Finite Bisimulations for Dynamical Systems with Overlapping Trajectories

Béatrice Bérard
Sorbonne Université, CNRS, Laboratoire d'Informatique de Paris 6,
LIP6, F-75005 Paris, France
beatrice.berard@lip6.fr

Patricia Bouyer[1]
LSV, CNRS, ENS Paris-Saclay, Univ. Paris-Saclay, France
bouyer@lsv.fr

Vincent Jugé
Université Paris-Est, LIGM (UMR 8049), CNRS, ENPC, ESIEE, UPEM, F-77454,
Marne-la-Vallée, France
vincent.juge@u-pem.fr

Abstract

Having a finite bisimulation is a good feature for a dynamical system, since it can lead to the decidability of the verification of reachability properties. We investigate a new class of o-minimal dynamical systems with very general flows, where the classical restrictions on trajectory intersections are partly lifted. We identify conditions, that we call *Finite* and *Uniform Crossing*: When Finite Crossing holds, the time-abstract bisimulation is computable and, under the stronger Uniform Crossing assumption, this bisimulation is finite and definable.

2012 ACM Subject Classification Theory of computation → Logic and verification

Keywords and phrases Reachability properties, dynamical systems, o-minimal structures, intersecting trajectories, finite bisimulations

Digital Object Identifier 10.4230/LIPIcs.CSL.2018.26

Acknowledgements We would like to thank Françoise Point, who pointed out to us reference [33], and anonymous referees for providing additional references.

1 Introduction

Hybrid automata. Hybrid systems [16] combine continuous dynamics, i.e. evolution of variables according to flow functions (possibly described by differential equations) in control locations, and discrete jumps between these locations, equipped with guards and variable updates. For this very expressive class of models, where the associated transition system has an uncountable state space, most verification questions are undecidable [19, 4], in particular the reachability of some error states. For the last twenty-five years, a large amount of research has been devoted to approximation methods [34, 12] and to the identification of subclasses with decidable properties obtained by restricting the continuous dynamics and/or the discrete behaviour of the systems [2]. Among these subclasses lie the well-known timed automata [1], where all variables are *clocks* evolving with rate 1 with respect to a global time, guards are comparisons of clocks with rational constants, and updates are resets. Decidability results

[1] Supported by ERC project EQualIS.

27th EACSL Annual Conference on Computer Science Logic (CSL 2018).
Editors: Dan Ghica and Achim Jung; Article No. 26; pp. 26:1–26:17
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

were also obtained for larger classes (see [18, 14, 25, 2, 8, 15, 10, 9, 3]), usually (but not always) by building a finite abstraction based on some *bisimulation* equivalence, preserving a specific class of properties, like reachability or those expressed by temporal logic formulas.

Ingredients for decidability results. We now describe the restrictions mentioned above. The first one consists in constraining the variable updates on discrete transitions between locations by some "strong reinitialization", to make the dynamics of locations independent from each other, hence decoupling the discrete and continuous components. Considering a single location with its dynamics is then sufficient; in the next step, the aim is to identify subclasses of the dynamical systems governing the variables on a fixed location, for which a finite bisimulation can be found.

With the decoupling conditions, powerful flows, like the linear flows considered in [17], become possible. The approach in [2] handles o-minimal hybrid systems, using o-minimal structures over the reals as time and variable domains. The first-order theory of reals is then exploited to produce a finite bisimulation. This direction was further explored in [25, 8, 22, 10, 9], where analytical or algebraic methods are proposed to extend the set of flow functions as well as the underlying o-minimal structures. In [10, 9], decidability of reachability is even obtained with the theory of reals while no finite bisimulation may exist. The work of [15] explores how to slightly lift the hypothesis on strong reinitialization.

A few cases feature hybrid automata with no decoupling between the discrete and the continuous parts, at the price of very simple dynamical systems: the first one is the class of timed automata, where clocks describe the most basic flow functions, and the second one is the (incomparable) class of Interrupt Timed Automata with polynomial constraints [3], where the variables are stopwatches (with rate 0 or 1 depending on the location) organized along hierarchical levels. In this latter case, classical polyhedron-based abstractions are not sufficient and the finite bisimulation is obtained via an adaptation of the cylindrical algebraic decomposition algorithm [13].

Contribution. We investigate a new class of o-minimal dynamical systems, where some classical restrictions on the trajectories are lifted: overlapping trajectories are possible, as depicted for instance in Figure 1. Our method involves a classification of intersection points, similar to the cylindrical decomposition, producing a time-abstract bisimulation leading to a finite abstraction under suitable hypotheses.

Outline. In Section 2, we recall the base properties of o-minimal structures used in our developments; we then define the dynamical systems we will study; we also define the technical tool of time-abstract bisimulation which is used to build a finite abstraction of the dynamical systems; we end up this section with a discussion on related work. In Section 3, we present the graph construction, which leads to abstract the original dynamical system with some partition of the state-space, on which we are able to check time-abstract bisimulation. In Section 4, we discuss definability and decidability issues, and show how our approach can be used to recover the original work [25]. We end up with some perspectives.

2 Definitions

We consider linearly ordered structures $\mathcal{M} = \langle M, <, \ldots \rangle$. These structures can be dense or discrete (or mixed), with or without endpoints (i.e. minimum or maximum). Classical examples without endpoints are the set $\mathbb{Z}$ of integers or the real line $\mathbb{R}$, while the sets $\mathbb{N}$

of natural numbers and $\mathbb{R}_+$ of the non negative real numbers have 0 as left endpoint. We will consider the first-order theory associated with $\mathcal{M}$: we say that some relation, subset or function is *definable* when it is first-order definable in the structure $\mathcal{M}$. Next we may abusively identify the structure $\mathcal{M}$ with its first-order theory. A general reference for first-order logic is [20]. Moreover, we will assume that the theory of $\mathcal{M}$ is o-minimal and we recall here the definition of o-minimality (references are [31, 21, 32, 33, 36]).

2.1 O-minimal structures

Recall that *intervals* of $\mathcal{M} = \langle M, <, \dots \rangle$ are convex sets with either a supremum in M or no upper bound, and either an infimum in M or no lower bound.

▶ **Definition 1.** A linearly ordered structure $\mathcal{M} = \langle M, <, \dots \rangle$ has an *o-minimal* theory if every definable subset of M is a finite union of intervals.

In other words, the definable subsets of M are the simplest possible. This assumption implies that definable subsets of M^n (in the sense of $\mathcal{M}$) admit very nice structure theorems (like the *cell decomposition* [21, 32]). Classical o-minimal structures are: the ordered group of rationals $\langle \mathbb{Q}, <, +, 0, 1 \rangle$, the ordered field of reals $\langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$, the field of reals with exponential function, the field of reals expanded by restricted pfaffian functions and the exponential function, and many more interesting structures (see [36, 37]). An example of non o-minimal structure is given by $\langle \mathbb{R}, <, \sin, 0 \rangle$, since the definable set $\{x \mid \sin(x) = 0\}$ is not a finite union of intervals. However, note that the structure[2] $\langle \mathbb{R}, +, \cdot, 0, 1, <, \sin_{|[0,2\pi]}, \cos_{|[0,2\pi]} \rangle$ is o-minimal (see [35]).

We recall here a standard base result of o-minimal structures, used to build the cell decomposition, and which will be useful in the subsequent developments. While initially proved for dense structures [31, Theorem 4.2], a version for discrete structures is provided in [32, Lemmas 1.3 and 1.5], and the result holds for general mixed structures as a consequence of [33, Proposition 2.3].

▶ **Theorem 2.** *Let $\mathcal{M} = \langle M, <, \dots \rangle$ be a linearly ordered structure with an o-minimal theory. Let $f : M \mapsto M$ be a definable function. The set M can be partitioned into finitely many intervals $I_1, \dots, I_k$ such that, for every interval I_j, (i) the restriction $f_{|I_j}$ is either constant or one-to-one and monotonic, and (ii) the set $f(I_j)$ is an interval of M.*

The other result on o-minimal structures used in the sequel is the following, restated from [33, Section 2], which provides a uniform bound on the partition size:

▶ **Theorem 3.** *Let $\mathcal{M} = \langle M, <, \dots \rangle$ be a linearly ordered structure with an o-minimal theory. Let φ be a formula with k variables. Then there exists an integer $\mathbf{N}_\varphi$ such that, for all $b_2, \dots, b_k \in M$, the set $\{a \in M \mid (a, b_2, \dots, b_k) \models \varphi\}$ can be partitioned into at most $\mathbf{N}_\varphi$ intervals.*

2.2 Dynamical systems

▶ **Definition 4.** A *dynamical system* is a pair $(\mathcal{M}, \gamma)$ where:
- $\mathcal{M} = \langle M, <, \dots \rangle$ is a linearly ordered structure,
- $\gamma : V_1 \times V \to V_2$ is a function definable in $\mathcal{M}$ (where $V_1 \subseteq M^{k_1}$, $V \subseteq M$ and $V_2 \subseteq M^{k_2}$ are definable subsets).[3]

[2] $\sin_{|[0,2\pi]}$ and $\cos_{|[0,2\pi]}$ correspond to the sine and cosine functions restricted to interval $[0, 2\pi]$.
[3] We use these notations in the rest of the paper.

