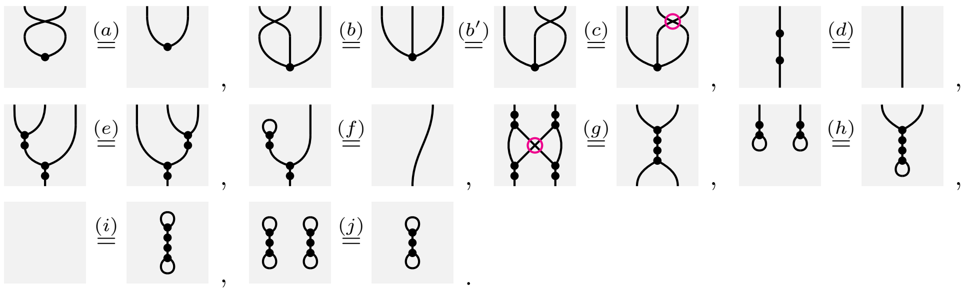


These axioms say that the fermionic swap behaves like a symmetric braiding in $F[T/E]$, except for the fact that sliding the black vertices (that is, the only odd generators) through a wire induces a fermionic "self-crossing" on it.

Moreover, the axioms on the interplay between the structural and fermionic swaps imply that only the number of fermionic swaps between two wires matters, and not their direction; which, as we will see, also implies that a sequence of two fermionic self-crossings on either side of a wire can be straightened.

Altogether, the result of the other axioms is that any diagram containing an *even* number of black vertices can slide past a wire through fermionic swaps with no other effect, while any diagram containing an *odd* number of black vertices can do the same by introducing a fermionic self-crossing on the wire. As with the structural axioms, we will make use of this fact implicitly most of the time.

▶ **Axioms 10.** *Axioms for black vertices.*

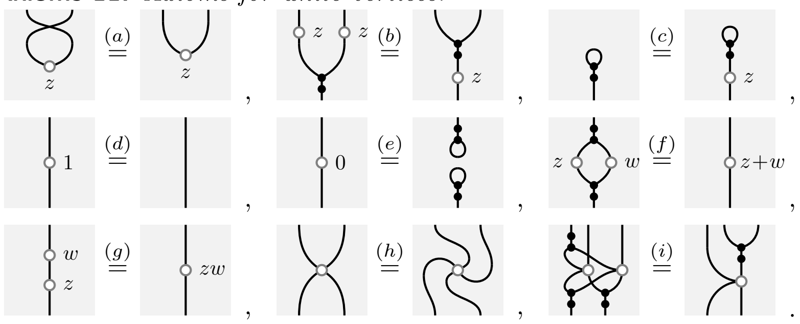


These axioms say that the black vertices are symmetric under permutation of wires (which justifies, a posteriori, their arbitrary transposition), and that they can be assembled to form a (co)commutative (co)monoid. This (co)monoid has the property of forming a bialgebra (in fact, a Hopf algebra) with its own transpose.

In the interpretation, this is the Hopf algebra known as *fermionic line* in the theory of quantum groups [27, Example 14.6], whose comultiplication is given by $|\,0\,\rangle \mapsto |\,00\,\rangle$, $|\,1\,\rangle \mapsto |\,10\,\rangle + |\,01\,\rangle$. As discussed in [17, Section 5.3], the fermionic line has "anyonic" and "bosonic" analogues in every countable dimension, with the same self-duality property.

The final axiom says that 0 times 0 is 0; it will serve to ensure that there is a unique zero map between any two systems, rather than an "even" and an "odd" zero map.

▶ **Axioms 11.** *Axioms for white vertices.*

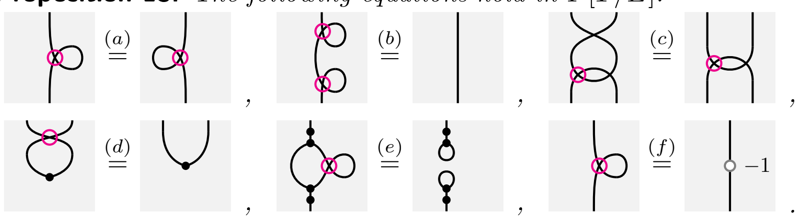


These axioms say that the binary white vertices are endomorphisms for the fermionic line algebra, and that composition and convolution by the algebra correspond to product and sum, respectively, of their complex parameters. Finally, the projector is symmetric under cyclic permutation of its wires, and it determines a kind of mixed action/coaction of the algebra on itself.

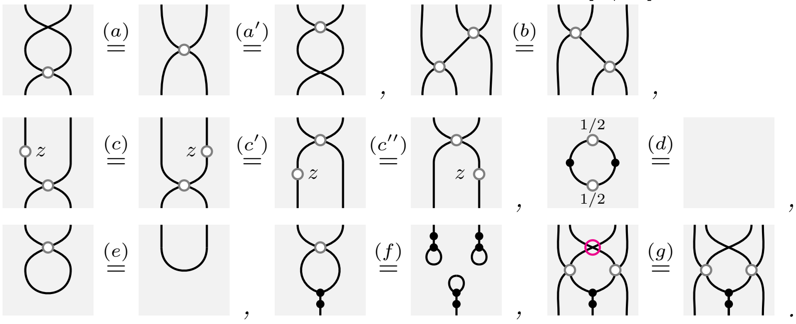
▶ Remark 12. Because **LFM** is a subcategory of the category **Qubit** of [18], all the axioms of the fermionic ZW calculus are sound for the original ZW calculus. Moreover, adding either the ternary or the unary white vertex from the original ZW calculus to our language would make it universal for **Qubit**. We have not yet investigated, however, what axioms would need to be added to E in the extended signature to make it complete.

