On the Complexity of Stable Fractional Hypergraph Matching

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Abstract

In this paper, we consider the complexity of the problem of finding a stable fractional matching in a hypergraphic preference system. Aharoni and Fleiner proved that there exists a stable fractional matching in every hypergraphic preference system. Furthermore, Kintali, Poplawski, Rajaraman, Sundaram, and Teng proved that the problem of finding a stable fractional matching in a hypergraphic preference system is **PPAD**-complete. In this paper, we consider the complexity of the problem of finding a stable fractional matching in a hypergraphic preference system whose maximum degree is bounded by some constant. The proof by Kintali, Poplawski, Rajaraman, Sundaram, and Teng implies the **PPAD**-completeness of the problem of finding a stable fractional matching in a hypergraphic preference system whose maximum degree is 5. In this paper, we prove that (i) this problem is **PPAD**-complete even if the maximum degree is 3, and (ii) if the maximum degree is 2, then this problem can be solved in polynomial time. Furthermore, we prove that the problem of finding an approximate stable fractional matching in a hypergraphic preference system is **PPAD**-complete.

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1 Introduction

The stable matching model introduced by Gale and Shapley [7] is one of the most important mathematical models for matching problems. The classical stable matching model is defined on undirected graphs. Thus, this model is naturally generalized to hypergraphs. It is not difficult to see that there exists an instance of the stable matching problem in hypergraphs that has no stable hypergraph matching. Thus, in this paper, we consider the following relaxation concept called a fractional matching. In the ordinary stable matching problem, the value 0 or 1 is assigned to each edge. On the other hand, in a fractional matching, a real number between 0 and 1 is assigned to each edge. Fortunately, it is known [1] that there exists a stable fractional matching in every hypergraph. The proof of this result in [1] was based on Scarf's Lemma [16]. For example, the concept of stable fractional matchings in hypergraphs is used in [2, 3, 13]. It should be noted that stable fractional matchings

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in hyergraphs are closely related to the stable matching problem with couples [4] that is a practically and theoretically important variant of the stable matching problem (see, e.g., [3, 13]). In this paper, we consider the problem of finding a stable fractional matching in a hypergraph.

For considering the computational complexity of a problem for which every instance is guaranteed to have a solution, Megiddo and Papadimitriou [11] introduced the complexity class **TFNP** that consists of all search problems in **NP** for which every instance is guaranteed to have a solution. The class **PPAD** introduced by Papadimitriou [14] is the class of all search problems such that the above property (i.e., every instance is guaranteed to have a solution) is proved by using a directed parity argument. Some problem A in **PPAD** is said to be ***PPAD**-complete*, if every problem in **PPAD** is reducible to A in polynomial time.[2] The assumption that **PPAD** contains hard problems is considered as a reasonable hypothesis (see e.g., [15, Section 2.4.1]). Thus, it is reasonable to consider that a **PPAD**-complete problem is hard. For example, it is known [5, 6] that the problem of finding a Nash equilibrium [12] is **PPAD**-complete.

Kintali, Poplawski, Rajaraman, Sundaram, and Teng [10] proved that the problem of finding a stable fractional matching in a hypergraphic preference system is **PPAD**-complete. In this paper, we consider the complexity of the problem of finding a stable fractional matching in a hypergraphic preference system whose maximum degree is bounded by some constant. It is natural to consider that in many practical applications, the length of a preference list (i.e., the degree of a vertex) is constant. Thus, it is important to reveal the complexity of this problem with low constant degree. The proof by Kintali, Poplawski, Rajaraman, Sundaram, and Teng [10] implies the **PPAD**-completeness of the problem of finding a stable fractional matching in a hypergraphic preference system whose maximum degree is 5. However, to the best of our knowledge, the complexity of the problem of finding a stable fractional matching in a hypergraphic preference system whose maximum degree is at most 4 is open. In this paper, we prove that (i) this problem is **PPAD**-complete even if the maximum degree is 3, and (ii) if the maximum degree is 2, then this problem can be solved in polynomial time. Furthermore, we prove that the problem of finding an approximate stable fractional matching in a hypergraphic preference system is **PPAD**-complete.

2 Problem Formulation and Main Results

A *hypergraphic preference system* P consists of the following two components. The first component is a finite hypergraph (V, E). The second component is a set of strict total orders $\succ_v$ for vertices v in V such that for each vertex v in V, $\succ_v$ is a strict total order on $E(v)$, where for each vertex v in V, we denote by $E(v)$ the set of hyperedges e in E such that $v \in e$. We denote by $P = (V, E, \{\succ_v\})$ this hypergraphic preference system P. Notice that if $|e| = 2$ for every hyperedge e in E, then P is just an instance of the well-known stable roommate problem (see, e.g., [8]). Define $\deg(P) := \max_{v \in V} |E(v)|$.

