Reachability for Two-Counter Machines with One Test and One Reset

Alain Finkel

LSV, ENS Paris-Saclay, CNRS, Université Paris-Saclay, France
alain.finkel@lsv.ens-paris-saclay.fr

Jérôme Leroux

LaBRI, Université de Bordeaux, CNRS, Bordeaux-INP, Talence, France
jerome.leroux@labri.fr

Grégoire Sutre

LaBRI, Université de Bordeaux, CNRS, Bordeaux-INP, Talence, France
gregoire.sutre@labri.fr

Abstract

We prove that the reachability relation of two-counter machines with one zero-test and one reset is Presburger-definable and effectively computable. Our proof is based on the introduction of two classes of Presburger-definable relations effectively stable by transitive closure. This approach generalizes and simplifies the existing different proofs and it solves an open problem introduced by Finkel and Sutre in 2000.

2012 ACM Subject Classification Theory of computation → Logic and verification

Keywords and phrases Counter machine, Vector addition system, Reachability problem, Formal verification, Presburger arithmetic, Infinite-state system

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2018.31

Funding This work was supported by the grant ANR-17-CE40-0028 of the French National Research Agency ANR (project BRAVAS).

Acknowledgements The work reported was carried out in the framework of ReLaX, UMI2000 (ENS Paris-Saclay, CNRS, Univ. Bordeaux, CMI, IMSc).

1 Introduction

Context. Vector addition systems with states (VASS) are equivalent to Petri nets and to counter machines without the ability to test counters for zero. Although VASS have been studied since the 1970's, they remain fascinating since there are still some important open problems like the complexity of reachability (known between ExpSpace and cubic-Ackermannian) or even an efficient (in practice) algorithm to solve reachability. In 1979, Hopcroft and Pansiot [13] gave an algorithm that computes the Presburger-definable reachability set of a 2-dim VASS, hence VASS in dimension 2 are more easy to verify and they enjoy interesting properties like reachability and equivalence of reachability sets, for instance, are both decidable. Unfortunately, these results do not extend in dimension 3 or for 2-dim VASS with zero-tests on the two counters: the reachability set (hence also the reachability relation) is not Presburger-definable for 3-dim VASS [13]; reachability, and all non-trivial problems, are undecidable for 2-dim VASS extended with zero-tests on the two counters.

38th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2018).
Editors: Sumit Ganguly and Paritosh Pandya; Article No. 31; pp. 31:1–31:14
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Table 1 Reachability sets ($post^*$ and pre^*) and reachability relation ($\xrightarrow{*}$) for extensions of 2-dimensional VASS. We let $\simeq$ denote the existence of mutual reductions between two classes of machines that preserve the effective Preburger-definability of the reachability sets and relation. The contributions of this paper are indicated in boldface.

Class	*Post**	*Pre**	$\xrightarrow{*}$
$\mathbf{T}_1\mathbf{Tr}_2 \simeq \mathbf{T}_{1,2} \simeq \mathbf{T}_{1,2}\mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Not Recursive	Not Recursive	Not Recursive
$\mathbf{T}_1\mathbf{R}_2 \simeq \mathbf{T}_1\mathbf{R}_{1,2}\mathbf{Tr}_1$	**Eff.** Presburger	Eff. Presburger	**Eff. Presburger**
$\mathbf{R}_{1,2}\mathbf{Tr}_1 \simeq \mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Eff. Presburger	Eff. Presburger	**Eff. Presburger**
$\mathbf{T}_1 \simeq \mathbf{T}_1\mathbf{R}_1\mathbf{Tr}_1$	Eff. Presburger	Eff. Presburger	**Eff. Presburger**
2-dim VASS	Eff. Presburger	Eff. Presburger	Eff. Presburger

In 2004, Leroux and Sutre proved that the reachability relation of a 2-dim VASS is also effectively Presburger-definable [17] and this is not a consequence of the Presburger-definability of the reachability set. As a matter of fact, there exist counter machines (even 3-dim VASS) with a Presburger-definable reachability set but with a non Presburger-definable reachability relation [13, 17]. But, for all recursive 2-dim extended VASS, the reachability sets are Presburger-definable [11, 10]. More precisely, let us denote by $\mathbf{T}_I\mathbf{R}_J\mathbf{Tr}_K$, with $I, J, K \subseteq \{1, 2\}$, the class of 2-dim VASS extended with zero-tests on the I-counters, resets on the J-counters and transfers from the K-counters. For instance, $\mathbf{T}_{\{1\}}\mathbf{R}_{\{1,2\}}\mathbf{Tr}_\emptyset$, also written $\mathbf{T}_1\mathbf{R}_{1,2}$ for short, is the class of 2-dim VASS extended with zero-tests on the first counter, resets on both counters, and no transfer. The relations between classes from [11] are recalled in Figure 1 and the class $\mathbf{T}_1\mathbf{R}_2$ has been shown to be the "maximal" class having Presburger-definable $post^*$ and pre^* reachability sets [11]. However, it was unknown whether the Presburger-definable reachability set $post^*$ can be effectively computed or not. In fact, even the boundedness problem (is the reachability set $post^*$ finite?) was open for this class.

Contributions. Our main contribution is a proof that the reachability relation of counter machines in $\mathbf{T}_1\mathbf{R}_2$ is effectively Presburger-definable. Our proof relies on the effective Presburger-definability of the reachability relation for 2-dim VASS [17]. The impact of our result is threefold.

- We solve the main *open problem* in [11] which was the question of the existence of an algorithm that computes the Presburger-definable reachability set for two-counter machines in $\mathbf{T}_1\mathbf{R}_2$.
- In fact, we prove a stronger result, namely that the *reachability relation* of counter machines in $\mathbf{T}_1\mathbf{R}_2$ is Presburger-definable and computable. This completes the decidability picture of 2-dim extended VASS.
- We provide a *simple* proof of the effective Presburger-definability of the reachability relation in $\mathbf{T}_1\mathbf{R}_2$. As an immediate consequence, one may deduce all existing results [11, 10] for 2-dim extended VASS and our proof unifies all different existing proofs on 2-dim extended VASS, including the proof in [6] that the boundedness problem is decidable for the class $\mathbf{R}_{1,2}$ of 2-dim VASS extended with resets on both counters.

Related work. VASS have been extended with resets, transfers and zero-tests. Extended VASS with resets and transfers are well structured transition systems [9] hence termination and coverability are decidable; but reachability and boundedness are undecidable (except boundedness which is decidable for extended VASS with transfers) [5, 6]. The reachability and place-boundedness problems are decidable for extended VASS with *one* zero-test [19, 3, 8, 4].

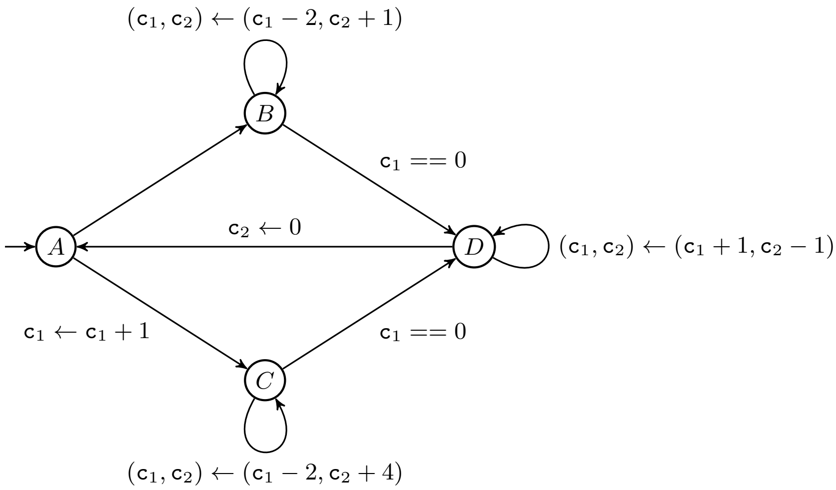

Figure 1 A 2-dimensional VASS extended with zero-tests on the first counter and resets on the second counter (shortly called TRVASS).