We state some useful derived equations, leaving the proofs to the Appendix.

▶ **Proposition 13.** *The following equations hold in $F[T/E]$:*



▶ **Proposition 14.** *The following equations hold in $F[T/E]$:*

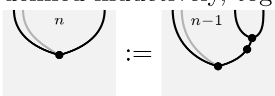


Together with its invariance under cyclic permutation of wires, the first two equations justify the arbitrary transposition of inputs and outputs of the quaternary white vertex.

Our axioms form a sound and complete set of equations for **LFM**, so in principle any equation of diagrams whose interpretations are equal can be derived from them. In practice, however, it is convenient to introduce further short-hand notation, including black vertices with n wires and white vertices with $2n$ wires for all $n \in \mathbb{N}$, and derive inductive equation schemes to use directly in proofs.

1. *Black vertices.* The nullary and unary black vertices are defined as follows:

We already have binary and ternary black vertices. For $n > 3$, the n-ary black vertex is defined inductively, together with its interpretation in **LFM**, by

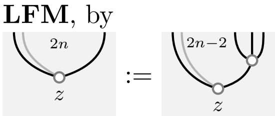


$$1 \mapsto \sum_{k=1}^{n} |\underbrace{0\dots0}_{k-1}1\underbrace{0\dots0}_{n-k}\rangle.$$

Here, and in what follows, lighter wires and vertices indicate the repetition of a pattern for a number of times, which may or may not be specified. This is similar to the the way that "…" is often used, and can be formalised using !-boxes in pattern graphs, as developed in [23].

2. *White vertices.* The nullary white vertex with parameter $z \in \mathbb{C}$ is defined by

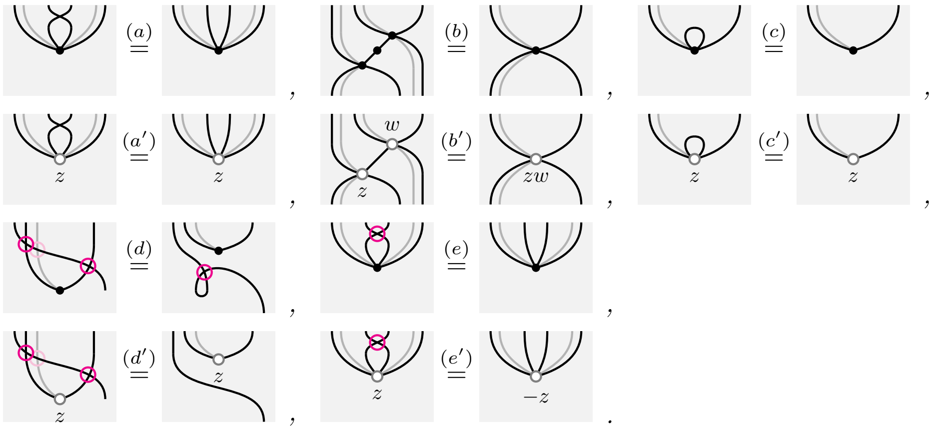
We already have binary white vertices with parameters. For $n > 1$, the $2n$-ary white vertex with parameter $z \in \mathbb{C}$ is defined inductively, together with its interpretation in **LFM**, by



$$1 \mapsto |\underbrace{0\dots0}_{2n}\rangle + z\,|\underbrace{1\dots1}_{2n}\rangle.$$

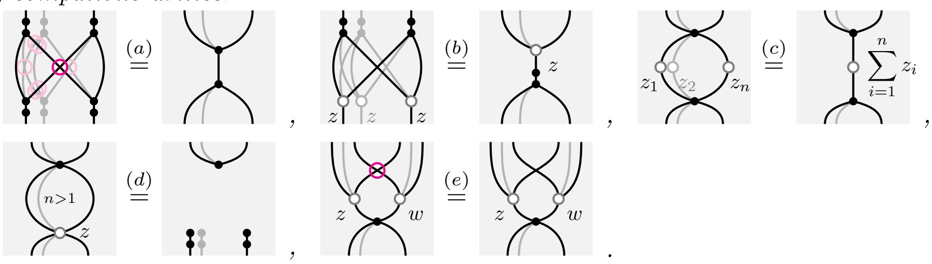
We state some basic properties of black and white vertices. Both are symmetric under permutation of wires, which allows us to write vertices with different numbers of inputs and outputs, transposing some of them with no ambiguity. Most importantly, they satisfy certain "fusion" equations, as shown on the first two lines. All black vertices correspond to odd maps, while white vertices correspond to even maps, as reflected in their sliding through fermionic swaps, on the fourth line; finally, black vertices are unaffected by fermionic swaps of their wires, whereas the sign of the white vertex parameter is flipped.

▶ **Proposition 15.** *The following equations hold in $F[T/E]$ for black and white vertices of any arity:*



Several other equations, both axioms and derived, admit inductive generalisations; we list them in the following Proposition.

▶ **Proposition 16.** *The following equations hold in $F[T/E]$ for black and white vertices of any compatibile arities:*



▶ Remark 17. Some of these inductive schemes subsume several axioms at once: for example, Proposition 16.(a) has Axioms 10.(g), 10.(h), and 10.(i) as special cases, and Proposition 16.(b) has Axioms 11.(b), 11.(c), and 11.(i) as special cases.

