


On Compositionality of Dinatural Transformations

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Abstract

Natural transformations are ubiquitous in mathematics, logic and computer science. For operations of mixed variance, such as currying and evaluation in the lambda-calculus, Eilenberg and Kelly's notion of extranatural transformation, and often the even more general dinatural transformation, is required. Unfortunately dinaturals are not closed under composition except in special circumstances. This paper presents a new sufficient condition for composability.

We propose a generalised notion of dinatural transformation in many variables, and extend the Eilenberg-Kelly account of composition for extranaturals to these transformations. Our main result is that a composition of dinatural transformations which creates no cyclic connections between arguments yields a dinatural transformation.

We also extend the classical notion of horizontal composition to our generalized dinaturals and demonstrate that it is associative and has identities.

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1 Introduction

Natural transformations are a ubiquitous notion in mathematics, logic and computer science. They are used to interpret logical rules, program forming operations, adjointness conditions and free constructions. Naturality is an equational property that expresses the idea that the transformation operates on structure, independent of the underlying data. Given functors $F, G: \mathbb{C} \rightarrow \mathbb{D}$, a natural transformation $\varphi: F \rightarrow G$ comprises a family of morphisms $\varphi_A: F(A) \rightarrow G(A)$ in \mathbb{D} . The naturality condition is specified as a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \varphi_A \downarrow & & \downarrow \varphi_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

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which may alternatively be pictured as follows:

$$\begin{array}{ccc}
 F\left(\boxed{f}\right) & & F\left(\boxed{id}\right) \\
 \varphi \downarrow & = & \varphi \downarrow \\
 G\left(\boxed{id}\right) & & G\left(\boxed{f}\right)
 \end{array}$$

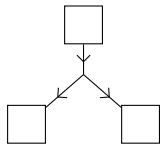
In this picture, each box represents an argument of a functor, while the vertical lines instances of the transformation. With this reading, the diagrammatic equation above is the same as the standard naturality square. Later we will give a precise meaning to pictures of this kind and make use of them in our proofs. For now, they will provide useful intuition.

When dealing with logic, type theory and programming languages, often we encounter transformations between functors of mixed variance, like the evaluation map $ev_{A,B}: A \times (A \Rightarrow B) \rightarrow B$. Eilenberg and Kelly [5] developed the notion of extranatural transformation to account for this. The equational property of ev is explained by means of graphs:

$$\begin{array}{ccc}
 A \times (A' \Rightarrow B) & \xrightarrow{f \times (id \Rightarrow g)} & A' \times (A' \Rightarrow B) \\
 \downarrow id \times (f \Rightarrow id) & & \downarrow ev_{A', B'} \\
 A \times (A \Rightarrow B') & \xrightarrow{ev_{A, B}} & B'
 \end{array}
 \quad \Leftarrow \quad
 \begin{array}{ccc}
 \boxed{f} \times (\boxed{id} \Rightarrow \boxed{g}) & & \boxed{id} \times (\boxed{f} \Rightarrow \boxed{id}) \\
 \downarrow A' & & \downarrow A \\
 \boxed{id} & & \boxed{g}
 \end{array}$$

(Grey boxes indicate contravariant arguments of the functors involved.) $ev = (ev_{A,B})$ is said to be extranatural in A and natural in B . Note that the connections show which arguments of the functors involved must be set equal in order for an equational property to hold. There is one connected component in the graph for each such collection of arguments. Eilenberg and Kelly show that a composite of extranatural transformations is again extranatural, provided the graph obtained by pasting the graphs together along the common interface is acyclic.

In the graphs of all the transformations we have seen so far, arguments are linked in pairs. For transformations such as the diagonal $\delta = (\delta_A: A \rightarrow A \times A)$, this is a limitation. Though its equational properties are adequately described using the naturality of a transformation from the identity functor $id_{\mathbb{C}}$ to the diagonal functor Δ , this account becomes clumsy if we attempt to discuss the associativity of the diagonal operation, for example. One would prefer to picture it as:



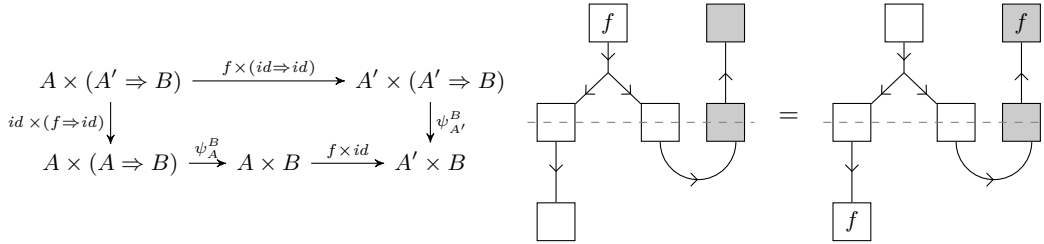
Kelly [13] points this out, and suggests that a more general notion of natural transformation, in which graphs have ramifications, may be available, but does not go on to develop it.

Such ramifications have ramifications. One source of difficulty is that composing these generalised natural and extranatural transformations quickly leads to *dinatural transformations* [4]. For example, working in a cartesian closed category, by composing the diagonal δ and evaluation $ev^B = (ev_A^B: A \times (A \Rightarrow B) \rightarrow B)_{A \in \mathbb{C}}$, we can construct morphisms

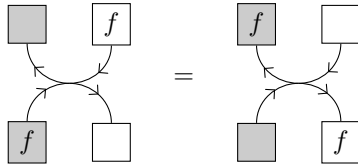
$$\psi_A^B = \delta_A \times id_{A \Rightarrow B}; id_A \times ev_A^B: A \times (A \Rightarrow B) \rightarrow A \times B.$$

By pasting together the graphs of the transformations $\delta_A \times id_{A \Rightarrow B}$ and $id_A \times ev_A^B$ we obtain the following depiction of the appropriate “naturality” in A for this family of morphisms.

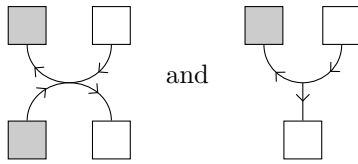
(From now on we drop the name of the functors involved and retain only the boxes and the lines, an empty box being the same as a box containing an identity.)



This is not a natural nor an extranatural transformation, but a dinatural transformation. Dinatural transformations are families of morphisms between functors of the form $\mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbb{C}$ where the dinaturality condition can be drawn as follows:



Dinatural transformations arise often in a computer science context. For instance the Church numerals $n = (n_A : (A \Rightarrow A) \rightarrow (A \Rightarrow A))$ and the fixed point combinator $Y = (Y_A : (A \Rightarrow A) \rightarrow A)$ are dinatural transformations, with graphs

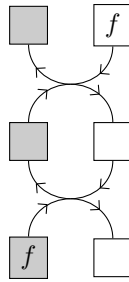


(We note Curry’s prescience in his naming of the Y combinator.) More generally, dinatural transformations have been proposed as a suitable way to understand parametric polymorphism [1] and as an interpretation of cut-free proofs, or equivalently typed lambda terms [8]. But dinatural transformations suffer from a troublesome shortcoming: they do not compose.

Our pictorial representation makes it clear that there is no reason to expect these to be closed under composition: starting from the situation as pictured on Figure 1 below, there is no way to reach a situation where the dinaturality of either transformation may be applied. Under special circumstances, such as when certain squares of morphisms are pullbacks or pushouts, the composite may turn out to be dinatural, but not as a direct consequence of the dinaturality of the two transformations.

In contrast to Eilenberg and Kelly’s treatment of extranatural transformations, the usual description of dinatural transformations concerns functors of one argument (strictly speaking two arguments, of different variance, that are required to be equal). A consequence of this is that any composition of dinaturals appears to have a cyclic dependency among arguments, as seen in Figure 1.

In this paper we introduce a generalised notion of dinatural transformation. As in [13], our transformations are equipped with a graph as part of their data whose composition does not always form a cycle. These transformations enjoy a similar compositionality property to the extranaturals: as long as no cycles are created, dinaturality is preserved by composition.



■ **Figure 1** Cycles and impossibility to apply dinaturality.

Thus, one is freed from the burden of conducting ad hoc verification of dinaturality conditions. For example, the dinaturality theorems of [8] can readily be proved by drawing the graphs of the transformations interpreting cut-free proofs and observing that they are acyclic. The proof of our result is significantly more demanding than Eilenberg and Kelly’s case, because of the ramifications in the dependency graphs. But to a computer scientist these graphs have a familiar appearance: they look and behave like Petri nets. Our argument proceeds by formalising a correspondence between morphisms built from functors and dinatural transformations and configurations of Petri nets. The desired dinaturality equation reduces to a question of reachability of one configuration from another, which is readily settled using the theory of Petri nets. In this way we not only discover a helpful sufficient condition for composability of dinaturals but also turn an intuitive diagrammatic reasoning method into a formal tool. Moreover, one can show that ours is also an “essentially necessary” condition: if the dinaturality of a composite transformation $\varphi; \psi$ may be derived using only the dinaturality of φ and ψ , then the composite graph is acyclic (cf. [12, §1.3]). For lack of space we do not present the proof of this fact here.