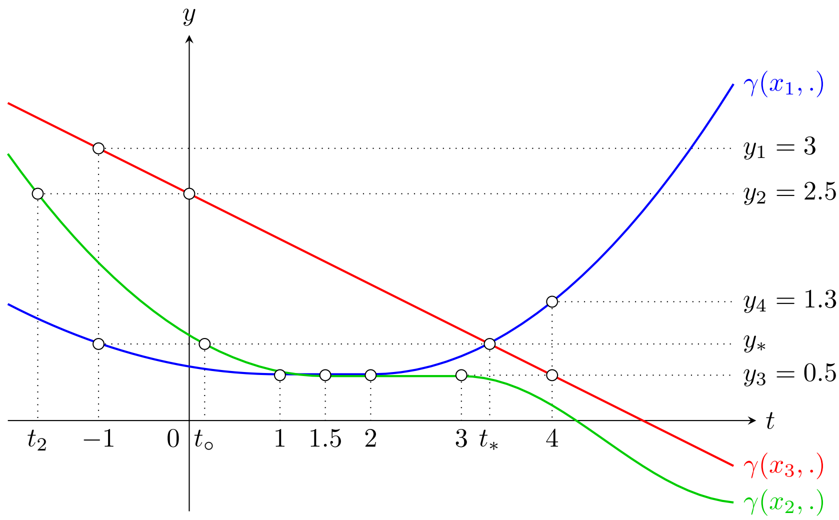

Figure 1 A dynamical system with three trajectories.

The function γ is called the *dynamics* of the system and $(\mathcal{M}, \gamma)$ is said to be o-minimal when the theory of $\mathcal{M}$ is itself o-minimal.

Classically, we see V as the time, V_1 as the input space, or set of parameters, $V_1 \times V$ as the space-time and V_2 as the output, or geometrical, space.

▶ **Definition 5.** For a dynamical system $(\mathcal{M}, \gamma)$, if we fix a point $x \in V_1$, the set $\Gamma_x = \{\gamma(x,t) \mid t \in V\} \subseteq V_2$ is called the *trajectory* determined by x.

We define a transition system associated with the dynamical system. This definition is an adaptation to our context of the classical *continuous transition system* in the case of hybrid systems (see [25] for example).

▶ **Definition 6.** Given $(\mathcal{M}, \gamma)$ a dynamical system, the associated *transition system* $T_\gamma = (Q, \rightarrow)$ is defined by:

- its set of states $Q = V_2$;
- its transition relation $\rightarrow$, which is defined by: $y \rightarrow y'$ if $\exists x \in V_1,\ \exists t, t' \in V$ such that $t \leq t'$ and $\gamma(x,t) = y,\ \gamma(x,t') = y'$.

As usual, an execution is a sequence of consecutive transitions. Note that it is possible to switch between trajectories, as illustrated below.

▶ **Example 7.** The dynamical system depicted in Figure 1 is composed of three trajectories (with $\gamma(x_1,.)$ in blue, $\gamma(x_2,.)$ in green and $\gamma(x_3,.)$ in red), with set of parameters $V_1 = \{x_1, x_2, x_3\}$ and $V = V_2 = \mathbb{R}$. Executions take place in $\mathbb{R}$, according to the trajectories. For instance: $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4$ with $y_1 = 3 = \gamma(x_3, -1)$ and $y_2 = 2.5 = \gamma(x_3, 0)$ on the red curve, then switching to the green curve since $y_2 = \gamma(x_2, t_2)$ for some $t_2 < 0$, $y_3 = 0.5 = \gamma(x_2, 2)$, and finally jumping to the blue curve since $y_3 = \gamma(x_1, 2)$, leading to $y_4 = \gamma(x_1, 4)$.

The definition of *dynamical system* encompasses a lot of different behaviours, examples of which can be obtained with structures enriched by additional operations like addition, multiplication (or the exponential function).

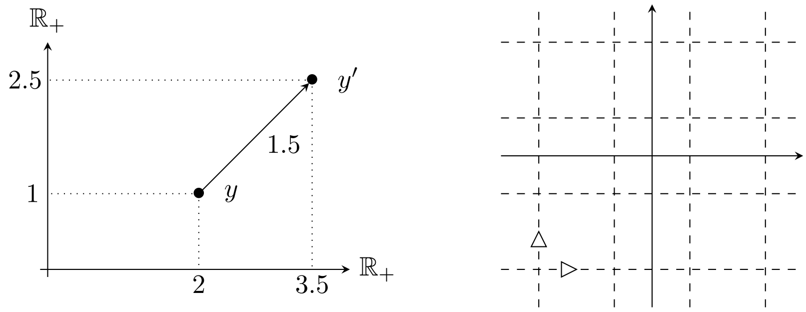

Figure 2 Dynamical systems: timed automata in dimension 2 (left) and example from [6] (right).

▶ **Example 8.** A classical one is the continuous dynamics of *timed automata* [1]: In this case, $\mathcal{M} = \langle \mathbb{R}, <, + \rangle$ and the dynamics $\gamma : \mathbb{R}_+^n \times [0, +\infty[\to \mathbb{R}_+^n$ is defined by $\gamma(x_1, \dots, x_n, t) = (x_1 + t, \dots, x_n + t)$. The standard graphical view, displayed in Figure 2 left, represents the dynamical system directly on the output space: $y \to y'$ with $y = (2, 1) = \gamma((2, 1), 0)$ and $y' = (3.5, 2.5) = \gamma((2, 1), 1.5)$.

▶ **Example 9.** Another example, borrowed from [6] and illustrated in Figure 2 right, features a dynamical system where each point of the plane has two possible behaviours: going right or going up. The dynamics $\gamma : \mathbb{R}^2 \times \{-1, +1\} \times \mathbb{R} \to \mathbb{R}^2$ is defined by:

$$\gamma(x_1, x_2, p, t) = \begin{cases} (x_1 + t, x_2) & \text{if } p = +1 \\ (x_1, x_2 + t) & \text{if } p = -1 \end{cases}$$

Then $y_1 \to y_2 \to y_3$ for the three points $y_1 = (0, 0)$, $y_2 = (0, 1)$ and $y_3 = (1, 1)$, since $y_1 = \gamma(0, 0, -1, 0)$, $y_2 = \gamma(0, 0, -1, 1) = \gamma(0, 1, 1, 0)$, and $y_3 = \gamma(0, 1, 1, 1)$.

In hybrid automata, such behaviours are combined with a finite set of discrete locations, each one having its own dynamics with respect to a common structure $\mathcal{M}$; jumps between locations are constrained by guards and equipped with updates. As mentioned in the introduction, basic verification problems like reachability checking are undecidable in the general case, and solutions to recover decidability are often to impose strong reinitializations of trajectories at jumps (we will come back to that in subsection 2.4), which amounts to concentrating on the analysis of a single dynamical system.

2.3 Time-abstract bisimulation

Time-abstract bisimulation [18, 14, 2, 25] is a behavioural relation often used to obtain a quotient of the original transition system. When this quotient is finite, a large class of properties can be verified, notably reachability properties.

We associate with a dynamical system $(\mathcal{M}, \gamma)$ a finite set G of guards, which are definable subsets of V_2. For every $y \in V_2$, we define the set $G_y \stackrel{\text{def}}{=} \{g \in G \mid y \in g\}$ of guards that are "satisfied" by y, thus producing a finite partition of V_2 into subsets satisfying the same sets of guards.

▶ **Definition 10.** Consider a dynamical system $(\mathcal{M}, \gamma)$, a finite set G of definable guards and an integer $k \geq -1$. A *k-step time-abstract bisimulation* is an equivalence relation $R_k \subseteq V_2 \times V_2$ such that either (i) $k = -1$, or (ii) $k \geq 0$ and there exists a $(k-1)$-step time-abstract bisimulation R_{k-1} such that, if $(y_1, y_2) \in R_k$, then:

(a) $G_{y_1} = G_{y_2}$;
(b) if $y_1 \to y_1'$ then there exists y_2' such that $y_2 \to y_2'$ and $(y_1', y_2') \in R_{k-1}$;
(c) if $y_2 \to y_2'$ then there exists y_1' such that $y_1 \to y_1'$ and $(y_1', y_2') \in R_{k-1}$.

