An Algorithm for the Maximum Weight Strongly Stable Matching Problem

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Abstract

An instance of the maximum weight strongly stable matching problem with incomplete lists and ties is an undirected bipartite graph $G = (A \cup B, E)$, with an adjacency list being a linearly ordered list of ties, which are vertices equally good for a given vertex. We are also given a weight function w on the set E. An edge $(x, y) \in E \setminus M$ is a *blocking edge* for M if by getting matched to each other neither of the vertices x and y would become worse off and at least one of them would become better off. A matching is *strongly stable* if there is no blocking edge with respect to it. The goal is to compute a strongly stable matching of maximum weight with respect to w.

We give a polyhedral characterisation of the problem and prove that the strongly stable matching polytope is integral. This result implies that the maximum weight strongly stable matching problem can be solved in polynomial time. Thereby answering an open question by Gusfield and Irving [6]. The main result of this paper is an efficient $O(nm \log(Wn))$ time algorithm for computing a maximum weight strongly stable matching, where we denote $n = |V|$, $m = |E|$ and W is a maximum weight of an edge in G. For small edge weights we show that the problem can be solved in $O(nm)$ time. Note that the fastest known algorithm for the unweighted version of the problem has $O(nm)$ runtime [9]. Our algorithm is based on the rotation structure which was constructed for strongly stable matchings in [12].

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1 Introduction

An instance of the Stable marriage problem with ties and incomplete lists (smti) is an undirected bipartite graph $G = (A \cup B, E)$, with an adjacency list being a linearly ordered list of ties, which are vertices equally good for a given vertex. Ties are disjoint and may contain one vertex. Let b_1 and b_2 be two vertices incident to a in G. Depending on the preference of a one of the following holds. (1) a (strictly) prefers b_1 to b_2 - denoted as $b_1 \succ_a b_2$, (2) a is indifferent between b_1 and b_2 - denoted as $b_1 =_a b_2$, (3) a (strictly) prefers b_2 to b_1 - denoted as $b_1 \prec_a b_2$. If a prefers b_1 to b_2 or is indifferent between them then we say that a *weakly prefers* b_1 to b_2 and denote it as $b_1 \succeq_a b_2$.

An edge $(a, b) \in E \setminus M$ is a *blocking edge* with respect to M if by getting matched with each other neither of the vertices a and b would become worse off and at least one of them would become better off than in M. Formally an edge $(a, b) \in E \setminus M$ is blocking if either $a \succ_b M(b)$ and $b \succeq_a M(a)$ or $a \succeq_b M(b)$ and $b \succ_a M(a)$ hold.

By $M(a)$ we denote a partner of a in the matching M. If a is unmatched in M we abuse the notation and write $b \succ_a M(a)$ for each $(a, b) \in E$. We assume that every vertex prefers

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to be matched to its neightbour in G rather than to remain unmatched. We say that a matching is *strongly stable* if there is no blocking edge with respect to it.

We study a version of the problem where besides the graph G and preference lists we are also given a weight function $w : E \to \mathbb{N}$. We define the weight of a matching M to be $w(M) = \sum_{e \in M} w(e)$. The goal is to find a strongly stable matching M maximising $w(M)$.

Motivation. The stable matching problem and its extensions have widespread application to matching schemes [19]. One of the most known examples are the labor market for medical interns and the college admissions market.

It is known that the deferred acceptance algorithm [5] calculates a stable matching optimal for one side of the market. The extension to the weighted variant of the problem allows us to define suitable objective functions and use them to obtain various optimal stable matchings.

The notion of strong stability allows us to prevent the following scenarios. Suppose that agent a is matched to $M(a)$ and a is indifferent between $M(a)$ and b. Also assume that b prefers a over $M(b)$. The agent b to improve their situation may be inclined to use an action, like bribery, to convince a to accept them. Since a would not get worse and b would get better by getting matched to each other, they might undermine the current assignment.

Previous results. The variant of the problem with strict preferences known as the stable marriage problem (SMI) has been extensively studied in the literature. In their seminal paper Gale and Shapley [5] showed that every instance of the problem admits a stable matching and described an $O(n + m)$ time algorithm for computing such a matching. Many structural properties of the problem have been described over the years. In [6] Gusfield and Irving have proven that the set of stable marriage solutions forms a distributive lattice. They also show that even though the lattice can be of exponential size, it can be compactly represented as a set of closed sets of a certain partial order on $O(m)$ elements. The representation can be built in $O(m)$ time based on the notion of rotation.

Vande Vate [25] initiated the study of the stable marriage problem using the polyhedral approach. He described a stable marriage polytope and showed its integrality. His description has been extended by Rothblum [21] to the case of incomplete preference lists. In subsequent papers several simpler proofs of the integrality of the stable marriage polytope have been given [20], [24]. It has been also proven that any fractional solution in the stable marriage polytope can be expressed as a convex combination of integral solutions [24]. These results imply that the maximum weight stable marriage problem can be solved in polynomial time.

Several efficient algorithms for this problem have been developed over the years. Gusfield and Irving [6] described an $O(m^2 \log n)$ algorithm. The authors exploit the rotation structure and reduce the problem to finding a maximum weight closed subset of a poset. This classical problem can in turn be reduced to computing a maximum flow. The flow network obtained from the reduction consists of $O(m)$ nodes and $O(m)$ edges. Gusfield and Irving use $O(nm \log n)$ algorithm by Sleator and Tarjan [23] to solve the maximum flow problem and obtain $O(m^2 \log n)$ complexity. A faster maximum flow algorithm would lead to the improvement in their algorithm. Feder [2] showed that if $K = O((m/\log^2 m)^2)$ then the weighted stable marriage problem can be solved in $O(m\sqrt{K})$ time and $O(nm \log K)$ for arbitrary K where K is the weight of the solution. Note that algorithms by Gusfield and Irving and by Feder assume a certain monotonicity condition on edge weights, however in the case of bipartite graphs this condition can be dropped as we show later.