The above discussion concerns only the “vertical” composition of transformations. Natural transformations may also be composed horizontally; this operation is needed when one wishes to substitute functors for the arguments of other functors, and apply transformations between them. Kelly already noticed this in his generalisation of Godement calculus for functors and natural transformations in many variables [13]. To date we are not aware of any generalisation of this operation to dinatural transformations. Our second contribution is to develop a notion of horizontal composition for our dinatural transformations that extends the well known version for natural transformations and establish that it is associative and has identities. Unfortunately, we seem to have lost one of the fundamental properties of horizontal composition of natural transformations: compatibility with the vertical one. Indeed, an analogous version of interchange law for the natural case does not (*cannot*) hold with dinatural transformations, even when we restrict ourselves to simple cases. The problem stems from the “shape”, as it were, of the dinaturality condition, that prevents the vertical composability of the two horizontal compositions. A different kind of interchange law seems to be needed, but as yet we have not been able to find one that works, not even for Eilenberg and Kelly’s transformations with no ramifications. We shall dedicate the near future to investigate this matter and hopefully we shall solve this rather *natural* problem.

Related work. Our interest in this topic arose from a desire to understand better the algebraic properties of Guglielmi and Gundersen’s atomic flows [10, 9], which are an abstraction of information flow in classical logic proofs. The graphical structures we use extend so-called Kelly-Mac Lane graphs [14] which originated with Eilenberg and Kelly [5] and may be seen

as string diagrams for closed categories; the wide variety of string diagrams is surveyed in [19]. They are closely related to proof nets encountered in the proof theory of linear logic [7]. Blute [2] studies dinatural transformations corresponding to proofs of multiplicative linear logic and establishes a compositionality result for that case. Freyd, Robinson and Rosolini [6] studied dinaturality in the category of PERs. The close relationship between dinaturality and fixed point combinators was studied by Mulry [16] and Simpson [20].

► **Notation.** We denote by \mathbb{I} the category with one object and one morphism. Let $\alpha \in \text{List}\{+, -\}$, $|\alpha| = n$. We refer to the i -th element of α as α_i . We denote by $\bar{\alpha}$ the list obtained from α by swapping the signs. Given a category \mathbb{C} , if $n \geq 1$, then we define $\mathbb{C}^\alpha := \mathbb{C}^{\alpha_1} \times \dots \times \mathbb{C}^{\alpha_n}$, with $\mathbb{C}^+ = \mathbb{C}$ and $\mathbb{C}^- = \mathbb{C}^{\text{op}}$, otherwise $\mathbb{C}^\alpha := \mathbb{I}$. Composition of morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ will be denoted by $g \circ f$, gf or also $f; g$. \mathbb{N} is the set of natural numbers, including 0. Given $n \in \mathbb{N}$, we ambiguously denote n for both the number n and the set $\{1, \dots, n\}$.

2 Dinatural transformations, types and vertical composition

To make precise the graphical ideas introduced above, we employ a notion of *type* for our transformations, following Kelly [13]. The type indicates how the various arguments of the domain and codomain functors are related by naturality conditions.

The category $\mathbb{T}\text{ypes}$. Let $\mathbb{T}\text{ypes}$ be the category of cospans [3] of finite sets and functions; that is, $\mathbb{T}\text{ypes}$ has \mathbb{N} as its set of objects, and a morphism $f: n \rightarrow m$ is a cospan $f = (n \xrightarrow{\sigma} k \xleftarrow{\tau} m)$; different cospans counting as the same morphism if they differ only by an automorphism, that is a permutation, of k . Given $n \in \mathbb{N}$, the identity morphism on n is the cospan of id_n . Composition of f and $g = (m \xrightarrow{\sigma'} p \xleftarrow{\tau'} t)$ is the cospan $gf = (n \rightarrow q \leftarrow t)$ got by computing the pushout of τ against σ' as functions:

$$\begin{array}{ccc}
 & & t \\
 & & \downarrow \tau' \\
 m & \xrightarrow{\sigma'} & p \\
 \tau \downarrow & \lrcorner & \downarrow \xi \\
 n & \xrightarrow{\sigma} & k \xrightarrow{\zeta} q
 \end{array} \tag{1}$$

Transformations. Throughout this section, we fix a category \mathbb{C} .

► **Definition 1.** Let $\alpha, \beta \in \text{List}\{+, -\}$, $T: \mathbb{C}^\alpha \rightarrow \mathbb{C}$, $S: \mathbb{C}^\beta \rightarrow \mathbb{C}$ functors. A *transformation* $\varphi: T \rightarrow S$ of *type* $f = (|\alpha| \xrightarrow{\sigma} k \xleftarrow{\tau} |\beta|)$ (with k positive integer) is a family of morphisms

$$\left(\varphi_{A_1, \dots, A_k}: T(A_{\sigma_1}, \dots, A_{\sigma_{|\alpha|}}) \rightarrow S(A_{\tau_1}, \dots, A_{\tau_{|\beta|}}) \right)_{(A_1 \dots A_k) \in \mathbb{C}^k}.$$

Functions σ and τ tell us which of the $|\alpha|$ arguments of T and the $|\beta|$ arguments of S must be equated, and also which among A_1, \dots, A_k to use in each “slot”. Notice that σ and τ need not be surjective, so we can define transformations with “unused variables”.

► **Definition 2.** Let $\varphi: T \rightarrow S$ of type f be a transformation as in Definition 1, $R: \mathbb{C}^\gamma \rightarrow \mathbb{C}$ and $\psi: S \rightarrow R$ a transformation of type $g = (|\beta| \xrightarrow{\sigma'} p \xleftarrow{\tau'} |\gamma|)$, so that we have, for all B_1, \dots, B_p ,

$$\psi_{B_1, \dots, B_p}: S(B_{\sigma'_1}, \dots, B_{\sigma'_p}) \rightarrow R(B_{\tau'_1}, \dots, B_{\tau'_p}).$$

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The *vertical composition* $\psi \circ \varphi$ is defined as the transformation of type

$$gf = |\alpha| \xrightarrow{\zeta\sigma} q \xleftarrow{\xi\tau'} |\gamma|$$

where ζ and ξ are given by (1) and $(\psi \circ \varphi)_{C_1, \dots, C_q}$ is the composite:

$$\begin{aligned} T(C_{\zeta\sigma 1}, \dots, C_{\zeta\sigma|\alpha|}) &\xrightarrow{\varphi_{C_{\zeta\sigma 1}, \dots, C_{\zeta\sigma|\alpha|}}} S(C_{\zeta\tau 1}, \dots, C_{\zeta\tau|\beta|}) \\ &\parallel \\ S(C_{\xi\sigma' 1}, \dots, C_{\xi\sigma'|\beta|}) &\xrightarrow{\psi_{C_{\xi\sigma' 1}, \dots, C_{\xi\sigma'|\beta|}}} R(C_{\xi\tau' 1}, \dots, C_{\xi\tau'|\gamma|}) \end{aligned}$$

(Notice that by definition $\varphi_{C_{\zeta\sigma 1}, \dots, C_{\zeta\sigma|\alpha|}}$ requires that the i -th variable of T be the σi -th element of the list $(C_{\zeta\sigma 1}, \dots, C_{\zeta\sigma|\alpha|})$, which is indeed $C_{\zeta\sigma i}$.)

► **Definition 3.** Consider $T: \mathbb{C}^\alpha \rightarrow \mathbb{C}$, $S: \mathbb{C}^\beta \rightarrow \mathbb{C}$, $\varphi: T \rightarrow S$ a transformation of type $|\alpha| \xrightarrow{\sigma} k \xleftarrow{\tau} |\beta|$ as in Definition 1. For $i \in \{1, \dots, k\}$, we say that φ is *dinatural in A_i* (or, more precisely, *in its i -th variable*) if and only if for all $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_k$ objects of \mathbb{C} and for all $f: A \rightarrow B$ in \mathbb{C} the following diagram commutes:

$$\begin{array}{ccccc} & & \cdot & \xrightarrow{\varphi_{A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_k}} & \cdot & & \\ & \nearrow & & & & \searrow & \\ T(x_1, \dots, x_{|\alpha|}) & & & & & & S(y_1, \dots, y_{|\beta|}) \\ & \searrow & & & & \nearrow & \\ & & \cdot & \xrightarrow{\varphi_{A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_k}} & \cdot & & \\ & & T(x'_1, \dots, x'_{|\alpha|}) & & & & S(y'_1, \dots, y'_{|\beta|}) \end{array}$$

where

$$x_j = \begin{cases} f & \sigma j = i \wedge \alpha_j = + \\ id_B & \sigma j = i \wedge \alpha_j = - \\ id_{A_{\sigma j}} & \sigma j \neq i \end{cases} \quad y_j = \begin{cases} id_B & \tau j = i \wedge \beta_j = + \\ f & \tau j = i \wedge \beta_j = - \\ id_{A_{\tau j}} & \tau j \neq i \end{cases}$$

$$x'_j = \begin{cases} id_A & \sigma j = i \wedge \alpha_j = + \\ f & \sigma j = i \wedge \alpha_j = - \\ id_{A_{\sigma j}} & \sigma j \neq i \end{cases} \quad y'_j = \begin{cases} f & \tau j = i \wedge \beta_j = + \\ id_A & \tau j = i \wedge \beta_j = - \\ id_{A_{\tau j}} & \tau j \neq i \end{cases}$$

► **Remark.** Definition 3 is a generalisation of the well known notion of dinatural transformation, which we can obtain when $\alpha = \beta = [-, +]$ and $k = 1$. Here we are allowing multiple variables at once and the possibility for T and S of having an arbitrary number of copies of \mathbb{C} and \mathbb{C}^{op} in their domain, for each variable $i \in \{1, \dots, k\}$.

It is known that dinatural transformations generalise natural and extranatural ones. Here we make this fact explicit by defining the latter as particular cases of dinatural transformations where the functors and the type have a special shape: essentially, a dinatural transformation $\varphi: T \rightarrow S$ is natural in A_i if T and S are both covariant or both contravariant in the variables involved by A_i ; φ is extranatural in A_i if one of the functors T and S does not involve the variable A_i while A_i appears both covariantly and contravariantly in the other.

