

Lower Bounds for Dynamic Programming on Planar Graphs of Bounded Cutwidth

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Abstract

Many combinatorial problems can be solved in time $\mathcal{O}^*(c^{tw})$ on graphs of treewidth tw , for a problem-specific constant c . In several cases, matching upper and lower bounds on c are known based on the Strong Exponential Time Hypothesis (SETH). In this paper we investigate the complexity of solving problems on graphs of bounded cutwidth, a graph parameter that takes larger values than treewidth. We strengthen earlier treewidth-based lower bounds to show that, assuming SETH, INDEPENDENT SET cannot be solved in $\mathcal{O}^*((2 - \varepsilon)^{ctw})$ time, and DOMINATING SET cannot be solved in $\mathcal{O}^*((3 - \varepsilon)^{ctw})$ time. By designing a new crossover gadget, we extend these lower bounds even to planar graphs of bounded cutwidth or treewidth. Hence planarity does not help when solving INDEPENDENT SET or DOMINATING SET on graphs of bounded width. This sharply contrasts the fact that in many settings, planarity allows problems to be solved much more efficiently.

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1 Introduction

Dynamic programming on graphs of bounded treewidth is a powerful tool in the algorithm designer’s toolbox, which has many applications (cf. [5]) and is captured by several meta-theorems [7, 25]. Through clever use of techniques such as Möbius transformation, fast subset convolution [2, 28], cut & count [9], and representative sets [4, 8, 14], algorithms were developed that can solve numerous combinatorial problems on graphs of treewidth tw in $\mathcal{O}^*(c^{tw})$ time, for a problem-specific constant c . In recent work [23], it was shown that under the *Strong Exponential Time Hypothesis* (SETH, see [18, 19]), the base of the exponent c achieved by the best-known algorithm is actually optimal for DOMINATING SET ($c = 3$) and INDEPENDENT SET ($c = 2$), amongst others. This prompts the following questions:

1. Do faster algorithms exist for bounded-treewidth graphs that are *planar*?
2. Do faster algorithms exist for a more restrictive graph parameter, such as cutwidth?

It turns out that these questions are related, because the nature of cutwidth allows crossover gadgets to be inserted to planarize a graph without increasing its width significantly.

Before going into our results, we briefly motivate these questions. There is a rich bidimensionality theory (cf. [10]) of how the planarity of a graph can be exploited to obtain better algorithms than in the nonplanar case, leading to what has been called the *square-root phenomenon* [24]: in several settings, parameterized algorithms on planar graphs can be faster by a square-root factor in the exponent, compared to their nonplanar counterparts. Hence it may be tempting to believe that problems on bounded-width graphs can be solved more efficiently when they are planar. Lokshstanov et al. [23, §9] explicitly ask whether their SETH-based lower bounds continue to apply for planar graphs. The same problem is posed by Baste and Sau [1, p. 3] in their investigation on the influence of planarity when solving connectivity problems parameterized by treewidth. This motivates question 1.

When faced with lower bounds for the parameterization by treewidth, it is natural to investigate whether these continue to hold for more restrictive graph parameters. We work with the parameter cutwidth since it is one of the classic graph layout parameters (cf. [11]) which takes larger values than treewidth [22], and has been the subject of frequent study [17, 26, 27]. In their original work, Lokshstanov et al. [23] showed that their lower bounds also hold for pathwidth instead of treewidth. However, the parameterization by *cutwidth* has so far not been considered, which leads us to question 2. (See Section 2 for the definition of cutwidth.)

Our results. We answer questions 1 and 2 for the problems INDEPENDENT SET and DOMINATING SET, which are formally defined in Section 2. Our conceptual contribution towards answering question 1 comes from the following insight: any graph G can be drawn in the plane (generally with crossings) such that the graph G' obtained by replacing each crossing by a vertex of degree four, does not have larger cutwidth than G . Hence the property of having bounded cutwidth can be preserved while planarizing the graph, which was independently² discovered by Eppstein [13]. When we planarize by replacing each crossing by a planar crossover gadget H instead of a single vertex, then we obtain $\text{ctw}(G') \leq \text{ctw}(G) + \text{ctw}(H) + 4$ if the endpoints of the crossing edges each obtain at most one neighbor in the crossover gadget. This gives a means to reduce a problem instance on a general graph of bounded

² We learned of Eppstein’s result while a previous version of this work was under submission at a different venue; see Footnote 2 in [13]. Our previous manuscript, cited by Eppstein, was later split into two separate parts due to its excessive length. The present paper is one part, and [21] is the other.

cutwidth to a planar graph of bounded cutwidth, if a suitable crossover gadget is available. The parameter cutwidth is special in this regard: one cannot planarize a drawing of $K_{3,n}$ while keeping the pathwidth or treewidth constant [12, 13].