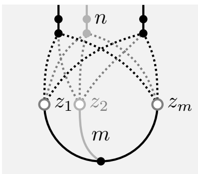
4 Normal form and completeness

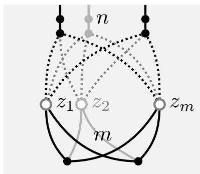
We prove completeness in three stages:

1. First, we associate to any state $v : \mathbb{C} \to B^{\otimes n}$ of **LFM** a diagram $g(v) : 0 \to n$ in $F[T]$ such that $f(g(v)) = v$. Because both categories are compact closed, and the dualities of **LFM** are in the image of f, this assignment can be extended to any map of **LFM**, proving universality of our interpretation. We say that a diagram is in *normal form* if it is of the form $g(v)$ for some v.
2. Then, we show that any composite of diagrams in normal form can be rewritten in normal form using the equations in E, proving that g determines a monoidal functor from **LFM** to $F[T/E]$.
3. Finally, we show that all the generators of $F[T]$ can be rewritten in normal form using the equations in E, proving that g and $f_E : F[T/E] \to$ **LFM** are two sides of a monoidal equivalence between $F[T/E]$ and **LFM**.

▶ **Theorem 18** (Universality)**.** *The functor $f : F[T] \to$ **LFM** is full.*

Proof. Write an arbitrary state $v : \mathbb{C} \to B^{\otimes n}$ in the form $1 \mapsto \sum_{i=1}^{m} z_i | b_{i1} \dots b_{in} \rangle$, where $z_i \neq 0$ for all i, and no pair of n-tuples $(b_{i1}, \dots, b_{in})$ is equal; we can fix an ordering (for example, lexicographic) on n-tuples of bits to make this unique. Then, define

$g(v) :=$ 

if v is odd, 

if v is even, (1)

where, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, the dotted wire connecting the i-th white vertex to the j-th output is present if and only if $b_{ij} = 1$. The definition is only ambiguous if $v = 0$, in which case we arbitrarily pick one of the two forms; they will be equal in $F[T/E]$ by Axiom 10.(j).

Because for all summands of an odd (respectively, even) state v, we have $b_{ij} = 1$ for an odd (respectively, even) number of bits, the white vertices in $g(v)$ have an odd (respectively, even) number of outputs. The two distinct forms of $g(v)$ for odd and even states ensure that only white vertices with an even arity appear.

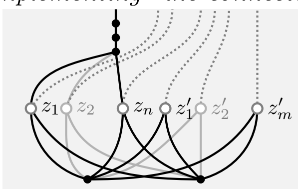
It can then be checked that $f(g(v)) = v$, which, by our earlier remark, suffices to prove the statement. ◀

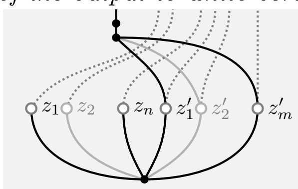
▶ **Definition 19.** A string diagram of $F[T]$ is in *normal form* if it is $g(v)$ for some state v of **LFM**. It is in *pre-normal form* if it has one of the two forms in (1), where the following are also allowed:

- the white vertices can be in an arbitrary order;
- two or more white vertices may be connected to the exact same outputs;
- z_i may be 0 for some i.

The completeness proof closely follows that of the qubit ZW calculus [17, Section 5.2]. The one proof that is significantly different is the following. We take the liberty of "zooming in" on a certain portion of a diagram, which may require some reshuffling of vertices, using swapping or transposition of wires, with the implicit understanding that this can always be reversed later.

▶ **Lemma 20** (Negation). *The composition of one output of a diagram in pre-normal form with a binary black vertex can be rewritten in pre-normal form, and that has the effect of "complementing" the connections of the output to white vertices: that is, locally,*

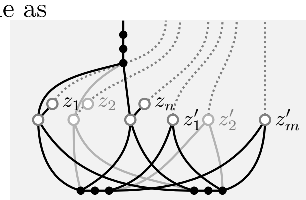


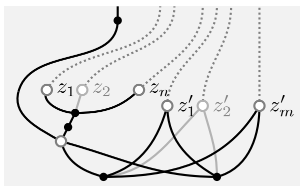
$=$


.

▶ Remark. In the picture, the dotted wires can stand for a multitude of wires. The version where the original diagram is odd, rather than even, is obtained by composing again both sides with a binary black vertex and using Axiom 10.(d).

Proof. Using the "fusion equations" Proposition 15.(b) and (b'), we rewrite the left-hand side as



$\overset{16.(b)}{=}$


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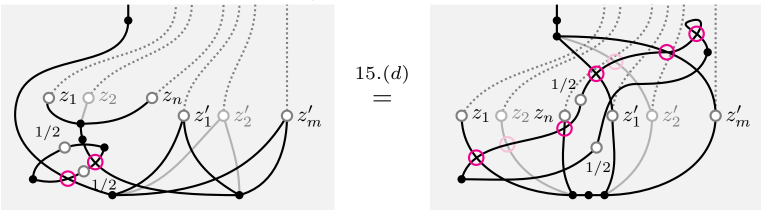
By definition of the quaternary white vertex, this is equal to