We further say that an equivalence relation $R \subseteq V_2 \times V_2$ is a *time-abstract bisimulation* if R is a k-step time-abstract bisimulation for all $k \geq -1$. We also say that y_1 and y_2 are *(k-step) time-abstract bisimilar* whenever there is a (k-step) time-abstract bisimulation $R \subseteq V_2 \times V_2$ such that $(y_1, y_2) \in R$.

Note that, for every k, the class of k-step time-abstract bisimulations is closed under union, and therefore there is a largest k-step time-abstract bisimulation, which can be obtained as the union of all such relations. In particular, the relation R_{k-1} used in items (b) and (c) when defining R_k can be taken as the largest $(k-1)$-step time-abstract bisimulation. Similarly, there is a largest time-abstract bisimulation.

2.4 Problem and existing results

We focus here on the construction of finite (time-abstract) bisimulation relations, which is a standard and powerful tool to prove decidability of classes of hybrid systems [18].

Existence of such relations is, for instance, the key property satisfied by timed automata [1], a well-established model for real-time systems. However, for hybrid systems with more complex dynamics, proving that there is a finite bisimulation might be difficult and is not possible in general. In several works willing to better understand rich continuous dynamics in a system, the idea has been to decouple the continuous and the discrete parts of the system by assuming (possibly non-deterministic) reinitializations of the trajectories when a jump between locations is performed, see e.g. [14, 24, 26, 23, 25]. This leads to only focus on bisimulation relations within a discrete location. In this work, we follow this idea, and therefore only focus on bisimulations generated by a single dynamical system.

A standard methodology for proving that there is a finite time-abstract bisimulation is to compute successive approximations of the bisimulation relation (see [18, 24, 26, 23, 25, 7, 5]), and show that the procedure terminates. In (almost) all the references mentioned below, this is the way the problem is attacked. While the methodology seems rather universal, it is amazing to see the variety of arguments which are used to show termination of the procedure. They range from analytical and geometrical arguments [24, 26, 25] to model theory arguments [23], algebraic and topological arguments [14] or, more recently, arguments based on word combinatorics [7, 5].

While the precise domains of applicability of the approaches might vary, in most mentioned related works (except [8, 7, 5]), time-determinism is assumed, in the sense that there is a single trajectory going through some point of the output space. In [8, 7, 5], several trajectories may intersect or self-intersect, but rather strong assumptions need to be made. For instance, in the *suffix-determinism* assumption, all trajectories starting from a given point of the output space visit the same pieces of the initial partition in a similar way; in the *loop-determinism* assumption, two trajectories cannot intersect each other, but a trajectory can intersect itself in finitely many points.

In our work, lots of self-intersecting and overlapping trajectories are possible, but we bound the number of trajectories one can reach by switching between them (we will formalize this later). For example, the dynamical system of Figure 1 does not satisfy any of the above assumptions, but typically fits our framework.

The generic symbolic approach of [5] (and in particular the 2-subword refinement procedure) is a semi-procedure for finding finite bisimulations of o-minimal dynamical systems: it finds a finite bisimulation relation if there is one, but cannot tell that there is no finite bisimulation. But only the two above-mentioned assumptions (suffix-determinism and loop-determinism) guarantee termination of the computation. For instance, even though it does not satisfy any of the sufficient conditions above, the system of Example 9 has a finite bisimulation, which can be computed by the refinement procedure with respect to the single guard $y = (0, 0)$. Since there is no bound on the intersections of trajectories, this system does not belong to our class. On the other hand, both the present work and the approach of [5] encompass the original result [25].

Also, while the theory of o-minimality has been developed in any linearly ordered structure [30, 31, 21, 33], initial settings [25, 14] assume expansion of the reals. Here, similarly to [5], our results hold in the general setting.

In this paper we provide a method which is only based on geometrical properties of o-minimal systems. It does not assume the field of real numbers, nor dense or discrete structures. Furthermore, we are able to deal not only with (self-)crossing trajectories but also with partly stationary trajectories.

3 The graph construction

In this section, we fix an o-minimal dynamical system $(\mathcal{M}, \gamma)$ and a finite set G of guards as defined above, and we build a graph representing the time-abstract behaviour of γ.

We define a relation $\sim$ on V_1, where $x \sim x'$ if and only if the trajectories Γ_x and $\Gamma_{x'}$ cross each other, i.e. if there exist $t, t' \in V$ such that $\gamma(x, t) = \gamma(x', t')$. We also set $V_1(x) \stackrel{\text{def}}{=} \{x' \in V_1 : x \sim x'\}$.

To build the graph we distinguish between points of V_2 with (at most) finitely many predecessors by γ on any trajectory and points of V_2 with infinitely many predecessors on some trajectory. We will show that those two sets are definable, and that they can be used to provide a nice finite decomposition of the state-space, fine enough to characterize the time-abstract bisimulation. After defining suitable notions of intervals, we independently provide a finite decomposition result and the construction of the graph itself.

3.1 Towards a decomposition

In what follows, we need to distinguish two kinds of intervals: singletons, i.e. intervals with one unique element, and intervals with at least two elements, which we call *large* intervals.

▶ **Definition 11.** An interval $I \subseteq V$ is called *x-static* if either (i) I is large and $|\gamma(x, I)| = 1$, or (ii) I is a singleton and there exist a parameter $x' \in V_1$ and a large interval $J \subseteq V$ such that $\gamma(x', J) = \gamma(x, I)$. We further say that an element t of V is *x-static* if t belongs to some x-static interval, and that a state $y \in V_2$ is *static* if there exists $x \in V_1$ and $t \in V$ such that t is x-static and $y = \gamma(x, t)$.

On the contrary, we say that an element t of V is *x-dynamic* if t is not x-static, and we say that an interval I is *x-dynamic* if every element of I is x-dynamic. We further say that I is *x-suitable* if (i) I is x-dynamic, (ii) the function $t \mapsto G_{\gamma(x,t)}$ is constant on I, and (iii) the function $\gamma(x, \cdot)$ is one-to-one on I.

This produces a classification of points in V_2: static, if some trajectory stops at that position, or dynamic. It also induces a classification of timepoints and intervals along trajectories: a static point y of V_2 generates x-static timepoints on Γ_x, even though the trajectory Γ_x may not be responsible for making y static.

▶ **Example 12.** We illustrate the various notions on the example of Figure 1. Value $y = 0.5$ is static, because of x_1 and x_2. In particular, interval $(1, 2)$ is x_1-static and interval $(1.5, 3)$ is x_2-static. Time $t = 4$ is x_3-static because $\gamma(x_3, 4) = 0.5$ is static, even though $\gamma(x_3, \cdot)$ itself crosses $y = 0.5$ only in one point. And thus, interval $\{4\}$ is also x_3-static but not large.

Note that $y = y_*$ is dynamic, since no trajectory of the dynamical system is constant on a large interval on which its value is y_*.

Assuming no guard in the system (or a single guard $y = 0.5$), the intervals $(-\infty, 1)$ and $(2, +\infty)$ are x_1-suitable (and maximal for that condition). Similarly, the intervals $(-\infty, 1.5)$ and $(3, +\infty)$ are x_2-suitable; the intervals $(-\infty, 4)$ and $(4, +\infty)$ are x_3-suitable.

Then, since we want a finite representation of important points of the dynamical system, we need to get uniform (definable) descriptions of the above classification of points.

First, we gather all portions of trajectories corresponding to dynamic parts of the system. Note that such trajectories, while they visit the same state-space (in V_2), might follow different directions (hence the value $\varepsilon = \pm 1$ below).

▶ **Definition 13.** Consider two parameters $x, x' \in V_1$, one x-suitable interval $I \subseteq V$ and one x'-suitable interval $I' \subseteq V$. We say that the pairs (x, I) and (x', I') are *adapted* to each other if:

(i) the sets $\{\gamma(x, t) \mid t \in I\}$ and $\{\gamma(x', t') \mid t' \in I'\}$ are equal to each other;
(ii) there exists $\varepsilon = \pm 1$ such that: for all $t, u \in I$ with $t < u$, there exist $t', u' \in I'$ such that $\gamma(x, t) = \gamma(x', t')$, $\gamma(x, u) = \gamma(x', u')$, and $t' < u' \Leftrightarrow \varepsilon = 1$.

