Quantitative Foundations for Resource Theories

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Abstract

Considering resource usage is a powerful insight in the analysis of many phenomena in the sciences. Much of the current research on these resource theories focuses on the analysis of specific resources such quantum entanglement, purity, randomness or asymmetry. However, the mathematical foundations of resource theories are at a much earlier stage, and there has been no satisfactory account of quantitative aspects such as costs, rates or probabilities.

We present a categorical foundation for *quantitative* resource theories, derived from enriched category theory. Our approach is compositional, with rich algebraic structure facilitating calculations. The resulting theory is parameterized, both in the quantities under consideration, for example costs or probabilities, and in the structural features of the resources such as whether they can be freely copied or deleted. We also achieve a clear separation of concerns between the resource conversions that are freely available, and the costly resources that are typically the object of study. By using an abstract categorical approach, our framework is naturally open to extension. We provide many examples throughout, emphasising the resource theoretic intuitions for each of the mathematical objects under consideration.

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1 Introduction

The importance of analyzing phenomena from the perspective of resource conversions and consumption is an insight that pervades many disciplines. Logicians have long understood the significance of this point of view. For example, strong resource based intuitions underlie linear logic [14] and the resource and differential lambda calculi [2, 6].

In the natural sciences, many aspects of physics are now investigated using what are loosely termed *resource theories*. There are many different resource theories, for example, for quantum information alone, researchers have considered a multitude of possibilities, including

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asymmetry [24], non-uniformity [15], athermality [3] and superposition [29]. Much of the current work on resource theories focuses on specific situations. An exception is [4], where a pleasing categorical abstraction of resource theories is proposed.

In order to facilitate discussions, we describe a very simple culinary example, which hopefully does not require any domain specific expertise. Consider the "recipe":

$$\text{egg} + \text{egg} + \text{cream} + \text{sugar} \to \text{custard} \tag{1}$$

We read this as saying if we take two eggs, a standard unit of both cream and sugar, we can produce one unit of custard. Obviously we would like to combine such conversions, for example as a second step, we may want to combine our custard with an apple pie to form a pleasant dessert. Therefore a model of resource conversions should be *compositional*.

The recipe (1) already encodes some simple quantitative data about *resources* - two eggs are required as an input. In this paper we are interested not in quantifying the resources themselves, but in adding the ability to provide quantitative data about the *conversions* that can take place. For example:

- There may be a *cost* to producing custard, in elapsed time, energy consumed, or simply in paying a chef to do the cooking.
- Producing custard is unfortunately probabilistic, the custard may split or get burnt during cooking. We may therefore wish to quantify the *success probability* of a conversion taking place.
- If we are running a restaurant we may be interested in the *rate* of production so that we can keep our customers happy.

One can imagine quantifying similar features for chemical and biological reactions, economic behaviour, network communications, physical interactions and so on. Refining these ideas, resource theories typically separate resources into "free" resources conversions that are readily available, and "costly" processes that are often the focus of attention. An abstract model of resources should provide a clear separation of concerns between these two classes of resources.

Although some specific quantitative elements of resource theories are touched upon towards the end of [4], the approach is ad-hoc and no general purpose account of quantitative aspects is provided. They also fix the structural aspects of resources once and for all, rather than identifying this as a parameter of their theory. We provide a more general framework that allows variation in both the quantitative and structural aspects of resource theories.

We propose a foundation for *quantitative* resource theories, in which quantitative data can be attached to resource conversions. Our approach is based on two central ideas:

1. Exploiting enriched category theory allows us to incorporate quantitative data in a categorical framework. This is a classical idea, originating in Lawvere's seminal paper on generalized metric spaces [22]. By varying the base of enrichment, we can then adjust our quantities to the needs of a given application.
2. More recent theory on generalized algebraic structures [17, 23, 9] allows us to incorporate structural aspects of resources, such as whether they can always be deleted, or copied, or if the order in which they are provided matters. These models of generalized algebraic structures are closely related to relational models of linear logic, and many of the structures we exploit can intuitively be viewed as *generalized binary relations*.

By successfully combining these two elements, and systematically applying categorical methods, a satisfactory mathematical theory emerges. Pleasingly, many meaningful resource theoretic features emerge naturally as standard categorical structures such as monads, profunctors and bimodules.

Providing a general purpose foundation for quantitative resource theories opens up the opportunity for the unification and transfer of ideas between many fields of mathematics and the sciences. It also allows us to analyze such models in the abstract, letting us compare theories and understand their essential features, uncluttered by application specific details. At this level of abstraction, unexpected connections become apparent, for example there is clearly a link with Pavlovic's quantitative formal concept analysis [27] that should be explored.

1.1 Features

We highlight the following key features of our framework:

- **Modularity**: Our approach is parametric in two key directions. Firstly, how resource conversions are quantified can be configured to suit application needs, for example probabilities, rates or costs. Secondly, we can choose the structural aspects of resources, does their order matter? Can they be copied or deleted?
- **Compositionality**: The ability to compose and combine resources is intrinsic to our categorical approach. As we develop the underlying mathematics a great deal of algebraic structure emerges. This structure enables a calculational approach to reasoning about resource theories.
- **Separation of concerns**: We provide a clear separation between the "free" resource conversions that are readily available to everybody, and the "costly" conversions that are typically the main object of study.
- **Extensibility**: A categorical framework is naturally open to further extensions. This is a necessary feature of any realistic approach to quantifying resources. Given the breadth of potential applications, it is unrealistic to expect to anticipate every possible model of resources, their composition and quantification.
- **Practicality**: Although we work with abstractions such as enriched categories, monads and bicategories, in the special cases we deal with they have simple concrete descriptions as special sorts of matrices. This means that calculations in particular instances of our framework should be straightforward, and will not require advanced mathematical techniques.