Assume that we are given a hypergraphic preference system $P = (V, E, \{\succ_v\})$. Then a vector x in $\mathbb{R}_+^E$ is called a *fractional matching in P*, if $\sum_{e \in E(v)} x(e) \leq 1$ for every vertex v in

[2] A polynomial-time computable function f is called a polynomial-time reduction from a problem B in **PPAD** to a problem A in **PPAD**, if for every instance I_B of B, $f(I_B)$ is an instance of A, and furthermore there exists a polynomial-time computable function g such that for every solution y of $f(I_B)$, $g(y)$ is a solution of I_B. A problem A in **PPAD** is said to be **PPAD**-complete, if for every problem B in **PPAD**, there exists a polynomial-time reduction from B to A.

V, where $\mathbb{R}_+$ is the set of non-negative real numbers. Furthermore, a fractional matching x in $\mathbb{R}_+^E$ is said to be *stable*, if for every hyperedge e in E, there exists a vertex v in e such that

$$x(e) + \sum_{f \in E(v)\colon f \succ_v e} x(f) = 1.$$

It is known [1, Theorem 2.1] that there exists a stable fractional matching in every hypergraphic preference system. The problem called Fractional Hypergraph Matching is defined as follows. In this problem, we are given a hypergraphic preference system P. Then the goal of this problem is to find a stable fractional matching in P. The following result about the computational complexity of Fractional Hypergraph Matching is known.

▶ **Theorem 1** (Kintali, Poplawski, Rajaraman, Sundaram, and Teng [10, Theorem 5.7])**.** Fractional Hypergraph Matching *is **PPAD**-complete.*

The proof by Kintali, Poplawski, Rajaraman, Sundaram, and Teng [10] implies the **PPAD**-completeness of the problem of finding a stable fractional matching in a hypergraphic preference system P such that $\deg(P) = 5$. However, to the best of our knowledge, the complexity of the problem of finding a stable fractional matching in a hypergraphic preference system P such that $2 \leq \deg(P) \leq 4$ is open. (If $\deg(P) = 1$, then the answer of Fractional Hypergraph Matching is trivial.) In this paper, we prove the following theorems.

▶ **Theorem 2.** Fractional Hypergraph Matching *in a hypergraphic preference system P such that* $\deg(P) = 3$ *is **PPAD**-complete.*

It should be noted that Theorem 2 implies the **PPAD**-completeness of Fractional Hypergraph Matching in a hypergraphic preference system P such that $\deg(P) = 4$. (It is sufficient to add a vertex with degree 4 to the instance used in the proof of Theorem 2.)

▶ **Theorem 3.** Fractional Hypergraph Matching *in a hypergraphic preference system P such that* $\deg(P) = 2$ *can be solved in polynomial time.*

Furthermore, we consider Approximate Fractional Hypergraph Matching that is an approximate variant of Fractional Hypergraph Matching. In this problem, we are given a hypergraphic preference system $P = (V, E, \{\succ_v\})$ and a positive rational number ϵ that may depend on $|V|$ and $|E|$. Then a fractional matching x in $\mathbb{R}_+^E$ is said to be *ϵ-stable*, if for every hyperedge e in E, there exists a vertex v in e such that

$$x(e) + \sum_{f \in E(v)\colon f \succ_v e} x(f) \geq 1 - \epsilon.$$

Notice that since a stable fractional matching in P always exists, an ϵ-stable fractional matching in P always exists. The goal of this problem is to find an ϵ-stable fractional matching in P. We prove the following theorem.

▶ **Theorem 4.** Approximate Fractional Hypergraph Matching *is **PPAD**-complete.*

3 Proof of Theorem 2

For proving Theorem 2, we need the following lemma.

▶ **Lemma 5.** *Assume that we are given a hypergraphic preference system P such that* $\deg(P) \geq 4$*. Then there exists a hypergraphic preference system Q such that* (i) $\deg(Q) = 3$ *and* (ii) *we can construct a stable fractional matching in P from a stable fractional matching in Q in polynomial time. Furthermore, we can construct Q in polynomial time.*

Before proving Lemma 5, we prove Theorem 2 by using this lemma.

Proof of Theorem 2. It follows from Theorem 1 that Fractional Hypergraph Matching in a hypergraphic preference system P such that $\deg(P) = 3$ is in **PPAD**. Furthermore, Theorem 1 and Lemma 5 imply that every problem in **PPAD** is reducible to Fractional Hypergraph Matching in a hypergraphic preference system P such that $\deg(P) = 3$ in polynomial time. This completes the proof. ◀

3.1 Proof of Lemma 5

In this subsection, we prove Lemma 5. The following proof is inspired by the proof of the **PPAD**-completeness of Preference Game with degree 3 by Kintali, Poplawski, Rajaraman, Sundaram, and Teng [10].