In general, we say that a family of pairs $(x_k, I_k)_{k \in K}$ is *strongly adapted* if:

(iii) every two pairs (x_k, I_k) and (x_ℓ, I_ℓ) are adapted to each other;
(iv) for all $k \in K$, $\{(x, t) \in V_1 \times M \mid \gamma(x, t) \in \gamma(x_k, I_k)\} = \bigcup_{\ell \in K} \{x_\ell\} \times I_\ell$.

Finally, we say that an interval I is *x-adaptable* if the pair (x, I) belongs to a strongly adapted family.

▶ **Example 14.** Going back to the previous example:
- the pairs $(x_1, (-\infty, 1))$ and $(x_2, (-\infty, 1.5))$ are adapted to each other (with $\varepsilon = +1$);
- the pairs $(x_1, (2, +\infty))$ and $(x_2, (-\infty, 1.5))$ are also adapted to each other (with $\varepsilon = -1$);
- the pairs $(x_2, (3, +\infty))$ and $(x_3, (4, +\infty))$ are adapted to each other (with $\varepsilon = +1$).

By extension, we get that:
- the pairs $(x_1, (-\infty, 1))$, $(x_1, (2, +\infty))$, $(x_2, (-\infty, 1.5))$ and $(x_3, (-\infty, 4))$ form a strongly adapted family;
- the interval $(2, +\infty)$ is x_1-adaptable, due to the strongly adapted family above;
- the interval $(3, +\infty)$ is x_2-adaptable, due to the family formed of $(x_2, (3, +\infty))$ and $(x_3, (4, +\infty))$;
- the singleton $\{t_*\}$ is both x_1-adaptable and x_3-adaptable, due to the family formed of $(x_1, \{-1\})$, $(x_2, \{t_\circ\})$, $(x_1, \{t_*\})$ and $(x_3, \{t_*\})$.

An interval I is said maximal x-static (resp. maximal x-adaptable), whenever it is x-static (resp. x-adaptable), and is contained in no larger x-static (resp. x-adaptable) interval.

It turns out that maximal x-static and x-adaptable intervals form a covering of the time domain V.

▶ **Lemma 15.** *Consider a parameter $x \in V_1$ and a timepoint $t \in V$. There exists an interval $I \subseteq V$, which contains t, and such that I is a maximal x-static interval (if t is x-static) or a maximal x-adaptable interval (if t is x-dynamic).*

Proof. If t is x-static, then the singleton $\{t\}$ is x-static. If t is x-dynamic, then the family of pairs $(x', \{t'\})$ such that $\gamma(x', t') = \gamma(x, t)$ is strongly adapted, and therefore the singleton $\{t\}$ is x-adaptable. Moreover, both the class of x-static intervals and the class of x-adaptable intervals are closed under increasing union: this is clear for x-static intervals, and can be argued as follows for x-adaptable intervals.

Let $(I^\alpha)_\alpha$ be an increasing family of x-adaptable intervals. For every α, let $\mathcal{F}^\alpha = (x_k^\alpha, I_k^\alpha)_{k \in K^\alpha}$ be a corresponding strongly adapted family. There is an obvious one-to-one correspondence between elements of $\mathcal{F}^\alpha$ and elements of $\mathcal{F}^{\alpha'}$ for any pair of indices (α, α'), hence one can rewrite the family $\mathcal{F}^\alpha$ uniformly as $(x_k, I_k^\alpha)_{k \in K}$. One can therefore take $\mathcal{F} = (x_k, \bigcup_\alpha I_k^\alpha)_{k \in K}$ as a strongly adapted family for $(x, \bigcup_\alpha I^\alpha)$. The result follows. ◀

3.2 Finite decomposition result

Our goal here is to prove the following decomposition, which refines Lemma 15.

▶ **Proposition 16.** *Consider a parameter $x \in V_1$ such that $V_1(x)$ is finite. Then, the set V is a finite, disjoint and definable union of intervals $I_1, \ldots, I_k$ such that every interval I_j is either*

1. *a maximal x-static interval, or*
2. *a maximal x-adaptable interval.*

We first focus on static (geometrical, i.e. in V_2) points and show that there can only be finitely many such points along a trajectory.

▶ **Lemma 17.** *There exists an integer $\mathbf{K}$ such that, for every parameter $x \in V_1$, there exist at most $\mathbf{K}$ large maximal x-static intervals.*

Proof. We first observe that, if I_1 and I_2 are maximal large x-static intervals, with $I_1 \neq I_2$, then $I_1 \cap I_2 = \emptyset$. Otherwise, the union $I_1 \cup I_2$ would also be x-static, contradicting the maximality of I_1 and I_2. Henceforth, we denote by $\prec$ the linear order on maximal large x-static intervals, defined by

$$I_1 \prec I_2 \text{ if and only if } \forall t \in I_1, \forall t' \in I_2, t < t'.$$

If $I_1 \prec I_2$, and if I_1 and I_2 have respective lower bounds ℓ_1 and ℓ_2, then $t \leq \ell_2$ for all $t \in I_1$ and therefore $\ell_1 < \ell_2$ (since I_1 is large). Consequently, if $\ell_2 \in I_1$, then I_1 must have a maximal element, and $\ell_2 = \max(I_1)$.

Now, let $\mathcal{L}(x)$ be the set of lower bounds of maximal x-static intervals. Observe that $\mathcal{L}(x)$ is definable, and therefore by Theorem 3, there exists an integer $\mathbf{K}_1$ such that, for all $x \in V_1$, $\mathcal{L}(x)$ is a disjoint union of at most $\mathbf{K}_1$ intervals. We claim that each of these intervals has (strictly) less than three elements.

Assume on the contrary that there exists a sub-interval J of $\mathcal{L}(x)$ containing three elements $\ell_1 < \ell_2 < \ell_3$. For all $t \in J$, we denote by $I(t)$ the maximal large x-static interval with lower bound t. Since $I(\ell_1)$ is large, it contains some element t such that $\ell_1 < t$. Up to replacing both t and ℓ_2 by $\min\{t, \ell_2\}$, we assume that $t = \ell_2$. It follows, as noted above, that $\ell_2 = \max(I(\ell_1))$. Since $I(\ell_2)$ is large too, consider some element u of $I(\ell_2)$ that is not maximal in $I(\ell_2)$. Since $\ell_1 \in I(\ell_1)$ and $I(\ell_1) \prec I(\ell_2)$, we know that $\ell_2 < u$. Up to replacing both u and ℓ_3 by $\min\{u, \ell_3\}$, we also assume that $u = \ell_3$, hence that $\ell_3 \in I(\ell_2)$. However, since $\ell_2 < \ell_3$, our initial remark proves that $u = \ell_3$ must be the maximal element of $I(\ell_2)$, contradicting the definition of u. This proves our claim.

The set $\mathcal{L}(x)$ is therefore of cardinality at most $2\mathbf{K}_1$. Observing that at most one maximal large x-static interval has no lower bound proves that there exist at most $\mathbf{K}$ large maximal x-static intervals, where $\mathbf{K} = 2\mathbf{K}_1 + 1$. ◀

▶ **Lemma 18.** *There exists an integer $\mathbf{L}$ such that, for every parameter $x \in V_1$, the set of x-static elements of V is a disjoint union of at most $\mathbf{L}\ |V_1(x)|$ maximal x-static intervals.*

Proof. Fix $x \in V_1$. Let $\mathcal{S}$ denote the set of static elements of $\gamma(x, V)$. With each element y of $\mathcal{S}$ we can associate a pair (x', I), where $x' \in V_1(x)$ and I is a maximal large x'-static interval such that $\gamma(x', I) = \{y\}$. This association is one-to-one, and therefore $|\mathcal{S}| \leq \mathbf{K}\ |V_1(x)|$.

Moreover, there exists an integer $\mathbf{L}_1$ such that, for every $y \in V_2$, the definable set $\{t \in V \mid \gamma(x, t) = y\}$ is a finite union of at most $\mathbf{L}_1$ intervals (Theorem 3). Assuming, without loss of generality, that these intervals are pairwise disjoint, proves Lemma 18 for $\mathbf{L} = \mathbf{K}\ \mathbf{L}_1$. ◀

We now turn to the case of dynamic elements. We start with the following combinatorial lemma, whose proof is immediate by induction on $k + \ell$.