► **Definition 4.** Let $\varphi: T \rightarrow S$ be a transformation as in Definition 1. $\varphi = (\varphi_{A_1, \dots, A_k})$ is said to be *natural in A_i* if and only if

- it is dinatural in A_i ;
- $\forall u \in \sigma^{-1}\{i\}. \forall v \in \tau^{-1}\{i\}. (\alpha_u = \beta_v = +) \vee (\alpha_u = \beta_v = -)$.

φ is called *extranatural* in A_i if and only if

- it is dinatural in A_i ;
- $(\sigma^{-1}\{i\} = \emptyset \wedge \exists j_1, j_2 \in \tau^{-1}\{i\}. \beta_{j_1} \neq \beta_{j_2}) \vee (\tau^{-1}\{i\} = \emptyset \wedge \exists i_1, i_2 \in \sigma^{-1}\{i\}. \alpha_{i_1} \neq \alpha_{i_2})$.

Notice that our notion of (extra)natural transformations is more general than the one given by Eilenberg and Kelly in [5], as we allow the arguments of T and S to be equated not just in pairs, but in an arbitrary number, according to σ and τ .

► **Example 5.** Suppose that \mathbb{C} is a cartesian category, with $\times: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ the product functor, and consider the diagonal transformation $\delta = (\delta_A: A \rightarrow A \times A)_{A \in \mathbb{C}}: id_{\mathbb{C}} \rightarrow \times$ of type $1 \rightarrow 1 \leftarrow 2$. We have that δ is natural in its only variable.

► **Example 6.** Suppose that \mathbb{C} is a cartesian closed category, fix an object R in \mathbb{C} , and consider the functor

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}^{\text{op}} & \xrightarrow{T} & \mathbb{C} \\ (A, A') & \longmapsto & (A' \Rightarrow R) \times A \end{array}$$

The evaluation $ev^R = (ev_A^R: T(A, A) \rightarrow R)_{A \in \mathbb{C}}: T \rightarrow R$ is a transformation of type $2 \rightarrow 1 \leftarrow 0$ which is extranatural in its only variable.

We proceed now to study the composability problem for dinatural transformations. Let $\varphi: F_1 \rightarrow F_2$ and $\psi: F_2 \rightarrow F_3$ be transformations where

- $F_i: \mathbb{C}^{\alpha^i} \rightarrow \mathbb{C}$ is a functor for all $i \in \{1, 2, 3\}$,
- φ and ψ have type, respectively,

$$|\alpha^1| \xrightarrow{\sigma_1} k_1 \xleftarrow{\tau_1} |\alpha^2| \quad \text{and} \quad |\alpha^2| \xrightarrow{\sigma_2} k_2 \xleftarrow{\tau_2} |\alpha^3|.$$

We shall establish conditions under which $\psi \circ \varphi$ is dinatural in some of its variables. In order to do so, we associate to $\psi \circ \varphi$ a graph which somehow reflects the signature of φ and ψ .

The graph of $\psi \circ \varphi$. We assign to $\psi \circ \varphi$ a directed bipartite graph $\Gamma(\psi \circ \varphi)$ whose vertices are given by (distinct) finite sets P and T , while $\bullet-, -\bullet: T \rightarrow \mathcal{P}(P)$ are the input and output functions for elements in T (that is, there is an arc from p to t if and only if $p \in \bullet t$, and there is an arc from t to p if and only if $p \in -t$), as follows: $P = |\alpha^1| + |\alpha^2| + |\alpha^3|$, $T = k_1 + k_2$ and, indicating with $\iota_i: |\alpha^i| \rightarrow P$ and $\rho_i: k_i \rightarrow T$ the canonical injections,

$$\bullet(\rho_i(t)) = \{\iota_i(p) \mid \sigma_i(p) = t, \alpha_p^i = +\} \cup \{\iota_{i+1}(p) \mid \tau_i(p) = t, \alpha_p^{i+1} = -\}$$

$$(\rho_i(t))\bullet = \{\iota_i(p) \mid \sigma_i(p) = t, \alpha_p^i = -\} \cup \{\iota_{i+1}(p) \mid \tau_i(p) = t, \alpha_p^{i+1} = +\}$$

In other words, the inputs of a variable t of transformation φ are the covariant arguments of F_1 and the contravariant arguments of F_2 which are mapped by σ_1 and τ_1 , respectively, to t ; similarly for outputs of t (swapping ‘covariant’ and ‘contravariant’) and for variables of ψ . Graphically, we draw elements of P as white or grey boxes (if corresponding to a covariant or contravariant argument of an F_i , respectively), and elements of T as black squares, as in the following example.

► **Example 7.** Suppose that \mathbb{C} is cartesian closed, fix an object R in \mathbb{C} , consider functors

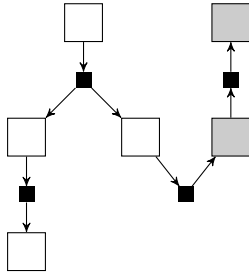
$$\begin{array}{ccccc} \mathbb{C} \times \mathbb{C}^{\text{op}} & \xrightarrow{F} & \mathbb{C} & \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{\text{op}} & \xrightarrow{G} & \mathbb{C} & \mathbb{C} & \xrightarrow{H} & \mathbb{C} \\ (A, B) & \longmapsto & A \times (B \Rightarrow R) & (A, B, C) & \longmapsto & A \times B \times (C \Rightarrow R) & A & \longmapsto & A \times R \end{array}$$

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and transformations $\varphi = \delta \times id_{(-)\Rightarrow R}: F \rightarrow G$ and $\psi = id_C \times ev^R: G \rightarrow H$ of types, respectively,

$$\begin{array}{ccc}
 2 \xrightarrow{\sigma} 2 & \xleftarrow{\tau} & 3 \\
 1 \xrightarrow{\quad} 1 & \xleftarrow{\quad} & 1 \\
 2 \xrightarrow{\quad} 2 & \xleftarrow{\quad} & 3
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 3 \xrightarrow{\eta} 2 & \xleftarrow{\theta} & 1 \\
 1 \xrightarrow{\quad} 1 & \xleftarrow{\quad} & 1 \\
 2 \xrightarrow{\quad} 2 & & \\
 3 \xrightarrow{\quad} & &
 \end{array}
 .$$

Then $\psi \circ \varphi$ has type $2 \rightarrow 1 \leftarrow 1$ and its graph is:



► **Remark.** Each connected component of $\Gamma(\psi \circ \varphi)$ corresponds to a variable of $\psi \circ \varphi$. This is due to how the pushout of τ_1 against σ_2 is computed when we calculate the type of $\psi \circ \varphi$: if p is the result of the pushout, then p is isomorphic, in Set , to the quotient set of T modulo the least equivalence relation \sim such that for all $\rho_1(x)$ and $\rho_2(y)$, $\rho_1(x) \sim \rho_2(y)$ if and only if there exists $z \in |\alpha^2|$ such that $\tau_1(z) = x$ and $\sigma_2(z) = y$; in other words, if they are connected in $\Gamma(\psi \circ \varphi)$ (by means of an undirected path).

Since we want to discuss the dinaturality of $\psi \circ \varphi$ in each of its variables separately, we start by assuming that $\psi \circ \varphi$ is “connected”, that is has type $|\alpha^1| \rightarrow 1 \leftarrow |\alpha^3|$, and that φ and ψ are dinatural in all their variables. The result we want to prove is then the following.

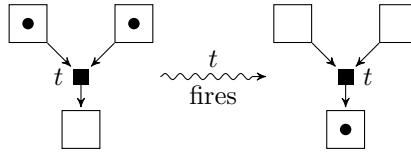
► **Theorem 8.** *Let φ and ψ be transformations which are dinatural in all their variables and such that $\psi \circ \varphi$ depends on only one variable. If $\Gamma(\psi \circ \varphi)$ is acyclic, then $\psi \circ \varphi$ is a dinatural transformation.*

We shall prove this theorem by interpreting $\Gamma(\psi \circ \varphi)$ as a Petri Net [18], whose set of places is P and of transitions is T . Places can host tokens, and recall that a marking for $\Gamma(\psi \circ \varphi)$ is a function $M: P \rightarrow \mathbb{N}$, that is, a distribution of tokens. A transition t is enabled in M if $M(p) > 0$ for all $p \in \bullet t$; an enabled transition t can fire, and the firing of t removes one token from each of its inputs and adds one token to each of its outputs, that is it generates a new marking M' defined as follows:

$$M'(p) = \begin{cases} M(p) - 1 & p \in \bullet t \\ M(p) + 1 & p \in t \bullet \\ M(p) & \text{otherwise} \end{cases}$$

Graphically, we draw tokens as black dots, see Figure 2.

The reason for which we use Petri Nets to prove Theorem 8 is that the firing of an enabled transition in $\Gamma(\psi \circ \varphi)$ corresponds to applying the dinaturality of φ or ψ in the corresponding variable, thus giving rise to an equation of morphisms in \mathbb{C} . It follows that a sequence of firings corresponds to a chain of equations. Since we are interested in proving that two certain morphisms, corresponding to the two legs of the hexagon that we want to show is



■ **Figure 2** The firing of an enabled transition t .

commutative (to prove that $\psi \circ \varphi$ is dinatural), are equal, we shall individuate two markings M_0 and M_d for $\Gamma(\psi \circ \varphi)$ that correspond to those morphisms, and prove that M_d is *reachable* from M_0 , that is that there is a sequence of firings of enabled transitions that transforms M_0 into M_d . This reduction to Petri nets not only provides an intuitive reasoning tool that corresponds directly to the diagrams we have been drawing, but also allows us to make use of the well-developed theory of Petri nets. Indeed our compositionality result will follow from a theorem about reachability in acyclic Petri nets.