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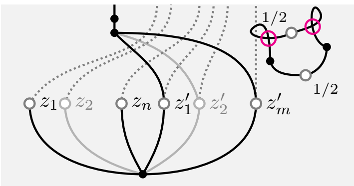
where we made implicit use of some symmetry properties of vertices. Now, fusing black vertices, and using Proposition 15.(d) and (d') to move the closed loop to the outside of the main diagram, we see that this is equal to



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and we can conclude by Proposition 13.(b) and 14.(d). ◀

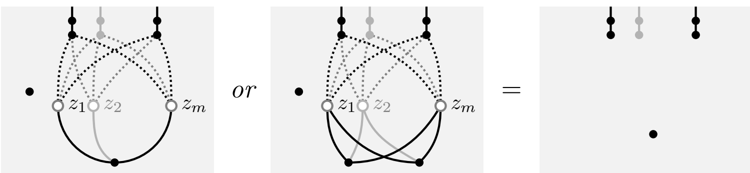
In the following, and later statements, "plugging one output of a diagram into another" means a post-composition with $\mathsf{dual}^\dagger : 2 \to 0$, possibly after some swapping of wires.

▶ **Lemma 21** (Trace). *The plugging of two outputs of a diagram in pre-normal form into each other can be rewritten in pre-normal form.*

Proof. Essentially the same as [17, Lemma 5.24]. ◀

The nullary black vertex is interpreted as the scalar 0; the following lemma shows that it acts as an "absorbing element" for diagrams in pre-normal form.

▶ **Lemma 22** (Absorption). *For all diagrams in pre-normal form, the following equation holds in $F[T/E]$:*



. (2)

Proof. If the diagram is even, expanding the nullary black vertex, we can treat it as an additional output of the diagram, with no connections to the white vertices, composed with a unary black vertex; then the proof is the same as [17, Lemma 5.25].

Suppose the diagram is odd. If it has at least one output wire, we can freely introduce two binary black vertices on it; applying the negation lemma once, we obtain a negated even diagram, to which the first part of the proof can be applied. Another application of the negation lemma, followed by Axiom 10.(j), produces the desired equation. If the diagram has no outputs, it necessarily consists of a single nullary black vertex, and the statement follows immediately from Axiom 10.(j). ◀

▶ **Lemma 23** (Functoriality of the normal form). *Any composition of two diagrams in pre-normal form can be rewritten in pre-normal form.*

Proof. We can factorise any composition of diagrams in pre-normal form as a tensor product followed by a sequence of "self-pluggings"; thus, by the trace lemma, it suffices to prove that a tensor product – diagrammatically, the juxtaposition of two diagrams in pre-normal form – can be rewritten in pre-normal form.

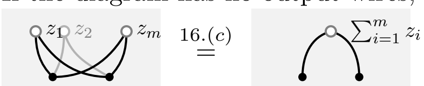
Suppose first that the two diagrams are both even. We can create a pair of unary black vertices connected by a wire by Axiom 10.(i), and treat them as additional outputs, one for each diagram. In that case, the proof proceeds exactly as [17, Theorem 5.26].

Now, suppose one diagram is odd, or they both are odd. If the odd diagrams have at least one output wire, we can introduce a pair of black vertices on it, and apply the negation lemma to produce negated even diagrams. We can then apply the first part of the proof to obtain a diagram in pre-normal form negated once or twice, then apply the negation lemma again to conclude. If one of the odd diagrams has no outputs, it necessarily consists of a single nullary black vertex, and we can conclude with an application of the absorption lemma. ◀

▶ **Lemma 24.** *Any diagram in pre-normal form can be rewritten in normal form.*

Proof. If the diagram is odd, the proof of [17, Lemma 5.22] goes through. If the diagram is even, and has at least one output wire, we can introduce a pair of binary black vertices, apply the negation lemma once to produce a negated odd diagram, reduce that to normal form, and apply the negation lemma again; it is easy to see that negation turns diagrams in normal form into diagrams in normal form, *modulo* a reshuffling of white vertices.

If the diagram has no output wires, then it is of the form

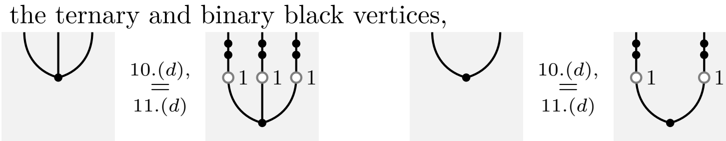


,

where the right-hand side is in normal form. This concludes the proof. ◀

▶ **Theorem 25** (Completeness). *The functor $f_E : F[T/E] \to$ **LFM** induced by the soundness of E for the interpretation $f : F[T] \to$ **LFM** is a monoidal equivalence.*

Proof. By the combination of the previous two lemmas, it suffices to show that all the generators (with all wires transposed to output wires) can be rewritten in pre-normal form. For the ternary and binary black vertices,



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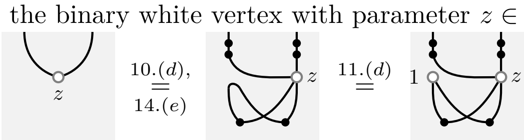
For the binary white vertex with parameter $z \in \mathbb{C}$,



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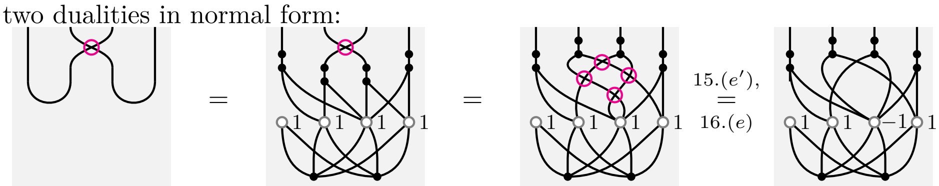
By Axiom 11.(d), the rewriting of dualities in normal form follows as a special case of the binary white vertex with parameter 1.