▶ **Lemma 19.** *Let $\mathcal{I} = (I_1, \ldots, I_k)$ and $\mathcal{J} = (J_1, \ldots, J_\ell)$ be two partitions of V into sub-intervals. There exists a partition $\mathcal{K} = (K_1, \ldots, K_m)$ of V into sub-intervals that refines both $\mathcal{I}$ and $\mathcal{J}$, and such that $m + 1 \leq k + \ell$.*

▶ **Lemma 20.** *There exists an integer $\mathbf{M}$ such that, for every parameter $x \in V_1$, every maximal x-dynamic interval of V is a disjoint union of at most $\mathbf{M}(1 + |V_1(x)|)$ maximal x-adaptable intervals.*

Proof. First, recall that there exists an integer $\mathbf{L}_1$ such that, for all $x \in V_1$ and all $y \in V_2$, the definable set $\{t \in V \mid \gamma(x', t) = y\}$ is a disjoint union of at most $\mathbf{L}_1$ intervals. If y is not static, then these intervals must be singletons, and therefore $|\{t \in V \mid \gamma(x', t) = y\}| \leq \mathbf{L}_1$.

Now, for all $t \in V$ and $x, x' \in V_1$, we denote by $f_1(x, x', t) < \ldots < f_{\mathbf{L}_1}(x, x', t)$ the elements of the set $\{t' \in V \mid \gamma(x, t) = \gamma(x', t')\}$, where $f_i(x', t)$ is undefined if $|\{t' \in V \mid \gamma(x, t) = \gamma(x', t')\}|$ is either smaller than i or greater than $\mathbf{L}_1$ (in the latter case, $\gamma(x, t)$ must be static). Observe that every function f_i is definable. Consequently, there exists an integer $\mathbf{M}_1$ such that, for all $x \in V_1$ and $x' \in V_1(x)$, there exists a partition $\mathcal{P}_i(x, x')$ of V into at most $\mathbf{M}_1$ intervals on which the function $t \mapsto f_i(x, x', t)$ is either undefined, constant, or continuous and strictly monotonic (Theorems 2 and 3).

Similarly, since the function $(x, t) \mapsto G_{\gamma(x,t)}$ is definable, there exists an integer $\mathbf{M}_2$ such that, for all $x \in V_1$, there exists a partition $\mathcal{P}'(x)$ of V in at most $\mathbf{M}_2$ intervals on which $t \mapsto G_{\gamma(x,t)}$ is constant.

Now, consider some $x \in V_1$. By Lemma 19, there exists a partition $\mathcal{P}$ of V, which refines $\mathcal{P}'(x)$ and every partition $\mathcal{P}_i(x, x')$, for $i \leq \mathbf{L}_1$ and $x' \in V_1(x)$, and which contains at most $\mathbf{M}_1\ \mathbf{L}_1\ |V_1(x)| + \mathbf{M}_2$ intervals. By construction, every interval of the partition $\mathcal{P}$ is either x-adaptable or x-static, and by choosing $\mathcal{P}$ to contain as few intervals as possible, these intervals are guaranteed to be maximal x-adaptable intervals. Lemma 20 follows, by choosing $\mathbf{M} = \max\{\mathbf{M}_1\ \mathbf{L}_1, \mathbf{M}_2\}$. ◀

Since maximal x-adaptable intervals and maximal x-static intervals are definable, we derive from Lemmas 18 and 20 the targeted Proposition 16.

3.3 Construction of the graph

▶ **Definition 21.** We call *bisimulation graph* for the o-minimal dynamical system $(\mathcal{M}, \gamma)$ and the set of definable guards G the (possibly infinite) labeled graph with ε-transitions $\mathcal{G} = (N, E, E_\varepsilon, L)$ defined as follows:

- the set of nodes is

$$N = \{(x, I) : x \in V_1,\ I \text{ is a maximal } x\text{-static or } x\text{-adaptable interval }\};$$

- the set of edges is

$$E = \{((x, I), (x, J)) \in N \times N : \exists t \in I, \exists t' \in J,\ t \leq t'\};$$

- the set of ε-transitions is

$$E_\varepsilon = \{((x, I), (x', I')) \in N \times N : \exists t \in I, \exists t' \in I', \gamma(x, t) = \gamma(x', t')\};$$

- the labeling function is $L : (x, I) \mapsto \{g : \exists t \in I,\ g \in G_{\gamma(x,t)}\}$.

Next, we write $\rightarrow$ (resp. $\rightarrow_\varepsilon$) the transition relation defined by E (resp. E_ε), and we denote by $\rightsquigarrow$ the relation defined by: $n_1 \rightsquigarrow n_4$ if there exist nodes n_2 and n_3 such that $n_1 \rightarrow_\varepsilon n_2 \rightarrow n_3 \rightarrow_\varepsilon n_4$.

▶ **Definition 22.** Consider an integer $k \geq -1$. A *k-step ε-bisimulation* is an equivalence relation $\mathfrak{R}_k \subseteq N \times N$ such that either (i) $k = -1$, or (ii) $k \geq 0$ and there exists a $(k-1)$-step ε-bisimulation $\mathfrak{R}_{k-1}$ such that, if $n_1 \mathfrak{R}_k n_2$, then:

(a) $L(n_1) = L(n_2)$;
(b) if $n_1 \rightsquigarrow n'_1$ then there exists n'_2 such that $n_2 \rightsquigarrow n'_2$ and $n'_1\ \mathfrak{R}_{k-1}\ n'_2$;
(c) if $n_2 \rightsquigarrow n'_2$ then there exists n'_1 such that $n_1 \rightsquigarrow n'_1$ and $n'_1\ \mathfrak{R}_{k-1}\ n'_2$.

We further say that an equivalence relation $\mathfrak{R} \subseteq N \times N$ is a *ε-bisimulation* if $\mathfrak{R}$ is a k-step ε-bisimulation for all $k \geq -1$. We also say that two nodes n_1 and n_2 are *(k-step) ε-bisimilar* whenever there is a (k-step) ε-bisimulation $\mathfrak{R} \subseteq N \times N$ such that $n_1\ \mathfrak{R}\ n_2$.

Like time-abstract bisimulation, the class of (k-step) ε-bisimulations is closed under union, hence there is a largest (k-step) ε-bisimulation, which can be obtained as the union of all such relations. In particular, the relation $\mathfrak{R}_{k-1}$ used in items (b) and (c) when defining $\mathfrak{R}_k$ can be taken as the largest $(k-1)$-step ε-bisimulation.

▶ **Lemma 23.** *Let $n = (x, I)$ and $n' = (x', I')$ be nodes of the bisimulation graph $\mathcal{G}$. The following statements are equivalent: (i) $n \rightarrow_\varepsilon n'$, (ii) $\gamma(x, I) \cap \gamma(x', I') \neq \emptyset$, and (iii) $\gamma(x, I) = \gamma(x', I')$.*

Proof. The equivalence between (i) and (ii) follows directly from the definition of the set E_ε of ε-transitions, and the implication (iii) $\Rightarrow$ (ii) is obvious.

It remains to prove (iii), under the assumption that (ii) holds. If I is x-static, then $\gamma(x, I)$ is a singleton, hence I' contains an x'-static element, and therefore I' is not x'-suitable. This proves that I' is x'-static, hence that $\gamma(x', I')$ is a singleton too, and (iii) follows.

If I is maximal x-adaptable, then I' cannot be x'-static, hence I' is maximal x'-adaptable too. Let I'' be an interval such that (x, I) and (x', I'') are adapted, with $I' \cap I'' \neq \emptyset$. Since maximal x'-adaptable intervals are disjoint, it follows that $I'' \subseteq I'$, whence $\gamma(x, I) \subseteq \gamma(x', I')$. Similarly, we have $\gamma(x', I') \subseteq \gamma(x, I)$, which completes the proof. ◀

▶ **Lemma 24.** *Let $n = (x, I)$ and $n' = (x', I')$ be nodes of the bisimulation graph $\mathcal{G}$. The following statements are equivalent: (i) $n \rightsquigarrow n'$, (ii) $\exists y \in \gamma(x, I), \exists y' \in \gamma(x', I')$ s.t. $y \to y'$, and (iii) $\forall y \in \gamma(x, I), \exists y' \in \gamma(x', I')$ s.t. $y \to y'$.*

Proof. We first prove that (i) ⇒ (iii). Assume that $n \rightsquigarrow n'$, and let $n_1 = (x_1, I_1)$, $n_2 = (x_2, I_2)$ be nodes such that $n \to_\varepsilon n_1 \to n_2 \to_\varepsilon n'$. Let also $y \in \gamma(x, I)$. By Lemma 23, there exists $t \in I_1$ such that $y = \gamma(x_1, t)$. Let us prove that there exists $t' \in I_2$ such that $t \leq t'$. Indeed, if $I_1 = I_2$, we may choose $t' = t$. Otherwise, recall that I_1 and I_2, as maximal x_1-static or x_1-adaptable intervals, must be disjoint, and that there exist $t_1 \in I_1$ and $t_2 \in I_2$ such that $t_1 \leq t_2$; this proves in fact that $t_1 < t_2$ for all $t_1 \in I_1$ and $t_1 \in I_2$, and therefore that every $t' \in I_2$ is greater than t. Finally, let $y' = \gamma(x_2, t')$. Since $x_1 = x_2$ and $t \leq t'$, we know that $y \to y'$, and since $n_2 \to_\varepsilon n'$, Lemma 23 proves that $y' \in \gamma(x', I')$, which proves (iii).