The problem of computing a strongly stable matching in instances of SMTI has received a significant attention in the literature. Irving [7] gave an $O(n^4)$ algorithm for the problem under the assumption that the graph is complete and there is an equal number of men and women. In [14] Manlove extended this algorithm to incomplete bipartite graphs. His algorithm has $O(m^2)$ time complexity. Kavitha et al. [9] gave an $O(nm)$ algorithm for the problem. The structure of the set of solutions to the problem has been proven to be similar to the structure of the case of no ties. In [15] Manlove has proven that the set of solutions forms a distributive lattice. Recently, Kunysz et al. [12] gave an $O(nm)$ algorithm for constructing a compact representation of the lattice and generalized the notion of rotation to the strong stability setting. To the best of our knowledge the weighted version of the strongly stable matching problem has not been studied in the literature yet.

Our results. Gusfield and Irving [6] asked whether there is an LP representation of an instance of SMTI under strong stability similar to the case of no ties. The problem was again posed by Manlove [16] in his recent book. We solve this problem, adapting techniques used in [24] to our setting. We prove that any fractional solution to the polytope can be expressed as a convex combination of integral solutions. Thus the polytope is integral and the maximum weight strongly stable matching problem can be solved in polynomial time.

A natural question is whether the rotation structure can be exploited to obtain a faster algorithm. We answer this question affirmatively and give an $O(nm \log(Wn))$ algorithm, where W is the maximum weight of an edge. We also show that if W is sufficiently small then the problem can be solved in $O(nm)$ time. The technique of Gusfield and Irving cannot be directly applied to our problem. In the setting without ties the authors base their algorithm on the fact that there is a one-to-one correspondence between stable matchings and closed sets of a certain poset of size $O(m)$. In our problem a similar one-to-one correspondence exists between equivalence classes of strongly stable matchings under a certain equivalence relation and closed sets of a poset of size $O(m)$. The correspondence allows us to represent exactly one matching from each equivalence class based on a computation of so called maximal sequence of strongly stable matchings. The main obstacle is that each equivalence class may contain exponentially many matchings and there is a possibility that a represented matching is not of maximum weight within its class. The primary novelty of this paper is an algorithm for computing so called heavy maximal sequence of strongly stable matchings, which allows us to represent a matching of maximum weight from each equivalence class. As a result we reduce our problem to finding a maximum weight closed set of a poset, and solve this problem using Feder algorithm [2].

Related work. Stable matchings have been extensively studied in non-bipartite instances with strict preferences. Feder [1] has shown that in this setting the maximum weight stable matching problem is NP-hard and he gave a 2-approximation algorithm for the problem.

In SMTI instances three different notions of stability can be defined depending on the definition of a blocking edge. Namely weak, strong and super stability. Weakly stable matchings can be of different sizes. Iwama et al. [8] have proven that the problem of finding a maximum size weakly stable matching is NP-hard. Several approximation algorithms are known for the problem [17], [10], [18]. It is also known that the weighted version of the problem is NP-hard and it is not approximable within a factor $n^{1-\epsilon}$ for any $\epsilon > 0$ unless $P = NP$ [13]. The structure of stable matchings under the notion of super stability is well understood. In [3] Fleiner et al. gave a reduction to the 2-SAT problem which results in fast algorithms for a range of problems related to finding "optimal" super stable matchings.

2 Preliminaries

Let $\mathcal{I}$ be an instance of SMTI. Denote the set of all strongly stable matchings in $\mathcal{I}$ by $\mathcal{M}(\mathcal{I})$. Let $V(\mathcal{I})$ and $E(\mathcal{I})$ be respectively sets of vertices and edges of the underlying bipartite graph $G = (A \cup B, E(\mathcal{I}))$ of $\mathcal{I}$. As is customary we call the vertices of A and B respectively men and women. We say that an instance $\mathcal{I}$ is solvable if there is a strongly stable matching in G. We define the rank of w in v's preference list, denoted by $rank(v, w)$, to be 1 plus the number of ties which are preferred to w by v. A matching is *man-optimal* if every man gets the best partner among all his possible partners in any strongly stable matching.

▶ **Theorem 1** ([9]). *There is an $O(nm)$ algorithm to determine a man-optimal strongly stable matching of the given instance or report that no strongly stable matching exists.*

2.1 Lattice Structure

In this subsection we give a brief overview of results related to the lattice structure of $\mathcal{M}(\mathcal{I})$. As we will see later the lattice can be of exponential size, however its representation of polynomial size can be constructed. Such a representation is described in the next subsection.

▶ **Theorem 2** (Rural Hospitals Theorem, [14]). *In a given instance of* SMTI*, the same vertices are matched in all strongly stable matchings.*

We define an equivalence relation $\sim$ on $\mathcal{M}(\mathcal{I})$ as follows. For two strongly stable matchings M and N, $M \sim N$ if and only if each man m is indifferent between $M(m)$ and $N(m)$. Denote by $[M]$ the equivalence class containing M and denote by $\mathcal{X}$ the set of equivalence classes of $\mathcal{M}(\mathcal{I})$ under $\sim$.