Recently, Akshay et al. studied extended Petri nets with a hierarchy on places and with resets, transfers and zero-tests [1]. As a counter is a particular case of a stack, it is natural to study counter machines with one stack. Termination and boundedness are decidable for VASS with one stack [16] but surprisingly, the decidability status of the reachability problem is open for VASS with one stack, both in arbitrary dimension and in dimension 1. We only know that reachability and coverability for VASS with one stack are TOWER-hard [14, 15].

Outline. We present in Section 2 an example of 2-dim extended VASS in $\mathbf{T}_1\mathbf{R}_2$. This example motivates the study of two classes of binary relations on natural numbers, namely diagonal relations in Section 3 and horizontal relations in Section 4. These two classes of relations are combined in Section 5 into a new class of one counter automata with effectively Presburger-definable reachability relations. These automata are used in Section 6 to compute the reachability relations of 2-dim extended VASS in $\mathbf{T}_1\mathbf{R}_2$.

For the remainder of the paper, 2-dim extended VASS in $\mathbf{T}_1\mathbf{R}_2$ are shortly called TRVASS.

2 Motivating Example

Figure 1 depicts an example of a TRVASS. There are four *states* A, B, C and D, and two *counters* c_1 and c_2. Following the standard semantics of vector addition systems, these counters range over natural numbers. The operations labeling the three loops and the edge from A to C are classical addition instructions of vector addition systems. In dimension 2, these addition instructions are always of the form $(c_1, c_2) \leftarrow (c_1 + a_1, c_2 + a_2)$ where a_1 and a_2 are integer constants. For instance, the instruction $(c_1, c_2) \leftarrow (c_1 - 2, c_2 + 1)$ labeling the loop on B means that c_1 is decremented by 2 and at the same time c_2 is incremented by 1. As the counters must remain nonnegative, this instruction may be executed (i.e., the loop on B may be taken) only if $c_1 \geq 2$. In addition to classical addition instructions, TRVASS may test the first counter for zero, written $c_1 == 0$, and reset the second counter to zero, written $c_2 \leftarrow 0$.

The operational semantics of a TRVASS is given, as for vector addition systems, by an infinite directed graph whose nodes are called *configurations* and whose edges are called *steps*. Formal definitions will be given in Section 6. For the TRVASS of Figure 1, configurations are triples $q(x_1, x_2)$ where $q \in \{A, B, C, D\}$ is a state and $x_1, x_2 \in \mathbb{N}$ are values of the counters c_1 and c_2, respectively. It is understood that $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \ldots\}$. There is a step from a configuration $p(x_1, x_2)$ to a configuration $q(y_1, y_2)$, written $p(x_1, x_2) \to q(y_1, y_2)$, if there is an edge from p to q labeled by an operation (1) that can be executed from the counter values (x_1, x_2) and (2) whose execution changes the counter values from (x_1, x_2) to (y_1, y_2). Here, we have the steps $B(5, 1) \to B(3, 2)$, $C(0, 2) \to D(0, 2)$ and $D(7, 3) \to A(7, 0)$. But there is no step from $C(1, 2)$ and there is no step to $A(7, 1)$.

The *reachability relation* of a TRVASS, written $\xrightarrow{*}$, is the reflexive-transitive closure of the step relation $\to$. The reachability relation is one of the main objects of interest for verification purposes. Coming back to our example of Figure 1, we have $A(1, 0) \xrightarrow{*} A(2, 0)$ since we have the following contiguous sequence of steps:

$$A(1, 0) \to C(2, 0) \to C(0, 4) \to D(0, 4) \to D(1, 3) \to D(2, 2) \to A(2, 0)$$

By removing the steps $\to D(1, 3) \to D(2, 2)$, we also get that $A(1, 0) \xrightarrow{*} A(0, 0)$. In fact, it can be shown that $A(1, 0) \xrightarrow{*} A(y, 0)$ for every $y \in \mathbb{N}$, thanks to the following pattern, where k denotes an odd natural number and $i \in \{1, 2\}$:

$$A(k, 0) \to C(k + 1, 0) \xrightarrow{*} D(2k + 2, 0) \xrightarrow{*} D(k + i, k + 2 - i) \xrightarrow{*} A(k + i, 0)$$

One may wonder whether it also holds that $A(x, 0) \xrightarrow{*} A(y, 0)$ for every $x, y \in \mathbb{N}$. A consequence of our main result (see Theorem 14) is that we can do even better: we can compute the set of pairs $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $A(x, 0) \xrightarrow{*} A(y, 0)$, as a formula in Presburger arithmetic[1].

▶ Remark. It is well-known that zero-tests are more expressive than resets. Indeed, a reset $\mathsf{c}_1 \leftarrow 0$ can be simulated by a loop $\mathsf{c}_1 \leftarrow \mathsf{c}_1 - 1$ followed by a zero-test $\mathsf{c}_1 == 0$. A crucial difference between resets and zero-tests is *monotony*. In a 2-dimensional VASS extended with resets on both counters (shortly called RRVASS), larger counter values are always better, in the sense that every behavior from a configuration $q(x_1, x_2)$ can be reproduced from a configuration $q(x'_1, x'_2)$ with $x'_1 \geq x_1$ and $x'_2 \geq x_2$. This is not true anymore in presence of zero-tests. This difference makes the analysis of TRVASS more complex than that of RRVASS, as illustrated in the following example.

▶ **Example 1.** Consider the RRVASS obtained from the TRVASS of Figure 1 by replacing the two zero-tests (from B to D and from C to D) with resets $\mathsf{c}_1 \leftarrow 0$. Suppose that we want to show that c_1 is unbounded in state A from $A(1, 0)$, i.e., $A(1, 0) \xrightarrow{*} A(y, 0)$ for infinitely many $y \in \mathbb{N}$. A natural strategy is, starting from $A(x, 0)$ with $x \geq 1$, to reach $D(0, y)$ with y as large as possible (without visiting A on the way), and then to reach $A(y, 0)$ by taking the "transfer" loop on D as much as possible. By iterating this strategy, we get

$$A(1, 0) \xrightarrow{*} D(0, 4) \xrightarrow{*} A(4, 0) \xrightarrow{*} D(0, 8) \xrightarrow{*} A(8, 0) \xrightarrow{*} D(0, 16) \xrightarrow{*} A(16, 0) \cdots$$

This witnesses that c_1 is unbounded in state A from $A(1, 0)$. In comparison, this strategy does not work for the original TRVASS of Figure 1. Indeed, we get

$$A(1, 0) \xrightarrow{*} D(0, 4) \xrightarrow{*} A(4, 0) \xrightarrow{*} D(0, 2) \xrightarrow{*} A(2, 0) \xrightarrow{*} D(0, 1) \xrightarrow{*} A(1, 0)$$

by following this strategy. This is because the only way to reach D from a configuration $A(x, 0)$ with x even is via B.

[1] Recall that Presburger arithmetic [18] is the first-order theory of the natural numbers with addition.