Assume that we are given a hypergraphic preference system $P = (V, E, \{\succ_v\})$ such that $\deg(P) \geq 4$. Then we construct a new hypergraphic preference system $Q = (W, F, \{\rhd_v\})$ as follows (see Figure 1). Define

$$W := \big\{v_i \mid v \in V, i \in \{1, 2, \dots, |E(v)|\}\big\} \cup \big\{\overline{v}_i \mid v \in V, i \in \{1, 2, \dots, |E(v)| - 1\}\big\}.$$

For each vertex v in V and each hyperedge e in $E(v)$, we define

$$r(v, e) := 1 + |\{f \in E(v) \mid f \succ_v e\}|.$$

For each hyperedge e in E, we define $\overline{e} := \{v_{r(v,e)} \mid v \in e\}$. Define $\overline{E} := \{\overline{e} \mid e \in E\}$ and

$$F := \overline{E} \cup \big\{\{v_i, \overline{v}_i\}, \{\overline{v}_i, v_{i+1}\} \mid v \in V, i \in \{1, 2, \dots, |E(v)| - 1\}\big\}.$$

For each vertex v in V and each integer i in $\{1, 2, \dots, |E(v)|\}$, we denote by h^v_i the hyperedge e in $\overline{E}$ such that $v_i \in e$. For each vertex w in W, we define the strict total order $\rhd_w$ as follows. We first consider the case where $w = v_i$ for some vertex v in V and some integer i in $\{1, 2, \dots, |E(v)|\}$. It suffices to consider the case where $|E(v)| \geq 2$. In this case, we define

$$\begin{cases} h^v_1 \rhd_w \{v_1, \overline{v}_1\} & \text{if } i = 1 \\ \{\overline{v}_{|E(v)|-1}, v_{|E(v)|}\} \rhd_w h^v_{|E(v)|} & \text{if } i = |E(v)| \\ \{\overline{v}_{i-1}, v_i\} \rhd_w h^v_i \rhd_w \{v_i, \overline{v}_i\} & \text{otherwise.} \end{cases}$$

Next we assume that $w = \overline{v}_i$ for some vertex v in V and some integer i in $\{1, 2, \dots, |E(v)| - 1\}$. In this case, we define $\{v_i, \overline{v}_i\} \rhd_w \{\overline{v}_i, v_{i+1}\}$. Since $|W| \leq 2|V||E|$ and $|F| \leq |E| + 2|V||E|$, Q can be constructed in polynomial time. Furthermore, $\deg(Q) = 3$.

In what follows, we prove that we can construct a stable fractional matching in P from a stable fractional matching in Q in polynomial time. Assume that we are given a stable fractional matching z in Q. Then we define the vector x in $\mathbb{R}^E_+$ by $x(e) := z(\overline{e})$. Clearly, we can construct x from z in polynomial time. What remains is to prove that x is a stable fractional matching in P. For proving this, we need the following lemma.

▶ **Lemma 6.** *For every vertex v in V and every integer i in $\{1, 2, \dots, |E(v)| - 1\}$,*

(T1) $z(\{v_i, \overline{v}_i\}) = 1 - \sum_{j=1}^{i} z(h^v_j)$, *and*

(T2) $z(\{\overline{v}_i, v_{i+1}\}) = \sum_{j=1}^{i} z(h^v_j)$.

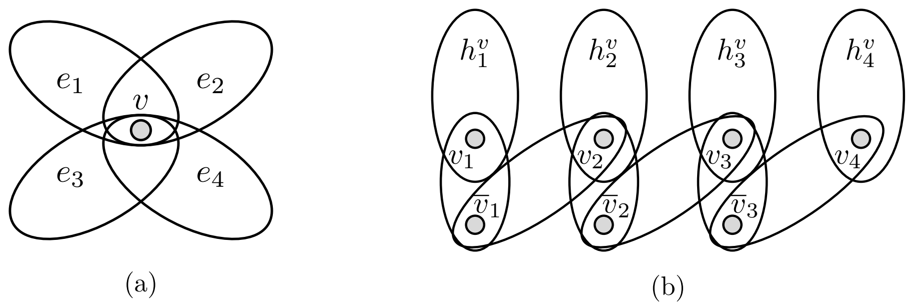

Figure 1 (a) A vertex v and the hyperedges containing v. We assume that $e_1 \succ_v e_2 \succ_v e_3 \succ_v e_4$. (b) The copies of v in Q and the hyperedges containing these copies. For every integer i in $\{1,2,3,4\}$, we have $h_i^v = \overline{e_i}$.