For the INDEPENDENT SET problem, the crossover gadget developed by Garey, Johnson, and Stockmeyer [15] can be used in the process described above. Together with the observation that the SETH-based lower bound construction by Lokshtanov et al. [23] for the treewidth parameterization also works for the cutwidth parameterization, this yields our first result.³

► **Theorem 1.** *Assuming SETH, there is no $\varepsilon > 0$ such that INDEPENDENT SET on a planar graph G given along with a linear layout of cutwidth k can be solved in time $\mathcal{O}^*((2 - \varepsilon)^k)$.*

For the DOMINATING SET problem, more work is needed to obtain a lower bound for planar graphs of bounded cutwidth. While the lower bound construction of Lokshtanov et al. [23] also works for the parameter cutwidth after a minor tweak, no crossover gadget for the DOMINATING SET problem was known. Our main technical contribution therefore consists of the design of a crossover gadget for DOMINATING SET, which we believe to be of independent interest. Together with the framework above, this gives our second result.

► **Theorem 2.** *Assuming SETH, there is no $\varepsilon > 0$ such that DOMINATING SET on a planar graph G given along with a linear layout of cutwidth k can be solved in time $\mathcal{O}^*((3 - \varepsilon)^k)$.*

Since any linear ordering of cutwidth k can be transformed into a tree decomposition of width at most k in polynomial time (cf. [3, Theorem 47]), the lower bounds of Theorems 1 and 2 also apply to the parameterization by treewidth. Hence our work resolves the question raised by Lokshtanov et al. [23] and by Baste and Sau [1] whether the SETH-lower bounds for INDEPENDENT SET and DOMINATING SET parameterized by treewidth also apply for planar graphs.

Organization. In Section 2 we provide preliminaries. In Section 3 we present a general theorem for planarizing graphs of bounded cutwidth, using a crossover gadget. It leads to a proof of Theorem 1. In Section 4 we prove Theorem 2. Finally, we provide some conclusions in Section 5. Due to space restrictions, proofs for statements marked (★) have been deferred to the full version [16].

2 Preliminaries

We use \mathbb{N} to denote the natural numbers, including 0. For a positive integer n and a set X we use $\binom{X}{n}$ to denote the collection of all subsets of X of size n . The *power set* of X is denoted 2^X . The set $\{1, \dots, n\}$ is abbreviated as $[n]$. The \mathcal{O}^* notation suppresses polynomial factors in the input size n , such that $\mathcal{O}^*(f(k))$ is shorthand for $\mathcal{O}(f(k)n^{\mathcal{O}(1)})$. All our logarithms have base two.

We consider finite, simple, and undirected graphs G , consisting of a vertex set $V(G)$ and edge set $E(G) \subseteq \binom{V(G)}{2}$. The neighbors of a vertex v in G are denoted $N_G(v)$. The closed neighborhood of v is $N_G[v] := N_G(v) \cup \{v\}$. For a vertex set $S \subseteq V(G)$ the open neighborhood is $N_G(S) := \bigcup_{v \in S} N_G(v) \setminus S$ and the closed neighborhood is $N_G[S] := N_G(S) \cup S$. The subgraph of G induced by a vertex subset $U \subseteq V(G)$ is denoted $G[U]$. The operation of *identifying vertices u and v* in a graph G results in the graph G' that is obtained from G by replacing the two vertices u and v by a new vertex w with $N_{G'}(w) = N_G(\{u, v\})$.

³ The analogous lower bound of $\Omega((2 - \varepsilon)^k)$ for solving INDEPENDENT SET on planar graphs of pathwidth k was already observed by Jansen and Wulms [20], based on an elaborate ad-hoc argument.

An *independent set* is a set of pairwise nonadjacent vertices. A *vertex cover* in a graph G is a set $S \subseteq V(G)$ such that S contains at least one endpoint from every edge. A set $S \subseteq V(G)$ *dominates* the vertices $N_G[S]$. A *dominating set* is a vertex set S such that $N_G[S] = V(G)$. The associated decision problems ask, given a graph G and integer t , whether an independent set (dominating set) of size t exists in G . The size of a maximum independent set (resp. minimum dominating set) in G is denoted $\text{OPT}_{\text{IS}}(G)$ (resp. $\text{OPT}_{\text{DS}}(G)$). The q -SAT problem asks whether a given Boolean formula, in conjunctive normal form with clauses of size at most q , has a satisfying assignment.

► **Strong Exponential Time Hypothesis** ([18, 19]). *For every $\varepsilon > 0$, there is a constant q such that q -SAT on n variables cannot be solved in time $\mathcal{O}^*((2 - \varepsilon)^n)$.*

Drawings. A *drawing* of a graph G is a function ψ that assigns a unique point $\psi(v) \in \mathbb{R}^2$ to each vertex $v \in V(G)$, and a curve $\psi(e) \subseteq \mathbb{R}^2$ to each edge $e \in E(G)$, such that the following four conditions hold. (1) For $e = \{u, v\} \in E(G)$, the endpoints of $\psi(e)$ are exactly $\psi(u)$ and $\psi(v)$. (2) The interior of a curve $\psi(e)$ does not contain the image of any vertex. (3) No three curves representing edges intersect in a common point, except possibly at their endpoints. (4) The interiors of the curves $\psi(e), \psi(e')$ for distinct edges intersect in at most one point. If the interiors of all the curves representing edges are pairwise-disjoint, then we have a *planar drawing*. In this paper we combine (nonplanar) drawings with crossover gadgets to build planar drawings. A graph is *planar* if it admits a planar drawing.