Strongly stable matchings belonging to the same equivalence class can be easily characterised. For a given strongly stable matching M we define an auxiliary graph $H_M = (V', E')$ where V' is the set of vertices matched in M and $E' = \{(a, b) \in E : a, b \in V' \wedge b =_a M(a) \wedge a =_b M(b)\}$. The following lemma characterises the set $[M]$.

▶ **Lemma 3** ([15]). *Let $M \in \mathcal{M}(\mathcal{I})$. Then M' is a strongly stable matching such that $M' \sim M$ if and only if M' is a perfect matching in H_M.*

For two strongly stable matchings M and N we say that M dominates N and write $N \preceq M$ if each man m weakly prefers $M(m)$ to $N(m)$. If M dominates N and there exists a man m who strictly prefers $M(m)$ to $N(m)$ then we say that M strictly dominates N, denote it by $N \prec M$ and we call N a successor of M. Next we define a partial order $\preceq^*$ on $\mathcal{X}$. For any two equivalence classes $[M]$ and $[N]$, we define $[M] \preceq^* [N]$ if and only if $M \preceq N$.

Let M and N be two strongly stable matchings. Consider the symmetric difference $M \oplus N$. Theorem 2 implies that this set contains only alternating cycles.

▶ **Lemma 4** ([15]). *Let M and N be two strongly stable matchings. Consider any alternating cycle C of $M \oplus N$. Let $(m_0, w_0, m_1, w_1, ..., m_{k-1}, w_{k-1})$ be the sequence of vertices of C where m_i are men and w_i are women. Then there are only three possibilities:*

- $(\forall m_i) w_i =_{m_i} w_{i+1}$ *and* $(\forall w_i) m_i =_{w_i} m_{i-1}$
- $(\forall m_i) w_i \prec_{m_i} w_{i+1}$ *and* $(\forall w_i) m_i \succ_{w_i} m_{i-1}$
- $(\forall m_i) w_i \succ_{m_i} w_{i+1}$ *and* $(\forall w_i) m_i \prec_{w_i} m_{i-1}$

Subscripts are taken modulo k.

Below we introduce two operations transforming pairs of strongly stable matchings into other strongly stable matchings. Let M and N be two strongly stable matchings. Consider any man m and his partners $M(m)$ and $N(m)$. By $M \wedge N$ we denote the matching such that if $M(m) \succeq_m N(m)$ then $(m, M(m)) \in M \wedge N$ and if $M(m) \prec_m N(m)$ then $(m, N(m)) \in M \wedge N$. Similarly by $M \vee N$ we denote the matching such that if $M(m) \succ_m N(m)$ then $(m, N(m)) \in M \vee N$ and if $M(m) \preceq_m N(m)$ then $(m, M(m)) \in M \vee N$.

It is proven in [15] that both $M \vee N$ and $M \wedge N$ are strongly stable matchings, and $M, N \succeq M \vee N$ and $M, N \preceq M \wedge N$. Operations $\vee$ and $\wedge$ can be extended to the set $\mathcal{X}$. For $[M], [N] \in \mathcal{X}$ we simply define $[M] \vee [N] = [M \vee N]$, $[M] \wedge [N] = [M \wedge N]$.

A *lattice* is a partially ordered set in which every two elements a, b have a unique infimum (denoted $a \vee b$) and a unique supremum (denoted $a \wedge b$). A lattice L with operations join $\vee$ and meet $\wedge$ is *distributive* if for any three elements x, y, z of L the following holds: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

▶ **Theorem 5** ([15]). *The partial order $(\mathcal{X}, \preceq^*)$ with operations meet $\vee$ and join $\wedge$ defined above forms a distributive lattice.*

2.2 Rotations

In this subsection we review known theorems about rotations in instances of SMTI under strong stability. These results were previously given in [12] and [11].

Let M and N be two strongly stable matchings such that $N \prec M$. We say that N is a *strict successor* of M if and only if there is no strongly stable matching M' such that $N \prec M' \prec M$. Let M_0 be a man-optimal strongly stable matching, and let M_z be a woman optimal strongly stable matching. We call a sequence $(M_0, M_1, \dots, M_z)$ such that $M_0 \succ M_1 \succ \dots \succ M_z$ and M_{i+1} is a strict successor of M_i, a *maximal sequence of strongly stable matchings*.

▶ **Theorem 6** ([12]). *There is an $O(nm)$ time algorithm to compute a maximal sequence of strongly stable matchings.*

Let M and N be two strongly stable matchings such that N is a strict successor of M. Consider some matchings $M' \in [M]$, $N' \in [N]$. Note that from the definition of $\sim$ it follows that for every vertex v we have $rank(v, M(v)) - rank(v, N(v)) = rank(v, M'(v)) - rank(v, N'(v))$. In other words when we transform a matching from $[M]$ into some matching from $[N]$, the change of v's rank does not depend on the choice of matchings from equivalence classes. This observation motivates the definition of *rotation*.

Let M and N be two strongly stable matchings such that N is a strict successor of M. For any vertex v denote $r_v = rank(v, M(v))$ and $r'_v = rank(v, N(v))$. We say that a set of triples $\rho([M], [N]) = \{(v, r_v, r'_v) : v \in V(\mathcal{I}), r_v \neq r'_v\}$ is a *rotation* transforming $[M]$ into $[N]$.