For the fermionic swap, we use the fact that we know how to rewrite the tensor product of two dualities in normal form:



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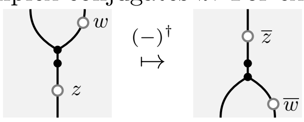
The case of the structural swap is similar, and easier. This concludes the proof. ◀

▶ Remark 26. We can make this an equivalence of dagger compact closed categories, by defining the dagger of a morphism in $F[T/E]$, represented by a diagram in $F[T]$, to be the vertical reflection of that diagram, with parameters $z \in \mathbb{C}$ of white vertices turned into their complex conjugates $\overline{z}$. For example,



.

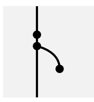
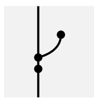
▶ Remark 27. The only properties of complex numbers that are used in the proof are that they form a commutative ring, and that they contain an element z such that $z + z = 1$ (namely, $1/2$). Thus, we can replace $\mathbb{C}$ with any commutative ring R that has the latter property (for example, $\mathbb{Z}_{2n+1}$, for each $n \in \mathbb{N}$), and obtain a similar completeness result for "LFMs with coefficients in R".

Moreover, for any such ring, instead of introducing binary white vertices with arbitrary parameters $r \in R$, we can introduce one binary white vertex for each element of a family of generators of R, together with one axiom for each relation that they satisfy. Then, in the normal form, instead of having a white vertex labelled $r \in R$ at each end of the bottom black vertex or vertices, we will need to have some expression of r by sums and products of generators, encoded by composition and convolution by the fermionic line algebra.

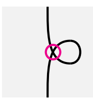
The completeness proof still goes through: we work with diagrams in pre-normal form, where terms in a sum of products of generators are decomposed into different legs of the bottom vertex or vertices, until the very end; then Lemma 24 can be adapted to combine white vertices with the same connections into a fixed expression of the sum of their parameters.

For example, in the complex case, it may be convenient to have separate phase gates, that is, white vertices with parameter $e^{i\vartheta}$, for $\vartheta \in [0, 2\pi)$, and "resistor" gates, with real parameter $r > 0$.

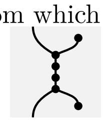
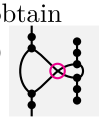
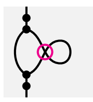
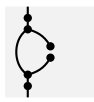
▶ Remark 28. It is customary to describe the fermionic behaviour of a multi-particle system in terms of a pair of operators $a^\dagger$ (creation) and a (annihilation) that satisfy the anti-commutation relation $aa^\dagger = 1 - a^\dagger a$; see for example [32, Chapter 27]. In our language, these operators can be defined as

$a^\dagger :=$  , $a :=$  .

We can see the anti-commutation relation as subsumed by the axioms in the following way: pulling back the linear structure of **LFM** to $F[T/E]$ through the equivalence, we have

 = − , (3)

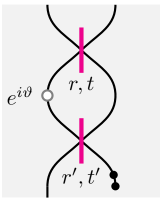
from which we obtain

 $\overset{10.(g)}{=}$  =  $\overset{(3)}{=}$ −  ,

which can be read as the equation $aa^\dagger = 1 - a^\dagger a$.

5 An application: the Mach-Zehnder interferometer

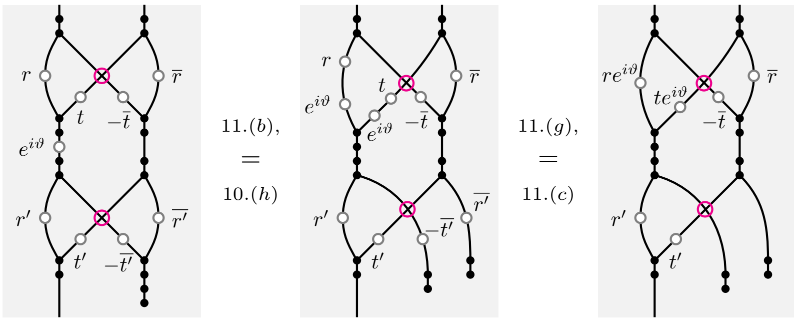
The Mach-Zehnder interferometer is a classic quantum optical setup (see for example [30, Chapter 4]), which, despite its simplicity, can demonstrate interesting features of quantum mechanics, as in the Elitzur-Vaidman bomb tester experiment [14]. The theoretical setup can be straightforwardly imported into FQC, with the same statistics as long as single-particle experiments are concerned; an electronic analogue of the Mach-Zehnder interferometer has also been realised in practice [19].



With the graphical notation introduced in Section 2, the experimental setup is represented by the diagram on the left, where $r, t, r', t' \in \mathbb{C}$ and $\vartheta \in [0, 2\pi)$ are parameters subject to $|r|^2 + |t|^2 = |r'|^2 + |t'|^2 = 1$. In practice, it would also include "mirrors", or beam splitters with $|r| = 1$, which we omit in the picture, instead taking the liberty of bending wires at will.

As a first application of the fermionic ZW calculus, we show how this circuit diagram can be simplified in just a few steps using our axioms, in such a way that its statistics become immediately readable from the diagram.