Cutwidth. For an n -vertex graph G , a *linear layout* of G is a linear ordering of its vertex set, as given by a bijection $\pi: V(G) \rightarrow [n]$. The *cutwidth* of G with respect to the layout π is:

$$\text{ctw}_\pi(G) = \max_{1 \leq i < n} \left| \{ \{u, v\} \in E(G) \mid \pi(u) \leq i \wedge \pi(v) > i \} \right|.$$

The cutwidth $\text{ctw}(G)$ of a graph G is the minimum cutwidth attained by any linear layout. It is well-known that $\text{ctw}(G) \geq \text{pw}(G) \geq \text{tw}(G)$, where the latter denote the pathwidth and treewidth of G , respectively (cf. [3]).

3 Planarizing graphs while preserving cutwidth

In this section we show how to planarize a graph without blowing up its cutwidth. An intuitive way to think about cutwidth is to consider the vertices as being placed on a horizontal line in the order dictated by the layout π , with edges drawn as x -monotone curves. For any position i we consider the gap between vertex $\pi^{-1}(i)$ and $\pi^{-1}(i + 1)$, and count the edges that *cross* the gap by having one endpoint at position at most i and the other at position after i . The cutwidth of a layout is the maximum number of edges crossing any single gap; see Figure 1. The simple but useful fact on which our approach hinges is the following. If we obtain G' by replacing a crossing in the drawing by a new vertex of degree four, and we let π' be the left-to-right order of the vertices in the resulting drawing, then $\text{ctw}_\pi(G) = \text{ctw}_{\pi'}(G')$. Hence by repeating this procedure we can eliminate all crossings to obtain a planarized version of G without increasing the cutwidth. To utilize this idea in reductions, we formalize a version of this approach where we planarize the graph by inserting gadgets, rather than simply replacing crossings by degree-four vertices.

► **Definition 3.** A *crossover gadget* is a graph H with terminal vertices u, u', v, v' such that:

1. there is a planar drawing ψ of H in which all terminals lie on the outer face, and
2. there is a closed curve intersecting the drawing ψ only in the terminals, which visits the terminals in the order u, v, u', v' .

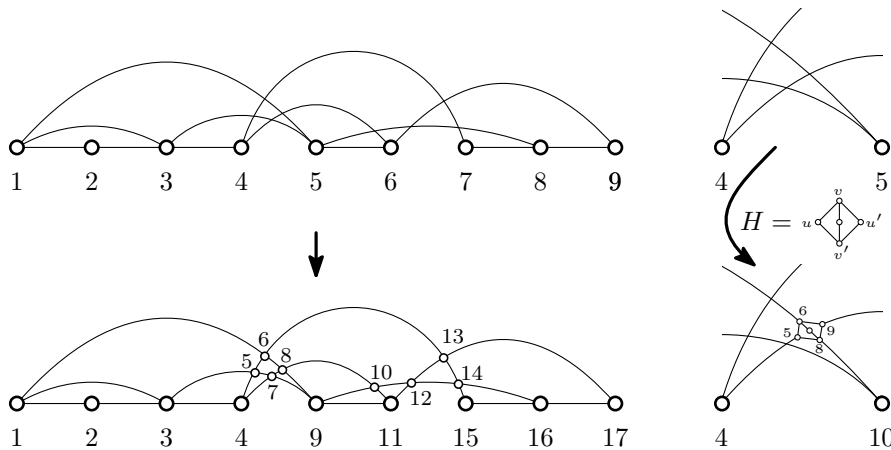


Figure 1 Top-left: a linear layout π of a graph G with $\text{ctw}_\pi(G) = 4$. The largest cutsize is attained after vertices 4 and 5. Bottom-left: after inserting vertices at the crossings to obtain G' and extending π to π' based on the x -coordinates of the inserted vertices, we have $\text{ctw}_\pi(G) = \text{ctw}_{\pi'}(G')$. Top-right: enlarged view. Bottom-right: replacing a crossing by gadget H .

► **Definition 4.** Let $\{a, b\}$ and $\{c, d\}$ be disjoint edges of a graph G , and let H be a crossover gadget. The operation of *replacing* $\{\{a, b\}, \{c, d\}\}$ by H removes edges $\{a, b\}$ and $\{c, d\}$, inserts a new copy of the graph H , and inserts the edges $\{a, u\}, \{u', b\}, \{c, v\}, \{v', d\}$.

For a crossover gadget to be useful to planarize instances of a decision problem, a replacement should have a predictable effect on the answer. To formalize this, we say that a *decision problem Π on graphs* is a decision problem whose input consists of a graph G and integer t .

► **Definition 5.** A crossover gadget H is *useful for a decision problem Π on graphs* if there exists an integer c_Π such that the following holds. If (G, t) is an instance of Π containing disjoint edges $\{a, b\}, \{c, d\}$, and G' is the result of replacing these edges by H , then (G, t) is a YES-instance of Π if and only if $(G', t + c_\Pi)$ is a YES-instance of Π .

The following theorem proves that a useful crossover gadget can be used to efficiently planarize instances without blowing up their cutwidth.