Let ρ be a rotation and M, N be two strongly stable matchings such that N is a strict successor of M. We say that the set of alternating cycles $M \oplus N$ *realizes* a rotation ρ if $\rho = \rho([M], [N])$. There are potentially many sets of cycles realizing a given rotation. A rotation ρ is *exposed* in $[M]$ if $\rho = \rho([M], [N])$ for some N which is a strict successor of M. We say that $\rho = \rho([M], [N])$ *transforms* M' into N' if $M' \in [M]$ and $N' \in [N]$.

▶ **Theorem 7** ([12]). *Let $S = (M_0, M_1, \dots, M_z)$ be a maximal sequence of strongly stable matchings. For $i \in \{0, 1, \dots, z-1\}$ denote $\rho_i = \rho([M_i], [M_{i+1}])$. Then the set $D(\mathcal{I}) = \{\rho_0, \rho_1, \dots, \rho_{z-1}\}$ does not depend on the choice of S, and $\rho_i \neq \rho_j$ for $i \neq j$.*

Note that given a maximal sequence of strongly stable matchings $S = (M_0, M_1, \ldots, M_z)$ we can easily compute rotations $(\rho_0, \rho_1, \ldots, \rho_{z-1})$ where $\rho_i = \rho([M_i], [M_{i+1}])$. Moreover the set $C_S(\rho_i) = M_i \oplus M_{i+1}$ realizes ρ_i for each i.

▶ **Definition 8.** Let $D(\mathcal{I})$ be the set of all rotations in $\mathcal{I}$. We define the order $\prec$ on elements of $D(\mathcal{I})$ as follows. We say that a rotation ρ precedes rotation ρ' and write $\rho \prec \rho'$ if and only if for every maximal sequence $S = (M_0, M_1, \ldots, M_z)$ of strongly stable matchings we have $\rho = \rho([M_i], [M_{i+1}])$ and $\rho' = \rho([M_j], [M_{j+1}])$ for some i, j such that $i < j$.

Let Z be a subset of $D(\mathcal{I})$. We say that Z is a *closed set* if there is no $\rho \in D(\mathcal{I}) \setminus Z$ such that $\rho \prec \rho'$ for some $\rho' \in Z$. It turns out that each closed set corresponds to an equivalence class of $\sim$. Given Z we can efficiently find an equivalence class corresponding to it.

Assume that we are given a maximal sequence $S = (M_0, M_1, \ldots, M_z)$ of strongly stable matchings, the set of rotations $D(\mathcal{I})$, and for each rotation $\rho_i = \rho([M_i], [M_{i+1}])$ a set of cycles $C_S(\rho_i) = M_i \oplus M_{i+1}$ realizing it. Let $Z = \{\rho_{a_0}, \rho_{a_1}, \ldots, \rho_{a_{k-1}}\}$ be a closed set. We order its elements so that there are no i, j such that $i < j$ and $\rho_{a_i} \succ \rho_{a_j}$. We define a sequence of strongly stable matchings $N_0 = M_0$, $N_{i+1} = N_i \oplus C_S(\rho_{a_i})$. We denote $f_S(Z) = N_k$. Note that the sequence $\{N_i\}$ depends on the ordering of elements of Z, however its last element $f_S(Z) = M_0 \oplus C_S(\rho_{a_0}) \oplus C_S(\rho_{a_1}) \oplus \ldots \oplus C_S(\rho_{a_{k-1}})$ is the same regardless of the ordering.

▶ **Lemma 9.** *For each equivalence class $[M]$ there is a closed set X such that $f_S(X) \in [M]$. Let Z_1 and Z_2 be closed sets. Then $Z_1 \neq Z_2$ implies that $[f_S(Z_1)] \neq [f_S(Z_2)]$.*

For each closed set Z we define $g_S(Z) = [f_S(Z)]$. It can be proven that g_S does not depend on the choice of S and that g_S is a bijection between closed sets of $D(\mathcal{I}, \prec)$ and the set $\mathcal{X}$. The above discussion is summarized in the following theorem.

▶ **Theorem 10** ([12]). *There is a one-to-one correspondence between the set $\mathcal{X}$ of equivalence classes of $\sim$ and the closed sets of $(D(\mathcal{I}), \prec)$.*

It is important to note that given the function $f_{\mathcal{S}}$ we can get one strongly stable matching from each equivalence class and that depending on the choice of $\mathcal{S}$ these matchings may differ. In other words if $\mathcal{S} \neq \mathcal{S}'$ then it may happen that $f_{\mathcal{S}}(Z) \neq f_{\mathcal{S}'}(Z)$ for some Z, however regardless of the choice of $\mathcal{S}$ and $\mathcal{S}'$ we have $[f_{\mathcal{S}}(Z)] = [f_{\mathcal{S}'}(Z)]$.

Note that from Definition 8 alone it is non-trivial how to efficiently construct the relation $\prec$ on $D(\mathcal{I})$. Construction of an explicit representation of the relation $\prec$ would take $\Omega(m^2)$ time, because $D(\mathcal{I})$ might have $\Omega(m)$ elements.