The rest of the paper is devoted to the proof that the reachability relation of a TRVASS is *effectively Presburger-definable*, i.e., there is an algorithm that, given a TRVASS and two states p and q, computes a formula $\varphi(x_1, x_2, y_1, y_2)$ in Presburger arithmetic whose models are precisely the quadruples (x_1, x_2, y_1, y_2) of natural numbers such that $p(x_1, x_2) \xrightarrow{*} q(y_1, y_2)$. It is already known that the reachability relation is effectively Presburger-definable in the absence of zero-tests and resets [17]. Obviously, the counter c_1 is zero after a zero-test $\mathsf{c}_1 == 0$ and, similarly, the counter c_2 is zero after a reset $\mathsf{c}_2 \leftarrow 0$. So we focus on the reachability subrelations between configurations where at least one of the counters is zero, for instance, $\{(x, 0, 0, y) \mid p(x, 0) \xrightarrow{*} q(0, y)\}$. Such a subrelation can be seen as a (binary) relation on $\mathbb{N}$. This motivates our study in Sections 3 and 4 of two classes of relations on $\mathbb{N}$ that naturally stem from the operational semantics of TRVASS.

3 Diagonal Relations

We call a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ *diagonal* when $(x, y) \in R$ implies $(x + c, y + c) \in R$ for every $c \in \mathbb{N}$. For instance, the *identity relation* on $\mathbb{N}$, namely $\{(x, x) \mid x \in \mathbb{N}\}$, is a diagonal relation. The usual order $\leq$ on natural numbers is also a diagonal relation. It is readily seen that the class of diagonal relations is closed under union, intersection, composition, and transitive closure. In this section, we show that the transitive closure of a diagonal Presburger-definable relation is effectively Presburger-definable. Our study of diagonal relations is motivated by the following observation.

▶ Remark. The reachability subrelations $\{(x, y) \mid p(0, x) \xrightarrow{*} q(0, y)\}$, where p and q are states, are diagonal in a TRVASS with no reset. Analogously, the reachability subrelations $\{(x, y) \mid p(x, 0) \xrightarrow{*} q(y, 0)\}$ are diagonal in a TRVASS with no zero-test.

▶ **Example 2.** Let us consider the diagonal relation $R \subseteq \mathbb{N} \times \mathbb{N}$ defined by $(x, y) \in R$ if, and only if, the Presburger formula $x \leq y \wedge y \leq 2x$ holds. It is routinely checked that the transitive closure R^+ of R satisfies $(x, y) \in R^+$ if, and only if, the Presburger formula $(x = 0 \Leftrightarrow y = 0) \wedge x \leq y$ holds.

We fix, for the remainder of this section, a diagonal relation $R \subseteq \mathbb{N} \times \mathbb{N}$. Consider the subsets I_R and D_R of $\mathbb{N}$ defined by

$$I_R \stackrel{\text{def}}{=} \{x \mid \exists y : (x, y) \in R \wedge x < y\} \qquad D_R \stackrel{\text{def}}{=} \{y \mid \exists x : (x, y) \in R \wedge x > y\}$$

Since R is diagonal, the sets I_R and D_R are upward-closed, meaning that $x \in I_R$ implies $x' \in I_R$ for every $x' \geq x$ (and similarly for D_R). If $x \in I_R$ then $(x, x + \delta) \in R$ for some positive integer $\delta > 0$. Since R is diagonal, $(x', x' + \delta) \in R$ for every $x' \geq x$. So the pair $(x, x + \delta)$ can be viewed as an "increasing loop" that applies to every $x' \geq x$. Similarly, if $y \in D_R$ then there is a "decreasing loop" $(y + \delta, y) \in R$ that applies to every $y' \geq y$. We are mostly interested in increasing and decreasing loops that apply to every element of I_R and D_R, respectively. This leads us to the following definitions:

$$\alpha \stackrel{\text{def}}{=} \begin{cases} \min\{\delta > 0 \mid \forall x \in I_R : (x, x + \delta) \in R\} & \text{if } I_R \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\beta \stackrel{\text{def}}{=} \begin{cases} \min\{\delta > 0 \mid \forall y \in D_R : (y + \delta, y) \in R\} & \text{if } D_R \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Let us explain why the natural numbers α and β are well-defined. If $I_R \neq \emptyset$ then there exists $\delta > 0$ such that $(m, m + \delta) \in R$ where $m = \min I_R$. It follows from diagonality of R that $(x, x+\delta) \in R$ for every $x \geq m$, hence, for every $x \in I_R$. Therefore the set $\{\delta > 0 \mid \forall x \in I_R : (x, x+\delta) \in R\}$ is non-empty, and so it has a minimum. A similar argument shows that $\{\delta > 0 \mid \forall y \in D_R : (y+\delta, y) \in R\}$ is non-empty when $D_R \neq \emptyset$.

We are now almost ready to provide a characterization of the transitive closure of R^+. To do so, we introduce the relations $\mathtt{Inc}_R$ and $\mathtt{Dec}_R$ on $\mathbb{N}$ defined by

$$\mathtt{Inc}_R(x, y) \stackrel{\text{def}}{=} (x = y) \vee (x \in I_R \wedge \exists h \in \mathbb{N} : y = x + h\alpha)$$
$$\mathtt{Dec}_R(x, y) \stackrel{\text{def}}{=} (x = y) \vee (y \in D_R \wedge \exists k \in \mathbb{N} : x = y + k\beta)$$

We let $\fatsemi$ denote *relational composition* ($S \fatsemi R \stackrel{\text{def}}{=} \{(x, z) \mid \exists y : x \, S \, y \, R \, z\}$). The *powers* of a relation R are inductively defined by $R^1 \stackrel{\text{def}}{=} R$ and $R^{n+1} \stackrel{\text{def}}{=} R \fatsemi R^n$.

▶ **Lemma 3.** *It holds that* $R^+ = \mathtt{Inc}_R \fatsemi (R \cup \cdots \cup R^{\alpha+\beta+1}) \fatsemi \mathtt{Dec}_R$.

Proof. We introduce the relation $C = \mathtt{Inc}_R \fatsemi (R \cup \cdots \cup R^{\alpha+\beta+1}) \fatsemi \mathtt{Dec}_R$, so as to reduce clutter. To prove that $C \subseteq R^+$, we show that $\mathtt{Inc}_R$ and $\mathtt{Dec}_R$ are both contained in R^*. Let $(x, y) \in \mathtt{Inc}_R$. If $x = y$ then $(x, y) \in R^*$. Otherwise, $x \in I_R$ and there exists $h \in \mathbb{N}$ such that $y = x + h\alpha$. Moreover, h and α are positive as $x \neq y$. It follows from $x \in I_R$ and $\alpha > 0$ that $(x, x+\alpha) \in R$. Since R is diagonal, we derive that $(x, x+\alpha), \ldots, (x+(h-1)\alpha, x+h\alpha)$ are all in R. Hence, $(x, y) \in R^+$. We have shown that $\mathtt{Inc}_R \subseteq R^*$. Now let $(x, y) \in \mathtt{Dec}_R$. If $x = y$ then $(x, y) \in R^*$. Otherwise, $y \in D_R$ and there exists $k \in \mathbb{N}$ such that $x = y + k\beta$. Moreover, k and β are positive as $x \neq y$. It follows from $y \in D_R$ and $\beta > 0$ that $(y+\beta, y) \in R$. Since R is diagonal, we derive that $(y+k\beta, y+(k-1)\beta), \ldots, (y+\beta, y)$ are all in R. Hence, $(x, y) \in R^+$. We have shown that $\mathtt{Dec}_R \subseteq R^*$. We derive from $\mathtt{Inc}_R \subseteq R^*$ and $\mathtt{Dec}_R \subseteq R^*$ that $C \subseteq R^+$.