► **Theorem 6.** *If H is a crossover gadget that is useful for decision problem Π on graphs, then there is a polynomial-time algorithm that, given an instance (G, t) of Π and a linear layout π of G , outputs an instance (G', t') and a linear layout π' of G' such that:*

1. G' is planar,
2. $\text{ctw}_{\pi'}(G') \leq \text{ctw}_\pi(G) + \text{ctw}(H) + 4$, and
3. (G, t) is a YES-instance of Π if and only if (G', t') is a YES-instance of Π .

Proof. Consider the crossover gadget H for Π with terminals u, u', v, v' . Let π_H be a linear layout of H of minimum cutwidth, which is hardcoded into the algorithm along with the integer c_Π described in Definition 5.

Let (G, t) be an instance of Π with a linear layout π . We start by constructing a (nonplanar) drawing ψ of G with the following properties.

1. The vertices of G are placed on the x -axis, in the order dictated by π .
2. The image of each edge of G is a strictly x -monotone curve.
3. If the drawings of two edges intersect in their interior, then their endpoints are all distinct and the corresponding curves properly cross; they do not only touch.

4. For each pair of edges, their drawings intersect in at most one point.
5. The x -coordinates of all crossings are distinct from each other, and from the x -coordinates of the vertices.

It is easy to see that such a drawing always exists and can be found in polynomial time; we omit the details as they are not interesting. Properties 1 and 2 together ensure that for any $i \in [|V(G)| - 1]$, the set of edges that cross the gap after vertex $\pi^{-1}(i)$ in the linear layout is exactly the set of edges intersected by a vertical line between $\pi^{-1}(i)$ and $\pi^{-1}(i + 1)$, which therefore has size at most $\text{ctw}_\pi(G)$. We will use this property later.

The algorithm replaces the crossings one by one. If two edges $\{a, b\}$ and $\{c, d\}$ intersect in their interior, then their endpoints are all distinct by (3) and they properly cross. Hence we can replace these two edges by a copy of H as in Definition 4. Since there is a planar drawing of H with the terminals alternating along the outer face, after possibly swapping the labels of a and b , and of c and d , the drawing can be updated so that the crossing between $\{a, b\}$ and $\{c, d\}$ is eliminated. Since each of a, b, c, d is made adjacent to exactly one vertex of H , the replacement can be done such that the remaining crossings are in exactly the same locations as before; see the right side of Figure 1. When inserting the crossover gadget, we scale it down sufficiently far that the following holds: all vertices and crossings that were originally on the left of the crossing between $\{a, b\}$ and $\{c, d\}$ lie to the left of all vertices that are inserted to replace this crossing; and all vertices and crossings that were on the right of the $\{\{a, b\}, \{c, d\}\}$ crossing, lie to the right of all vertices inserted for its replacement.

Since each pair of edges intersects at most once by (4), the number of crossings is $\mathcal{O}(|V(G)|^2)$. Hence in polynomial time we can replace all crossings by copies of H to arrive at a graph G' . If ℓ is the number of replaced crossings, then we set $t' := t + \ell \cdot c_\Pi$. By Definition 5 and transitivity it follows that (G, t) is a YES-instance of Π if and only if (G', t') is a YES-instance. By construction, G' is planar. It remains to define a linear layout of G' and bound its cutwidth.

The layout π' of G' is defined as follows. Let the *elements* of the original drawing ψ of G consist of its vertices and its crossings. The elements of ψ are linearly ordered by their x -coordinates, by (5). The linear layout π' of G' has one block per element of ψ , and these blocks are ordered according to the x -coordinates of the corresponding element. For elements that consist of a vertex v , the block simply consists of v . For elements that consist of a crossing X , the block consists of the vertices of the copy of H that was inserted to replace X , in the order dictated by π_H . It is easy to see that π' can be constructed in polynomial time.

We classify the edges of G' into two types. We have *internal* edges, which are edges within an inserted copy of H , and we have *external edges* which connect two different copies of H , or which connect a vertex of $V(G) \cap V(G')$ to a copy of H . Using this classification we argue that for an arbitrary vertex v^* of G' , the cut crossing the gap after vertex v^* in π' contains at most $\text{ctw}_\pi(G) + \text{ctw}(H) + 4$ edges. To do so, we distinguish two cases depending on whether v^* is an original vertex from G , or was inserted as part of a copy of H .

► **Claim 7.** *If $v^* \in V(G) \cap V(G')$, then the size of the cut after v^* in π' is at most $\text{ctw}_\pi(G)$.*

Proof. The layout π' consists of blocks, and $v^* \in V(G) \cap V(G')$ is a block. So for each copy C of a crossover gadget, the vertices of C all appear on the same side of v^* in the ordering. Hence no internal edge of C crosses the cut after v^* , implying that no internal edge is in the cut. Each external edge crossing the cut is (a segment of) an edge of G that is intersected by a vertical line after v^* in the drawing ψ ; see Figure 1. As such a line intersects at most $\text{ctw}_\pi(G)$ edges as observed above, the cut after v^* has size at most $\text{ctw}_\pi(G)$. ◻

► **Claim 8.** *If $v^* \in V(G')$ is a vertex of a copy C of a crossover gadget that was inserted to replace a crossing X , then the size of the cut after v^* in π' is at most $\text{ctw}_\pi(G) + \text{ctw}(H) + 4$.*