▶ **Theorem 11** ([12]). *There is a graph $G' = (D(\mathcal{I}), E')$ such that $|E'| = O(m)$, and the closed sets in G' are exactly the same as the closed sets in the poset $(D(\mathcal{I}), \prec)$. Such a graph can be constructed in $O(nm)$ time.*

3 Strongly Stable Matching Polytope

Let us denote the set of men as $A = \{a_1, a_2, \ldots, a_p\}$ and the set of women as $B = \{b_1, b_2, \ldots, b_q\}$. Additionally by P_{SSM} we denote a *strongly stable matching polytope* described by the following set of inequalities.

$$\sum_{j=1}^{q} x_{i,j} \leq 1 \qquad \forall i (1 \leq i \leq p) \tag{1}$$

$$\sum_{i=1}^{p} x_{i,j} \leq 1 \qquad \forall j (1 \leq j \leq q) \tag{2}$$

$$x_{i,j} \geq 0 \qquad \forall (i,j)(1 \leq i \leq p, 1 \leq j \leq q) \tag{3}$$

$$\sum_{k: b_k \succ_{a_i} b_j} x_{i,k} + \sum_{k: a_k \succ_{b_j} a_i} x_{k,j} + \sum_{k: b_k =_{a_i} b_j} x_{i,k} \geq 1 \qquad \forall (i,j)(a_i, b_j) \in E \tag{4}$$

$$\sum_{k: b_k \succ_{a_i} b_j} x_{i,k} + \sum_{k: a_k \succ_{b_j} a_i} x_{k,j} + \sum_{k: a_k =_{b_j} a_i} x_{k,j} \geq 1 \qquad \forall (i,j)(a_i, b_j) \in E \tag{5}$$

Inequalities (1), (2) and (3) are standard matching constraints. If $x \in P_{SSM}$ is an integral solution, then constraints (4) and (5) for an edge (a_i, b_j) imply that (a_i, b_j) does not block the matching associated with x. Thus integral solutions of P_{SSM} are exactly strongly stable matchings of G. We call such solutions *strongly stable matching solutions*.

Note that if there are no ties in the instance then the terms $\sum_{k: a_k =_{b_j} a_i} x_{k,j}$ and $\sum_{k: b_k =_{a_i} b_j} x_{i,k}$ in (4) and (5) reduce to $x_{i,j}$ and the description of the polytope is identical to the well known description of the stable marriage polytope (see [24]). The proof of the next lemma is based on self-duality of the associated linear program and complementary slackness conditions.

▶ **Lemma 12.** *Let $x \in P_{SSM}$ be a feasible solution. Then for each $1 \leq i \leq p, 1 \leq j \leq q$ the following hold:*

$$x_{i,j} > 0 \Rightarrow \sum_{k: b_k \succ_{a_i} b_j} x_{i,k} + \sum_{k: a_k \succ_{b_j} a_i} x_{k,j} + \sum_{k: b_k =_{a_i} b_j} x_{i,k} = 1$$

$$x_{i,j} > 0 \Rightarrow \sum_{k: b_k \succ_{a_i} b_j} x_{i,k} + \sum_{k: a_k \succ_{b_j} a_i} x_{k,j} + \sum_{k: a_k =_{b_j} a_i} x_{k,j} = 1$$

$$x_{i,j} > 0 \Rightarrow \sum_{k=1}^{q} x_{i,k} = 1$$

$$x_{i,j} > 0 \Rightarrow \sum_{k=1}^{p} x_{k,j} = 1$$

It is important to note that for each feasible solution x if $x_{i,j} > 0$ then $\sum_{k: a_k =_{b_j} a_i} x_{k,j} = \sum_{k: b_k =_{a_i} b_j} x_{i,k}$. Lemma 12 allows us to prove Theorem 13 which shows that each fractional solution to P_{SSM} can be expressed as a convex combination of strongly stable matchings. The proof is constructive and given a fractional solution one can obtain matchings constituting such a convex combination. Theorem 13 also implies that P_{SSM} is integral.

▶ **Theorem 13.** *The polytope P_{SSM} is the convex hull of the strongly stable matching solutions.*

Proof. Let $x \in P_{SSM}$ be a feasible solution. For each man a_i such that $x_{i,j} > 0$ for some j we perform the following construction. From Lemma 12 it follows that $\sum_{k=1}^{q} x_{i,k} = 1$. For a_i we arrange all the $x_{i,k}$ for $k = 1, 2, \ldots, q$ in order of decreasing preference for a_i. If there are any ties we pick an arbitrary order amongst tied variables. We cover the interval

$(0,1]$ with smaller intervals $(v_{i,k}, v_{i,k}+x_{i,k}]$ where intervals are arranged in the same order as variables $x_{i,k}$. We slightly abuse the notation here and by $x_{i,k}$ we denote corresponding interval $(v, v+x_{i,k}]$. We denote such an arrangement as X_i. Similarly for each woman b_j such that $x_{i,j} > 0$ for some i we construct an arrangement Y_j. The difference is that for women we order intervals in the increasing order of preference and we again order tied variables arbitrarily. Let us by $T_i(j)$ denote the interval spanned by all intervals $x_{i,k}$ such that $b_k =_{a_i} b_j$. Note that such intervals are next to each other in the arrangement. Similarly by $T'_j(i)$ we denote the interval spanned by $x_{k,j}$ such that $a_k =_{b_j} a_i$.

Let u be any real number belonging to $(0,1]$. We first construct an auxiliary graph $H_u = (A' \cup B', F)$ as follows. Let $A' \subseteq A$ and $B' \subseteq B$ be sets of men and women for which we created arrangements X_i, Y_j, i.e., $A' = \{a_i : 1 \le i \le p \wedge \exists j(x_{i,j} > 0)\}$ and $B' = \{b_j : 1 \le j \le q \wedge \exists i(x_{i,j} > 0)\}$. For each man a_i if u lies in the subinterval spanned by $x_{i,j}$, we add to F edges corresponding to variables in the tie $T_i(j)$ in X_i. Obviously each man is indifferent between all the edges incident to him. We now prove that this holds for women as well. Note that from Lemma 12 it follows that if $x_{i,j} > 0$ then intervals $T_i(j)$ and $T'_j(i)$ coincide in arrangements X_i and Y_j. Let us assume that there are two edges (a_i, b_j), (a_k, b_j) in H_u. Then u lies in the subintervals spanned by $T_i(j)$ and $T_k(j)$. So in particular u lies in the subintervals spanned by $T'_j(i)$ and $T'_j(k)$. This implies that $T'_j(i)$ and $T'_j(k)$ are identical so we have $a_i =_{b_j} a_k$. Hence each woman is indifferent between edges incident to her in H_u.