Proof. Let v be a vertex in V such that $|E(v)| \geq 2$. We prove by induction on i.

We first consider the case of $i = 1$. Since z is a fractional matching in Q, we have

$$1 \geq \sum_{e \in F(v_1)} z(e) = z(h_1^v) + z(\{v_1, \overline{v}_1\}).$$

This implies that $z(\{v_1, \overline{v}_1\}) \leq 1 - z(h_1^v)$. For proving (T1) by contradiction, we assume that $z(\{v_1, \overline{v}_1\}) < 1 - z(h_1^v)$. Since z is a stable fractional matching in Q, at least one of the following statements holds.

$$1 = z(\{v_1, \overline{v}_1\}) + \sum_{e \in F(v_1)\colon\, e \rhd_{v_1} \{v_1, \overline{v}_1\}} z(e) = z(\{v_1, \overline{v}_1\}) + z(h_1^v). \tag{1}$$

$$1 = z(\{v_1, \overline{v}_1\}) + \sum_{e \in F(\overline{v}_1)\colon\, e \rhd_{\overline{v}_1} \{v_1, \overline{v}_1\}} z(e) = z(\{v_1, \overline{v}_1\}). \tag{2}$$

However, since $z(h_1^v) \geq 0$, the above assumption implies that $z(\{v_1, \overline{v}_1\}) + z(h_1^v) < 1$ and $z(\{v_1, \overline{v}_1\}) < 1$. These observations contradict (1) and (2). Thus, $z(\{v_1, \overline{v}_1\}) = 1 - z(h_1^v)$.

Next we consider (T2). Since z is a fractional matching in Q, we have

$$1 \geq \sum_{e \in F(\overline{v}_1)} z(e) = z(\{v_1, \overline{v}_1\}) + z(\{\overline{v}_1, v_2\}).$$

Since (T1) for the case of $i = 1$ implies that $z(\{v_1, \overline{v}_1\}) = 1 - z(h_1^v)$, we have $z(\{\overline{v}_1, v_2\}) \leq z(h_1^v)$. For proving (T2) by contradiction, we assume that $z(\{\overline{v}_1, v_2\}) < z(h_1^v)$. Since z is a stable fractional matching in Q, at least one of the following statements holds.

$$1 = z(\{\overline{v}_1, v_2\}) + \sum_{e \in F(\overline{v}_1)\colon\, e \rhd_{\overline{v}_1} \{\overline{v}_1, v_2\}} z(e) = z(\{\overline{v}_1, v_2\}) + z(\{v_1, \overline{v}_1\}). \tag{3}$$

$$1 = z(\{\overline{v}_1, v_2\}) + \sum_{e \in F(v_2)\colon\, e \rhd_{v_2} \{\overline{v}_1, v_2\}} z(e) = z(\{\overline{v}_1, v_2\}). \tag{4}$$

Since (T1) for the case of $i = 1$ implies that $z(\{v_1, \overline{v}_1\}) = 1 - z(h_1^v)$, the above assumption implies that

$$z(\{\overline{v}_1, v_2\}) + z(\{v_1, \overline{v}_1\}) = z(\{\overline{v}_1, v_2\}) + 1 - z(h_1^v) < z(h_1^v) + 1 - z(h_1^v) = 1.$$

This contradicts (3). Furthermore, since z is a fractional matching in Q, we have $z(h_1^v) \leq 1$, Thus, the above assumption implies that $z(\{\overline{v}_1, v_2\}) < 1$. This contradicts (4), and completes the proof of $z(\{\overline{v}_1, v_2\}) = z(h_1^v)$.

Let k be an integer in $\{2, 3, \ldots, |E(v)| - 1\}$, and we assume that this lemma holds in the case of $i = k - 1$. Then we prove that this lemma holds in the case of $i = k$. Since z is a fractional matching in Q, we have

$$1 \geq \sum_{e \in F(v_k)} z(e) = z(\{\overline{v}_{k-1}, v_k\}) + z(h_k^v) + z(\{v_k, \overline{v}_k\}).$$

Since the induction hypothesis implies that

$$z(\{\overline{v}_{k-1}, v_k\}) + z(h_k^v) + z(\{v_k, \overline{v}_k\}) = \sum_{j=1}^{k-1} z(h_j^v) + z(h_k^v) + z(\{v_k, \overline{v}_k\}),$$

we have

$$z(\{v_k, \overline{v}_k\}) \leq 1 - \sum_{j=1}^{k} z(h_j^v). \tag{5}$$

For proving (T1) by contradiction, we assume that the inequality in (5) strictly holds. Since z is a stable fractional matching in Q, at least one of the following statements holds.