Second, observe that the implication (iii) ⇒ (ii) is immediate. It remains to prove that (ii) ⇒ (i). Assume that (ii) holds. Let $x_1 \in V_1$ be a parameter, and $t_1 \leq t_2$ be elements of V such that $y = \gamma(x_1, t_1)$ and $y' = \gamma(x_1, t_2)$. Let I_1 and I_2 be the maximal x_1-static or x_1-adaptable intervals to which belong t_1 and t_2, and let $n_1 = (x_1, t_1)$ and $n_2 = (x_1, t_2)$. By construction, and using Lemma 23, we have $n \to_\varepsilon n_1 \to n_2 \to_\varepsilon n'$, which proves (i). ◀

▶ **Theorem 25.** *For all integers $k \geq -1$, two elements y_1 and y_2 in V_2 are (k-step) time-abstract bisimilar if and only if there exist (k-step) ε-bisimilar nodes $n_1 = (x_1, I_1)$ and $n_2 = (x_2, I_2)$ of the bisimulation graph $\mathcal{G}$ such that $y_i \in \gamma(x_i, I_i)$.*

Proof. In the following, we conveniently write $\gamma(n)$ instead of $\gamma(x, I)$ when n is the node (x, I).

For every $k \geq -1$, define R_k as the largest k-step time-abstract bisimulation over V_2. We define the relation $\mathfrak{R}_k$ over N as follows:

$n_1 \mathfrak{R}_k n_2$ iff $\exists y_i \in \gamma(n_i)$ such that $y_1 \; R_k \; y_2$.

Let us prove, by induction on k, that $\mathfrak{R}_k$ is a k-step ε-bisimulation relation. The case $k = -1$ is immediate, hence we assume that $k \geq 0$ and that $\mathfrak{R}_{k-1}$ is a $(k-1)$-step time-abstract bisimulation.

Let $n_1 = (x_1, I_1)$ and $n_2 = (x_2, I_2)$ be two nodes such that $n_1 \mathfrak{R}_k n_2$, and let $y_1 \in \gamma(n_1)$ and $y_2 \in \gamma(n_2)$ such that $y_1 \; R_k \; y_2$. First, since I_1 is either x_1-static or x_1-suitable, we know that the function $t \mapsto G_{\gamma(x_1,t)}$ is constant on I_1. Similarly, the function $t \mapsto G_{\gamma(x_2,t)}$ is constant on I_2 and therefore $L(n_1) = G_y = L(n_2)$.

Then, let $n'_1 = (x'_1, I'_1)$ be a node such that $n_1 \rightsquigarrow n'_1$. By Lemma 24, there exists $y'_1 \in \gamma(n'_1)$ such that $y_1 \to y'_1$. Since $y_1 \; R_k \; y_2$, there also exists y'_2 such that $y_2 \to y'_2$ and $y'_1 \; R_{k-1} \; y'_2$. Let $n'_2 = (x'_2, I'_2)$ be a node such that $y'_2 \in \gamma(n'_2)$. By construction, we have $y'_1 \mathfrak{R}_{k-1} y'_2$. Since n_1 and n_2 play symmetric roles, $\mathfrak{R}_k$ is a k-step ε-bisimulation relation.

Likewise, if R is the largest time-abstract bisimulation over V_2, the relation $\mathfrak{R}$ over N defined by $n_1 \mathfrak{R} n_2$ iff $\exists y_i \in \gamma(n_i)$ such that $y_1 \; R \; y_2$ is an ε-bisimulation relation.

Consequently, if y_1 and y_2 are (k-step) time-abstract bisimilar, constructing the relation ($\mathfrak{R}_k$ or) $\mathfrak{R}$ as above proves that there exist (k-step) ε-bisimilar nodes n_1 and n_2 of the bisimulation graph $\mathcal{G}$ such that $y_i \in \gamma(n_i)$.

Conversely, for every $k \geq -1$, define $\mathfrak{R}_k$ as the largest k-step ε-bisimulation over N. We define the relation R_k over V_2 as follows:

$y_1 \; R_k \; y_2$ iff $\exists n_i \in N$ such that $y_i \in \gamma(n_i)$ and $n_1 \mathfrak{R}_k n_2$.

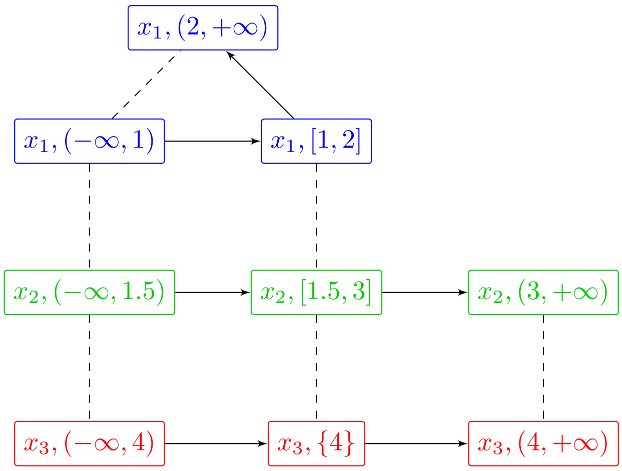

Figure 3 The bisimulation graph of the previous example ($\rightarrow_\varepsilon$ is obtained by reflexive and transitive closure of dashed lines; $\rightarrow$ is represented by normal edges).

Let us prove, by induction on k, that R_k is a k-step ε-bisimulation relation. The case $k = -1$ is immediate, hence we assume that $k \geq 0$ and that R_{k-1} is a $(k-1)$-step time-abstract bisimulation.

Consider two states $y_1, y_2 \in V_2$ such that $y_1 \; R_k \; y_2$, and let $n_1 = (x_1, I_1)$ and $n_2 = (x_2, I_2)$ be two nodes such that $y_i \in \gamma(n_i)$ and $n_1 \mathfrak{R}_k n_2$. Once again, the function $t \mapsto G_{\gamma(x_1,t)}$ is constant on I_1, and $t \mapsto G_{\gamma(x_2,t)}$ is constant on I_2, hence $G_{y_1} = L(n_1) = L(n_2) = G_{y_2}$.

Then, let y'_1 be a state such that $y_1 \rightarrow y'_1$, and let $n'_1 = (x'_1, I'_1)$ be a node such that $y'_1 \in \gamma(n'_1)$. Lemma 24 proves that $n_1 \rightsquigarrow n'_1$, and since $\mathfrak{R}_k$ is a k-step ε-bisimulation relation there exists a node $n'_2 = (x'_2, I'_2)$ such that $n_2 \rightsquigarrow n'_2$ and $n'_1 \mathfrak{R}_{k-1} n'_2$. Lemma 24 proves that $y_2 \rightarrow y'_2$ for some $y'_2 \in \gamma(n'_2)$, and we have $y'_1 \; R_{k-1} \; y'_2$ by construction. Since y_1 and y_2 play symmetric roles, $\mathfrak{R}_k$ is a k-step ε-bisimulation relation.

Likewise, if $\mathfrak{R}$ is the largest ε-bisimulation over N, the relation R over V_2 defined by $y_1 \; R \; y_2$ iff $\exists n_i \in N$ such that $y_i \in \gamma(n_i)$ and $n_1 \mathfrak{R} n_2$ is a time-abstract bisimulation relation. In particular, if $\mathfrak{R}$ is an ε-bisimulation relation, then R is a time-abstract bisimulation relation.