We are going to show that there exists a perfect matching in H_u. Let us first create a variable y. For each $i \in A'$ we consider X_i, and assume that u lies in the subinterval spanned by $x_{i,j}$. For each k such that $x_{i,k} > 0$ and $b_k =_{a_i} b_j$ we set $y_{i,k} = \frac{x_{i,k}}{|T_i(j)|}$, where $|T_i(j)|$ is the length of $T_i(j)$. From the definition we know that for each i we have $\sum_j y_{i,j} = 1$ and similarly for each j we have $\sum_i y_{i,j} = 1$. Thus y is a fractional perfect matching in H_u and there exists a perfect matching M_u in H_u (see [22] for the details of the construction).

We now show that M_u is strongly stable. Let $a_i \in A$ be a man matched in M_u to some b_j. Assume that $b_k \succ_{a_i} b_j$. In X_i the tie corresponding to $x_{i,k}$ lies to the left of the tie corresponding to $x_{i,j}$. Recall that the tie corresponding to $x_{i,k}$ coincides in X_i and Y_k, thus from the construction of Y_k it follows that b_k strictly prefers $M_u(b_k)$ to a_i, hence (a_i, b_k) does not block the matching. We can analogously prove that if there exists a_k such that $a_k \succ_{b_j} a_i$ then (a_k, b_j) does not block the matching. Thus M_u is strongly stable.

It remains to show how to express x as a convex combination of strongly stable matchings. Note that as we move u from 0 to 1 graphs H_u change. We denote a sequence of graphs that we can obtain in this way by $H_1, H_2, \ldots, H_q$ and let $(I_i, I_{i+1}]$ be an interval corresponding to H_i for each i. From the discussion above we know that each of the graphs H_i admits a perfect matching M_i. Let y_i be the incidence vector of M_i. One can easily see that $x = \sum_{i=1}^{q-1}(I_{i+1} - I_i)y_i$, thus the theorem holds. ◀

4 Maximum Weight Strongly Stable Matching

In this section we give an efficient algorithm for computing a maximum weight strongly stable matching. We first show that given a matching M we can easily find a maximum weight matching amongst the ones belonging to $[M]$.

▶ **Definition 14.** We say that a strongly stable matching M is *heavy* if for each strongly stable matching M' such that $M' \in [M]$ we have $w(M) \ge w(M')$.

In order to characterise heavy matchings belonging to $[M]$ we first extend the definition of H_M (see Section 2) so that each edge is of the same weight as in G. The following lemma is a direct consequence of Lemma 3 and allows us to find a heavy matching belonging to a given equivalence class.

▶ **Lemma 15.** *Let $M \in \mathcal{M}(\mathcal{I})$. Then M' is a heavy strongly stable matching such that $M' \sim M$ if and only if M' is a maximum weight perfect matching in H_M.*

In order to solve the general problem we need the following definition.

▶ **Definition 16.** Let $S = (M_0, M_1, \ldots, M_z)$ be a maximal sequence of strongly stable matchings. We say that a sequence S is a *heavy maximal sequence of strongly stable matchings* if M_i is heavy for each $0 \leq i \leq z$.

It turns out that once a heavy maximal sequence of strongly stable matchings is computed, we are able to efficiently find a heavy matching in each equivalence class.

▶ **Theorem 17.** *Let $S = (M_0, M_1, \ldots, M_z)$ be a heavy maximal sequence of strongly stable matchings of $\mathcal{I}$. Then for each closed subset of rotations $X \subseteq D(\mathcal{I})$ the matching $f_S(X)$ is heavy.*

Before we prove Theorem 17 we need to introduce a few more definitions.

Let M and N be two strongly stable matchings such that N is a strict successor of M. We denote by $\rho = \rho([M], [N])$ a rotation transforming $[M]$ into $[N]$ and by $V_\rho = \{v : \exists(a, b)(v, a, b) \in \rho\}$ we denote the set of all vertices that change their rank when ρ is applied.

Now we define two auxiliary graphs $K_\rho = (V_\rho, E_\rho)$ and $L_\rho = (V_\rho, F_\rho)$. The intuition behind these two graphs is as follows. The graph L_ρ contains all the edges of the original graph that have both endpoints in V_ρ and can potentially belong to matchings from $[M]$. The graph K_ρ fulfills a similar role for the class $[N]$. The set E_ρ is defined as $E_\rho = \{(a, b) \in E(\mathcal{I}) : \exists(c, d)((a, c, rank(a, b)) \in \rho \wedge (b, d, rank(b, a)) \in \rho)\}$. Similarly we define $F_\rho = \{(a, b) \in E(\mathcal{I}) : \exists(c, d)((a, rank(a, b), c) \in \rho \wedge (b, rank(b, a), d) \in \rho)\}$.