1.2 Contribution

We outline our contribution:

- We provide a consistent resource theoretic interpretation of all the mathematical structure under consideration, building upon classical ideas of Lawvere [22]. This begins with material that will be familiar to some in the community, as we introduce mathematical background in sections 2 and 3, and continues with the newer concepts in later sections.
- In section 4 we give concrete descriptions of a hierarchy of five different free constructions on quantale enriched categories, that can be used to model the structural aspects of resources.
- Also in section 4, we show that each of the monads corresponding to the hierarchy of free constructions distributes over the free cocompletion monad. This allows us to extend our notions of resource interaction with new structural features.
- In section 5 we demonstrate how the resulting comonads yield "thin" variations on the notion of multicategory or operad, suitable for quantitative reasoning.
- In section 6 we show bimodules are the correct mathematical framework for incorporating freely available conversions requiring multiple components.

- In section 7 we address practical methods for closing resource conversions under composition in various ways. We establish that these constructions are canonical, by showing that each of them yields a free internal monad in an appropriate bicategory.

2 Quantale Enriched Categories

This section sets up standard technical background and notation. Throughout the paper, we aim for a self contained account with respect to enriched category theory. We will assume some basic knowledge of category theory, at the level of categories, functors, natural transformations, and (co)monads and their (co)Kleisli categories. The ideas in this section are well known, and the basic resource theoretic interpretations will be familiar to some in the community.

Throughout the document, we will specialize definitions to our situation of interest, without spelling out the details in full generality, as this will often significantly reduce the complexity involved. This applies to notions such as enriched categories, free constructions, bimodules and internal monads that occur in later sections. Experts will be able to recover our definitions from the more abstract formulations.

2.1 Quantales

We will use quantales to describe the abstract mathematical structure needed to quantify the costs of resource conversions.

▶ **Definition 1** (Quantale). A **quantale** is a complete join semilattice with a monoid structure $(\otimes, k)$ such that the following axioms hold [1]:

$$p \otimes \left(\bigvee_i q_i \right) = \bigvee_i p \otimes q_i \qquad \text{and} \qquad \left(\bigvee_i p_i \right) \otimes q = \bigvee_i p_i \otimes q$$

A **commutative quantale** is a quantale whose underlying monoid is commutative. All the quantales we consider in this paper will be commutative. Throughout, we shall use the symbol $\mathbb{Q}$ to denote an arbitrary commutative quantale.

The structure of a commutative quantale has a clear resource theoretic interpretation, with two key components:

1. The monoid structure allows us to combine quantities across the different steps of a process or algorithm, for example costs, success probabilities or connection strengths.
2. The join semilattice structure is then an optimizer. Having calculated aggregate values for various candidate procedures to achieve a desired aim, we can then quantify the best value attainable. For example, this might be the cheapest price, highest success probability or best connection strength achievable.

We introduce four quantales that will be used repeatedly in examples throughout the paper.

▶ **Example 2.** The **Boolean quantale** $\mathbb{B}$ has the two Boolean truth values as its underlying set, with logical disjunction and conjunction providing the join semilattice and monoid structure respectively.

[1] Sometimes the term *unital quantale* is used, but we will have no interest in the case without a unit.

▶ **Example 3.** The **interval quantale** $\mathbb{I}$ has underlying set the closed real interval $[0, 1]$. The join semilattice structure is given by the usual supremum, and the binary monoid operation takes the minimum of two elements.

▶ **Example 4.** The **Lawvere quantale** $\mathbb{L}$ has underlying set the extended positive reals $[0, \infty]$ with the join semilattice structure given by *infima* and the monoid structure given by addition of real numbers.

▶ **Example 5.** The **multiplicative quantale** $\mathbb{M}$ has underlying set the closed real interval $[0, 1]$. The join semilattice is given by suprema, and the binary monoid is ordinary multiplication of real numbers.

Finally, we remark that there are many more examples of commutative quantales. In particular, every locale [18] is a commutative quantale, including all complete Boolean algebras, finite distributive lattices and complete chains.

From a categorical perspective, a commutative quantale is a (small, thin, skeletal) complete and cocomplete symmetric monoidal closed category. It is this structure that makes them very pleasant to work with in enriched category theory.

2.2 Quantale Enriched Category Theory

The use of enriched category theory will be an essential tool for this paper. The standard source for enriched category theory is [19], but as we suggested earlier, the general definitions simplify significantly in the quantale enriched case. This is because there are many axioms to enforce structure, such as composition being associative or functors preserving identities, that are phrased in terms of certain diagrams commuting. As quantales are thin categories, all these axioms become trivial. We therefore provide concrete descriptions of the various enriched mathematical objects that we use, specialized to the simpler quantale enriched setting. Via examples, we take the opportunity to introduce our resource theoretic perspective on each of the various notions.

All our quantale enriched categories will be small, that is, we will require that they have a *set* of objects.

▶ **Definition 6** ($\mathbb{Q}$-enriched Categories). A **$\mathbb{Q}$-enriched category** $\mathcal{A}$ consists of:
- A set of **objects obj** $\mathcal{A}$. We will typically denote these objects as $a, b, c, \ldots$.
- For each pair of objects, there is a **hom object** $\mathcal{A}(a, b) \in \mathbb{Q}$.