Let us now prove the converse inclusion $R^+ \subseteq C$. We first observe that $\mathtt{Inc}_R = \mathtt{Inc}_R^*$ and $\mathtt{Dec}_R = \mathtt{Dec}_R^*$. These equalities easily follow from the definitions of $\mathtt{Inc}_R$ and $\mathtt{Dec}_R$. As a consequence, we get that

$$C \;=\; \mathtt{Inc}_R^* \fatsemi (R \cup \cdots \cup R^{\alpha+\beta+1}) \fatsemi \mathtt{Dec}_R^* \tag{3}$$

Let us prove by induction on n that $R^n \subseteq C$ for all $n \geq 1$. The base cases $n = 1, \ldots, \alpha+\beta+1$ are trivial. Assume that $R^m \subseteq C$ for all $1 \leq m < n$, where $n \geq \alpha+\beta+2$, and let us show that this inclusion also holds for $m = n$. Let $(x, y) \in R^n$. There exists $x_0, \ldots, x_n$ such that $x = x_0 \, R \, x_1 \, R \cdots R \, x_n = y$. We start by showing the two following properties, as they will be crucial for the rest of the proof.

$$x \notin I_R \implies x_0 \geq x_1 \geq \cdots \geq x_n \qquad \text{and} \qquad y \notin D_R \implies x_0 \leq x_1 \leq \cdots \leq x_n$$

We prove these properties by contraposition. If $x_i < x_{i+1}$ for some $0 \leq i < n$, then we may, w.l.o.g., choose the first such i. This entails that $x_0 \geq \cdots \geq x_i$. Moreover, $x_i \in I_R$ since $x_i < x_{i+1}$ and $x_i \, R \, x_{i+1}$. It follows that $x = x_0 \in I_R$ as I_R is upward-closed. Similarly, if $x_{i-1} > x_i$ for some $0 < i \leq n$, then we may, w.l.o.g., choose the last such i. This entails that $x_i \leq \cdots \leq x_n$. Moreover, $x_i \in D_R$ since $x_{i-1} > x_i$ and $x_{i-1} \, R \, x_i$. It follows that $y = x_n \in D_R$ as D_R is upward-closed.

To prove that $(x, y) \in C$, we consider four cases, depending on the membership of x in I_R and on the membership of y in D_R.

If $x \notin I_R$ and $y \notin D_R$ then $x_0 = x_1 = \cdots = x_n$. This means in particular that $x_0 \, R \, x_n$, hence, $x = x_0 \, C \, x_n = y$.

If $x \notin I_R$ and $y \in D_R$ then $x_0 \geq x_1 \geq \cdots \geq x_n$. Note that $\beta > 0$ since D_R is non-empty. Since $n \geq \beta$, there exists $0 \leq i < j \leq n$ such that $x_i \equiv x_j \pmod{\beta}$, hence, $x_i = x_j + k\beta$ for some $k \in \mathbb{N}$. Recall that $x = x_0 \, R^i \, x_i$ and $x_j \, R^{n-j} \, x_n = y$. As R is diagonal, we derive that $x_i \, R^{n-j} \, y'$ where $y' = y + k\beta$. We obtain that $x \, R^{n+i-j} \, y'$. It follows from the induction hypothesis that $x \, C \, y'$. Moreover, we have $(y', y) \in \mathtt{Dec}_R$ since $y \in D_R$ and $y' = y + k\beta$. Hence, $x \, (C \fatsemi \mathtt{Dec}_R) \, y$ and we derive from Equation 3 that $x \, C \, y$.

If $x \in I_R$ and $y \notin D_R$ then $x_0 \leq x_1 \leq \cdots \leq x_n$. Note that $\alpha > 0$ since I_R is non-empty. Since $n \geq \alpha$, there exists $0 \leq i < j \leq n$ such that $x_i \equiv x_j \pmod{\alpha}$, hence, $x_j = x_i + h\alpha$ for some $h \in \mathbb{N}$. Recall that $x = x_0 \, R^i \, x_i$ and $x_j \, R^{n-j} \, x_n = y$. As R is diagonal, we derive that $x' \, R^i \, x_j$ where $x' = x + h\alpha$. We obtain that $x' \, R^{n+i-j} \, y$. It follows from the induction hypothesis that $x' \, C \, y$. Moreover, we have $(x, x') \in \mathtt{Inc}_R$ since $x \in I_R$ and $x' = x + h\alpha$. Hence, $x \, (\mathtt{Inc}_R \fatsemi C) \, y$ and we derive from Equation 3 that $x \, C \, y$.

If $x \in I_R$ and $y \in D_R$ then both α and β are positive. Since $n \geq \alpha$, there exists $0 \leq i < j \leq n$ such that $x_i \equiv x_j \pmod{\alpha}$. If $x_i \leq x_j$ then $x_j = x_i + h\alpha$ for some $h \in \mathbb{N}$ and we may proceed as in the case $x \in I_R \wedge y \notin D_R$ to show that $x \, C \, y$. Otherwise, $x_i = x_j + k\alpha$ for some $k \in \mathbb{N}$. Recall that $x = x_0 \, R^i \, x_i$ and $x_j \, R^{n-j} \, x_n = y$. As R is diagonal, we derive that $x' \, R^i \, z' \, R^{n-j} \, y'$ where $x' = x + k\alpha(\beta - 1)$, $z' = x_i + k\alpha(\beta - 1) = x_j + k\alpha\beta$ and $y' = y + k\alpha\beta$. We obtain that $x' \, R^{n+i-j} \, y'$. It follows from the induction hypothesis that $x' \, C \, y'$. Moreover, we have $(x, x') \in \mathtt{Inc}_R$ since $x \in I_R$ and $x' = x + k\alpha(\beta - 1)$, and we also have $(y', y) \in \mathtt{Dec}_R$ since $y \in D_R$ and $y' = y + k\alpha\beta$. Hence, $x \, (\mathtt{Inc}_R \fatsemi C \fatsemi \mathtt{Dec}_R) \, y$ and we derive from Equation 3 that $x \, C \, y$. ◀

We derive the following theorem.

▶ **Theorem 4.** *The transitive closure of a diagonal Presburger-definable relation is effectively Presburger-definable.*

Proof. Assume that $\varphi_R(x, y)$ is a Presburger formula denoting a diagonal relation R. The sets I_R and D_R are defined by the Presburger formulas $\exists y : \varphi_R(x, y) \wedge x < y$ and $\exists x : \varphi_R(x, y) \wedge x > y$, respectively. The natural numbers α and β defined in Equations 1 and 2 are obviously computable from φ_R. So the characterization given in Lemma 3 immediately provides a computable Presburger formula denoting R^+. ◀

4 Horizontal Relations

A relation $R \subseteq \mathbb{N} \times \mathbb{N}$ is said to be *horizontal* if $(x, y) \in R$ implies $(x + c, y) \in R$ for every $c \in \mathbb{N}$. The class of horizontal relations is clearly stable by union, intersection, composition, and transitive closure. In this section we prove that the transitive closure of a horizontal Presburger-definable relation is effectively Presburger-definable. Our study of horizontal relations is motivated by the following observation.

▶ Remark. The reachability subrelations $\{(x, y) \mid p(0, x) \xrightarrow{*} \xrightarrow{\mathsf{c}_2 \leftarrow 0} \xrightarrow{*} q(y, 0)\}$, where p and q are states, are horizontal in a TRVASS.