▶ **Lemma 18.** *Let M, N be two strongly stable matchings such that N is a strict successor of M. Assume that M is a heavy matching and $\rho = \rho([M], [N])$ is a rotation transforming $[M]$ into $[N]$. Additionally let $X \in [N]$.*

Then X is a heavy matching if and only if the following hold:

1. *Edges of the set $X \cap E_\rho$ form a maximum weight perfect matching of K_ρ.*
2. *$w(\{(a, b) \in M : a, b \notin V_\rho\}) = w(\{(a, b) \in X : a, b \notin V_\rho\})$.*

Note that given a heavy matching M we can obtain a heavy matching $N' \in [N]$. In order to do so we first compute a maximum weight perfect matching X in K_ρ and then simply take $N' = M \cup X \setminus (M \cap (V_\rho \times V_\rho))$. The above lemma implies that N' is heavy. We are now ready to present the proof of Theorem 17.

Proof of Theorem 17. Let us assume by contradiction that there is a subset $Y \in D(\mathcal{I})$ of rotations such that $f_S(Y)$ is not heavy. Let $Y = \{\rho_1, \rho_2, \ldots, \rho_k\}$. We can assume without the loss of generality that rotations of Y are ordered so that there are no i, j such that $i < j$ and $\rho_i \succ \rho_j$.

We first define a sequence $N_0, N_1, \ldots, N_k$ of strongly stable matchings. Let $N_0 = M_0$ and $N_i = N_{i-1} \oplus C_S(\rho_i)$ for $0 < i \leq k$. From the initial assumptions we know that $N_k = f_S(Y)$. Moreover we can assume without the loss of generality that N_k is the first matching in the sequence $N_0, N_1, \ldots, N_k$ which is not heavy. Let us denote $\rho' = \rho([N_{k-1}], [N_k])$. From the definition of S we know that there exists j such that $\rho([M_{j-1}], [M_j]) = \rho'$.

From Lemma 18 we know that $M_j \cap E_{\rho'}$ is a maximum weight perfect matching in $K_{\rho'}$. Additionally since N_{k-1} is a heavy matching and N_k is not a heavy matching, we know that at least one of conditions (1) and (2) of Lemma 18 does not hold for N_{k-1} and N_k. We are going to prove that (2) holds for N_{k-1} and N_k, i.e., $w(\{(a, b) \in N_{k-1} : a, b \notin V_\rho\}) =$

$w(\{(a,b) \in N_k : a,b \notin V_\rho\})$. Recall that $N_{k-1} \oplus N_k = C_S(\rho') = M_{j-1} \oplus M_j$. Let C be any cycle belonging to $C_S(\rho')$ such that $C \cap (V_{\rho'} \times V_{\rho'}) = \emptyset$ (Note Lemma 4 and the definition of ρ' imply that each cycle of $C_S(\rho')$ is either contained in $V_{\rho'}$ or disjoint with this set). Each vertex of C is indifferent between edges of C since this cycle does not belong to ρ'. The cycle C can be partitioned into two matchings $C \cap M_{j-1}$ and $C \cap M_j$. One can easily see that we have $w(C \cap M_{j-1}) = w(C \cap M_j)$ as otherwise either $w(M_{j-1} \oplus C) > w(M_{j-1})$ or $w(M_j \oplus C) > w(M_j)$ would hold and this would contradict the assumption that M_{j-1} and M_j are both heavy matchings. This implies that $w(N_{k-1}) = w(N_{k-1} \oplus C)$ and the weight of the matching does not change when cycles of $C_S(\rho')$ which do not belong to the rotation are applied, thus $w(\{(a,b) \in N_{k-1} : a,b \notin V_\rho\}) = w(\{(a,b) \in N_k : a,b \notin V_\rho\})$ holds.

From Lemma 18 it follows that $N_k \cap E_{\rho'}$ is not a maximum weight perfect matching in $K_{\rho'}$. Thus we have $w(M_j \cap E_{\rho'}) > w(N_k \cap E_{\rho'})$.

Let C be any cycle of $C_S(\rho')$ belonging to the rotation ρ'. We will prove that $C \cap N_k = C \cap M_j$. To see this consider any man m belonging to C. Exactly two edges (m, w_1), (m, w_2) of $N_{k-1} \oplus N_k$ are incident to m. Since $m \in V_{\rho'}$ we can assume without the loss of generality that $w_1 \succ_m w_2$. From $N_{k-1} \succ N_k$ it follows that $(m, w_2) \in N_k$. We can similarly prove that $(m, w_2) \in M_j$. This implies that $C \cap N_k = C \cap M_j$ holds. Hence we also have $M_j \cap E_{\rho'} = N_k \cap E_{\rho'}$ - a contradiction with the fact that $w(M_j \cap E_{\rho'}) > w(N_k \cap E_{\rho'})$.

From the above discussion it follows that the lemma holds. ◀

Below we explain how a heavy maximum sequence of strongly stable matchings can be exploited to solve the maximum weight strongly stable matching problem. It turns out that given such a sequence, our problem can be reduced to computing a maximum weight closed subset of a poset, similarly as in the case of no ties.

Let us consider the poset of rotations $D(\mathcal{I}, \prec)$. We are going to assign a weight to each element of $D(\mathcal{I})$. Let $S' = (M'_0, M'_1, \ldots, M'_z)$ be a heavy maximal sequence of strongly stable matchings. Assume that $\rho' \in D(\mathcal{I})$ is a rotation such that $\rho' = \rho([M'_{i-1}], [M'_i])$. Let us denote $w_{S'}(\rho') = w(M'_i) - w(M'_{i-1})$. We first show that the weight of a rotation does not depend on the choice of a maximal heavy sequence of strongly stable matchings.