$$1 = z(\{v_k, \overline{v}_k\}) + \sum_{e \in F(v_k) \colon e \rhd_{v_k} \{v_k, \overline{v}_k\}} z(e) = z(\{v_k, \overline{v}_k\}) + z(\{\overline{v}_{k-1}, v_k\}) + z(h_k^v). \tag{6}$$

$$1 = z(\{v_k, \overline{v}_k\}) + \sum_{e \in F(\overline{v}_k) \colon e \rhd_{\overline{v}_k} \{v_k, \overline{v}_k\}} z(e) = z(\{v_k, \overline{v}_k\}). \tag{7}$$

However, the above assumption and the induction hypothesis imply that

$$\begin{aligned} z(\{v_k, \overline{v}_k\}) + z(\{\overline{v}_{k-1}, v_k\}) + z(h_k^v) &= z(\{v_k, \overline{v}_k\}) + \sum_{j=1}^{k-1} z(h_j^v) + z(h_k^v) \\ &< 1 - \sum_{j=1}^{k} z(h_j^v) + \sum_{j=1}^{k} z(h_j^v) = 1. \end{aligned}$$

This contradicts (6). Furthermore, since $z \in \mathbb{R}_+^F$, $z(\{v_k, \overline{v}_k\}) < 1$ follows from the above assumption. This contradicts (7). This completes the proof of (T1).

Next we consider (T2). Since z is a fractional matching in Q, we have

$$1 \geq \sum_{e \in F(\overline{v}_k)} z(e) = z(\{v_k, \overline{v}_k\}) + z(\{\overline{v}_k, v_{k+1}\}).$$

Since (T1) for the case of $i = k$ implies that

$$z(\{v_k, \overline{v}_k\}) = 1 - \sum_{j=1}^{k} z(h_j^v), \tag{8}$$

we have

$$z(\{\overline{v}_k, v_{k+1}\}) \leq \sum_{k=1}^{k} z(h_j^v). \tag{9}$$

For proving (T2) by contradiction, we assume that the inequality in (9) strictly holds. Since z is a stable fractional matching in Q, at least one of the following statements holds.

$$1 = z(\{\overline{v}_k, v_{k+1}\}) + \sum_{e \in F(\overline{v}_k)\colon e \rhd_{\overline{v}_k} \{\overline{v}_k, v_{k+1}\}} z(e) = z(\{\overline{v}_k, v_{k+1}\}) + z(\{v_k, \overline{v}_k\}). \tag{10}$$

$$1 = z(\{\overline{v}_k, v_{k+1}\}) + \sum_{e \in F(v_{k+1})\colon e \rhd_{v_{k+1}} \{\overline{v}_k, v_{k+1}\}} z(e) = z(\{\overline{v}_k, v_{k+1}\}). \tag{11}$$

Notice that (8) and the above assumption implies that

$$z(\{\overline{v}_k, v_{k+1}\}) + z(\{v_k, \overline{v}_k\}) < \sum_{j=1}^{k} z(h_j^v) + 1 - \sum_{j=1}^{k} z(h_j^v) = 1.$$

This contradicts (10). Furthermore, (8) and $z \in \mathbb{R}_+^F$ imply that $\sum_{j=1}^{k} z(h_j^v) \leq 1$. This and the above assumption imply that $z(\{\overline{v}_k, v_{k+1}\}) < 1$. This contradicts (11), and completes the proof. ◀

We are now ready to prove that x is a stable fractional matching in P. We first prove that x is a fractional matching in P. Let v be a vertex in V. Define $k := |E(v)|$. If $k = 1$, then

$$\sum_{e \in E(v)} x(e) = z(h_1^v) \leq 1.$$

If $k > 1$, then

$$\begin{aligned}
\sum_{e \in E(v)} x(e) &= \sum_{i=1}^{k} z(h_i^v) \\
&= \sum_{i=1}^{k-1} z(h_i^v) + z(h_k^v) \\
&= z(\{\overline{v}_{k-1}, v_k\}) + z(h_k^v) \quad \text{(by (T2) of Lemma 6)} \\
&= \sum_{e \in F(v_k)} z(e) \leq 1,
\end{aligned}$$

where the inequality follows from the fact that z is a fractional matching in Q.