In our language, the diagram becomes



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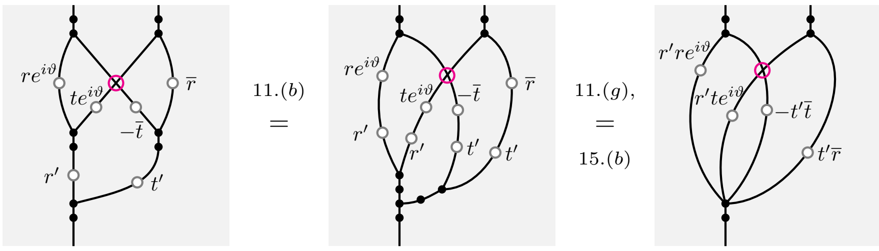
which, sliding the leftmost empty state past the fermionic swap, and using Axiom 10.(f) twice, becomes



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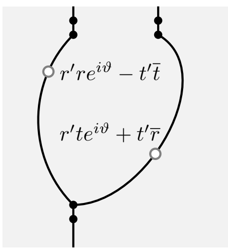
Finally, using the fermionic swap symmetry of black vertices (Proposition 15.(e)), together with Proposition 16.(c), this simplifies to



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If we input one particle, after fusing the bottom black vertices, we obtain a diagram in normal form, whose interpretation in **LFM** we can read off as $1 \mapsto (r're^{i\vartheta} - t'\bar{t})\,|\,10\,\rangle + (r'te^{i\vartheta} + t'\bar{r})\,|\,01\,\rangle$.

So, the probability of detecting the particle at the left-hand output is $|r're^{i\vartheta} - t'\bar{t}|^2$, and the probability of detecting the particle at the right-hand output is $|r'te^{i\vartheta} + t'\bar{r}|^2$. If the beam splitters are symmetric, that is, $r = r' = \frac{1}{\sqrt{2}}$, and $t = t' = \frac{i}{\sqrt{2}}$, the probability amplitudes become

$$\frac{1}{2}(e^{i\vartheta} - 1) = e^{i(\frac{\vartheta+\pi}{2})} \sin\vartheta, \qquad \frac{i}{2}(e^{i\vartheta} + 1) = e^{i(\frac{\vartheta+\pi}{2})} \cos\vartheta,$$

leading to probabilities $\sin^2\vartheta$ of detecting the particle at the left-hand output, and $\cos^2\vartheta$ of detecting it at the right-hand output.

Arguably, given that this particular example involves at most binary gates, a matrix calculation would not have been considerably harder. On the other hand, the result appears here as the outcome of a short sequence of intuitive, algebraically motivated local steps, rather than the unexplained product of a large matrix multiplication. We expect the advantage to become clearer when implementations of rewrite strategies in graph rewriting software are used to simplify larger circuits.

6 Conclusions and outlook

In this paper, we introduced a string-diagrammatic language for circuits of local fermionic modes, together with equations that axiomatise their theory of extensional equality: that is, two diagrams represent the same linear map of local fermionic modes if and only if they are equal modulo the equations. We believe that these fermionic circuits are to the ZW calculus what Clifford circuits [1] are to the ZX calculus: not the largest family of circuits that can technically be represented, but the one whose basic gates have simple, natural representations in terms of the language's components.

There are still several open questions and directions on the "syntactic" side. We do not know whether all our axioms are independent, nor we have looked at rewrite strategies, or ways of orienting the equations, beyond the goal of proving completeness. There is, then, the question of variants and extensions: we have mentioned a potential extension to mixed-state processes, via a mixed quantum-classical calculus in the style of [7, Chapter 8]; moreover, both universality and completeness are open problems for anyonic and bosonic generalisations of the fermionic ZW calculus, in the style of [17, Section 5.3].

The greatest challenge, however, is finding "real-world" applications for the calculus. With the Mach-Zehnder interferometer, we have only given a toy example, perhaps useful for pedagogical purposes, but we have not even attempted to link our work to current research on algorithms or complexity in FQC. The first version of a ZW calculus was introduced in order to tackle open problems in the classification of multipartite entanglement [6]: as a first step, the fermionic ZW calculus, with its strong topological flavour, involving braidings and a single type of ternary vertices, may be a better testing ground for this approach.

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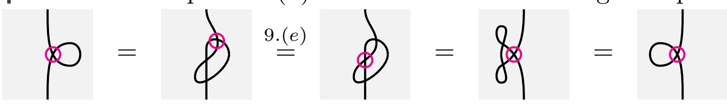
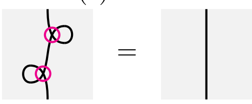
A Proofs of derived equations

Proposition 13. Equation (a) comes from the following manipulation:

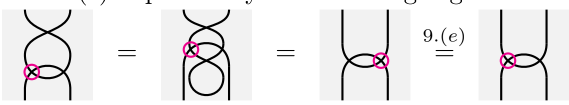


.

Equation (b) then follows from (a), combined with the equation

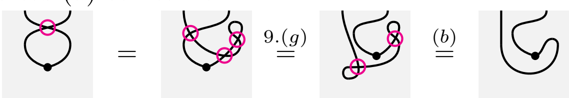

,

which is a consequence of the fermionic swap axioms by the Whitney trick [21, p. 484]. Equation (c) is proved by the following argument:



,

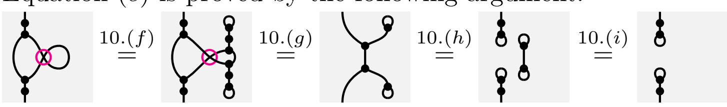
whereas (d) comes from



,

finally using the symmetry of the black vertex under the structural swap.