► **Notation.** We extend the input and output notation for places too, where

$$\bullet p = \{t \in T \mid p \in t\bullet\}, \quad p\bullet = \{t \in T \mid p \in t\bullet\}$$

► **Remark.** Since σ_i and τ_i are functions, we have that $|\bullet p|, |p\bullet| \leq 1$ and also that $|\bullet p \cup p\bullet| \geq 1$. With a little abuse of notation then, if $\bullet p = \{t\}$ then we shall simply write $\bullet p = t$, and similarly for $p\bullet$.

Labelled markings. Not all markings for $\Gamma(\psi \circ \varphi)$ correspond to a morphism in \mathbb{C} . In this section we shall individuate a class of them for which it is possible to define an associated morphism in \mathbb{C} .

► **Definition 9.** Consider $f: A \rightarrow B$ a morphism in \mathbb{C} . A *labelled marking* is a triple (M, L, f) where functions $M: P \rightarrow \{0, 1\}$ and $L: T \rightarrow \{A, B\}$ are such that for all $p \in P$

$$M(p) = 1 \implies L(\bullet p) = A, L(p\bullet) = B$$

$$M(p) = 0 \implies \begin{cases} p\bullet = \emptyset \implies L(\bullet p) = B \\ \bullet p = \emptyset \implies L(p\bullet) = A \\ \bullet p \neq \emptyset \neq p\bullet \implies L(\bullet p) = L(p\bullet) \end{cases}$$

For each labelled marking (M, L, f) we define a morphism in \mathbb{C} obtained by composing the functors F_i with appropriate components of φ and ψ . Each argument of F_i corresponds to a place in the graph. For each marked place the corresponding F_i 's argument will be f ; for unmarked places it will be id . The definition of labelled marking puts constraints on the marking itself, ensuring that the result of this operation is a well-formed morphism in \mathbb{C} .

► **Definition 10.** Let $f: A \rightarrow B$ in \mathbb{C} , (M, L, f) a labelled marking. We define a morphism $\mu(M, L, f)$ in \mathbb{C} as follows:

$$\mu(M, L, f) = F_1(x_1^1, \dots, x_{|\alpha_1|}^1); \varphi_{X_1^1 \dots X_{k_1}^1}; F_2(x_1^2, \dots, x_{|\alpha_2|}^2); \psi_{X_1^2 \dots X_{k_2}^2}; F_3(x_1^3, \dots, x_{|\alpha_3|}^3)$$

where

$$x_j^i = \begin{cases} f & M(\iota_i(j)) = 1 \\ id_{L(\iota_i(j))} & M(\iota_i(j)) = 0 \wedge t \in \bullet p \cup p\bullet \end{cases} \quad X_j^i = L(\rho_i(j)).$$

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We proceed now to show that the firing of an enabled, B -labelled transition in a labelled marking yields an equation between the associated morphisms. Consider then (M, L, f) a labelled marking, t in T such that $L(t) = B$ and $M(p) = 1$ for all $p \in \bullet t$. Notice that necessarily $M(p) = 0$ for all $p \in t \bullet$ (otherwise we would have $L(t) = A$ by definition of labelled marking). Define functions $M': P \rightarrow \{0, 1\}$ and $L': T \rightarrow \{A, B\}$ as follows, for all $p \in P$ and $s \in T$:

$$M'(p) = \begin{cases} 0 & p \in \bullet t \\ 1 & p \in t \bullet \\ M(p) & \text{otherwise} \end{cases} \quad L'(s) = \begin{cases} A & s = t \\ L(s) & \text{otherwise} \end{cases}$$

(M' is the marking obtained from M by firing t .) It is an immediate consequence of the definition that (M', L', f) is still a labelled marking.

► **Proposition 11.** *In the notations above, $\mu(M, L, f) = \mu(M', L', f)$.*

Proof. Since $t \in T$, we have $t = \rho_u(i)$ for some $u \in \{1, 2\}$ and $i \in \{1, \dots, k_u\}$. The fact that t is enabled ensures that, in the notations of Definition 10,

$$\begin{aligned} \sigma_u(j) = i \wedge \alpha_j^u = + &\implies x_j^u = f \\ \sigma_u(j) = i \wedge \alpha_j^u = - &\implies x_j^u = id_B \\ \tau_u(j) = i \wedge \alpha_j^{u+1} = + &\implies x_j^{u+1} = id_B \\ \tau_u(j) = i \wedge \alpha_j^{u+1} = - &\implies x_j^{u+1} = f \end{aligned}$$

hence we can apply the dinaturality of φ or ψ (if, respectively, $u = 1$ or $u = 2$) in its i -th variable and obtain therefore a new morphism, which a simple check can show is equal to $\mu(M', L', f)$. ◀

It immediately follows that a sequence of firings of B -labelled transitions gives rise to a labelled marking whose associated morphism is still equal to the original one, as the following Proposition states.

► **Proposition 12.** *Let (M, L, f) be a labelled marking, M_d a marking reachable from M by firing only B -labelled transitions t_1, \dots, t_m , $L_d: T \rightarrow \{A, B\}$ defined as:*

$$L_d(s) = \begin{cases} A & s = t_i \text{ for some } i \in \{1, \dots, m\} \\ L(s) & \text{otherwise} \end{cases}$$

Then (M_d, L_d, f) is a labelled marking and $\mu(M, L, f) = \mu(M_d, L_d, f)$.

We have now to individuate the two markings M_0 and M_d which correspond to the two morphisms we want to prove to be equal to show that $\psi \circ \varphi$ is dinatural, when $\Gamma(\psi \circ \varphi)$ is acyclic. Since we are assuming that $\psi \circ \varphi: F_1 \rightarrow F_3$ depends on only one variable, those morphisms are:

$$\begin{aligned} \delta_1 &= F_1(x_1, \dots, x_{|\alpha^1|}); [\psi \circ \varphi]_B; F_3(y_1, \dots, y_{|\alpha^3|}) \\ \delta_2 &= F_1(x'_1, \dots, x'_{|\alpha^1|}); [\psi \circ \varphi]_A; F_3(y'_1, \dots, y'_{|\alpha^3|}) \end{aligned}$$

where

$$x_i = \begin{cases} f & \alpha_i^1 = + \\ id_B & \alpha_i^1 = - \end{cases} \quad y_i = \begin{cases} id_B & \alpha_i^3 = + \\ f & \alpha_i^3 = - \end{cases}$$

$$x'_i = \begin{cases} id_B & \alpha_i^1 = + \\ f & \alpha_i^1 = - \end{cases} \quad y'_i = \begin{cases} f & \alpha_i^3 = + \\ id_B & \alpha_i^3 = - \end{cases}$$

Now, f appears in all the covariant arguments of F_1 and the contravariant ones of F_3 , in δ_1 , which correspond in $\Gamma(\psi \circ \varphi)$ to those places which have no inputs (in Petri nets terminology, *sources*), whereas f appears, in δ_2 , in those arguments corresponding to places with no outputs (*sinks*). The two markings we are interested into are, therefore,

$$M_0(p) = \begin{cases} 1 & \bullet p = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad M_d(p) = \begin{cases} 1 & p \bullet = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

What about the labelling? We have that $[\psi \circ \varphi]_B = \varphi_{B\dots B}; \psi_{B\dots B}$, hence we shall consider $L: T \rightarrow \{A, B\}$ constantly equal to B : it is easy to see that (M_0, L, f) is a labelled marking. Now all we have to show is that M_d is reachable from M_0 by only firing B -labelled transitions: it is enough to make sure that each transition is fired at most once to satisfy this condition. (Notice that since $\Gamma(\varphi)$ is acyclic, if a transition fires once than it will remain disabled for ever, hence no transition can fire more than once anyway.) In order to do that, we recall some general properties of Petri nets, see [17].

Every Petri Net N with n transitions and m places defines a $m \times n$ matrix of integers $A = [a_{pt}]$, called *incidence matrix* of N . In the case of a net with at most one arc between any two vertices (like $\Gamma(\psi \circ \varphi)$), we have

$$a_{pt} = \begin{cases} 1 & p \in t \bullet \\ -1 & p \in \bullet t \\ 0 & \text{otherwise} \end{cases}$$

It is not difficult to see that a_{pt} represents the number of tokens changed in place p when transition t fires once. If we represent an arbitrary marking M as a $m \times 1$ vector, we can state the following theorem [11], which gives a necessary and sufficient condition for reachability of a marking M_d from another marking M_0 in case N is acyclic.

► **Theorem 13.** *Let N be an acyclic Petri Net with m places and n transitions, A its incidence matrix, M_0, M_d two markings for N . Then M_d is reachable from M_0 if and only if there is a $n \times 1$ vector x of non-negative integers such that*

$$M_d = M_0 + Ax. \quad (3)$$

The “only if” part is easy to show, as x can be the vector which tells how many times each transition fires to transform M_0 into M_d . The interesting part is the vice versa: if we can find a vector of non-negative integers x that solves equation (3), then the proof of Theorem 13 ensures the existence of a firing sequence that transforms M_0 into M_d by firing each transition t exactly x_t times. (A constructive proof for Theorem 13 can be found in [21].)

We use these considerations to prove that $\psi \circ \varphi$ is a dinatural transformation by finding a vector x that solves equation (3) for $N = \Gamma(\psi \circ \varphi)$ and M_0 and M_d as in (2). Since we want to move the tokens from the sources to the sinks and $\Gamma(\psi \circ \varphi)$ is connected (Remark 2), we ought to fire each transition at least once; on the other hand, as already observed, the acyclicity of $\Gamma(\psi \circ \varphi)$ ensures that any transition cannot fire more than once. Hence $x = [1, \dots, 1]$ is the solution we are seeking.