Lastly, we prove that x is a stable fractional matching in P. Let e be a hyperedge in E. Then since z is a stable fractional matching in Q, there exists a vertex w in $\overline{e}$ such that

$$z(\overline{e}) + \sum_{f \in F(w)\colon f \rhd_w \overline{e}} z(f) = 1.$$

Assume that $w = v_k$ for some vertex v in e and some integer k in $\{1, 2, \ldots, |E(v)|\}$. Notice that $\overline{e} = h_k^v$. For each integer i in $\{1, 2, \ldots, k\}$, we assume that $h_i^v = \overline{e_i}$. Notice that $e_k = e$, $e_1 \succ_v e_2 \succ_v \cdots \succ_v e_k$, and $e \succ_v f$ holds for every hyperedge f in $E(v) \setminus \{e_1, e_2, \ldots, e_k\}$. For each integer i in $\{1, 2, \ldots, k\}$, $x(e_i) = z(h_i^v)$. If $k = 1$, then

$$1 = z(\overline{e}) + \sum_{f \in F(w)\colon f \rhd_w \overline{e}} z(f) = z(\overline{e}) = x(e) = x(e) + \sum_{f \in E(v)\colon f \succ_v e} x(f).$$

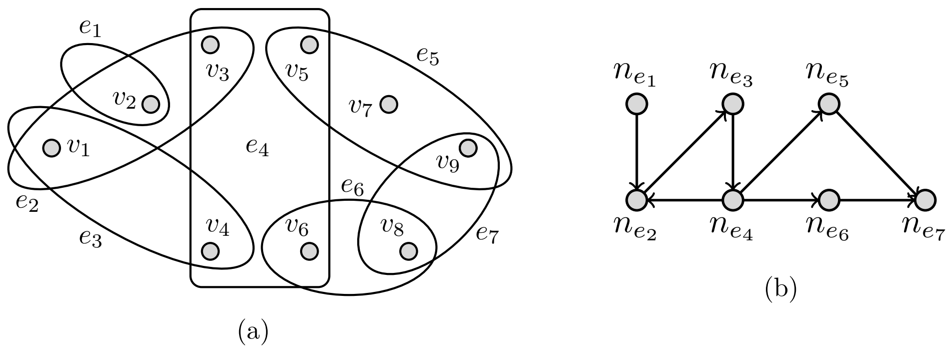

Figure 2 (a) A hypergraph $H = (V, E)$ such that $e_3 \succ_{v_1} e_2$, $e_2 \succ_{v_2} e_1$, $e_2 \succ_{v_3} e_4$, $e_4 \succ_{v_4} e_3$, $e_5 \succ_{v_5} e_4$, $e_6 \succ_{v_6} e_4$, $e_7 \succ_{v_8} e_6$, and $e_7 \succ_{v_9} e_5$. (b) The directed graph D constructed from H.

If $k > 1$, then

$$\begin{aligned}
1 &= z(\overline{e}) + \sum_{f \in F(w)\colon f \rhd_w \overline{e}} z(f) \\
&= z(h_k^v) + z(\{\overline{v}_{k-1}, v_k\}) \\
&= z(h_k^v) + \sum_{i=1}^{k-1} z(h_i^v) \quad \text{(by (T2) of Lemma 6)} \\
&= x(e) + \sum_{i=1}^{k-1} x(e_i) \\
&= x(e) + \sum_{f \in E(v)\colon f \succ_v e} x(f).
\end{aligned}$$

These imply that x is a stable fractional matching in P. This completes the proof.

4 Proof of Theorem 3

Throughout this section, we assume that we are given a hypergraphic preference system P such that $\deg(P) = 2$. Define V^* as the set of vertices v in V such that $|E(v)| = 2$. In addition, we define the directed graph $D = (N, A)$ as follows. For each hyperedge e in E, N contains a vertex n_e. For each vertex v in V^*, A contains an arc from n_f to n_e, where we assume that distinct hyperedges e, f in E contain v and $e \succ_v f$. See Figure 2 for an example of D.

Our algorithm is described in Algorithm 1. This algorithm can be intuitively explained as follows. If there exists a vertex n_e in N such that any arc in A does not leave n_e, then the hyperedge e is most preferred by every vertex v in e. Thus, we set the value for e to be 1. For every arc $a = (n_f, n_e)$ in A, since some vertex in V is contained in e, f, we must set the value for f to be 0. Then we can remove vertices in N whose value is determined from D. We repeat this. Finally, we obtain a directed graph D' in which the out-degree of every vertex is at least one. Thus, by setting the value for each vertex of D' to be $1/2$, we can construct a stable fractional matching in P.