Proof. The number of internal edges in the cut after v^* is at most $\text{ctw}(H)$, since the only internal edges in the cut all belong to the same copy C that contains v^* and we ordered them according to an optimal layout π_H . There are at most four external edges incident on a vertex of C , which contribute at most four to the cut. Finally, for each of the remaining external edges in the cut there is a unique edge of G intersected by a vertical line through crossing X in the drawing ψ . As at most $\text{ctw}_\pi(G)$ edges are intersected by any vertical line, as observed above, it follows that the size of the cut is at most $\text{ctw}_\pi(G) + \text{ctw}(H) + 4$. ◻

The two claims together show that any gap in the ordering π' is crossed by at most $\text{ctw}_\pi(G) + \text{ctw}(H) + 4$ edges, which bounds the cutwidth of G' as required. ◀

Using Theorem 6 we can now elegantly prove Theorem 1 by combining two known results.

► **Theorem 1.** *Assuming SETH, there is no $\varepsilon > 0$ such that INDEPENDENT SET on a planar graph G given along with a linear layout of cutwidth k can be solved in time $\mathcal{O}^*((2 - \varepsilon)^k)$.*

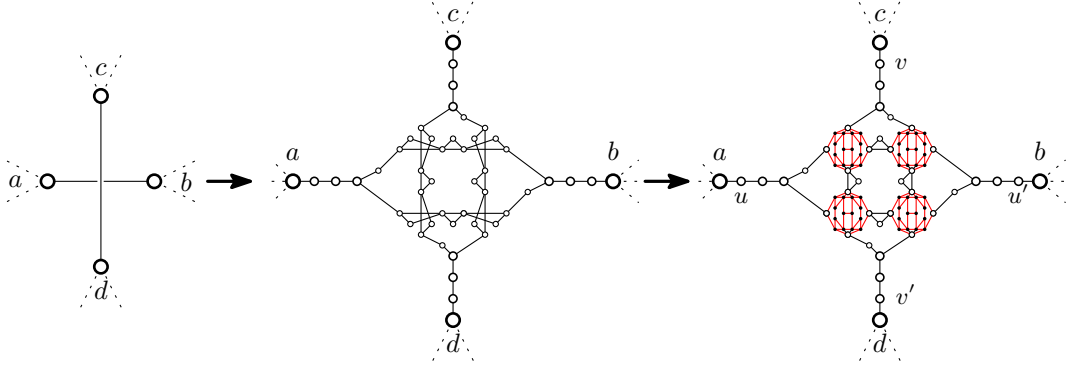
Proof. First, we observe that the crossover gadget for VERTEX COVER due to Garey, Johnson, and Stockmeyer [15, Thm. 2.7] satisfies our conditions of a useful crossover gadget. Since an n -vertex graph has a vertex cover of size k if and only if it has an independent set of size $n - k$, it also acts as a useful crossover gadget for INDEPENDENT SET with $c_\pi = 9$ (cf. [20, Proposition 20]). Second, we observe that by a different analysis of a construction due to Lokshtanov et al. [23], it follows that (assuming SETH) there is no $\varepsilon > 0$ such that INDEPENDENT SET on a graph G with a linear layout of cutwidth k can be solved in time $\mathcal{O}^*((2 - \varepsilon)^k)$. We prove this in the full version [16]. By Theorem 6, if such a runtime could be achieved on *planar* graphs of a given cutwidth, it could be achieved for a general graph as well, since the insertion of crossover gadgets increases the cutwidth by only a constant. Hence Theorem 1 follows. ◀

4 Lower bound for dominating set on planar graphs of bounded cutwidth

In this section we prove a runtime lower bound for solving DOMINATING SET on planar graphs of bounded cutwidth. Our starting point is the insight that through a minor modification, the lower bound by Lokshtanov et al. [23] for the parameterization by pathwidth can be lifted to apply to the parameterization by cutwidth as well.

► **Theorem 9 (★).** *Assuming SETH, there is no $\varepsilon > 0$ such that DOMINATING SET on a (nonplanar) graph G given along with a linear layout of cutwidth k can be solved in time $\mathcal{O}^*((3 - \varepsilon)^k)$.*

Our contribution is to extend the lower bound of Theorem 9 to apply to *planar* graphs. Following the strategy outlined in Section 3, to achieve this it suffices to develop a useful crossover gadget for DOMINATING SET as per Definition 5. Since our crossover gadget is fairly complicated (it has more than 100 vertices), we describe its design in steps. The main idea is as follows. We first show that an edge in a DOMINATING SET instance can be replaced by a longer double-path structure, which contains several triangles. Then we show that when two triangles cross, we can replace their crossing by a suitable adaptation of the VERTEX COVER crossover gadget due to Garey, Johnson, and Stockmeyer [15, Thm. 2.7].



■ **Figure 2** Overview of the method to eliminate an edge crossing in an instance of DOMINATING SET. Each edge is transformed into a double-path structure, to turn a single crossing edge into four crossing triangles (middle figure). Then each crossing triangle is replaced by a planar gadget (right figure). Some vertices have been omitted for readability. For each red edge $\{u, v\}$, there is a hidden degree-two vertex in the graph that forms a triangle with u and v .