Consequently, if n_1 and n_2 are (k-step) ε-bisimilar, constructing the relation (R_k or) R as above proves that, for all states $y_i \in \gamma(n_i)$, y_1 and y_2 are (k-step) time-abstract bisimilar, which completes the proof. ◀

▶ **Example 26.** The bisimulation graph for the dynamical system of Figure 1 is depicted on Figure 3. We infer that:

- all points of the interval $(-\infty, y_3) = \gamma(x_2, (3, +\infty)) = \gamma(x_3, (4, +\infty))$ are time-abstract bisimilar;
- the singleton $\{y_3\} = \gamma(x_1, [1, 2]) = \gamma(x_2, [1.5, 3]) = \gamma(x_3, \{4\})$ forms a class of the time-abstract bisimulation;
- all points of the interval $(y_3, +\infty) = \gamma(x_1, (-\infty, 1)) = \gamma(x_1, (2, +\infty)) = \gamma(x_2, (-\infty, 1.5))$ $= \gamma(x_3, (-\infty, 4))$ are time-abstract bisimilar.

4 Definability and decidability

In this section, we discuss definability and decidability issues.

We say that a theory $\mathcal{M} = \langle M, <, \ldots \rangle$ is *decidable* whenever for every first-order formula φ, for every $t \in M$, one can decide whether $t \models \varphi$ holds.

So far we have not assumed any decidability of the structures, and, indeed, not all o-minimal structures are decidable. For instance, it is not known whether the o-minimal structure $\langle \mathbb{R}, <, 0, 1, +, \cdot, \exp \rangle$ is decidable [27, 29]. Alternatively, if ω is a non-computable real number, such as Chaitin's constant [11], then the structure $\langle \mathbb{R}, <, 0, 1, \omega, + \rangle$ is o-minimal but not decidable.

In this section, we consider the relation $\sim^*$, which is the (reflexive and) transitive closure of $\sim$, with $V_1^*(x) \stackrel{\text{def}}{=} \{x' \in V_1 : x \sim^* x'\}$. We introduce the following assumption, called *Finite Crossing*: every equivalence class of the relation $\sim^*$ (i.e. every set $V_1^*(x)$) is finite. The stronger condition obtained when there is a uniform bound on the size of equivalence classes is called *Uniform Crossing*.

▶ **Theorem 27.** *Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system. Under the Uniform Crossing assumption, the relation of time-abstract bisimulation is definable, and it contains finitely many equivalence classes.*

Proof. Let y_1, y_2 be elements of V_2 and let $x_1, x_2 \in V_1$ be parameters such that $y_i \in \Gamma_{x_i}$. Let also $\mathbf{P}$ be a positive integer such that $|V_1^*(x)| \leq \mathbf{P}$ for all $x \in V_1$. Consider the sub-graph $\mathcal{G}'$ of the bisimulation graph $\mathcal{G}$ that consists in those nodes (x', I) with $x' \in V_1^*(x_1) \cup V_1^*(x_2)$. This sub-graph is finite, and Lemmas 18 and 20 prove that it contains at most $k(\mathbf{L}+\mathbf{M}+\mathbf{M}\,k)$ nodes, where $k = |V_1^*(x_1) \cup V_1^*(x_2)|$. Since $k \leq 2\mathbf{P}$, it follows that $\mathcal{G}'$ contains at most $\mathbf{N} = 2\mathbf{P}(\mathbf{L} + \mathbf{M} + 2\mathbf{M}\,\mathbf{P})$ nodes.

It is well-known that, on $\mathcal{G}'$, the relations of ε-bisimulation and of $\mathbf{N}$-step ε-bisimulation are equal to each other. Hence, it follows from Theorem 25 that y_1 and y_2 are time-abstract bisimilar if and only if they are $\mathbf{N}$-step time-abstract bisimilar. In particular, the latter relation has finitely many equivalence classes, and is definable, which proves Theorem 27. ◀

▶ **Theorem 28.** *Let $(\mathcal{M}, \gamma)$ be a decidable o-minimal dynamical system. Under the Finite Crossing assumption, the relation of time-abstract bisimulation is decidable: given $y_1, y_2 \in V_2$, one can decide whether y_1 and y_2 are time-abstract bisimilar.*

Proof. For all $k \geq 0$ and $x \in V_1$, let $\mathbf{V}(k, x) = \{x_k \in V_1 : \exists x_1, \ldots, x_k \in V_1 \text{ s.t. } x \sim x_1, x_1 \sim x_2, \ldots, x_{k-1} \sim x_k\}$, where we recall that the relation $\sim$ is defined by: $x \sim x'$ iff $\Gamma_x \cap \Gamma_{x'} \neq \emptyset$. By construction, the set $\mathbf{V}(k, x)$ is definable and is a subset of $V_1^*(x)$. Moreover, since $V_1^*(x)$ is finite, there exists a minimal integer $k \geq 0$ such that $\mathbf{V}(k, x) = \mathbf{V}(k+1, x)$, and we have $V_1^*(x) = \mathbf{V}(k, x)$. Since the equality of definable sets is decidable, the set $V_1^*(x)$ is therefore computable for every parameter $x \in V_1$.

Now, let y_1, y_2 be elements of V_2 and let $x_1, x_2 \in V_1$ be parameters such that $y_i \in \Gamma_{x_i}$. We just showed how to compute the set $V_1' = V_1^*(x_1) \cup V_1^*(x_2)$. Then, let $\mathcal{R}$ and $\mathcal{R}'$ be the respective time-abstract bisimulation relations in $(\mathcal{M}, \gamma)$ and of $(\mathcal{M}, \gamma')$, where γ' is the restriction of γ to the set $V_1' \times V$. Since $\mathcal{R}'$ coincides with the restriction of $\mathcal{R}$ to $\{y \in V_2 : \exists x \in V_1', y \in \Gamma_x\}$, it remains to compute the relation $\mathcal{R}'$.

Since V_1' is finite, we may apply Theorem 27 to $(\mathcal{M}, \gamma')$. We thereby prove that $\mathcal{R}'$ has finitely many equivalence classes, and therefore is equal to the $\mathbf{N}$-step time-abstract bisimulation in $(\mathcal{M}, \gamma')$, for some integer $\mathbf{N}$. Consequently, the standard partition refinement procedure (see e.g. [7, p. 6]) will terminate, since there are finitely many classes, and we will be able to detect termination, since the theory is decidable. The partition refinement procedure is therefore an effective algorithm which allows to compute the time-abstract bisimulation $\mathcal{R}'$, which completes the proof. ◀

Remark that all results still hold if we replace the conditions on the sets $V_1^*(x)$, $x \in V_1$ by a finer semantical definition:

$$V_1^*(x) = \{x' \in V_1 : \exists y_1, \ldots, y_k \in V_2 \text{ s.t. } y_1 \in \Gamma_x, y_k \in \Gamma_{x'} \text{ and } y_1 \to \ldots \to y_k\}.$$

Notice, however, that the assumption on the size of $V_1^*(x)$ could not be relaxed, due to the undecidability result of [4, Theorem 3.1].

Recovering the main result of [25]

The use of restricted dynamical systems also allows us to encompass the main result of [25].

▶ **Theorem 29.** *Let $\mathcal{M} = \langle \mathbb{R}, <, \ldots \rangle$ be an expansion of the ordered set of the reals with an o-minimal theory, and let $V = \mathbb{R}$ and $V_1 = V_2 = \mathbb{R}^n$ for some integer $n \geq 1$. Assume that there exists a smooth, complete vector field F over $\mathbb{R}^n$ such that the dynamics (called* flow *in [25]) $\gamma : (x,t) \to \gamma(x,t)$, which is defined by: $\gamma(x,0) = x$ and $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(x,t) = F(\gamma(x,t))$, is definable in $\mathcal{M}$. Then, the relation of time-abstract bisimulation is definable, and it contains finitely many equivalence classes.*

Proof. By construction, if two trajectories Γ_x and $\Gamma_{x'}$ have a non-empty intersection, then there exists a real number t such that $x' = \gamma(x,t)$, and we have $\gamma(x',u) = \gamma(x,t+u)$ for all $u \in \mathbb{R}$, so that the trajectories Γ_x and $\Gamma_{x'}$ are equal to each other. Hence, the relation $\sim$ is an equivalence relation.

Then, due to [28, Corollary 3.3.28], there exists a definable set V_1' such that every equivalence class of $\sim$ contains a unique element in V_1'. Consider the restricted dynamical system $(\mathcal{M}, \gamma')$, where γ' is the restriction of γ to the set $V_1' \times V$. This restricted dynamical system satisfies the hypothesis of Theorem 27, and therefore there exists an integer $\mathbf{N} \geq 0$ such that the time-abstract bisimulation relation and the $\mathbf{N}$-step time-abstract bisimulation relation in $(\mathcal{M}, \gamma')$ are equal to each other. Since the transition systems associated with $(\mathcal{M}, \gamma)$ and $(\mathcal{M}, \gamma')$ are equal to each other, the result follows. ◀

5 Conclusion

In this paper, we have proposed a new approach for the analysis of o-minimal dynamical systems. Our approach allows us to treat trajectories with overlapping portions, and with possibly rich intersections. There is however a restriction, which is that trajectory switches should remain within a finite family of trajectories, once the initial trajectory has been chosen. It is important to notice that, as mentioned in the end of Section 4, it would not be possible to arbitrarily relax that assumption, since the reachability problem is undecidable for dynamical systems allowing arbitrarily many switches, as proved in [4, Theorem 3.1].