Here we apply Algorithm 1 for the example in Figure 2. Since n_{e_7} is the only vertex such that any arc in A_t does not leave this vertex, we set $\xi_2(n_{e_7}) := 1$ and the value of ξ_2 for other vertex is equal to 0. Then the vertices $n_{e_5}, n_{e_6}, n_{e_7}$ (and the arcs around them) are

Algorithm 1:

Define $D_1 := D$ and $N_1 := N$.
Define the vector ξ_1 in $\mathbb{R}_+^N$ by $\xi_1(v) := 0$ for each vertex v in N.
Set $t := 1$.
while *there exists a vertex v in N_t such that any arc of D_t does not leave v* **do**
  Define S_t as the set of vertices v in N_t such that any arc of D_t does not leave v.
  Define the vector ξ_{t+1} in $\mathbb{R}_+^N$ by $\xi_{t+1}(v) := 1$ for each vertex v in S_t and $\xi_{t+1}(v) := \xi_t(v)$ for each vertex v in $N \setminus S_t$.
  Define T_t as the set of vertices v in N_t such that there exists an arc of D_t from v to some vertex in S_t.
  Define $N_{t+1} := N_t \setminus (S_t \cup T_t)$, and D_{t+1} as the subgraph of D_t induced by N_{t+1}.
  Set $t := t + 1$.
end
Define the vector ξ^* in $\mathbb{R}_+^N$ by $\xi^*(v) := 1/2$ for each vertex v in N_t and $\xi^*(v) := \xi_t(v)$ for each vertex v in $N \setminus N_t$.
Define the vector x in $\mathbb{R}_+^E$ by $x(e) := \xi^*(n_e)$ for each hyperedge e in E.
Output x, and halt.

removed. In the remaining graph, for every vertex, at least one arc leaves it. Thus, the value $1/2$ are assigned to the remaining vertices, and the algorithm halts. In the obtained stable fractional matching x, $x(e_i) = 1/2$ for every integer i in $\{1, 2, 3, 4\}$, $x(e_5) = 0$, $x(e_6) = 0$, and $x(e_7) = 1$.

▶ **Lemma 7.** *The output of Algorithm 1 is a stable fractional matching in P.*

Proof. Assume that Algorithm 1 halts when $t = k$. For proving this lemma, it suffices to prove the following conditions are satisfied.

(P1) For every arc $a = (u, v)$ in A, we have $\xi^*(u) + \xi^*(v) \leq 1$.
(P2) For every vertex v in N such that $\xi^*(v) \neq 1$, there exist a vertex w in N such that an arc from v to w is contained in A and $\xi^*(v) + \xi^*(w) = 1$.

We first prove (P1). Assume that we are given an arc $a = (u, v)$ in A. If $\xi^*(u) = 0$, then (P1) clearly holds. Next we assume that $\xi^*(u) = 1$. Then there exists a positive integer t such that $u \in N_t$ and any arc of D_t does not leave u. Notice that $v \notin N_t$. This implies that $\xi^*(v) \in \{0, 1\}$. If $\xi^*(v) = 1$, then then there exists a positive integer t' such that $t' < t$, $v \in N_{t'}$, and any arc of $D_{t'}$ does not leave v. Furthermore, the definition of $T_{t'}$ implies that $u \in T_{t'}$. This implies that $u \notin N_t$, which contradicts the fact that $u \in N_t$. Thus, we have $\xi^*(v) = 0$. Lastly, we consider the case where $\xi^*(u) = 1/2$, i.e., $u \in N_k$. If $\xi^*(v) = 1$, then $u \notin N_k$, which contradicts the fact that $u \in N_k$. This implies that $\xi^*(v) \in \{0, 1/2\}$. This completes the proof of (P1).

Next we prove (P2). Assume that we are given a vertex v in N such that $\xi^*(v) \neq 1$. Assume that $\xi^*(v) = 0$. In this case, there exists a positive integer t such that $v \in T_t$. That is, there exists a vertex w in S_t such that there exists an arc of D_t from v to w. Since $w \in S_t$, $\xi^*(w) = 1$. This implies that $\xi^*(v) + \xi^*(w) = 1$. Next we assume that $\xi^*(v) = 1/2$. In this case, there exists a vertex w in N_k such that there exists an arc of D_k from v to w. Since $\xi^*(w) = 1/2$, we have $\xi^*(v) + \xi^*(w) = 1$. This completes the proof. ◀

Proof of Theorem 3. This theorem immediately follows from Lemma 7. ◀

5 Proof of Theorem 4

In this section, we prove Theorem 4. Since a stable fractional matching is clearly an ϵ-stable fractional matching for any positive rational number ϵ, Theorem 1 (i.e., the fact that Fractional Hypergraph Matching is in **PPAD**) implies that Approximate Fractional Hypergraph Matching is in **PPAD**. What remains is to prove that every problem in **PPAD** is reducible to Approximate Fractiona Hypergraph Matching in polynomial time. For this, Theorem 1 implies that it is sufficient to prove that Fractional Hypergraph Matching is reducible to Approximate Fractional Hypergraph Matching in polynomial time. This fact immediately follows from the following lemma.