The hom objects are required to satisfy two axioms:
- The **identity axiom**, for all a:

$$k \leq \mathcal{A}(a, a)$$

- The **composition axiom**, for all a, b, c:

$$\mathcal{A}(b, c) \otimes \mathcal{A}(a, b) \leq \mathcal{A}(a, c)$$

Enrichment over each of our example quantales has a natural resource theoretic interpretation.

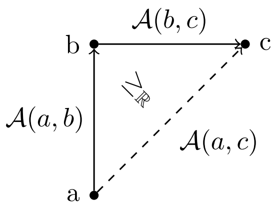
▶ **Example 7** (Boolean Quantale Enrichment). A $\mathbb{B}$-enriched category is the same thing as a preorder. We can interpret $a \leq b$ as meaning it is possible to convert resource a to b. The identity axiom corresponds to reflexivity, we can always convert a resource to itself. The composition axiom corresponds to transitivity, and captures the idea that if we can convert resource a to b and we can convert b to c, then we can combine these conversions to convert a to c.

▶ **Example 8** (Interval Quantale Enrichment). An $\mathbb{I}$-enriched category $\mathcal{A}$ is a "fuzzy" generalization of a preorder. From a resource perspective, we interpret $\mathcal{A}(a,b)$ as a connection strength between a and b. Connection strengths are valued in a worst case manner, a composite connection is only as good as its weakest link. Then:

- The identity axiom tells us we can always connect any a to itself with maximum strength.
- The composition axiom tell us that if we can connect a to b and b to c, we should be able to connect a to c at least as strongly as going via the intermediate b.

▶ **Example 9** (Lawvere Quantale Enrichment). For an $\mathbb{L}$-enriched category $\mathcal{A}$, $\mathcal{A}(a,b)$ can be seen as the cost of converting a to b.

- The identity axiom tells us that we can freely convert a to itself. In Lawvere's original metric space reading [22] the absence of the axiom $\mathcal{A}(a,b) = 0 \Rightarrow a = b$ is inconvenient. However, from a resource conversion perspective it is entirely natural that two distinct resources could be interconvertible.
- The composition axiom is a triangle inequality, saying that the cost of converting from a to c should be at least as cheap as converting via any intermediate resource b.



▶ **Example 10** (Multiplicative Quantale Enrichment). For an $\mathbb{M}$-enriched category $\mathcal{A}$, we interpret $\mathcal{A}(a,b)$ as the probability of successfully converting a to b. Conversion probabilities are assumed to be independent, so they multiply.

- The identity axiom tells us we can always convert a resource to itself with certainty.
- The composition axiom tells us that we can convert a to c with a success probability at least as high as that achievable by chaining two conversions via any intermediate resource b.

This concludes our examples for this section. It remains to define the enriched notions of $\mathbb{Q}$-functors and $\mathbb{Q}$-natural transformations in preparation for later sections.

▶ **Definition 11** ($\mathbb{Q}$-enriched Functor). Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{Q}$-enriched categories. A **$\mathbb{Q}$-enriched functor** F of type $\mathcal{A} \to \mathcal{B}$ consists of an object assignment function:

$$F : \mathbf{obj}\,\mathcal{A} \to \mathbf{obj}\,\mathcal{B}$$

such that:

$$\mathcal{A}(a,b) \leq \mathcal{B}(Fa, Fb)$$

Identity and composite functors are given in the obvious way, and the resulting structure yields a category $\mathbf{Cat}(\mathbb{Q})$ of $\mathbb{Q}$-categories and functors between them.

▶ **Definition 12** ($\mathbb{Q}$-enriched Natural Transformations). Let $F, G : \mathcal{A} \to \mathcal{B}$ be parallel $\mathbb{Q}$-enriched functors. The existence of a **$\mathbb{Q}$-enriched natural transformation** α of type $F \Rightarrow G$ simply states that the following inequalities hold for all objects of $\mathcal{A}$:

$$k \leq \mathcal{B}(Fa, Ga)$$

That is, there can be at most one $\mathbb{Q}$-natural transformation between two such functors.

We do not dwell on examples of functors and natural transformations now, as there will be many examples later in cases of particular importance.

3 Presheaves and Profunctors

We first introduce some constructions on quantale enriched categories.

▶ **Definition 13.** Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{Q}$-enriched categories.

- There is a **unit** $\mathbb{Q}$-category $\mathcal{I}$ with a single object and the quantale unit as the unique hom object.
- The **tensor category** $\mathcal{A} \otimes \mathcal{B}$ has set of objects $\mathbf{obj}\,\mathcal{A} \times \mathbf{obj}\,\mathcal{B}$, and hom objects:

  $$(\mathcal{A} \otimes \mathcal{B})((a,b),(a',b')) = \mathcal{A}(a,a') \otimes \mathcal{B}(b,b')$$

- The **opposite category** $\mathcal{A}^{op}$ has the same objects as $\mathcal{A}$, and hom objects:

  $$\mathcal{A}^{op}(a,a') = \mathcal{A}(a',a)$$

- A quantale $\mathbb{Q}$ also carries a canonical structure as a $\mathbb{Q}$-category, with objects the elements of $\mathbb{Q}$, and hom objects:

  $$\mathbb{Q}(q,q') = q \multimap q'$$

  Where $q \multimap q'$ denotes the internal hom in $\mathbb{Q}$.

▶ **Definition 14** (Presheaf)**.** Let $\mathcal{A}$ be a $\mathbb{Q}$-category.