Proof of Theorem 8. Consider $x = [1, \dots, 1]$ of length $|T|$. A simple computation shows that, if A is the incidence matrix of $\Gamma(\psi \circ \varphi)$ and M_0 and M_d are as in (2), $M_d = M_0 + Ax$:

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it is enough to notice that A 's row corresponding to place p is made of all 0's except for exactly one 1 if p is a sink, exactly one -1 if p is a source, and exactly one 1 and one -1 if p is neither of them. Hence, by Theorem 13, M_d is reachable from M_0 , and by Proposition 12 with $M = M_0$ and $L: T \rightarrow \{A, B\}$ constantly equal to B , we obtain that $\mu(M_0, L, f) = \mu(M_0, L_d, f)$. By the arbitrariness of the morphism $f: A \rightarrow B$ we have chosen, we get the dinaturality of $\psi \circ \varphi$. \blacktriangleleft

It is not difficult to generalise Theorem 8 to the case in which $\psi \circ \varphi$ depends on more than one variable: it is enough to apply the same argument to one connected component of $\Gamma(\psi \circ \varphi)$ at a time.

► **Theorem 14.** *Let $\varphi: T \rightarrow S$ and $\psi: S \rightarrow R$ as in Definition 2, $i \in \{1, \dots, q\}$. If φ and ψ are dinatural in all their variables in, respectively, $\zeta^{-1}\{i\}$ and $\xi^{-1}\{i\}$ (with ζ and ξ given by the pushout (1)), and if the i -th connected component of $\Gamma(\psi \circ \varphi)$ is acyclic, then $\psi \circ \varphi$ is dinatural in its i -th variable.*

We conclude this section with a straightforward corollary:

► **Corollary 15.** *Let $\varphi: T \rightarrow S$ and $\psi: S \rightarrow R$ be transformations which are dinatural in all their variables. If $\Gamma(\psi \circ \varphi)$ is acyclic, then $\psi \circ \varphi$ is dinatural in all its variables.*

3 Horizontal composition

Horizontal composition of natural transformations [15] is a well known operation which is rich in interesting properties: it is associative, unitary, compatible with vertical composition. Also, it plays a crucial role in the calculus of substitution of functors and natural transformations developed by Kelly in [13]. An appropriate generalisation of this notion for dinatural transformations seems to be absent in the literature; here we propose a possible definition and prove some of its properties. First, we briefly recall the definition for the natural case.

► **Definition 16.** Consider (classical) natural transformations

$$\begin{array}{ccc} \mathbb{A} & \begin{array}{c} \xrightarrow{F} \\ \Downarrow \varphi \\ \xrightarrow{G} \end{array} & \mathbb{B} \\ & & \begin{array}{c} \xrightarrow{H} \\ \Downarrow \psi \\ \xrightarrow{K} \end{array} \\ & & \mathbb{C} \end{array}$$

The horizontal composition $\psi * \varphi: HF \rightarrow KG$ is the natural transformation whose A -th component, for $A \in \mathbb{A}$, is either leg of the following commutative square:

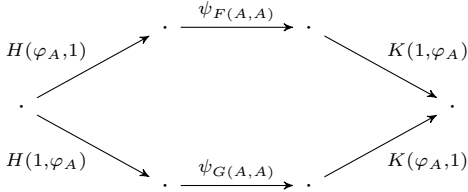
$$\begin{array}{ccc} HF(A) & \xrightarrow{\psi_{F(A)}} & KF(A) \\ H(\varphi_A) \downarrow & & \downarrow K(\varphi_A) \\ HG(A) & \xrightarrow{\psi_{G(A)}} & KG(A) \end{array} \quad (4)$$

Now, the commutativity of (4) is due to the naturality of ψ ; the fact that $\psi * \varphi$ is in turn a natural transformation is due to the naturality of both φ and ψ . However, in order to *define* the family of morphisms $\psi * \varphi$, all we have to do is to apply the naturality condition of ψ to the components of φ , one by one. We apply the very same idea to dinatural transformations, leading to the following preliminary definition for classical dinatural transformations.

► **Definition 17.** Let $\varphi: F \rightarrow G$ and $\psi: H \rightarrow K$ dinatural transformations of type $2 \rightarrow 1 \leftarrow 2$, where $F, G: \mathbb{A}^{\text{op}} \times \mathbb{A} \rightarrow \mathbb{B}$ and $H, K: \mathbb{B}^{\text{op}} \times \mathbb{B} \rightarrow \mathbb{C}$. The *horizontal composition* $\psi * \varphi$ is the family of morphisms

$$([\psi * \varphi]_A: H(G(A, A), F(A, A)) \rightarrow K(F(A, A), G(A, A)))_{A \in \mathbb{A}}$$

where the general component $[\psi * \varphi]_A$ is given, for any object $A \in \mathbb{A}$, by either leg of the following commutative hexagon:



► **Remark.** If functors F, G, H and K all factor through the projection $\mathbb{A}^{\text{op}} \times \mathbb{A} \rightarrow \mathbb{A}$ or $\mathbb{B}^{\text{op}} \times \mathbb{B} \rightarrow \mathbb{B}$, then φ and ψ are natural transformations and $\psi * \varphi$ coincides with the classical definition of horizontal composition of natural transformations.

It turns out that, as happens with classical natural transformations, the dinaturality of φ and ψ implies the dinaturality of their horizontal composition.

► **Theorem 18.** Let φ and ψ be dinatural transformations as in Definition 17. Then $\psi * \varphi$ is a dinatural transformation

$$\psi * \varphi: H(G^{\text{op}}, F) \rightarrow K(F^{\text{op}}, G)$$

of type $4 \rightarrow 1 \leftarrow 4$, where $H(G^{\text{op}}, F), K(F^{\text{op}}, G): \mathbb{A}^{[+, -, \cdot, +]} \rightarrow \mathbb{C}$ are defined on objects as

$$H(G^{\text{op}}, F)(A, B, C, D) = H(G^{\text{op}}(A, B), F(C, D))$$

$$K(F^{\text{op}}, G)(A, B, C, D) = K(F^{\text{op}}(A, B), G(C, D))$$

and similarly on morphisms.

Proof. The proof consists in showing that the diagram that asserts the dinaturality of $\psi * \varphi$ commutes: this is done in Figure 3, in the Appendix. ◀

We can now proceed with the general definition, which involves transformations of arbitrary type. As the idea behind Definition 17 is to apply the dinaturality of ψ on the general component of φ in order to define $\psi * \varphi$, if ψ is a transformation with many variables, then we have many dinaturality conditions we can apply to φ , namely one for each variable of ψ in which ψ is dinatural. Hence, the general definition will depend on the variable of ψ we want to use. For the sake of simplicity, we shall consider only the one-category case, that is when all functors in the definition involve one category \mathbb{C} , in line with our approach in Section 2; the general case follows with no substantial complications except for a much heavier notation. Indeed, when $\mathbb{A} = \mathbb{B} = \mathbb{C}$, Definition 17 is a special case of the following.

► **Definition 19.** Let $F: \mathbb{C}^\alpha \rightarrow \mathbb{C}, G: \mathbb{C}^\beta \rightarrow \mathbb{C}, H: \mathbb{C}^\gamma \rightarrow \mathbb{C}, K: \mathbb{C}^\delta \rightarrow \mathbb{C}$ be functors, $\varphi = (\varphi_{A_1, \dots, A_n}): F \rightarrow G$ be a transformation of type $|\alpha| \xrightarrow{\sigma} n \xleftarrow{\tau} |\beta|$ and $\psi = (\psi_{B_1, \dots, B_m}): H \rightarrow K$ of type $|\gamma| \xrightarrow{\eta} m \xleftarrow{\theta} |\delta|$ a transformation which is dinatural in its i -th variable. Denoting with $++$ the concatenation of a family of lists, let

$$H(X_1 \dots X_{|\gamma|}): \mathbb{C}^{++\lambda^u} \rightarrow \mathbb{C}, \quad K(Y_1 \dots Y_{|\delta|}): \mathbb{C}^{++\mu^v} \rightarrow \mathbb{C}$$

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be functors, defined similarly to $H(G^{\text{op}}, F)$ and $K(F^{\text{op}}, G)$ in Theorem 18, where for all $u \in \{1, \dots, |\gamma|\}$:

$$X_u = \begin{cases} F & \eta u = i \wedge \gamma_u = + \\ G^{\text{op}} & \eta u = i \wedge \gamma_u = - \\ id_{\mathbb{C}^{\gamma_u}} & \eta u \neq i \end{cases} \quad \lambda^u = \begin{cases} \alpha & \eta u = i \wedge \gamma_u = + \\ \bar{\beta}^3 & \eta u = i \wedge \gamma_u = - \\ [\gamma_u] & \eta u \neq i \end{cases}$$

$$a_u = \begin{cases} \iota_n \sigma & \eta u = i \wedge \gamma_u = + \\ \iota_n \tau & \eta u = i \wedge \gamma_u = - \\ \iota_m \eta_{\{u\}} & \eta u \neq i \end{cases}$$

with $\iota_n: n \rightarrow (i-1) + n + (m-i)$ and $\iota_m: m \rightarrow (i-1) + n + (m-i)$ fixed injections, and for all $v \in \{1, \dots, |\delta|\}$:

$$Y_v = \begin{cases} G & \theta v = i \wedge \delta_v = + \\ F^{\text{op}} & \theta v = i \wedge \delta_v = - \\ id_{\mathbb{C}^{\delta_v}} & \theta v \neq i \end{cases} \quad \mu^v = \begin{cases} \beta & \theta v = i \wedge \delta_v = + \\ \bar{\alpha} & \theta v = i \wedge \delta_v = - \\ [\delta_v] & \theta v \neq i \end{cases}$$

$$b_v = \begin{cases} \iota_n \tau & \theta v = i \wedge \delta_v = + \\ \iota_n \sigma & \theta v = i \wedge \delta_v = - \\ \iota_m \theta_{\{v\}} & \theta v \neq i \end{cases}$$