Adding the standard decoupling hypothesis, where jumps between locations reinitialize trajectories, we obtain a decidable class of hybrid systems.

Our future work will consist in trying to adapt the idea of interrupt timed automata of [3], where no reinitialization is assumed, to systems with richer (o-minimal) dynamics.

References

1 Rajeev Alur and David L. Dill. A theory of timed automata. *Theoretical Computer Science*, 126(2):183–235, 1994.

2 Rajeev Alur, Thomas A. Henzinger, Gerardo Lafferriere, and George J. Pappas. Discrete abstractions of hybrid systems. *Proc. of the IEEE*, 88:971–984, 2000.

3 Béatrice Bérard, Serge Haddad, Claudine Picaronny, Mohab Safey El Din, and Mathieu Sassolas. Polynomial Interrupt Timed Automata. In *Proceedings of the 9th International Workshop on Reachability problems (RP'15)*, volume 9328 of *LNCS*, pages 20–32. Springer, 2015.

4 Thomas Brihaye. A note on the undecidability of the reachability problem for o-minimal dynamical systems. *Math. Log. Q.*, 52(2):165–170, 2006. doi:10.1002/malq.200510024.

5 Thomas Brihaye. *Verification and Control of O-Minimal Hybrid Systems and Weighted Timed Automata*. PhD thesis, Université de Mons-Hainaut, Belgium, 2006.

6 Thomas Brihaye. Words and bisimulation of dynamical systems. *Discrete Mathematics and Theoretical Computer Science*, 9(2):11–31, 2007.

7 Thomas Brihaye and Christian Michaux. On the expressiveness and decidability of o-minimal hybrid systems. *Journal of Complexity*, 21(4):447–478, 2005.

8 Thomas Brihaye, Christian Michaux, Cédric Rivière, and Christophe Troestler. On o-minimal hybrid systems. In *Proc. 7th International Workshop on Hybrid Systems: Computation and Control (HSCC'04)*, volume 2993 of *Lecture Notes in Computer Science*, pages 219–233. Springer, 2004.

9 Alberto Casagrande, Pietro Corvaja, Carla Piazza, and Bud Mishra. Decidable compositions of o-minimal automata. In Sung Deok Cha, Jin-Young Choi, Moonzoo Kim, Insup Lee, and Mahesh Viswanathan, editors, *Proc. of 6th International Symposium on Automated Technology for Verification and Analysis, ATVA 2008*, volume 5311 of *Lecture Notes in Computer Science*, pages 274–288. Springer, 2008. doi:10.1007/978-3-540-88387-6_25.

10 Alberto Casagrande, Carla Piazza, Alberto Policriti, and Bud Mishra. Inclusion dynamics hybrid automata. *Inf. Comput.*, 206(12):1394–1424, 2008. doi:10.1016/j.ic.2008.09.001.

11 Gregory J Chaitin. A theory of program size formally identical to information theory. *Journal of the ACM (JACM)*, 22(3):329–340, 1975.

12 Xin Chen, Erika Ábrahám, and Sriram Sankaranarayanan. Taylor model flowpipe construction for non-linear hybrid systems. In *Proceedings of the 33rd IEEE Real-Time Systems Symposium, RTSS 2012*, pages 183–192. IEEE Computer Society, 2012. doi:10.1109/RTSS.2012.70.

13 G. E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In *Automata Theory and Formal Languages 2nd GI Conference*, volume 33 of *Lecture Notes in Computer Science*, pages 134–183. Springer, 1975.

14 Jennifer M. Davoren. Topologies, continuity and bisimulations. *Informatique Théorique et Applications*, 33(4-5):357–382, 1999.

15 Raffaella Gentilini. Reachability problems on extended o-minimal hybrid automata. In *Proc. 3rd International Conference on Formal Modeling and Analysis of Timed Systems (FORMATS'05)*, volume 3829 of *Lecture Notes in Computer Science*, pages 162–176. Springer, 2005.

16 R.L. Grossman, A. Nerode, A.P. Ravn, and H. Rischel, editors. *Hybrid systems*, volume 736 of *LNCS*. Springer, 1993.

17 Emmanuel Hainry. Reachability in linear dynamical systems. In *Logics and Theory of Algorithms, Proc. 4th Conference on Computability in Europe (CiE'08)*, volume 5028 of *Lecture Notes in Computer Science*, pages 241–250. Springer, 2008.

18 Thomas A. Henzinger. Hybrid automata with finite bisimulations. In *Proc. 22nd International Colloquium on Automata, Languages and Programming (ICALP'95)*, volume 944 of *Lecture Notes in Computer Science*, pages 324–335. Springer, 1995.

19 Thomas A. Henzinger, Peter W. Kopke, Anuj Puri, and Pravin Varaiya. What's decidable about hybrid automata? *Journal of Computer and System Sciences*, 57(1):94–124, 1998.

20 Wilfrid Hodges. *A Shorter Model Theory*. Cambridge University Press, 1997.

21 Julia F. Knight, Anand Pillay, and Charles Steinhorn. Definable sets in ordered structures. II. *Transactions of the American Mathematical Society*, 295(2):593–605, 1986.

22 Margarita V. Korovina and Nicolai Vorobjov. Upper and lower bounds on sizes of finite bisimulations of Pfaffian hybrid systems. In *CiE*, volume 3988 of *Lecture Notes in Computer Science*, pages 267–276. Springer, 2006.

23 Gerardo Lafferriere, George J. Pappas, and Shankar Sastry. Hybrid systems with finite bisimulations. In *Proc. Hybrid Systems V: Verification and Control (1997)*, volume 1567 of *Lecture Notes in Computer Science*, pages 186–203. Springer, 1997.

24 Gerardo Lafferriere, George J. Pappas, and Shankar Sastry. Subanalytic stratifications and bisimulations. In *Proc. 1st International Workshop on Hybrid Systems: Computation and Control (HSCC'98)*, volume 1386 of *Lecture Notes in Computer Science*, pages 205–220. Springer, 1998.

25 Gerardo Lafferriere, George J. Pappas, and Shankar Sastry. O-minimal hybrid systems. *Mathematics of Control, Signals, and Systems*, 13(1):1–21, 2000.

26 Gerardo Lafferriere, George J. Pappas, and Sergio Yovine. A new class of decidable hybrid systems. In *Proc. 2nd International Workshop on Hybrid Systems: Computation and Control (HSCC'99)*, volume 1569 of *Lecture Notes in Computer Science*, pages 137–151. Springer, 1999.

27 Angus Macintyre and Alex J. Wilkie. On the decidability of the real exponential field. In *Kreiseliana*, pages 441–467. A. K. Peters, 1996.

28 David Marker. *Model theory: an introduction*, volume 217. Springer Science & Business Media, 2006.

29 Daniel J Miller. Constructing o-minimal structures with decidable theories using generic families of functions from quasianalytic classes. Research Report math.LO/1008.2575, arXiv, 2010.

30 Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. *Bulletin of the American Mathematical Society*, 11(1), 1984.

31 Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. I. *Transactions of the American Mathematical Society*, 295(2):565–592, 1986.

32 Anand Pillay and Charles Steinhorn. Discrete o-minimal structures. *Ann. Pure Appl. Logic*, 34(3):275–289, 1987. `doi:10.1016/0168-0072(87)90004-2`.

33 Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. III. *Transactions of the American Mathematical Society*, 309(2):469–476, 1988.

34 Ashish Tiwari and Gaurav Khanna. Series of abstractions for hybrid automata. In C. J. Tomlin and M. R. Greenstreet, editors, *Hybrid Systems: Computation and Control HSCC*, volume 2289 of *LNCS*, pages 465–478. Springer, 2002.

35 Lou van den Dries. O-minimal structures. In *Proc. Logic, From Foundations to Applications*, pages 137–185. Oxford University Press, 1996.

36 Lou van den Dries. *Tame Topology and O-Minimal Structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1998.

37 Alex J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *Journal of the AMS*, 9(4):1051–1094, 1996.