This two-step approach is illustrated in Figure 2. We follow the same two steps in proving its correctness, starting with the insertion of the double-path structure.

► **Lemma 10.** *Let $\{x, y\}$ be an edge in a graph G . If G' is obtained from G by replacing $\{x, y\}$ by a double-path structure as shown in Figure 3, then $\text{OPT}_{\text{DS}}(G') = \text{OPT}_{\text{DS}}(G) + 6$.*

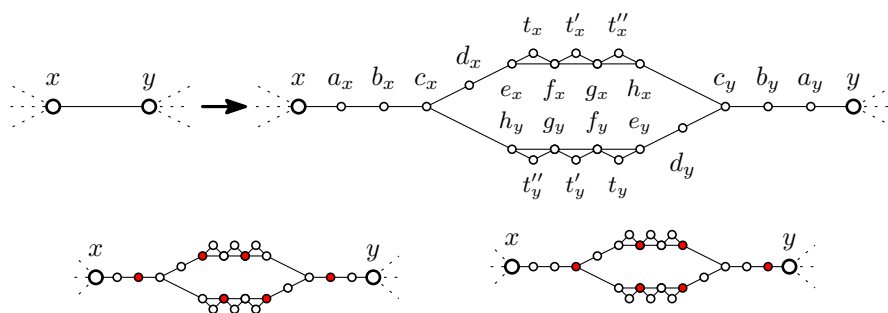
Proof. We prove equality by establishing matching upper- and lower bounds.

(\leq) To show $\text{OPT}_{\text{DS}}(G') \leq \text{OPT}_{\text{DS}}(G) + 6$, consider a minimum dominating set $S \subseteq V(G)$ of G . If $S \cap \{x, y\} = \emptyset$, or $S \cap \{x, y\} = \{x, y\}$, then the edge $\{x, y\}$ is not used to dominate vertex x or y , and therefore $S \cup \{b_x, b_y, e_x, e_y, g_x, g_y\}$ is a dominating set of size $|S| + 6 = \text{OPT}_{\text{DS}}(G) + 6$ in graph G' ; see Figure 3. If $S \cap \{x, y\} = \{x\}$, then $S \cup \{c_x, f_x, h_x, e_y, g_y, a_y\}$ is a dominating set of size $\text{OPT}_{\text{DS}}(G) + 6$ in G' : the vertex a_y takes over the role of dominating y after the direct edge $\{x, y\}$ is removed, while a_x is dominated from x . Symmetrically, if $S \cap \{x, y\} = \{y\}$, then $S \cup \{c_y, f_y, h_y, e_x, g_x, a_x\}$ is a dominating set in G' of size $\text{OPT}_{\text{DS}}(G) + 6$.

(\geq) To show $\text{OPT}_{\text{DS}}(G') \geq \text{OPT}_{\text{DS}}(G) + 6$, we instead prove $\text{OPT}_{\text{DS}}(G) \leq \text{OPT}_{\text{DS}}(G') - 6$. Consider a minimum dominating set $S' \subseteq V(G')$ of G' . Let B be the vertices in the interior of the double-path structure that was inserted into G' to replace edge $\{x, y\}$. If $|S' \cap B| \geq 7$, then $(S' \setminus B) \cup \{x\}$ is a dominating set of size at most $|S'| - 6 \leq \text{OPT}_{\text{DS}}(G') - 6$ in G , since x dominates itself and y using the edge $\{x, y\}$. We assume $|S' \cap B| \leq 6$ in the remainder. Then we have $|S' \cap B| = 6$: the closed neighborhoods of the six vertices $\{b_x, b_y, t_x, t_y, t'_x, t'_y\}$ are contained entirely within B , and are pairwise disjoint. Hence these six vertices must be dominated by six distinct vertices from B . If $S' \cap \{a_x, a_y\} = \emptyset$ then the vertices x and y are not dominated from within the double-path structure, implying that $S' \setminus B$ is a dominating set in G of size $|S'| - 6 \leq \text{OPT}_{\text{DS}}(G') - 6$. It remains to consider the case that S' contains a_x , or a_y , or both.

► **Claim 11.** *Let $B' \subseteq B$ be a set of size six that dominates the vertices $B \setminus \{a_x, a_y\}$. If B' contains a_x , then B' does not dominate a_y . Analogously, if B' contains a_y then it does not dominate a_x .*

Proof. We prove that if $a_y \in B'$, then B' does not dominate a_x . The other statement follows by symmetry. So assume for a contradiction that B' contains a_y and dominates a_x , which implies it contains a_x or b_x . Since B' dominates the interior of the double-path structure, it



■ **Figure 3** The double-path structure for DOMINATING SET is the subgraph on the top right minus the vertices x and y . Top: an edge $\{x, y\}$ is replaced by a double-path structure. Bottom-left: the interior of the double-path structure can be dominated by six vertices (in red). Bottom-right: there is a set of six vertices that dominates y and all vertices of the double-path structure except a_x .