- A **copresheaf** is a functor of type $\mathcal{A} \to \mathbb{Q}$. This is a function $F : \mathbf{obj}\,\mathcal{A} \to \mathbf{obj}\,\mathbb{Q}$ such that:

  $$F(a) \otimes \mathcal{A}(a,b) \leq F(b)$$

- A **presheaf** is a functor of type $\mathcal{A}^{op} \to \mathbb{Q}$. This is a function $F : \mathbf{obj}\,\mathcal{A} \to \mathbf{obj}\,\mathbb{Q}$ such that:

  $$\mathcal{A}(a,b) \otimes F(b) \leq F(a)$$

▶ **Definition 15** (Profunctor)**.** For a commutative quantale $\mathbb{Q}$, and $\mathbb{Q}$-enriched categories $\mathcal{A}$ and $\mathcal{B}$, a **profunctor** from $\mathcal{A}$ to $\mathcal{B}$ is a functor of type:

$$\mathcal{A}^{op} \otimes \mathcal{B} \to \mathbb{Q}$$

Concretely, this is a function $R : \mathbf{obj}\,\mathcal{A} \times \mathbf{obj}\,\mathcal{B} \to \mathbf{obj}\,\mathbb{Q}$ such that:

$$\mathcal{A}(a',a) \otimes R(a,b) \otimes \mathcal{B}(b,b') \leq R(a',b')$$

We write $R : \mathcal{A} \nrightarrow \mathcal{B}$ to indicate R is a profunctor from $\mathcal{A}$ to $\mathcal{B}$.
A profunctor can be thought of as a categorical generalization of the notion of binary relation, taking truth values in the underlying quantale. They generalize both presheaves and copresheaves, as they are profunctors of type $\mathcal{A} \nrightarrow \mathcal{I}$ and $\mathcal{I} \nrightarrow \mathcal{A}$ respectively.

▶ **Example 16.** (Co)presheaves have natural resource theoretic interpretations. For example, if we consider $\mathbb{L}$-enrichment:

- A copresheaf on $\mathcal{A}$ is a coherent set of costs for acquiring the resources in $\mathcal{A}$. The copresheaf condition:

  $$F(a) + \mathcal{A}(a,b) \geq_{\mathbb{R}} F(b)$$

  requires that it is always cheaper to buy a resource b directly, rather than purchase some other resource a and then pay $\mathcal{A}(a,b)$ to turn it into b.

- A presheaf on $\mathcal{A}$ is a coherent set of costs for disposing of the resource in $\mathcal{A}$. The presheaf condition:

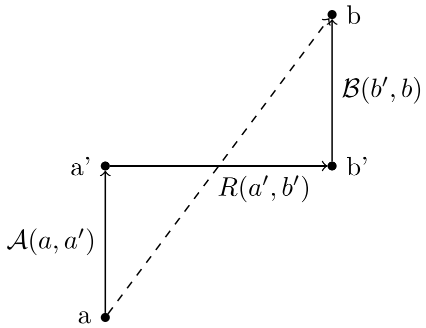
  $$\mathcal{A}(a,b) + F(b) \geq_{\mathbb{R}} F(a)$$

  requires that it is always cheaper to dispose of a resource a directly, rather than pay the cost $\mathcal{A}(a,b)$ to convert it to some b and then pay the cost to destroy b.

▶ **Example 17.** We consider profunctors from a resource perspective, using the multiplicative quantale. A profunctor $R : \mathcal{A} \nrightarrow \mathcal{B}$ satisfies:

$$\mathcal{A}(a',a) \times R(a,b) \times \mathcal{B}(b,b') \leq_{\mathbb{R}} R(a',b')$$

If we interpret R as describing a probabilistic device for converting $\mathcal{A}$ resources to $\mathcal{B}$ resources, the profunctor axiom says that the device will convert a' to b' with a success probability higher than the product of the probabilities of converting a' to a in $\mathcal{A}$, and then using R to convert a to b, and then converting b to b' in $\mathcal{B}$, as shown below:



Notice the probabilities here describe the chances of success of a chosen conversion, rather than which conversion will take place, as might be seen in stochastic relations for example.

▶ Remark (Separation of Concerns). Profunctors are the first point at which we see that the enriched categorical framework provides a clear separation of concerns between free and costly resources. The domain and codomain model the resources freely available. The transition costs encoded by the profunctor then provide additional resources conversions, with the profunctor axiom requiring that these all these conversions are better than can be achieved by additionally exploiting free resources.

▶ **Definition 18.** Given profunctors $R : \mathcal{A} \nrightarrow \mathcal{B}$ and $S : \mathcal{B} \nrightarrow \mathcal{C}$, we can form their composite $S \circ R : \mathcal{A} \nrightarrow \mathcal{C}$, defined pointwise as follows:

$$(S \circ R)(a,c) = \bigvee_b R(a,b) \otimes S(b,c)$$

This composition is associative, and has identity at $\mathcal{A}$ given by:

$$1_{\mathcal{A}}(a,a') = \mathcal{A}(a,a')$$

Therefore $\mathbb{Q}$-profunctors form a category $\mathbf{Prof}(\mathbb{Q})$.

▶ **Example 19.** Continuing example 17, we consider the composition of two $\mathbb{M}$-profunctors, $R : \mathcal{A} \nrightarrow \mathcal{B}$ and $S : \mathcal{B} \nrightarrow \mathcal{C}$. Intuitively, the value:

$$(S \circ R)(a,c) = \sup_b \{R(a,b) \times S(b,c)\}$$

describes the best probability achievable for converting a to c via some intermediate b using the two probabilistic devices described by R and S.

▶ Remark. In general, composition of profunctors is defined using colimits in the enriching category. Therefore we can only expect associativity and unitality of composition to hold up to isomorphism, pointing us in the more complicated direction of bicategories. Fortunately, the only isomorphisms in a quantale are the identities, and so composition is defined "on the nose", yielding a genuine category.