▶ **Example 5.** Let us consider the following horizontal relation R:

$$R \stackrel{\text{def}}{=} \{(x, y) \mid 2y \leq x \ \vee \ (y \in 4\mathbb{N} \wedge y \leq 2x + 2)\}$$

We prove that R^+ is equal to $C \stackrel{\text{def}}{=} \{(x, y) \mid x = 0 \Rightarrow y = 0\}$ as follows. Since $R \subseteq C$ and C is transitive, we get $R^+ \subseteq C$. Conversely, let $(x, y) \in C$. If $x = 0$ then $y = 0$ and from $(0, 0) \in R$ we derive $(x, y) \in R^+$. So, we can assume that $x \geq 1$. In that case $(x, 4) \in R$ and

$(4z, 4(z+1)) \in R$ for every $z > 0$. It follows that $(x, n) \in R^+$ for every $n \in 4+4\mathbb{N}$. Moreover, there exists such an n satisfying $2y \leq n$. For such an n, we have $(x, n) \in R^+$ and $(n, y) \in R$. We deduce that $(x, y) \in R^+$. It follows that $R^+ = C$.

The effective Presburger-definability of the transitive closure comes from the following characterization.

▶ **Lemma 6.** *For every horizontal relation R we have:*

$$R^+ \;=\; \{(x,y) \mid \exists z : (z,y) \in R \wedge \forall u : x \leq u < z \Rightarrow \exists v : (u,v) \in R \wedge u < v\} \tag{4}$$

Proof. Assume first that $(x, y) \in R^+$. There exists a sequence $x_0, \ldots x_k$ such that $x = x_0 \, R \, x_1 \ldots R \, x_k = y$ with $k \geq 1$. Let $z = x_{k-1}$ and let us prove that for every $u \in \{x, \ldots, z-1\}$ there exists $v > u$ such that $(u, v) \in R$. If $z \leq x$ we are done. So we can assume that $z > x$. Since $x_0 \leq u$, there exists a maximal $j \in \{1, \ldots, k\}$ such that $x_{j-1} \leq u$. Let $v = x_j$ and observe that $(u, v) \in R$. Since $x_{k-1} = z > u$, it follows that $j < k$ and by maximality of j we deduce that $x_j > u$. Therefore $v > u$.

Conversely, let us consider $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that there exists z satisfying $(z, y) \in R$ and such that for every $u \in \{x, \ldots, z-1\}$ there exists $v > u$ such that $(u, v) \in R$. Notice that there exists a sequence $x_0 < \cdots < x_k$ with $k \geq 0$ such that $x = x_0 \, R \, x_1 \ldots R \, x_k \geq z$. It follows that $(x, x_k) \in R^*$. Moreover, since $(z, y) \in R$, $z \leq x_k$, and R is horizontal we deduce that $(x_k, y) \in R$. It follows that $(x, y) \in R^+$. ◀

The previous lemma shows that the transitive closure of a horizontal relation R denoted by a Presburger formula φ_R is denoted by the Presburger formula obtained from (4) by replacing $(z, y) \in R$ and $(u, v) \in R$ by $\varphi_R(z, y)$ and $\varphi_R(u, v)$ respectively. We have proved the following theorem.

▶ **Theorem 7.** *The transitive closure of a horizontal Presburger-definable relation is effectively Presburger-definable.*

5 Presburger Automata

We exhibit in this section a general class of one counter automata with effectively Presburger-definable reachability relations. These automata will be used in the next section to compute the reachability relations of TRVASS.

A *Presburger automaton* is a pair $\mathcal{P} = (Q, \Delta)$ where Q is a finite set of *states*, and Δ is a finite set of *transitions* (p, R, q) where $p, q \in Q$ and $R \subseteq \mathbb{N} \times \mathbb{N}$ is a relation denoted by a Presburger formula (which is left implicit). A *configuration* is a pair $(q, x) \in Q \times \mathbb{N}$, also written as $q(x)$ in the sequel. The *one-step* relation $\rightarrow_{\mathcal{P}}$ is the binary relation over configurations defined by $p(x) \rightarrow_{\mathcal{P}} q(y)$ if there exists $(p, R, q) \in \Delta$ such that $(x, y) \in R$. The *reachability relation* $\xrightarrow{*}_{\mathcal{P}}$ is defined as the reflexive-transitive closure of $\rightarrow_{\mathcal{P}}$.

▶ Remark. The reflexive-transitive closure R^* of a Presburger-definable relation $R \subseteq \mathbb{N} \times \mathbb{N}$ need not be Presburger-definable, in general. For instance, if $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = 2x\}$ then R^* is the relation $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \exists k \in \mathbb{N} : y = 2^k x\}$, which is not definable in Presburger arithmetic. Worse, a simple reduction from the halting problem for Minsky machines shows that membership of a pair (x, y) in R^* is undecidable (where R is a Presburger-definable relation given as input along with x and y).

A consequence of the above remark is that the reachability problem for Presburger automata is undecidable, even if we restrict ourselves to Presburger automata with a single state and a single transition. This comes from the fact that transitions can use arbitrary Presburger-definable relations. We will exhibit a subclass of Presburger automata with effectively Presburger-definable reachability relations (hence, with a decidable reachability problem) by limiting the expressive power of the transitions occurring on cycles. We say that a transition (p, R, q) is *diagonal* if R is diagonal, *horizontal* if R is horizontal, and *ordinary* if it is neither diagonal nor horizontal. Note that a relation on $\mathbb{N}$ may be both diagonal and horizontal, for instance $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y \leq 2x\}$. A *cycle* is non-empty sequence of transitions $(p_1, R_1, q_1), \ldots, (p_n, R_n, q_n)$ such that $q_n = p_1$ and $q_i = p_{i+1}$ for all $1 \leq i < n$.

▶ **Lemma 8.** *Let $\mathcal{P}$ be a Presburger automaton. If every cycle of $\mathcal{P}$ contains only diagonal transitions then $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable.*

Proof. We first observe that $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable when $\mathcal{P} = (Q, \Delta)$ is a Presburger automaton whose transitions are all diagonal. Indeed, we may view $\mathcal{P}$ as a finite-state automaton over the finite alphabet $\{R \mid (p, R, q) \in \Delta\}$. For every states p and q, we may compute a regular expression denoting the language accepted by $\mathcal{P}$ with initial state p and final state q. The obvious evaluation of this regular expression (concatenation $\cdot$ becomes relational composition $\fatsemi$, sum $+$ becomes union $\cup$, and star $\star$ becomes reflexive-transitive closure $*$) yields the relation $\{(x, y) \mid p(x) \xrightarrow{*}_{\mathcal{P}} q(y)\}$. This evaluation is computable because Presburger-definable diagonal relations are effectively closed under union, composition and reflexive-transitive closure (as an immediate consequence of Theorem 4). We have shown that $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable when all transitions of $\mathcal{P}$ are diagonal.

We now prove the lemma. Let $\mathcal{P} = (Q, \Delta)$ be a Presburger automaton such that every cycle of $\mathcal{P}$ contains only diagonal transitions. Let $\mathcal{N}$ be the Presburger automaton obtained from $\mathcal{P}$ by keeping only diagonal transitions. Consider two configurations $p(x)$ and $q(y)$. It is readily seen that $p(x) \xrightarrow{*}_{\mathcal{P}} q(y)$ if, and only if, there exists $1 \leq k \leq |Q|$, $s_1, \ldots, s_k \in Q$ and $x_1, y_1, \ldots, x_k, y_k \in \mathbb{N}$ such that $p(x) = s_1(x_1)$, $s_k(y_k) = q(y)$ and

$$s_1(x_1) \xrightarrow{*}_{\mathcal{P}} s_1(y_1) \rightarrow_{\mathcal{P}} s_2(x_2) \xrightarrow{*}_{\mathcal{P}} s_2(y_2) \cdots s_{k-1}(y_{k-1}) \rightarrow_{\mathcal{P}} s_k(x_k) \xrightarrow{*}_{\mathcal{P}} s_k(y_k)$$

Observe that for every state $s \in Q$ and for every $x, y \in \mathbb{N}$, $s(x) \xrightarrow{*}_{\mathcal{P}} s(y)$ if, and only if, $s(x) \xrightarrow{*}_{\mathcal{N}} s(y)$. Moreover, $\xrightarrow{*}_{\mathcal{N}}$ is effectively Presburger-definable since all transitions of $\mathcal{N}$ are diagonal. We derive from the above characterization of $\xrightarrow{*}_{\mathcal{P}}$ that $\xrightarrow{*}_{\mathcal{P}}$ is also effectively Presburger-definable. ◀

We say that a Presburger automaton $\mathcal{P}$ is *shallow* if every cycle that contains an ordinary transition also contains a horizontal transition. Shallowness of Presburger automata is decidable. This follows from two easy observations. Firstly, diagonality and horizontality of Presburger-definable relations on $\mathbb{N}$ are decidable, since these properties can be expressed in Presburger arithmetic. Secondly, a Presburger automaton is shallow if, and only if, every *simple* cycle containing an ordinary transition also contains a horizontal transition. We now show the main result of this section.