The i -th horizontal composition $\psi \stackrel{i}{*} \varphi$ is a transformation

$$\psi \stackrel{i}{*} \varphi: H(X_1 \dots X_{|\gamma|}) \rightarrow K(Y_1 \dots Y_{|\delta|})$$

of type

$$\sum_{u=1}^{|\gamma|} |\lambda^u| \xrightarrow{[a_1 \dots a_{|\gamma|}]} (i-1) + n + (m-i) \xleftarrow{[b_1 \dots b_{|\delta|}]} \sum_{v=1}^{|\delta|} |\mu^v|$$

whose general component, $[\psi \stackrel{i}{*} \varphi]_{B_1 \dots B_{i-1}, A_1 \dots A_n, B_{i+1} \dots B_m}$, is either leg of the commutative hexagon obtained by applying the dinaturality of ψ in its i -th variable to $\varphi_{A_1, \dots, A_n}$, that is the morphism

$$H(x_1, \dots, x_{|\gamma|}); \psi_{B_1 \dots B_{i-1}, G(A_{\tau_1} \dots A_{\tau_{|\alpha|}}, B_{i+1} \dots B_m); K(y_1, \dots, y_{|\delta|})$$

where

$$x_u = \begin{cases} \varphi_{A_1, \dots, A_n} & \eta u = i \wedge \gamma_u = + \\ id_{G(A_{\tau_1} \dots A_{\tau_{|\alpha|}})} & \eta u = i \wedge \gamma_u = - \\ id_{B_{\eta u}} & \eta u \neq i \end{cases} \quad y_v = \begin{cases} id_{G(A_{\tau_1} \dots A_{\tau_{|\alpha|}})} & \theta v = i \wedge \delta_v = + \\ \varphi_{A_1, \dots, A_n} & \theta v = i \wedge \delta_v = - \\ id_{B_{\theta v}} & \theta v \neq i \end{cases}$$

► **Notation.** For the rest of this paper we shall denote the m variables of ψ as B_1, \dots, B_m and the n variables of φ as A_1, \dots, A_n , as in Definition 19. In this spirit, we shall sometimes write $\psi \stackrel{B_i}{*} \varphi$ instead of $\psi \stackrel{i}{*} \varphi$.

³ Remember that for any $\beta \in \text{List}\{+, -\}$ we denote $\bar{\beta}$ the list obtained from β by swapping the $+$'s with the $-$'s.

► Remark. $\psi \overset{i}{*} \varphi$ depends on all the variables of $\psi = (\psi_{B_1, \dots, B_m})$ where B_i has been substituted by the variables of $\varphi = (\varphi_{A_1, \dots, A_n})$.

As for the classical natural case, only the dinaturality of ψ in its i -th variable is needed to define the i -th horizontal composition of φ and ψ . It is immediate from the definitions that $\psi \overset{i}{*} \varphi$ is dinatural in all the “ B variables” (that is, those variables inherited from ψ) where also ψ is. Theorem 18 generalises to the following one, which states that if φ is dinatural in A_j , then $\psi \overset{i}{*} \varphi$ is also dinatural in A_j ; in other words, $\psi \overset{i}{*} \varphi$ is dinatural in all the “ A variables” where φ is dinatural.

► **Theorem 20.** *In the same notation as in Definition 19, if φ is dinatural in its j -th variable and ψ in its i -th one, then $\psi \overset{i}{*} \varphi$ is dinatural in its $(i - 1 + j)$ -th variable. In other words, if φ is dinatural in A_j and ψ in B_i , then $\psi \overset{B_i}{*} \varphi$ is dinatural in A_j .*

Unitarity. It is straightforward to see that horizontal composition has a unit, namely the identity (di)natural transformation of the identity functor.

► **Theorem 21.** *Let $T: \mathbb{C}^\alpha \rightarrow \mathbb{C}$ and $S: \mathbb{C}^\beta \rightarrow \mathbb{C}$ be functor, $\varphi: T \rightarrow S$ be a transformation of type $|\alpha| \xrightarrow{\sigma} k \xleftarrow{\tau} |\beta|$. Then $id_{id_{\mathbb{C}}} * \varphi = \varphi$. If φ is dinatural in its i -th variable, then also $\varphi \overset{i}{*} id_{id_{\mathbb{C}}} = \varphi$.*

Associativity. Throughout this section fix transformations $\varphi: F \rightarrow G$, $\psi: H \rightarrow K$ and $\chi: U \rightarrow V$. For sake of simplicity, denote with A_1, \dots, A_n , B_1, \dots, B_m and C_1, \dots, C_l the variables of, respectively, φ , ψ and χ . The theorem asserting associativity of horizontal composition, which we aim to prove here, is the following.

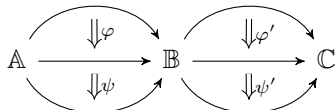
► **Theorem 22.** *Suppose ψ is dinatural in B_i and χ is dinatural in C_j . Then*

$$\chi \overset{j}{*} (\psi \overset{i}{*} \varphi) = (\chi \overset{j}{*} \psi) \overset{j-1+i}{*} \varphi \text{ or, in alternative notation, } \chi \overset{C_j}{*} (\psi \overset{B_i}{*} \varphi) = (\chi \overset{C_j}{*} \psi) \overset{B_i}{*} \varphi.$$

Proof. The proof that the two sides have the same signature is in the Appendix (Proposition 25). Regarding the single components, it is enough to consider the case in which φ , ψ and χ are all of type $2 \rightarrow 1 \leftarrow 2$, the general case follows as a consequence.

Fix then an object A in \mathbb{C} . Figure 4, in the Appendix, shows how to pass from $(\chi * \psi) * \varphi$ to $\chi * (\psi * \varphi)$ by pasting three commutative diagrams. In order to save space, we simply wrote “ $H(G, F)$ ” instead of the proper “ $H(G^{\text{op}}(A, A), F(A, A))$ ” and similarly for all the other instances of functors in the nodes of the diagram in Figure 4; we also dropped the subscript for components of φ , ψ and χ when they appear as arrows, that is we simply wrote φ instead of φ_A , since there is only one object involved and there is no risk of confusion. ◀

Incompatibility with vertical composition. It is well known that horizontal composition is compatible with the vertical one for classical natural transformations: in the following situation,



with φ, φ', ψ and ψ' natural transformations, we have:

$$(\psi' \circ \varphi') * (\psi \circ \varphi) = (\psi' * \psi) \circ (\varphi' * \varphi) \tag{†}$$

It is also well known that dinatural transformations do not vertically compose, in general; on the other hand, we have defined a notion of horizontal composition which is always possible. Are these two operations compatible, at least when vertical composition is defined?

The answer, unfortunately, is *No*, at least if by “compatible” we mean “compatible as in the natural case (\dagger)”. Indeed, consider dinatural transformations

$$\begin{array}{ccc}
 & F & \\
 \curvearrowright & \Downarrow \varphi & \curvearrowleft \\
 \mathbb{A}^{\text{op}} \times \mathbb{A} & \xrightarrow{G} & \mathbb{B} \\
 \curvearrowleft & \Downarrow \psi & \curvearrowright \\
 & H &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & J & \\
 \curvearrowright & \Downarrow \varphi' & \curvearrowleft \\
 \mathbb{B}^{\text{op}} \times \mathbb{B} & \xrightarrow{K} & \mathbb{C} \\
 \curvearrowleft & \Downarrow \psi' & \curvearrowright \\
 & L &
 \end{array}$$

such that $\varphi; \psi$ and $\varphi'; \psi'$ are dinatural. Then

$$\varphi' * \varphi: J(G, F) \rightarrow K(F, G) \qquad \psi' * \psi: K(H, G) \rightarrow L(G, H)$$

which means that $\varphi' * \varphi$ and $\psi' * \psi$ are not even composable as families of morphisms, as the codomain of the former is not the domain of the latter. The problem stems from the fact that the codomain of the horizontal composition $\varphi' * \varphi$ depends on the codomain of φ' and also the domain *and* codomain of φ , which are not the same as the domain and codomain of ψ : indeed, in order to be composable, φ and ψ must share only one functor, and not both. This does not happen in the natural case, and ultimately this is due to the difference between the naturality and the dinaturality conditions for a transformation.

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A Appendix

Regarding Theorem 20

The proof of this theorem relies on the fact that we can reduce ourselves, without loss of generality, to Theorem 18. In order to prove that, we introduce the notion of *focalisation* of a transformation on one of its variables.