The tensor structure of definition 13 gives $\mathbf{Prof}(\mathbb{Q})$ the structure of a symmetric monoidal category. In fact it is a compact closed category [20], and so has a powerful graphical calculus that can be exploited in calculations.
As profunctors are a generalization of binary relations, and relations are closed under taking unions, we may expect similar structure of profunctors.

▶ **Definition 20.** A complete join semilattice enriched category is an ordinary category such that the hom sets are complete join semilattices, and the following axioms hold:

$$\left(\bigsqcup_i S_i\right) \circ R = \bigsqcup_i (S_i \circ R) \qquad \text{and} \qquad S \circ \left(\bigsqcup_i R_i\right) = \bigsqcup_i (S \circ R_i)$$

Complete join semilattice enrichment also implies that hom sets have a partial order $\subseteq$ such that composition is monotone in both components.

The following then gives us a straightforward generalization of taking unions of ordinary binary relations.

▶ **Lemma 21.** *For a commutative quantale* $\mathbb{Q}$*, the category* $\mathbf{Prof}(\mathbb{Q})$ *is complete join semilattice enriched with:*

$$\left(\bigsqcup_i R_i\right)(a, b) = \bigvee_i R_i(a, b)$$

If we return to our resource theoretic perspective, $\bigsqcup_i R_i$ combines the best capabilities of a family of different resource conversion options. The induced order $R \subseteq S$ is equivalent to there being a $\mathbb{Q}$-natural transformation $R \Rightarrow S$. We require another specialized definition.

▶ **Definition 22** (Internal Monad)**.** An **internal monad** in a complete join semilattice enriched category is an endomorphism $R : A \to A$ such that both:

$$1_A \subseteq R \qquad \text{and} \qquad R \circ R \subseteq R$$

Internal monads are an important concept. From the point of view of resources, an internal monad captures closure under repeated application of the available conversions. We can think of an internal monad on $\mathcal{A}$ as describing a "better" $\mathbb{Q}$-enriched category structure on the objects of $\mathcal{A}$.

▶ **Example 23** (Internal Monads as Better Structures)**.** An internal monad $R : \mathcal{A} \nrightarrow \mathcal{A}$ in $\mathbf{Prof}(\mathbb{L})$ is a selection of resource conversion costs that is closed under composition. That is, the cost $R(a, a')$ will be cheaper than the cost of any iterated conversion:

$$a \to b_1 \to \ldots \to b_n \to a'$$

Such a monad provides resource conversion costs that are closed under composition, and better than those of the underlying category $\mathcal{A}$.

Similarly, an internal monad $P : \mathcal{A} \nrightarrow \mathcal{A}$ in $\mathbf{Prof}(\mathbb{B})$ is a preorder stronger than the original order on $\mathcal{A}$.

We shall encounter internal monads again in sections 5, 6 and 7 as we introduce richer structure to our resources.

4 A Hierarchy of Resource Structures

So far, we have considered only conversions between individual resources. In this section, we introduce additional structure that will allow us to consider conversions that require multiple inputs, such as the custard recipe of the introduction.

▶ **Definition 24.** A $\mathbb{Q}$-category $\mathcal{A}$ is:

- **Strictly monoidal** if the objects carry a monoid structure $\otimes, I$ such that:

  $$\mathcal{A}(a_1, b_1) \otimes \mathcal{A}(a_2, b_2) \leq \mathcal{A}(a_1 \otimes a_2, b_1 \otimes b_2)$$

  From here on, we will drop explicitly saying "strictly" and simply use the term monoidal $\mathbb{Q}$-category.
- **Symmetric monoidal** if it is monoidal and for all $a, b \in \mathcal{A}$:

  $$k \leq \mathcal{A}(a \otimes b, b \otimes a)$$

- **Deleting** if it is symmetric monoidal, and for all $a \in \mathcal{A}$:

  $$k \leq \mathcal{A}(a, I)$$

- **Copying** if it is symmetric monoidal, and for all $a \in \mathcal{A}$:

  $$k \leq \mathcal{A}(a, a \otimes a)$$

- **Cartesian** if it is both copying and deleting.

A homomorphism of each of these special sorts of $\mathbb{Q}$-categories is a $\mathbb{Q}$-functor that is a monoid homomorphism with respect to the monoid structure on objects.

Each of these structures has a resource theoretic reading. A monoidal $\mathbb{Q}$-category allows us to combine ordered collections of resources. This setting is very restrictive, we are not necessarily able to even adjust the order of the resources provided. A symmetric monoidal $\mathbb{Q}$-category allows us to cheaply interchange the order of resources. If a $\mathbb{Q}$-category is deleting, we can also delete resources we do not need, and if it is copying, we can copy available resources, effectively making them reusable. This perspective will be most apparent in the forthcoming free constructions, in which the objects are lists of resources.

We also introduce some additional properties of quantales that we will require, using terminology paralleling that used for $\mathbb{Q}$-categories.

▶ **Definition 25.** We say that a quantale $\mathbb{Q}$ is:

- **Deleting** if the monoid unit k is the top element.
- **Copying** if the for all $q \in \mathbb{Q}$, $q \leq q \otimes q$.
- **Cartesian** if it is both copying and deleting[2].