▶ **Theorem 9.** *The reachability relation of a shallow Presburger automaton is effectively Presburger-definable.*

Proof. By induction on the number of horizontal transitions. The base case follows from Lemma 8. Indeed, if $\mathcal{P}$ is a shallow Presburger automaton with no horizontal transition then every cycle of $\mathcal{P}$ contains only diagonal transitions. Assume that the theorem holds for every shallow Presburger automaton with n horizontal transitions, where $n \in \mathbb{N}$. Let $\mathcal{P} = (Q, \Delta)$ be a Presburger automaton with $n+1$ horizontal transitions. Pick a horizontal transition $(p, R, q) \in \Delta$ and let $\mathcal{N}$ be the Presburger automaton obtained from $\mathcal{P}$ by removing the transition (p, R, q). Let S denote the reachability relation from q to p in $\mathcal{N}$, namely the relation $S = \{(y, x) \mid q(y) \xrightarrow{*}_{\mathcal{N}} p(x)\}$. It is readily seen that, for every configurations $s(x)$ and $t(y)$ of $\mathcal{P}$, $s(x) \xrightarrow{*}_{\mathcal{P}} t(y)$ if, and only if, $s(x) \xrightarrow{*}_{\mathcal{N}} t(y)$ or there exists $x', y' \in \mathbb{N}$ such that

$$s(x) \xrightarrow{*}_{\mathcal{N}} p(x') \quad \wedge \quad (x', y') \in ((R \mathbin{;} S)^* \mathbin{;} R) \quad \wedge \quad q(y') \xrightarrow{*}_{\mathcal{N}} t(y)$$

By induction hypothesis, the relation $\xrightarrow{*}_{\mathcal{N}}$ is effectively Presburger-definable, and so is $R \mathbin{;} S$. Moreover, $R \mathbin{;} S$ is horizontal since R is horizontal. It follows from Theorem 7 that $(R \mathbin{;} S)^*$ is effectively Presburger-definable. We derive from the above characterization of $\xrightarrow{*}_{\mathcal{P}}$ that $\xrightarrow{*}_{\mathcal{P}}$ is also effectively Presburger-definable. ◀

▶ Remark. The notions of diagonal relations, horizontal relations and Presburger automata are extended to larger dimensions in the obvious way. A relation $R \subseteq \mathbb{N}^d \times \mathbb{N}^d$ is *diagonal* (resp. *horizontal*) if $(\mathbf{x}, \mathbf{y}) \in R$ implies $(\mathbf{x} + \mathbf{c}, \mathbf{y} + \mathbf{c}) \in R$ (resp. $(\mathbf{x} + \mathbf{c}, \mathbf{y}) \in R$) for every $\mathbf{c} \in \mathbb{N}^d$. But Theorem 9 does not extend to larger dimensions, even if we restrict ourselves to Presburger automata with a single state and a single transition. In fact, the reflexive-transitive closure of a Presburger-definable relation that is diagonal (resp. horizontal) need not be Presburger-definable. Consider the relation $R \subseteq \mathbb{N}^2 \times \mathbb{N}^2$ defined by $(x_1, x_2)\, R\, (y_1, y_2)$ if, and only if, the Presburger formula $y_1 \leq 2x_1 \wedge y_2 < x_2$ holds. The relation R is both diagonal and horizontal. It is routinely checked that the reflexive-transitive closure R^* is the set of pairs $((x_1, x_2), (y_1, y_2)) \in \mathbb{N}^2 \times \mathbb{N}^2$ such that $y_1 \leq 2^{x_2 - y_2} x_1$ and $y_2 \leq x_2$, which is not definable in Presburger arithmetic.

6 Reachability Relations of TRVASS

A TRVASS is a 2-dimensional vector addition system with states (2-dim VASS) such that the first counter can be tested for zero and the second one can be reset to zero. Formally, a *TRVASS* is a triple $\mathcal{V} = (Q, \Sigma, \Delta)$ where Q is a finite set of *states*, $\Sigma \subseteq \mathbb{Z}^2 \cup \{T, R\}$ is a finite set of *actions*, and $\Delta \subseteq Q \times \Sigma \times Q$ is a finite set of *transitions*. A *configuration* of $\mathcal{V}$ is a triple $(q, x_1, x_2) \in Q \times \mathbb{N} \times \mathbb{N}$ written as $q(x_1, x_2)$ in the sequel. The operational semantics of $\mathcal{V}$ is given by the binary relations $\xrightarrow{a}_{\mathcal{V}}$ over configurations, with $a \in \Sigma$, defined by $p(x_1, x_2) \xrightarrow{a}_{\mathcal{V}} q(y_1, y_2)$ if $(p, a, q) \in \Delta$ and

$$\begin{array}{lcll}
(y_1, y_2) & = & (x_1 + a_1, x_2 + a_2) & \text{if } a = (a_1, a_2) \in \mathbb{Z}^2 \\
(y_1, y_2) & = & (0, x_2) \wedge x_1 = 0 & \text{if } a = T \\
(y_1, y_2) & = & (x_1, 0) & \text{if } a = R
\end{array}$$

Given a word $w = a_1 \ldots a_k$ of actions $a_j \in \Sigma$, we denote by $\xrightarrow{w}_{\mathcal{V}}$ the binary relation over configurations defined as the relational composition $\xrightarrow{a_1}_{\mathcal{V}} \mathbin{;} \cdots \mathbin{;} \xrightarrow{a_k}_{\mathcal{V}}$. The relation $\xrightarrow{\varepsilon}_{\mathcal{V}}$ denotes the identity relation on configurations. Given a subset $W \subseteq \Sigma^*$, we let $\xrightarrow{W}_{\mathcal{V}}$ denote the union $\bigcup_{w \in W} \xrightarrow{w}_{\mathcal{V}}$. The relation $\xrightarrow{\Sigma^*}_{\mathcal{V}}$, also written $\xrightarrow{*}_{\mathcal{V}}$, is called the *reachability relation* of $\mathcal{V}$. Observe that $\xrightarrow{*}_{\mathcal{V}}$ is the reflexive-transitive closure of the *step* relation $\rightarrow_{\mathcal{V}} \stackrel{\text{def}}{=} \bigcup_{a \in \Sigma} \xrightarrow{a}_{\mathcal{V}}$.

The remainder of this section is devoted to the proof that TRVASS have effectively Presburger-definable reachability relations. Let us fix a TRVASS $\mathcal{V} = (Q, \Sigma, \Delta)$. We let A denote the set $\Sigma \cap \mathbb{Z}^2$ of *addition vectors*.