contains at least one vertex from the closed neighborhoods of t_x, t_y, t'_x, t'_y . Since these are pairwise disjoint, and do not contain a_y, a_x , or b_x , the set B' contains a vertex from the closed neighborhood of each of $\{t_x, t_y, t'_x, t'_y\}$. Since B' has size six, besides the four vertices from these closed neighborhoods, the vertex a_y , and the one vertex in $\{a_x, b_x\}$ there can be no further vertices in B' . Hence B' does not contain c_x or d_x , as these do not occur in the stated closed neighborhoods. This implies that to dominate d_x , the set B' contains e_x . Then vertex t'_x is not dominated by the vertex from $N_{G'}[t_x]$, and must therefore be dominated by the vertex in B' from $N_{G'}[t'_x]$, implying that $g_x \in B'$. But the vertices mentioned so far do not dominate c_y , and regardless of how a vertex is chosen from the closed neighborhoods of t_y and t'_y , the resulting choice does not dominate c_y since no vertex from the closed neighborhoods of t_y, t'_y is adjacent to c_y . So c_y is not dominated by B' ; a contradiction. \square

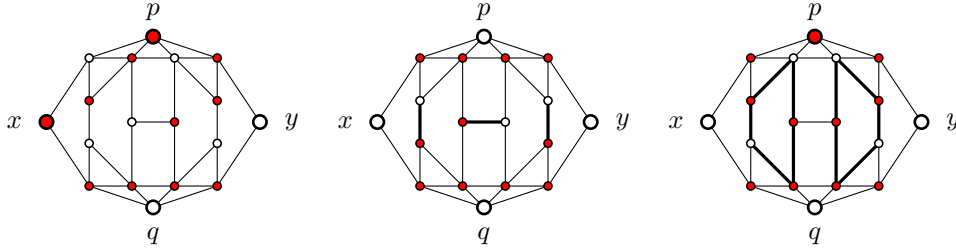
Using the claim we finish the proof. The set $B' := S \cap B$ has size six and dominates all of $B \setminus \{a_x, a_y\}$, since those vertices cannot be dominated from elsewhere. If B' contains a_x , then B' does not dominate a_y . Since S' is a dominating set, and y is the only neighbor of a_y outside of B , it follows that $y \in S'$. But then $S' \setminus B$ is a dominating set in G of size $|S'| - 6 = \text{OPT}_{\text{DS}}(G') - 6$ in G : the edge $\{x, y\}$ in G ensures that y dominates x . If B' contains a_y instead, then the symmetric argument applies. Hence $\text{OPT}_{\text{DS}}(G) \leq \text{OPT}_{\text{DS}}(G') - 6$. \blacktriangleleft

Using Lemma 10, we can replace a direct edge by a double-path structure while controlling the domination number. This allows two crossing edges to be reduced to four crossing triangles as in Figure 2. Even though more crossings are created in this way, these crossing triangles actually help to planarize the graph. The key point is that crossing triangles enforce a dominating set to locally act like a vertex cover, which allows us to exploit a known gadget for VERTEX COVER. The following two statements are useful to formalize these ideas. Recall that a vertex v is *simplicial* in a graph G if $N_G(v)$ forms a clique.

► **Observation 12.** *If I is an independent set of simplicial degree-two vertices in a graph G , then G has a minimum dominating set that contains no vertex of I .*

► **Proposition 13.** *Let U be a set of vertices in a graph G , such that for each edge $\{x, y\} \in E(G[U])$ there is a vertex $v \in V(G) \setminus U$ with $N_G(v) = \{x, y\}$. Then there is a minimum dominating set S of G such that S forms a vertex cover of $G[U]$.*

Proof. Construct a set I as follows. For each $\{x, y\} \in E(G[U])$, add a vertex $v \in V(G) \setminus U$ with $N_G(v) = \{x, y\}$ to I . Then I is an independent set of simplicial degree-two vertices.



■ **Figure 4** Three copies of the 18-vertex gadget graph H_{VC} , which has four terminals $\{x, y, p, q\}$. Left: A vertex cover for H_{VC} that contains p and q and has size eleven is shown in red. Middle: Any vertex cover for H_{VC} that does not contain p or q contains the neighbors of p and q and at least one endpoint of the three thick edges, and contains at least eleven non-terminals. Right: Any vertex cover for H_{VC} that does not contain x or y contains the four neighbors of x and y and at least three vertices from each of the two highlighted five-cycles, and contains at least ten non-terminals.

By Observation 12 there is a minimum dominating set S of G that contains no vertex of I . Then $S \cap U$ is a vertex cover of $G[U]$: for an arbitrary edge $\{x, y\} \in E(G[U])$ there is a vertex v in I whose open neighborhood is $\{x, y\}$. Since $I \cap S = \emptyset$, at least one of x and y belongs to S to dominate v . Hence the edge $\{x, y\}$ is covered by S . ◀

Proposition 13 relates minimum dominating sets to vertex covers. We therefore use a simplified version of a VERTEX COVER crossover gadget in our design. We exploit the graph H_{VC} with four terminals $\{x, y, p, q\}$ that is shown in Figure 4. It was obtained by applying the “folding” reduction rule for VERTEX COVER [6, Lemma 2.3] on the gadget by Garey et al. [15] and omitting two superfluous edges. We use the following property of the graph H_{VC} . It states that in H_{VC} , for every axis from which a vertex cover contains no terminal vertex, the number of non-terminal vertices used in a vertex cover increases.