In this section we describe a hierarchy of free constructions on $\mathbb{Q}$-enriched categories. This family of constructions is reminiscent of the Boom type hierarchy [26] familiar to the functional programming community, in which varying the axioms required of a construction of a particular shape results in a family of different datatypes. In our case, the objects of each free construction will be lists of resources. The interesting structure is in the hom objects, which will encode the resource conversions we wish to provide as standard. We will therefore frequently need to work with finite lists.

[2] A commutative quantale is Cartesian if and only if it is a locale.

▶ **Definition 26** (List Notation)**.** We will write $[a]$ for the singleton list. For a list of elements A, we will write A_i for the i^{th} element of the list and $\#A$ for the length of the list. We will also write $i : \#A$ to mean $1 \leq i \leq \#A$, and $\otimes_{i:\#A}\tau_i$ as shorthand for the iterated tensor product $\tau_1 \otimes ... \otimes \tau_{\#A}$. We will also abuse notation, and identify $\#A$ with the set $\{1, ..., \#A\}$.

▶ **Theorem 27.** *For a commutative quantale $\mathbb{Q}$, $\mathbb{Q}$-category $\mathcal{A}$, and lists of $\mathcal{A}$-objects A, B, define:*

$$\bigvee_{\psi:\#B\to\#A} \otimes_{i:\#B}\mathcal{A}(A_{\psi i}, B_i) \tag{2}$$

The following categories all have objects finite lists of elements from $\mathcal{A}$:

- *The free monoidal $\mathbb{Q}$-category $L(\mathcal{A})$ has hom objects $L(\mathcal{A})(A, B)$ given by expression* (2) *with ψ restricted to identity functions.*
- *The free symmetric monoidal $\mathbb{Q}$-category $M(\mathcal{A})$ has hom objects $M(\mathcal{A})(A, B)$ given by expression* (2) *with ψ restricted to permutations.*
- *If $\mathbb{Q}$ is deleting, the free deleting $\mathbb{Q}$-category $D(\mathcal{A})$ has hom objects $D(\mathcal{A})(A, B)$ given by expression* (2) *with ψ restricted to injective functions.*
- *If $\mathbb{Q}$ is copying, the free copying $\mathbb{Q}$-category $C(\mathcal{A})$ has hom objects $C(\mathcal{A})(A, B)$ given by expression* (2) *with ψ restricted to surjective functions.*
- *If $\mathbb{Q}$ is Cartesian, the free Cartesian $\mathbb{Q}$-category $K(\mathcal{A})$ has hom objects $K(\mathcal{A})(A, B)$ given by expression* (2) *with ψ ranging over all functions.*

Proof. We sketch the required argument. In each case, the universal morphism is given by the map to the singleton list, which can be verified to be a $\mathbb{Q}$-functor. It follows from the universal property of the free monoid construction on sets that there is a unique possible fill in $\mathbb{Q}$-functor. This can be confirmed by direct calculation, exploiting the additional properties of $\mathbb{Q}$ in the deleting, copying and Cartesian cases. ◀

Given they result from a free / forgetful adjunction, each of the constructions of theorem 27 yields a monad on **Cat**($\mathbb{Q}$). We wish to lift this structure to profunctors. As profunctors are analogous to binary relations, we might expect they arise as the Kleisli category of a generalization of the powerset monad. Recall [19] that the presheaves on a $\mathbb{Q}$-category form a $\mathbb{Q}$-category themselves. In fact, this is the free cocompletion, in the enriched sense. In general this construction does not induce a monad as there are size issues, leading to the need for more complex machinery [9]. In the case of quantale enrichment, we are fortunate as this problem goes away, and it can be shown that **Prof**($\mathbb{Q}$) is the Kleisli category of the free cocompletion monad. Lifting a monad to **Prof**($\mathbb{Q}$) can then be done by exhibiting an appropriate distributive law [1].

▶ **Theorem 28.** *Let $\mathbb{Q}$ be a commutative quantale, P the free cocompletion comonad, $\mathcal{A}$ a $\mathbb{Q}$-category, A a list of $\mathcal{A}$-objects, and F a list of presheaves on $\mathcal{A}$. Define:*

$$\bigvee_{\psi:\#F\to\#A} \otimes_{i:\#F}F_i A_{\psi i} \tag{3}$$

- *There is a distributive law $\lambda^L : LP \Rightarrow PL$ with $\lambda^L_{\mathcal{A}}(F)(A)$ given by expression* (3)*, with ψ restricted to identity functions.*
- *There is a distributive law $\lambda^M : MP \Rightarrow PM$ with $\lambda^M_{\mathcal{A}}(F)(A)$ give by expression* (3)*, with ψ restricted to permutations.*
- *If $\mathbb{Q}$ is deleting, there is a distributive law $\lambda^D : DP \Rightarrow PD$ with $\lambda^D_{\mathcal{A}}(F)(A)$ given by expression* (3)*, with ψ restricted to injective functions.*

- *If $\mathbb{Q}$ is copying, there is a distributive law $\lambda^C : CP \Rightarrow PC$ with $\lambda^C_{\mathcal{A}}(F)(A)$ given by expression* (3)*, with ψ restricted to surjective functions.*
- *If $\mathbb{Q}$ is Cartesian, there is a distributive law $\lambda^K : KP \Rightarrow PK$ with $\lambda^K_{\mathcal{A}}(F)(A)$ given by expression* (3)*, with ψ ranging over all functions.*

Proof. We can only sketch the proof. We first confirm that the components of each law are valid $\mathbb{Q}$-functors, and naturality of their components. With this in place, we verify Beck's axioms [1] by direct calculation. This is a long series of calculations to cover all the cases. Generally establishing the unit laws is routine. The naturality checks and multiplication laws are less straightforward, particularly in the deleting, copying and Cartesian cases. In these cases, we must carefully apply the additional quantale axioms to confirm the required properties, effectively by "copying" and "deleting" sub-terms in calculations. ◀