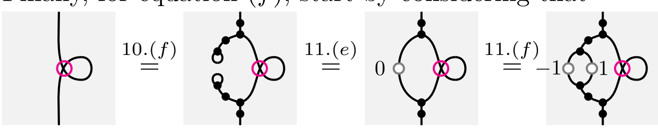
Equation (e) is proved by the following argument:



,

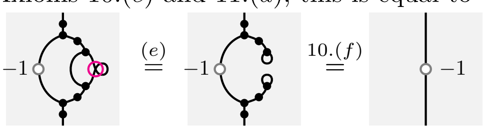
where we tacitly used Axiom 10.(d) to introduce or eliminate pairs of binary black vertices in several occasions.

Finally, for equation (f), start by considering that



;

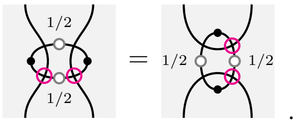
by Axioms 10.(e) and 11.(d), this is equal to



.

This completes the proof. ◀

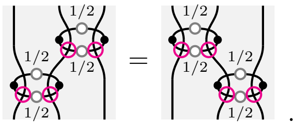
Proof of Proposition 14. Substituting the definition of the projector, Axiom 11.(h) becomes the following equation:



(4)

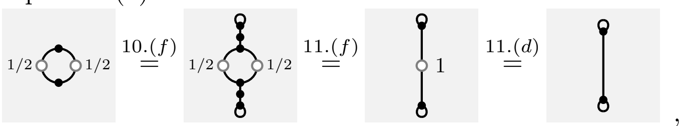
Equations (a) and (a') are then immediate consequences of Proposition 13.(c) and its transposes, applied to the right-hand side of (4).

Equation (b) is also immediate from the definition: because swaps slide through fermionic swaps and vice versa, we can slide one “circle” past another to get

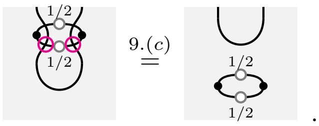


For equations (c), (c'), and (c''), we use either of the forms in equation (4) and slide the binary white vertex through a fermionic swap using Axiom 9.(i), to move it to a different wire.

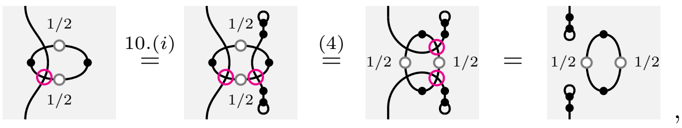
Equation (d) comes from



finally applying Axiom 10.(i). Then, equation (e) follows from it by

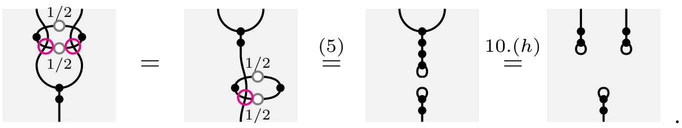


In order to prove equation (f), consider first that

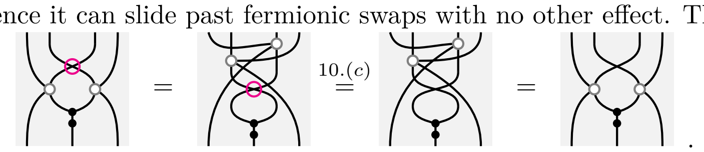


(5)

and we can eliminate the circle by equation (d). Then,



Finally, for equation (g), observe that the projector contains an even number of black vertices, hence it can slide past fermionic swaps with no other effect. Therefore,



This concludes the proof. ◀

Proof of Proposition 15. All the equations are proved by induction on the arity of the vertices involved.

For equation (a), let n be the number of outputs of the black vertex. For $n = 0, 1$ there is nothing to prove, and for $n = 2, 3$ these are Axioms 10.(a), (b), and (b'). For $n > 3$, if the two swapped wires are the rightmost ones, the equation follows immediately from the ternary case; otherwise, use Axiom 10.(e) on the three rightmost wires, and apply the inductive hypothesis.

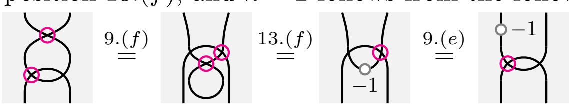
For equation (a'), let $2n$ be the number of outputs of the white vertex. For $n = 0$ there is nothing to prove, and $n = 1$ is Axiom 11.(a). For $n > 1$, observe that by Proposition 14.(c), (c') and (c''), we can always move the binary vertex with parameter z to a wire which is not swapped. The case $n = 2$ then follows from the combination of Axiom 11.(h) with Proposition 14.(a) and (a'). For $n > 2$, if the swapped wires are among the three rightmost ones, the equation follows from the case $n = 2$; otherwise, use Proposition 14.(b) (with some wires transposed) on the two rightmost quaternary white vertices, and apply the inductive hypothesis.