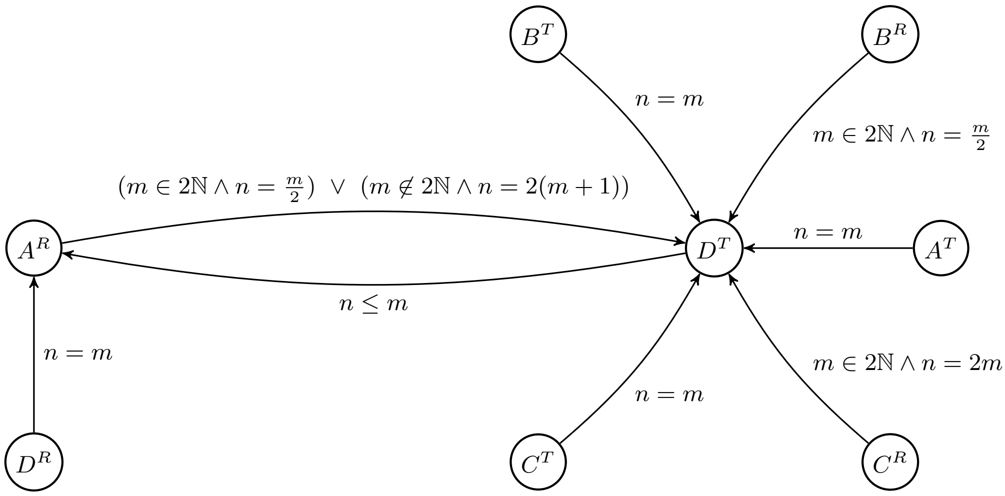

Figure 2 The Presburger automaton $\mathcal{P}$ associated to the TRVASS of Figure 1.

The reachability relation of $\mathcal{V}$ can be expressed in terms of the reachability relation of a Presburger automaton by observing that configurations reachable just after a zero-test T or a reset R are restricted to $q(0, n)$ or $q(n, 0)$, respectively, where $q \in Q$ and $n \in \mathbb{N}$. Those configurations are parametrized by introducing the set $S = \{q^T, q^R \mid q \in Q\}$ obtained as two disjoint copies of Q. Elements in $\{q^T \mid q \in Q\}$ are called *test states*, and those in $\{q^R \mid q \in Q\}$ are called *reset states*. Given $s \in S$ and $n \in \mathbb{N}$, we introduce the configuration $[\![s, n]\!]$ in $Q \times \mathbb{N}^2$ defined as follows:

$$[\![s, n]\!] \stackrel{\text{def}}{=} \begin{cases} q(0, n) & \text{if } s = q^T \\ q(n, 0) & \text{if } s = q^R \end{cases}$$

We also introduce, for each pair $(s, t) \in S \times S$, the binary relation $R_{s,t}$ defined by

$$R_{s,t} \stackrel{\text{def}}{=} \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid [\![s, m]\!] \xrightarrow{A^* X}_{\mathcal{V}} [\![t, n]\!]\}$$

where $X = T$ if t is a test state and $X = R$ if t is a reset state. It is known that the reachability relation of a 2-dim VASS is effectively Presburger-definable [17, 2]. This entails that the relation $\xrightarrow{A^*}_{\mathcal{V}}$ is effectively Presburger-definable, and it follows that the relations $R_{s,t}$ are also effectively Presburger-definable. We introduce the Presburger automaton $\mathcal{P}$ with set of states S and set of transitions $\{(s, R_{s,t}, t) \mid (s, t) \in S \times S\}$. Note that $\mathcal{P}$ is computable from $\mathcal{V}$.

► **Example 10.** Let us come back to the TRVASS of Figure 1. The relations $R_{s,t}$ are all empty except for R_{D^T,A^R}, R_{D^R,A^R} and R_{s,D^T} with $s \in \{A^T, A^R, B^T, B^R, C^T, C^R\}$. The corresponding automaton $\mathcal{P}$ is depicted in Figure 2. Each transition $(s, R_{s,t}, t)$ is depicted by an edge from s to t labeled by a Presburger formula $\varphi_{s,t}(m, n)$ denoting the relation $R_{s,t}$. The empty relations (which are both diagonal and horizontal) are not depicted. Notice that the transition from A^R to D^T is ordinary and the one from D^T to A^R is horizontal. It follows that $\mathcal{P}$ is shallow. We observe that the horizontal relation R defined as the composition $R_{D^T,A^R} \mathbin{⨾} R_{A^R,D^T}$ is the one introduced in Example 5.

We first show that the Presburger automaton $\mathcal{P}$ is shallow. By Theorem 9, this will entail that its reachability relation $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable.

▶ **Lemma 11.** *The Presburger automaton $\mathcal{P}$ is shallow.*

Proof. It is readily seen that $\mathcal{P}$ satisfies the following properties:

- Transitions from reset states to reset states are diagonal,
- Transitions from test states to reset states are horizontal,
- Transitions from test states to test states are diagonal.

It follows that an ordinary transition of $\mathcal{P}$ is a transition from a reset state to a test state. If a cycle contains such a transition then it must contain a transition from a test state to a reset state as well. Since such a transition is horizontal, we obtain that $\mathcal{P}$ is shallow. ◀

The two following lemmas show how to decompose the reachability relation of $\mathcal{V}$ in terms of the reachability relation of $\mathcal{P}$.

▶ **Lemma 12.** *For every $s, t \in S$ and $m, n \in \mathbb{N}$, if $s(m) \xrightarrow{*}_{\mathcal{P}} t(n)$ then $[\![s, m]\!] \xrightarrow{*}_{\mathcal{V}} [\![t, n]\!]$.*

Proof. It is easily seen that $s(m) \rightarrow_{\mathcal{P}} t(n)$ implies $[\![s, m]\!] \xrightarrow{*}_{\mathcal{V}} [\![t, n]\!]$, for every $s, t \in S$ and $m, n \in \mathbb{N}$. We derive, by an immediate induction on $k \geq 1$, that $s(m)\ (\rightarrow_{\mathcal{P}})^k\ t(n)$ implies $[\![s, m]\!] \xrightarrow{*}_{\mathcal{V}} [\![t, n]\!]$, for every $s, t \in S$ and $m, n \in \mathbb{N}$. The lemma follows. ◀

▶ **Lemma 13.** *Consider two configurations $p(x_1, x_2)$ and $q(y_1, y_2)$ of $\mathcal{V}$. It holds that $p(x_1, x_2) \xrightarrow{\Sigma^* \backslash A^*}_{\mathcal{V}} q(y_1, y_2)$ if, and only if, there exist $s, t \in S$ and $m, n \in \mathbb{N}$ such that:*

$$p(x_1, x_2) \xrightarrow{A^*\{T,R\}}_{\mathcal{V}} [\![s, m]\!] \quad \wedge \quad s(m) \xrightarrow{*}_{\mathcal{P}} t(n) \quad \wedge \quad [\![t, n]\!] \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$$

Proof. Lemma 12 shows the "if" direction of the equivalence. For the other direction, let $w \in \Sigma^* \backslash A^*$ such that $p(x_1, x_2) \xrightarrow{w}_{\mathcal{V}} q(y_1, y_2)$. By splitting w after each occurrence of an action in $\{T, R\}$, we deduce that $w = w_0 X_1 \ldots w_{k-1} X_k w_k$ where $k \geq 1$, and $w_0, \ldots, w_k \in A^*$. Let us introduce the configurations $c_1, \ldots, c_k$ satisfying the following relations:

$$p(x_1, x_2) \xrightarrow{w_0 X_1}_{\mathcal{V}} c_1 \cdots \xrightarrow{w_{k-1} X_k}_{\mathcal{V}} c_k \xrightarrow{w_k}_{\mathcal{V}} q(y_1, y_2)$$

Notice that $c_j = [\![q_j^{X_j}, n_j]\!]$ for some $q_j \in Q$ and some $n_j \in \mathbb{N}$. By definition of $\mathcal{P}$, we get $q_{j-1}^{X_{j-1}}(n_{j-1}) \rightarrow_{\mathcal{P}} q_j^{X_j}(n_j)$ for every $j \in \{1, \ldots, k\}$. We have proved the lemma. ◀

We deduce our main result.