▶ **Corollary 29.** *As* **Prof**$(\mathbb{Q})$ *is self-dual, each of the constructions L, M, D, C, K induces a comonad* $(!, \epsilon, \delta)$ [3] *on* **Prof**$(\mathbb{Q})$*, with action on morphisms:*

$$!R(A, B) = \bigvee_{\psi : \# B \to \# A} \otimes_{i : \# B} R(A_{\psi i}, B_i)$$

Where ψ is restricted appropriately as in theorem 27. The component of the counit and comultiplication at $\mathcal{A}$ are:

$$\epsilon_{\mathcal{A}} : !\mathcal{A} \nrightarrow \mathcal{A} \qquad\qquad \delta_{\mathcal{A}} : !\mathcal{A} \nrightarrow !!\mathcal{A}$$
$$\epsilon_{\mathcal{A}}(A, a) = !\mathcal{A}(A, [a]) \qquad\qquad \delta_{\mathcal{A}}(A, \underline{A}) = !\mathcal{A}(A, \operatorname{concat} \underline{A})$$

Here, concat *denotes list concatenation.*

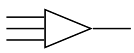
Proof. Although this is a natural construction, it is necessary to be careful with the various dualities involved, as some of the constructions on profunctors are necessarily oriented in nature. ◀

From the point of view of resources, the quantale value $\epsilon_{\mathcal{A}}(A, a)$ is the best way to convert the list A into the single $[a]$, using the structural features of $!\mathcal{A}$. Similarly, $\delta_{\mathcal{A}}(A, \underline{A})$ is the best way to convert the lists $\mathcal{A}$ into the concatenation of the list of list $\underline{A}$ using the structural features of $!\mathcal{A}$.

▶ Remark. These co-Kleisli categories carry a lot of additional structure that unfortunately we have insufficient space to exploit here. This includes further enrichment, various type constructors, and operations induced by the Day convolution [5]. Depending on the choice of comonad, there may also be higher order and differential structure [6]. This provides a rich algebra for calculations involving quantitative resources, formally similar to the calculus of generalized species presented in [7, 8].

5 Multicategories

If we examine a morphism $\mathcal{A} \to \mathcal{B}$ in the co-Kleisli category of one of the comonads in section 4, concretely, this is a profunctor of the form $!\mathcal{A} \nrightarrow \mathcal{B}$. From a resource perspective, we can read this as describing conversions from lists of $\mathcal{A}$-resources to $\mathcal{B}$-resources. So the comonad allows us to describe many-to-one resource conversions, diagrammatically:



[3] Our notation is a nod to connections with relational linear logic models.

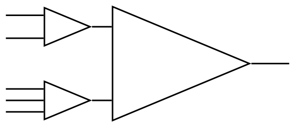
Depending on our choice of comonad, we can incorporate different structural aspects of the free conversions available, for example we may be able to cheaply reorder, copy or delete resources.

It is instructive to consider the co-Kleisli composition $S \bullet R$ of two such morphisms. This is given in $\mathbf{Prof}(\mathbb{Q})$ by the composite:

$$!\mathcal{A} \overset{\delta_{\mathcal{A}}}{\nrightarrow} !!\mathcal{A} \overset{!R}{\nrightarrow} !\mathcal{B} \overset{S}{\nrightarrow} \mathcal{C}$$

Intuitively, we can read the three steps as follows:

1. We first break our list of resources up into a list of lists, using the comultiplication $\delta_{\mathcal{A}}$.
2. We then use the resource conversions provided by R to process each of the sub-lists.
3. Finally, we process the resulting list using S, resulting in a two step multi-input conversion, which we might depict:



Then $(S \bullet R)(A, c)$ gives the best two stage conversion achievable converting the list of resources A to the resource c. The choice of comonad incorporates the structural aspects, such as copying or deleting, that we are prepared to permit.

What if we want to consider repeated many-to-one conversions? For that we must confirm a bit more structure is available.

▶ **Proposition 30.** *Each of the comonads of corollary 29 preserves non-empty joins.*

▶ **Corollary 31.** *Each of the co-Kleisli categories of these comonads is a non-empty join semilattice enriched category.*

It therefore makes sense to consider internal monads in our co-Kleisli categories. These internal monads quantify what we might call **multi-conversions**, in a manner that is closed under identities and composition. That is, they are generalizations of coloured operads [25], otherwise termed multicategories [21].

▶ Remark. This perspective on internal monads in such co-Kleisli bicategories is discussed in [8, 17]. There, they restrict to internal monads on discrete categories. However, in our setting, multicategories with non-discrete endpoints are a *virtue*. They describe the freely available resource conversions. The discrete case would say that the only freely available resource conversions are the trivial ones.

▶ **Example 32.** Even in the $\mathbb{B}$-enriched case, such multicategories are interesting objects. They are a multi-input generalization of preorders, describing the possibility of various multi-conversions being achievable. Possible conversions can be chained together, and trivial conversions are available. The choice of comonad introduces additional structure. For example in the deleting case, the list of resources $[a, b, c]$ is always convertible to $[a]$, by discarding the other resources.

6 Bimodules

In section 3 we showed how single input - single output resource conversions could be modelled as profunctors. In sections 4 and 5 we introduced additional comonadic structure that allowed us to introduce many-to-one costly resource conversions. In this section we show that an extra layer of abstraction allows us to model freely available many-to-one conversions with our categorical framework. We require the notion of bimodule between monads.