► **Definition 23.** Let $\phi = (\phi_{A_1, \dots, A_k}): T \rightarrow S$ be a transformation of type $|\alpha| \xrightarrow{\sigma} k \xleftarrow{\tau} |\beta|$ with $T: \mathbb{C}^\alpha \rightarrow \mathbb{C}$ and $S: \mathbb{C}^\beta \rightarrow \mathbb{C}$. Fix $j \in \{1, \dots, k\}$ and objects $A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_k$ in \mathbb{C} . Consider functors $\bar{T}^j, \bar{S}^j: \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$\bar{T}^j(A, B) = T(C_1, \dots, C_{|\alpha|}), \quad \bar{S}^j(A, B) = S(D_1, \dots, D_{|\beta|})$$

where

$$C_u = \begin{cases} B & \sigma u = j \wedge \alpha_u = + \\ A & \sigma u = j \wedge \alpha_u = - \\ A_{\sigma u} & \sigma u \neq j \end{cases} \quad D_v = \begin{cases} B & \tau v = j \wedge \beta_v = + \\ A & \tau v = j \wedge \beta_v = - \\ A_{\tau v} & \tau v \neq j \end{cases}$$

The focalisation of ϕ on its j -th variable is the transformation $\bar{\phi}^j: \bar{T}^j \rightarrow \bar{S}^j$ of type $2 \rightarrow 1 \leftarrow 2$ where

$$\bar{\phi}_X^j = \varphi_{A_1 \dots A_{j-1}, X, A_{j+1} \dots A_k}.$$

Sometimes we may write $\bar{\phi}^{A_j}: \bar{T}^{A_j} \rightarrow \bar{S}^{A_j}$ too, when we fix as A_1, \dots, A_k the name of the variables of ϕ .

► **Remark.** ϕ is dinatural in its j -th variable if and only if $\bar{\phi}^j$ is dinatural in its only variable for all objects $A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_k$ in \mathbb{C} fixed by the focalisation of ϕ .

The $(-)^j$ construction depends on the $k-1$ objects we fix, but not to make the notation too heavy, we shall always call those (arbitrary) objects $A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_n$ for $\bar{\varphi}^j$ and $B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_m$ for $\bar{\psi}^i$.

► **Lemma 24.** *Let φ and ψ be transformations as in Definition 19, with ψ dinatural in its i -th variable. It is the case that $\bar{\psi}^i * \varphi$ is dinatural in its $(i-1+j)$ -th variable if and only if $\bar{\psi}^i * \bar{\varphi}^j$ is dinatural in its only variable for all objects $B_1, \dots, B_{i-1}, A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_n, B_{i+1}, \dots, B_m$ in \mathbb{C} .*

Proof. Direct check that the equations between morphisms demanded by unpacking the two definitions are the same. ◀

Proof of Theorem 20. Consider transformations $\bar{\varphi}^j$ and $\bar{\psi}^i$. By Remark A, they are both dinatural in their only variable. Hence, by Theorem 18, $\bar{\psi}^i * \bar{\varphi}^j$ is dinatural and by Lemma 24 we conclude. ◀

Regarding the signature of $\chi * \psi * \varphi$

Suppose that $\varphi: F \rightarrow G$ has type $|\alpha| \xrightarrow{\sigma} n \xleftarrow{\tau} |\beta|$, $\psi: H \rightarrow K$ has type $|\gamma| \xrightarrow{\eta} m \xleftarrow{\theta} |\delta|$ and $\chi: U \rightarrow V$ has type $|\varepsilon| \xrightarrow{\pi} l \xleftarrow{\omega} |\zeta|$. First of all, notice how both $\chi^{C_j} * (\psi^{B_i} * \varphi)$ and $(\chi^{C_j} * \psi)^{B_i} * \varphi$ are families of morphisms depending on variables

$$C_1, \dots, C_{j-1}, B_1, \dots, B_{i-1}, A_1, \dots, A_n, B_{i+1}, \dots, B_m, C_{j+1}, \dots, C_l.$$

Next, we compute their domain and codomain functors. We have $\psi^{B_i} * \varphi: H(X_1, \dots, X_{|\gamma|}) \rightarrow K(Y_1, \dots, Y_{|\delta|})$ where we are using the same notations as in Definition 19. Hence

$$\chi^{C_j} * (\psi^{B_i} * \varphi): U(W_1, \dots, W_{|\varepsilon|}) \rightarrow V(Z_1, \dots, Z_{|\zeta|})$$

with $U(W_1, \dots, W_{|\varepsilon|}): \mathbb{C}_{u=1}^{|\varepsilon|} \nu^u \rightarrow \mathbb{C}$, $V(Z_1, \dots, Z_{|\zeta|}): \mathbb{C}_{u=1}^{|\zeta|} \xi^u \rightarrow \mathbb{C}$ where

$$W_u = \begin{cases} H(X_1, \dots, X_{|\gamma|}) & \pi u = j \wedge \varepsilon_u = + \\ K(Y_1, \dots, Y_{|\delta|})^{\text{op}} & \pi u = j \wedge \varepsilon_u = - \\ id_{\mathbb{C}^{\varepsilon_u}} & \pi u \neq j \end{cases} \quad \nu^u = \begin{cases} \frac{|\gamma|}{u=1} \lambda^u & \pi u = j \wedge \varepsilon_u = + \\ \frac{|\delta|}{u=1} \mu^u & \pi u = j \wedge \varepsilon_u = - \\ [\varepsilon_u] & \pi u \neq j \end{cases}$$

and similarly are defined Z_u and ξ^u (swapping $H(X_1, \dots, X_{|\gamma|})$ with $K(Y_1, \dots, Y_{|\delta|})$, ω with π , ε with ζ and so on).

On the other hand, we have

$$\chi \overset{C_j}{*} \psi: U(L_1, \dots, L_{|\varepsilon|}) \rightarrow V(M_1, \dots, M_{|\zeta|})$$

with $U(L_1, \dots, L_{|\varepsilon|}): \mathbb{C}_{u=1}^{|\varepsilon|} \rho^u \rightarrow \mathbb{C}$, $V(M_1, \dots, M_{|\zeta|}): \mathbb{C}_{u=1}^{|\zeta|} \vartheta^u \rightarrow \mathbb{C}$ where

$$L_u = \begin{cases} H & \pi u = j \wedge \varepsilon_u = + \\ K^{\text{op}} & \pi u = j \wedge \varepsilon_u = - \\ id_{\mathbb{C}^{\varepsilon_u}} & \pi u \neq j \end{cases} \quad \rho^u = \begin{cases} \gamma & \pi u = j \wedge \varepsilon_u = + \\ \bar{\delta} & \pi u = j \wedge \varepsilon_u = - \\ [\varepsilon_u] & \pi u \neq j \end{cases}$$

$$M_u = \begin{cases} K & \omega u = j \wedge \zeta_u = + \\ H^{\text{op}} & \omega u = j \wedge \zeta_u = - \\ id_{\mathbb{C}^{\zeta_u}} & \omega u \neq j \end{cases} \quad \vartheta^u = \begin{cases} \delta & \omega u = j \wedge \zeta_u = + \\ \bar{\gamma} & \omega u = j \wedge \zeta_u = - \\ [\zeta_u] & \omega u \neq j \end{cases}$$

$\chi \overset{C_j}{*} \psi$ has type $\sum_{u=1}^{|\varepsilon|} |\rho^u| \xrightarrow{[c_1, \dots, c_{|\varepsilon|}]} (j-1) + m + (l-j) \xleftarrow{[d_1, \dots, d_{|\zeta|}]} \sum_{u=1}^{|\zeta|} |\zeta^u|$ with

$$c_u = \begin{cases} \iota_m \eta & \pi u = j \wedge \varepsilon_u = + \\ \iota_m \theta & \pi u = j \wedge \varepsilon_u = - \\ \iota_l \pi_{\{i\}} & \pi u \neq j \end{cases} \quad d_u = \begin{cases} \iota_m \theta & \omega u = j \wedge \zeta_u = + \\ \iota_m \eta & \omega u = j \wedge \zeta_u = - \\ \iota_l \omega_{\{i\}} & \omega u \neq j \end{cases}$$

and $\iota_m: m \rightarrow (j-1) + m + (l-j)$, $\iota_l: l \rightarrow (j-1) + m + (l-j)$ defined as

$$\iota_m(x) = x + j - 1 \quad \iota_l(x) = \begin{cases} x & x \leq j \\ x + m - 1 & x > j \end{cases}$$

Therefore, the domain of $(\chi \overset{C_j}{*} \psi) \overset{B_i}{*} \varphi$ is $U(L_1, \dots, L_{|\varepsilon|})(P_1^1, \dots, P_{|\rho^1|}^1, \dots, P_1^{|\varepsilon|}, \dots, P_{|\rho^{|\varepsilon|}|}^{|\varepsilon|})$ while the codomain is $V(M_1, \dots, M_{|\zeta|})(Q_1^1, \dots, Q_{|\vartheta^1|}^1, \dots, Q_1^{|\zeta|}, \dots, Q_{|\vartheta^{|\zeta|}|}^{|\zeta|})$ where

$$P_v^u = \begin{cases} F & c_u(v) = j - 1 + i \wedge \rho_v^u = + \\ G^{\text{op}} & c_u(v) = j - 1 + i \wedge \rho_v^u = - \\ id_{\mathbb{C}^{\rho_v^u}} & c_u(v) \neq j - 1 + i \end{cases}$$

and similarly Q_v^u . Denoting the domain of $(\chi \overset{C_j}{*} \psi) \overset{B_i}{*} \varphi$ as $U(L(P))$, we have

$$U(L(P)): \mathbb{C}_{u=1}^{|\varepsilon|} \left(\prod_{v=1}^{|\rho^u|} w_v^u \right) \rightarrow \mathbb{C}$$

where

$$w_v^u = \begin{cases} \alpha & c_u(v) = j - 1 + i \wedge \rho_v^u = + \\ \bar{\beta} & c_u(v) = j - 1 + i \wedge \rho_v^u = - \\ [\rho_v^u] & c_u(v) \neq j - 1 + i \end{cases}$$

► **Proposition 25.** *Transformations $\chi \overset{C_j}{*} (\psi \overset{B_i}{*} \varphi)$ and $(\chi \overset{C_j}{*} \psi) \overset{B_i}{*} \varphi$ have the same domain, codomain, and type.*

Proof. One can prove that $\prod_{u=1}^{|\varepsilon|} \left(\prod_{v=1}^{|\rho^u|} w_v^u \right) = \prod_{u=1}^{|\varepsilon|} \nu^u$ by showing that $\prod_{v=1}^{|\rho^u|} w_v^u = \nu^u$ for all $u \in \{1, \dots, |\varepsilon|\}$, analysing each of the three cases for ηu that define ν^u .