▶ **Theorem 14.** *The reachability relation of a TRVASS is effectively Presburger-definable.*

Proof. Lemma 13 shows that $p(x_1, x_2) \xrightarrow{*}_{\mathcal{V}} q(y_1, y_2)$ if, and only if, $p(x_1, x_2) \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$ or there exists $s, t \in S$ and $m, n \in \mathbb{N}$ such that:

$$p(x_1, x_2) \xrightarrow{A^*\{T,R\}}_{\mathcal{V}} [\![s, m]\!] \quad \wedge \quad s(m) \xrightarrow{*}_{\mathcal{P}} t(n) \quad \wedge \quad [\![t, n]\!] \xrightarrow{A^*}_{\mathcal{V}} q(y_1, y_2)$$

From [17, 2], the relation $\xrightarrow{A^*}_{\mathcal{V}}$ is effectively Presburger-definable. From Lemma 11 and Theorem 9, the relation $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable as well. ◀

Coming back to the classes of 2-dim extended VASS discussed in the introduction (see Table 1), Theorem 14 means that the reachability relation is effectively Presburger-definable for the "maximal" class $\mathbf{T}_1\mathbf{R}_2$. This result also applies to 2-dim VASS extended with resets and transfers on both counters (i.e., the class $\mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$), since they can be simulated by machines in $\mathbf{T}_1\mathbf{R}_2$.

7 Conclusion and Open Problems

We have shown that the reachability relation of 2-dim VASS extended with tests on the first counter and resets on the second counter, is effectively Presburger-definable. This completes the decidability picture of 2-dim extended VASS initiated in [11]. Our proof techniques may also be used for other classes of counter machines where shallow Presburger automata would naturally appear. Many other problems on extensions of VASS are still interesting to solve.

- The reachability problem is NP-complete [12] for 1-dim VASS, PSPACE-complete [2] for 2-dim VASS, and NL-complete [7] for unary 2-dim VASS. But we do not know what are the complexities for the reachability problem, for the construction of the reachability set and for the reachability relation for all 2-dim extended VASS.
- The boundedness problem is undecidable for 3-dim VASS extended with resets on all counters [5] and it is decidable for arbitrary dimension VASS extended with resets on two counters [6]. Is boundedness decidable for arbitrary dimension TRVASS?

References

1. S. Akshay, Supratik Chakraborty, Ankush Das, Vishal Jagannath, and Sai Sandeep. On Petri Nets with Hierarchical Special Arcs. In *CONCUR*, volume 85 of *LIPIcs*, pages 40:1–40:17. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.
2. Michael Blondin, Alain Finkel, Stefan Göller, Christoph Haase, and Pierre McKenzie. Reachability in Two-Dimensional Vector Addition Systems with States Is PSPACE-Complete. In *LICS*, pages 32–43. IEEE Computer Society, 2015.
3. Rémi Bonnet. The reachability problem for Vector Addition Systems with one zero-test. In Filip Murlak and Piotr Sankowski, editors, *Proceedings of the 36th International Symposium on Mathematical Foundations of Computer Science (MFCS'11)*, volume 6907 of *Lecture Notes in Computer Science*, pages 145–157, Warsaw, Poland, August 2011. Springer.
4. Rémi Bonnet, Alain Finkel, Jérôme Leroux, and Marc Zeitoun. Place-Boundedness for Vector Addition Systems with one zero-test. In Kamal Lodaya and Meena Mahajan, editors, *Proceedings of the 30th Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS'10)*, volume 8 of *Leibniz International Proceedings in Informatics*, pages 192–203, Chennai, India, December 2010. Leibniz-Zentrum für Informatik.
5. Catherine Dufourd, Alain Finkel, and Philippe Schnoebelen. Reset Nets Between Decidability and Undecidability. In *ICALP*, volume 1443 of *Lecture Notes in Computer Science*, pages 103–115. Springer, 1998.
6. Catherine Dufourd, Petr Jancar, and Philippe Schnoebelen. Boundedness of Reset P/T Nets. In *ICALP*, volume 1644 of *Lecture Notes in Computer Science*, pages 301–310. Springer, 1999.
7. Matthias Englert, Ranko Lazic, and Patrick Totzke. Reachability in Two-Dimensional Unary Vector Addition Systems with States is NL-Complete. In *LICS*, pages 477–484. ACM, 2016.
8. Alain Finkel and Arnaud Sangnier. Mixing coverability and reachability to analyze VASS with one zero-test. In David Peleg and Anca Muscholl, editors, *Proceedings of the 36th International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM'10)*, volume 5901 of *Lecture Notes in Computer Science*, pages 394–406, Špindlerův Mlýn, Czech Republic, January 2010. Springer.
9. Alain Finkel and Philippe Schnoebelen. Well-Structured Transition Systems Everywhere! *Theoretical Computer Science*, 256(1-2):63–92, April 2001.

10 Alain Finkel and Grégoire Sutre. An Algorithm Constructing the Semilinear Post* for 2-Dim Reset/Transfer VASS. In *MFCS*, volume 1893 of *Lecture Notes in Computer Science*, pages 353–362. Springer, 2000.

11 Alain Finkel and Grégoire Sutre. Decidability of Reachability Problems for Classes of Two Counters Automata. In *STACS*, volume 1770 of *Lecture Notes in Computer Science*, pages 346–357. Springer, 2000.

12 Christoph Haase, Stephan Kreutzer, Joël Ouaknine, and James Worrell. Reachability in Succinct and Parametric One-Counter Automata. In *CONCUR*, volume 5710 of *Lecture Notes in Computer Science*, pages 369–383. Springer, 2009.

13 John Hopcroft and Jean-Jacques Pansiot. On the reachability problem for 5-dimensional vector addition systems. *Theoretical Computer Science*, 8(2):135–159, 1979.

14 Ranko Lazic. The reachability problem for vector addition systems with a stack is not elementary. *CoRR*, abs/1310.1767, 2013.

15 Ranko Lazic and Patrick Totzke. What Makes Petri Nets Harder to Verify: Stack or Data? In *Concurrency, Security, and Puzzles*, volume 10160 of *Lecture Notes in Computer Science*, pages 144–161. Springer, 2017.

16 Jérôme Leroux, M. Praveen, and Grégoire Sutre. Hyper-Ackermannian bounds for pushdown vector addition systems. In Thomas A. Henzinger and Dale Miller, editors, *Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), CSL-LICS '14, Vienna, Austria, July 14 - 18, 2014*, page 63. ACM, 2014.

17 Jérôme Leroux and Grégoire Sutre. On Flatness for 2-Dimensional Vector Addition Systems with States. In *CONCUR 2004 - Concurrency Theory, 15th International Conference, London, UK, August 31 - September 3, 2004, Proceedings*, volume 3170 of *Lecture Notes in Computer Science*, pages 402–416. Springer, 2004.

18 M. Presburger. Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. *Comptes Rendus du premier congrès de mathématiciens des Pays Slaves, Warszawa*, pages 92–101, 1929.

19 Klaus Reinhardt. Reachability in Petri Nets with Inhibitor Arcs. *Electr. Notes Theor. Comput. Sci.*, 223:239–264, 2008.