Equations (a) and (a') justify the unambiguous writing of n-ary vertices with inputs as well as outputs in equations (b) and (b'), and the latter will follow from the all-output case. In equation (b), let $n, m > 0$ be the arities of the leftmost and rightmost vertex, respectively. If $n = 1$, the equation follows from Axiom 10.(f), and if $n = 2$ from Axiom 10.(d). Suppose $n > 2$. Then, if $m = 1$, the equation follows from Axiom 10.(f), and if $m = 2$ from Axiom 10.(d). All other cases are just immediate from the definition. Equation (b') also follows from the definition, together with Proposition 14.(c), (c') and (c'') in order to move the vertex with parameter w to the wire where the vertex with parameter z is, and Axiom 11.(g) to multiply the two.

In equation (c), let $n > 1$ be the arity of the black vertex in the left-hand side. If $n = 2, 3$ the equation is true by definition. If $n > 3$, by equation (a), we can assume the two wires plugged into each other are the two rightmost ones; the equation then follows from Axiom 10.(f). For equation (c'), let $2n > 1$ be the arity of the white vertex in the left-hand side. If $n = 1$, the equation is true by definition, and if $n = 2$ it follows from Proposition 14.(d), together with Proposition 14.(c), (c') and (c'') to move the vertex with parameter z out of the way. For $n > 2$, by equation (a'), we can assume the two wires plugged into each other are the two rightmost ones, and the equation follows from the case $n = 2$.

Equation (d) is a consequence of Axioms 9.(g) and (h), together with Proposition 13.(b) to eliminate pairs of self-crossings. Equation (d') is a consequence of Axiom 9.(i) together with the definition of the quaternary white vertex.

Equation (e) follows from equation (a) by Axiom 10.(c) and Proposition 13.(d). For equation (e'), let $2n > 1$ be the arity of the white vertex. The case $n = 1$ is a consequence of Proposition 13.(f), and $n = 2$ follows from the following argument:



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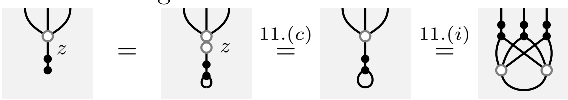
applied to the definition of the quaternary white vertex, as in the right-hand side of (4). All other cases follow from this one, by symmetry. ◀

Proof of Proposition 16. In equation (a), let n be the number of inputs, and m the number of outputs of the diagrams. The case $n = m = 0$ is Axiom 10.(i), and when either n or $m = 1$, the equation follows from Axiom 10.(d). The cases $n = 0, m > 1$ and $m = 0, n > 1$ are simple inductive generalisations of Axiom 10.(h). Finally, the case $n = m = 2$ is Axiom 10.(g), and from there we can proceed by double induction on n and m, using Proposition 15.(b).

In equation (b), let n be the number of inputs, and $2m - 1$, for $m > 0$, the number of outputs. Suppose first that $m = 1$. The case $n = 0$ is Axiom 11.(c), the case $n = 1$ follows from Axiom 10.(d), and the case $n = 2$ is Axiom 11.(b); then, for $n > 2$, it is a simple induction starting for the latter. In the case $n = 0$ and $m = 2$,



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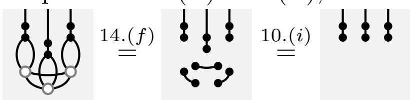
by Proposition 15.(b') and (c'), the latter is equal to



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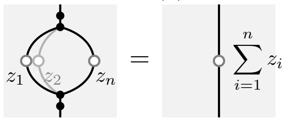
The cases $n = 0$, $m > 2$ are simple inductive generalisations of this one. All cases with $n = 1$ follow from Axiom 10.(d), and the case $n = m = 2$ is Axiom 11.(i). For $n, m > 2$, proceed by double induction, using Proposition 15.(b) and (b').

For equation (c), by Proposition 15.(b) it suffices to prove



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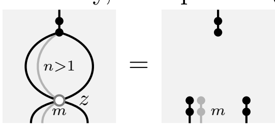
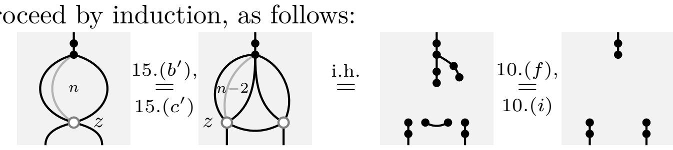
which for $n = 0$ is Axiom 11.(e), for $n = 1$ follows from Axiom 10.(d), for $n = 2$ is Axiom 10.(f), and for $n > 2$ is a simple inductive generalisation of the latter.

Similarly, for equation (d), it suffices, by Proposition 15.(b) and (b'), to prove

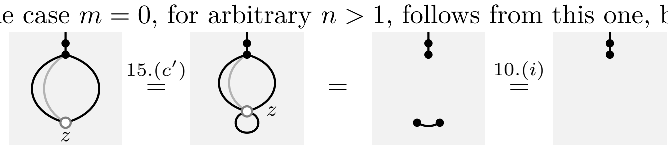


when $m = 0, 1, 2$. If $m = 2$, and $n = 2$, this is Proposition 14.(f), and for $n > 2$ we can proceed by induction, as follows:



.

The case $m = 0$, for arbitrary $n > 1$, follows from this one, by



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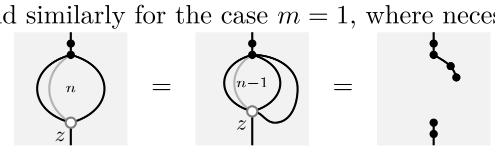
and similarly for the case $m = 1$, where necessarily $n > 2$, by



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Finally, equation (e) is an immediate generalisation of Proposition 14.(g), using Proposition 15.(b) and (b'). ◀