▶ **Lemma 19.** *Let S_1, S_2 be two heavy maximal sequences of strongly stable matchings and let $\rho \in D(\mathcal{I})$ be a rotation. Then $w_{S_1}(\rho) = w_{S_2}(\rho)$.*

From now on we are going to skip the subscript in the definition of w, i.e., we write $w(\rho)$ instead of $w_{S'}(\rho)$. We slightly abuse the notation here, but it should not cause any confusion. From Theorem 10 each closed subset of rotations $X \subseteq D(\mathcal{I})$ corresponds to a certain equivalence class $[M]$ of $\sim$. It turns out that given weights of rotations belonging to X we can determine the weight of a heavy strongly stable matching belonging to $[M]$.

▶ **Lemma 20.** *Assume that M is a heavy matching and that M_0 is a heavy man optimal matching. Let $X_M \subseteq D(\mathcal{I})$ be a subset of rotations corresponding to $[M]$. Then $w(M) = w(M_0) + \sum_{\rho \in X_M} w(\rho)$.*

The following theorem is a direct consequence of the above lemma.

▶ **Theorem 21.** *Let M be a heavy matching and let $X_M \subseteq D(\mathcal{I})$ be a subset of rotations corresponding to M. Then M is a maximum weight matching of $\mathcal{I}$ if and only if X_M is a maximum weight closed subset of $D(\mathcal{I}, \prec)$ with respect to the weight function w.*

A maximum weight closed subset of a poset is a classical problem. In [6] Gusfield and Irving show a reduction to the minimum s-t cut in a graph with $O(m)$ vertices and edges.

Algorithm 1 For computing a heavy maximal sequence of strongly stable matchings.

Input: $\mathcal{I}$ - a solvable instance of SMTI

```
compute a maximal sequence of strongly stable matchings S = (M_0, M_1, ..., M_z)
compute a heavy matching M'_0 ∈ [M_0]
for i = 1, 2, ..., z do
    let ρ_i = ρ([M_{i-1}], [M_i])
    compute a maximum weight perfect matching Y in K_{ρ_i}
    let M'_i = M'_{i-1} ∪ Y \ (M'_{i-1} ∩ (V_{ρ_i} × V_{ρ_i}))
return (M'_0, M'_1, ..., M'_z)
```

This problem can be solved with a standard maximum flow computation, however in the special case of posets obtained from instances of SMTI we can construct the minimum cut in $O(nm\log(Wn))$ time or in $O(nm)$ time if $W = O(\min\{n, \frac{m}{\log^2 m}\})$.

To achieve these complexity bounds we use algorithms of Feder [2]. The author shows that a maximum flow in an uncapacitated network with m edges and of explicit width q can be found in $O(qm\log(K))$ time. It can be shown that in our case we have $q \leq n$ and $log(K) \leq log(Wn)$, thus the runtime is $O(nm\log(Wn))$. Feder also shows that a maximum flow of value K in an uncapacitated network with m edges can be found in $O(m\sqrt{K} + K\log^2(m))$ time, implying an $O(nm)$ algorithm if $W = O(\min\{n, \frac{m}{\log^2 m}\})$.

More details about algorithms of Feder, the reduction to the minimum cut problem and missing proofs from this section are given in the full version of the paper.

It is important to note that none of the theorems in this section require any additional assumptions about the weight function w.

5 Computing a Heavy Sequence

We first show a very simple $O(mMWPM)$ algorithm for computing a heavy sequence where $MWPM$ is the time complexity of finding a maximum weight perfect matching. Then we improve its time complexity to either $O(nm\log n)$ or $O(nm + \sqrt{n}m\log(Wn))$ depending on whether we use classical $O(nm\log n)$ algorithm [22] or $O(\sqrt{n}m\log(Wn))$ algorithm by Gabow and Tarjan [4] for finding a maximum weight perfect matching.

We first compute a maximal sequence of strongly stable matchings $S = (M_0, M_1, \ldots, M_z)$. Recall that from Lemma 15 given a strongly stable matching M_i we can find a heavy matching $M'_i \in [M_i]$ with a single maximum weight perfect matching computation. We simply apply Lemma 15 to each of the matchings $M_0, M_1, \ldots, M_z$ and obtain a heavy maximal sequence of strongly stable matchings $M'_0, M'_1, \ldots, M'_z$. Such an algorithm obviously works in $O(mMWPM)$ time.

Let us now discuss Algorithm 1. We first compute a maximal sequence of strongly stable matchings $S = (M_0, M_1, \ldots, M_z)$. Then we find a heavy matching $M'_0 \in [M_0]$ using Lemma 15. In the next step we construct graphs K_{ρ_i} where $\rho_i = \rho([M_i], [M_{i+1}])$ for each $0 \leq i < z$. Then for each i we compute a maximum weight perfect matching of K_{ρ_i}. It can be easily proven that each edge of G may appear only in one of the graphs K_{ρ_i}, thus the following holds: $|E(K_{\rho_0})| + |E(K_{\rho_1})| + \ldots + |E(K_{\rho_{z-1}})| = O(m)$ and overall it takes either $O(nm\log n)$ or $O(nm + \sqrt{n}m\log(Wn))$ time to compute all maximum weight matchings.

With the aid of Lemma 18 we can construct a heavy maximal sequence of strongly stable matchings $(M'_0, M'_1, \ldots, M'_z)$. In order to do this we simply compute a heavy matching M'_i based on previously computed M'_{i-1} and a maximum weight matching of K_{ρ_i}.

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