▶ **Definition 33.** Let $\mathcal{C}$ be a preorder enriched category. For internal monads $(\mathcal{A}, R^{\mathcal{A}})$ and $(\mathcal{B}, R^{\mathcal{B}})$, a **bimodule** of type $(\mathcal{A}, R^{\mathcal{A}}) \nrightarrow (\mathcal{B}, R^{\mathcal{B}})$ is a $\mathcal{C}$-morphism $S : \mathcal{A} \to \mathcal{B}$ such that:

$$S \circ R^{\mathcal{A}} \subseteq S \qquad \text{and} \qquad R^{\mathcal{B}} \circ S \subseteq S$$

As with our previous mathematical structures, it is helpful to think of bimodules as binary relations respecting some additional structure.

▶ **Proposition 34.** *In a non-empty join semilattice enriched category $\mathcal{C}$, bimodules between monads include the identity morphisms, and are closed under both composition and joins in $\mathcal{C}$. They therefore form a non-empty join semilattice enriched category* **Bimod**$(\mathcal{C})$.

Bimodules between monads can be defined more generally, but their composition becomes more complicated, requiring a coequalizer construction not present in proposition 34. Fortunately, the quantale enriched setting circumvents this additional complexity. Resource theoretically, bimodules on our co-Kleisli categories have good properties:
- As **coKleisli**(!) morphisms, they model multi-conversions between $\mathcal{A}$ and $\mathcal{B}$ resources.
- These conversions are closed under precomposition with the multi-conversions described by the monad $(\mathcal{A}, R^{\mathcal{A}})$.
- The conversions are also closed under post composition with the multi-conversions described by the monad $(\mathcal{B}, R^{\mathcal{B}})$.

That is, they are exactly the right categorical object for describing resource conversions respecting freely available multi-conversions. As a corollary of proposition 34, we note that:

▶ **Corollary 35.** *The category* **coKleisli**(!) *is complete join semilattice enriched for any of the comonads introduced in section 4.*

Corollary 35 tells us that we can take composites and unions of bimodules to build more interesting structures. Also, we can consider internal monads in the categories of bimodules of interest. These can be seen as *multicategories that respect freely available multi-conversions*.

7 Reflexive Transitive Closure

Given the importance of internal monads in earlier sections, we briefly consider how they can be constructed from simpler data in complete join semilattice enriched categories. We require a new definition.

▶ **Definition 36.** Let $\mathcal{C}$ be a complete join semilattice enriched category. A monad $T : \mathcal{A} \to \mathcal{A}$ is **free** over an arbitrary endomorphism $R : \mathcal{A} \to \mathcal{A}$ if it is the least monad containing R.

We fall back on our intuition that each of our categories of interest can be interpreted as a category of generalized binary relations. It is therefore natural to ask if some operations on ordinary relations have analogues in this setting. The construction of immediate interest is a generalization of reflexive transitive closure.

▶ **Proposition 37.** *In a non-empty join semilattice enriched category $\mathcal{C}$, the free monad induced by an endomorphism $R : A \to A$ is given by:*

$$F(R) = \bigsqcup_i R^i \qquad \textit{where} \qquad R^0 = 1_A \qquad \textit{and} \qquad R^{n+1} = R \circ R^n$$

Recalling lemma 21, and corollaries 31 and 35, the categories $\mathbf{Prof}(\mathbb{Q})$, $\mathbf{coKleisli}(!)$ for any of the hierarchy of comonads of section 4, and the categories of bimodules on these co-Kleisli categories are all appropriately enriched. Therefore, we can conclude:

▶ **Corollary 38.** *Every endomorphism in our categories of interest can be used to construct a free internal monad using the reflexive transitive closure construction of proposition 37.*

In this way we can take some basic data specifying one-to-one or many-to-one resource conversions of interest. We can close them under composition in a canonical way.

8 Conclusion

We presented a flexible foundation for constructing compositional, quantitative models of resources, within which:

- There is a clear separation of concerns between freely available resource conversion, encoded as objects in our categories, and the costly conversions, encoded in the morphisms.
- Profunctors quantify one-to-one resource conversions, parameterized by a choice of quantity such as costs or probabilities.
- Morphisms in suitable co-Kleisli categories describe many-to-one resource conversions, parameterized by a choice of structural features such as copying and deleting.
- Bimodules model many-to-one resource conversions in which the freely available conversions may also include such multi-conversions.
- Throughout, internal monads capture closure under composition, yielding generalizations of categories or multicategories suitable for the quantitative setting.
- Free internal monads provide a convenient mechanism for building these (multi)categories from simpler data.
- The underlying objects can be considered as generalized relations, or just certain matrices of truth values, meaning calculations do not require difficult mathematical machinery.

Our approach is open to extension. For example, it is natural to also consider multi-input to multi-output conversions, in the style of polycategories [28]. These can be formulated in a similar manner to that used in sections 4 and 5. For the ordinary categorical setting this is technically complex, and has been developed by Garner [11, 10, 12]. Given the degeneracy of quantale enriched categories, we anticipate a more elementary approach will be feasible, and aim to develop this in later work.

We have focused on models. It would be interesting to develop corresponding syntactic aspects, in the form of a suitable metalanguage. Discussions in the related work of [16] and [8] suggest such a language will have a process algebraic feel, but we leave the details to later work.

Finally, a more speculative suggestion. Exciting recent categorical work on compositional game theory [13] has shown surprising applications of category theory in economic settings. Given the intrinsic interest of economists in both resources and costs, it would be interesting to explore applications of our approach in that setting.

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