▶ **Lemma 8.** *Assume that we are given a hypergraphic preference system $P = (V, E, \{\succ_v\})$. Furthermore, we define $\epsilon := 1/2^{20|E|^4}$. Then we can construct a stable fractional matching in P from an ϵ-stable fractional matching in P in polynomial time.*

Notice that the bit-length of ϵ in Lemma 8 is bounded by a polynomial in the size of P. More precisely, the bit-length of ϵ in Lemma 8 is $O(|E|^4)$.

What remains is to prove Lemma 8. We prove Lemma 8 by using the following known result called LP compactness. Assume that we are given positive integers m, n and vectors a in $\mathbb{Q}^{m\times n}$ and b in $\mathbb{Q}^m$, where $\mathbb{Q}$ is the set of rational numbers. Then we consider the following linear inequality system whose variable is a vector x in $\mathbb{R}^n$.

$$\sum_{j=1}^{n} a(i,j)\cdot x(j) \geq b(i) \quad (i \in \{1,2,\ldots,m\}). \tag{12}$$

For each positive real number δ and each vector y in $\mathbb{R}^n$, we say that y satisfies the linear inequality system (12) to within δ, if

$$\sum_{j=1}^{n} a(i,j)\cdot y(j) \geq b(i) - \delta$$

for every integer i in $\{1,2,\ldots,m\}$.

▶ **Theorem 9** (LP compactness (see [10, Lemma 4.11]))**.** *Assume that we are given positive integers m, n and vectors a in $\mathbb{Q}^{m\times n}$ and b in $\mathbb{Q}^m$. Furthermore, we assume that there exists a positive integer β satisfying the condition that for every pair of integers i in $\{1,2,\ldots,m\}$ and j in $\{1,2,\ldots,n\}$, there exist integers p, q, r, s such that $a(i,j) = p/q$, $b(i) = r/s$, and $|p|, |q|, |r|, |s| \leq 2^\beta$. Then we consider the following linear inequality system whose variable is a vector x in $\mathbb{R}^n$.*

$$\sum_{j=1}^{n} a(i,j)\cdot x(j) \geq b(i) \quad (i \in \{1,2,\ldots,m\}). \tag{13}$$

If there exists a vector y in $\mathbb{R}^n$ satisfying the linear inequality system (13) to within $1/2^{20n^4\beta}$, then there exists a vector x in $\mathbb{R}^n$ that is feasible for the linear inequality system (13).

We are now ready to prove Lemma 8.

Proof of Lemma 8. Assume that we are given an ϵ-stable fractional matching y in P. For each hyperedge e in E, we define set $U(e)$ as the set of vertices v in e such that

$$y(e) + \sum_{f\in E(v)\colon f\succ_v e} y(f) \geq 1-\epsilon.$$

Notice that since y in an ϵ-stable fractional matching in P, $U(e) \neq \emptyset$ for any hyperedge e in E. We consider the following linear inequality system whose variable is a vector x in $\mathbb{R}^E$.

$$\begin{aligned}
&-\sum_{e \in E(v)} x(e) \geq -1 \quad (v \in V)\\
&x(e) + \sum_{f \in E(v)\colon f \succ_v e} x(f) \geq 1 \quad (e \in E,\, v \in U(e))\\
&x(e) \geq 0 \quad (e \in E).
\end{aligned} \tag{14}$$

Notice that the number of constraints of the linear inequality system (14) is bounded by a polynomial in the input size of Fractional Hypergraph Matching.

Notice that y satisfies the linear inequality system (14) to within $1/2^{20|E|^4}$. Thus, by setting $n := |E|$ and $\beta := 1$, Theorem 9 implies that there exists a vector x in $\mathbb{R}^E$ that is feasible for the linear inequality system (14). Notice that we can find a vector x in $\mathbb{R}^E$ that is feasible for the linear inequality system (14) in polynomial time by using the ellipsoid method [9].

Let x be a vector in $\mathbb{R}^E$ that is feasible for the linear inequality system (14). Then we prove that x is a stable fractional matching in P. For this, it suffices to prove that for every hyperedge e in E, there exists a vertex v in e such that

$$x(e) + \sum_{f \in E(v)\colon f \succ_v e} x(f) = 1. \tag{15}$$

Let e be a hyperedge in E. The first constraint of (14) implies that

$$x(e) + \sum_{f \in E(v)\colon f \succ_v e} x(f) \leq 1$$

for every vertex v in $U(e)$. Thus, the second constraint of (14) implies that

$$x(e) + \sum_{f \in E(v)\colon f \succ_v e} x(f) = 1$$

for every vertex v in $U(e)$. Since $U(e) \neq \emptyset$, this implies that there exists a vertex v in e satisfying (15). This completes the proof. ◀

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