Next, we have that

$$U(L(P)) = U\left(L_1(P_1^1, \dots, P_{|\rho^1|}^1), \dots, L_{|\varepsilon|}(P_1^{|\varepsilon|}, \dots, P_{|\rho^{|\varepsilon|}|}^{|\varepsilon|})\right)$$

and by showing that $W_u = L_u(P_1^u, \dots, P_{|\rho^u|}^u)$ for all $u \in \{1, \dots, |\varepsilon|\}$, one proves that $\chi \overset{C_j}{*} (\psi \overset{B_i}{*} \varphi)$ and $(\chi \overset{C_j}{*} \psi) \overset{B_i}{*} \varphi$ have the same domain; an analogous procedure shows that they also share the same codomain.

Finally, we briefly analyse only the left hand sides of the types of $\chi \overset{C_j}{*} (\psi \overset{B_i}{*} \varphi)$ and $(\chi \overset{C_j}{*} \psi) \overset{B_i}{*} \varphi$; the right hand sides are handled analogously. For $\chi \overset{C_j}{*} (\psi \overset{B_i}{*} \varphi)$ we have

$$\sum_{u=1}^{|\varepsilon|} |\nu^u| \xrightarrow{[r_1, \dots, r_{|\varepsilon|}]} (j-1) + [(i-1) + k + (l-i)] + (m-j)$$

with

$$r_u = \begin{cases} ((\cdot) + j - 1) \circ [a_1, \dots, a_{|\gamma|}] & \eta u = j \wedge \varepsilon_u = + \\ ((\cdot) + j - 1) \circ [b_1, \dots, b_{|\delta|}] & \eta u = j \wedge \varepsilon_u = - \\ \iota_m \text{id}_{\mathbb{C}^{\eta u}} & \eta u \neq j \end{cases}$$

where function $((\cdot) + j - 1)$ merges $(i-1) + k + (l-i)$ into $N = (j-1) + [(i-1) + k + (l-i)] + (m-j)$, by adding $j - 1$ to its argument, and ι_m into N . For $(\chi \overset{C_j}{*} \psi) \overset{B_i}{*} \varphi$, which is the same as $(\chi \overset{j}{*} \psi) \overset{-1+i}{*} \varphi$, we have

$$\sum_{u=1}^{|\varepsilon|} \sum_{v=1}^{|\rho^u|} |w_v^u| \xrightarrow{[s_1^1, \dots, s_{|\rho^1|}^1, \dots, s_1^{|\varepsilon|}, \dots, s_{|\rho^{|\varepsilon|}|}^{|\varepsilon|}]} M$$

where $M = (j-1 + i-1) + k + [(j-1 + m + l - j) - (j-1 + i)] = N$ and

$$s_v^u = \begin{cases} ((\cdot) + j - 1 + i - 1) \circ \sigma & c_u(v) = j - 1 + i \wedge \rho_v^u = + \\ ((\cdot) + j - 1 + i - 1) \circ \tau & c_u(v) = j - 1 + i \wedge \rho_v^u = - \\ \iota_m d_{u \setminus \{v\}} & c_u(v) \neq j - 1 + i \end{cases}$$

Notice that here we are asserting an equality between natural numbers; in other words, we are just writing, in two different ways, the same set. Checking that $r_u = [s_1^u, \dots, s_{|\rho^u|}^u]$ and noticing that functions $[\dots r_u \dots]$ and $[\dots s_v^u \dots]$ coincide on every elements of their domain, we conclude. ◀

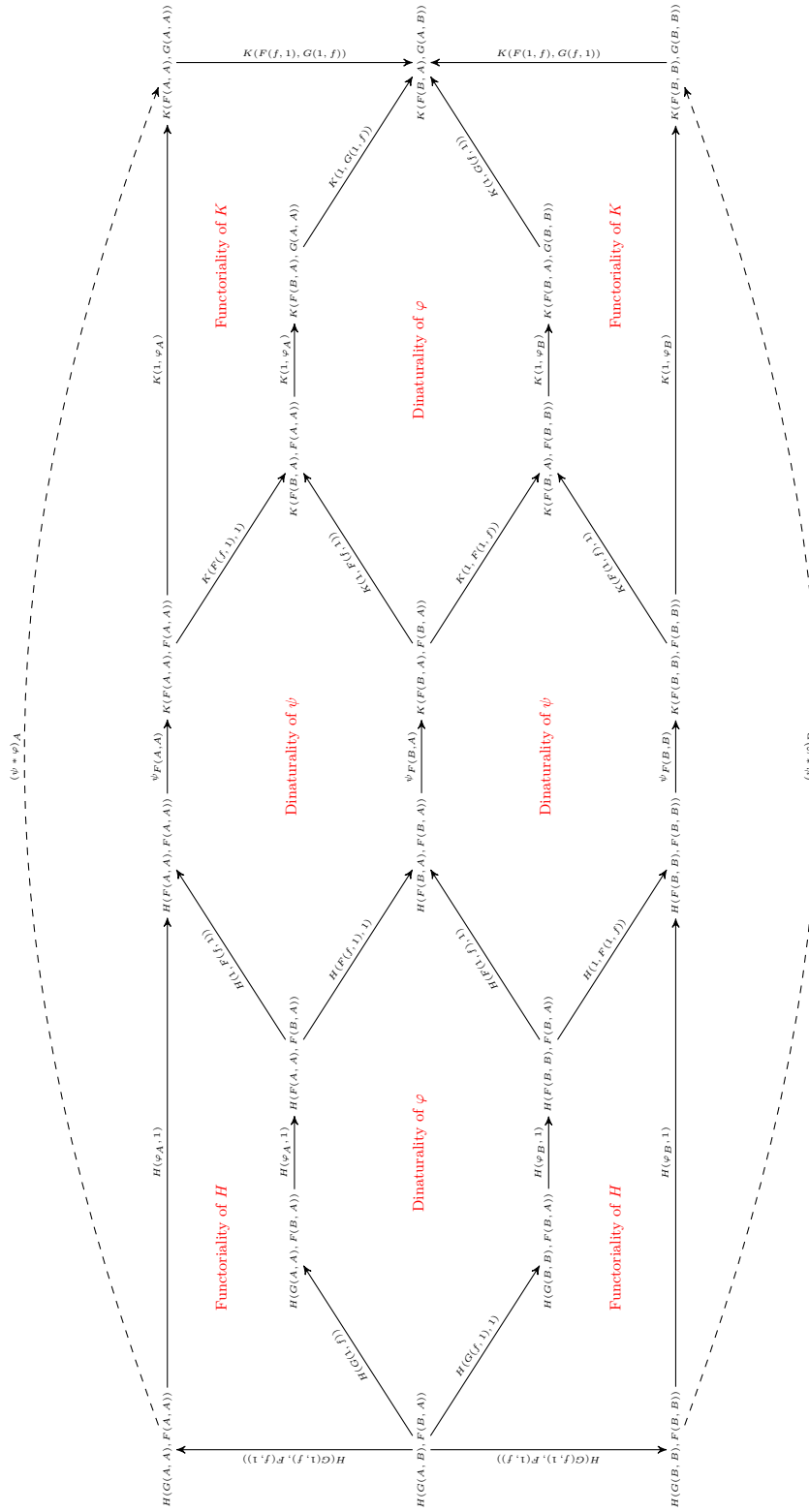


Figure 3 Proof of Theorem 18: dinaturnality of horizontal composition in the classical case. Here $f : A \rightarrow B$.

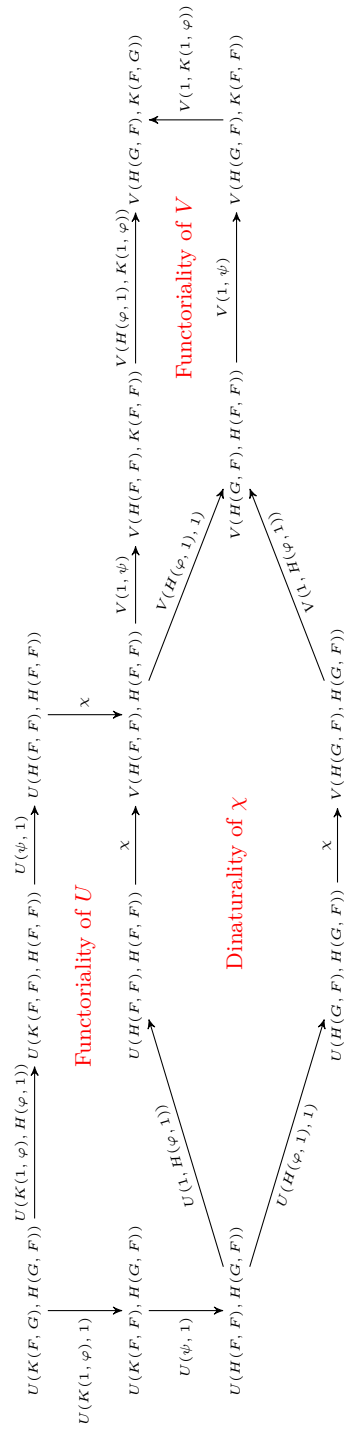


Figure 4 Associativity of horizontal composition in the classical case. The upper leg is $(\chi * \psi) * \varphi$, whereas the lower one is $\chi * (\psi * \varphi)$.