► **Proposition 14.** *Let S be a vertex cover of H_{VC} and let $\ell \in \{0, 1, 2\}$ be the number of pairs among $\{p, q\}$ and $\{x, y\}$ from which S contains no vertices. Then $|S \setminus \{p, q, x, y\}| \geq 9 + \ell$.*

Proof. We first show $|S \setminus \{p, q, x, y\}| \geq 9$ for any vertex cover S of H_{VC} , proving the claim for $\ell = 0$. The non-terminal vertices of H_{VC} can be partitioned into four vertex-disjoint triangles and an edge that is vertex-disjoint from the triangles. From any triangle, a vertex cover contains at least two vertices. From the remaining edge, it contains at least one vertex.

If S contains no vertex of $\{p, q\}$, then as illustrated in the middle of Figure 4, S contains at least eleven non-terminals. Hence $|S \setminus \{p, q, x, y\}| \geq 11 \geq 9 + \ell$.

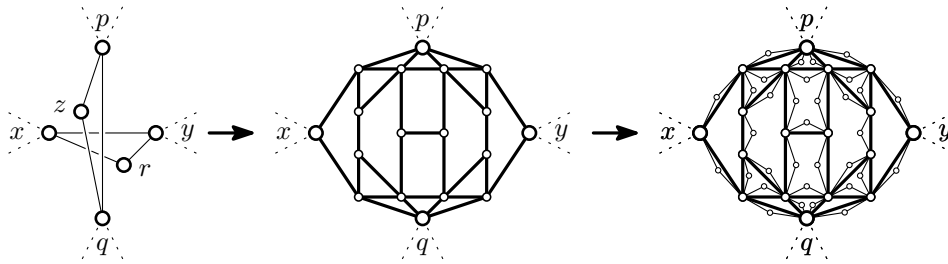
If the previous case does not apply, then $\ell \leq 1$ since S contains a vertex of $\{p, q\}$. If S contains no vertex of $\{x, y\}$, then as illustrated on the right of Figure 4, S contains at least ten non-terminals. Hence $|S \setminus \{p, q, x, y\}| \geq 10 \geq 9 + \ell$. ◀

Using Proposition 14 we prove that replacing two crossing triangles in a DOMINATING SET instance by the gadget, increases the optimum by exactly nine.

► **Lemma 15.** *Let G be a graph, and let $\{x, y, z\}$ and $\{p, q, r\}$ be two vertex-disjoint triangles in G such that z and r have degree two in G . Then the graph G' obtained from G by replacing z and r by H_{VC} as in Figure 5 satisfies $\text{OPT}_{DS}(G') = \text{OPT}_{DS}(G) + 9$.*

Proof. We prove equality by establishing matching upper- and lower bounds.

(\leq) Consider a minimum dominating set S in G that does not contain r or z , which exists by Observation 12. Then S contains at least one of $\{p, q\}$ to dominate z , and at least one



■ **Figure 5** Illustration of how a crossing between two triangles is eliminated in an instance of DOMINATING SET. A copy of H_{VC} is inserted, whose four terminals are identified with the four endpoints of the crossing edge. For each edge $\{u, v\}$ of the inserted copy of H_{VC} , an additional degree-two vertex is inserted that forms a triangle with u and v .

of $\{x, y\}$ to dominate r . We assume without loss of generality, by symmetry, that $p \in S$ and $x \in S$. As shown in Figure 4, there is a vertex cover for H_{VC} of size 11 that contains p and x , and therefore contains nine vertices from the interior of H_{VC} . Let T be this set of nine vertices, and note that T includes a neighbor of q and a neighbor of y . We claim that $S' := S \cup T$ is a dominating set for G' of size $|S| + 9 \leq \text{OPT}_{DS}(G) + 9$. Since $T \cup \{p, x\}$ is a vertex cover of H_{VC} and H_{VC} has no isolated vertices, each vertex of H_{VC} has a neighbor in S' and is dominated. The degree-two vertices that are inserted into G' in the last step are dominated by the vertex that covers the edge with which they form a triangle. Vertices q and y are dominated from their neighbors in T . Finally, the remaining vertices of G' are dominated in the same way as in G .

(\geq) To prove $\text{OPT}_{DS}(G') \geq \text{OPT}_{DS}(G) + 9$, we instead show $\text{OPT}_{DS}(G) \leq \text{OPT}_{DS}(G') - 9$. Let $U \subseteq V(G')$ denote the vertices from the copy of H_{VC} that was inserted; U contains p, q, x, y , but U does not contain the degree-two vertices that were inserted as the last step of the transformation. By Proposition 13, there is a minimum dominating set S' of G' such that $S' \cap U$ is a vertex cover of $G'[U]$. Let $\ell \in \{0, 1, 2\}$ be the number of pairs among $\{p, q\}$ and $\{x, y\}$ from which S' contains no vertices. Since $S' \cap U$ is a vertex cover of $G'[U]$, which is isomorphic to H_{VC} , by Proposition 14 we know that $|(S' \cap U) \setminus \{p, q, x, y\}| \geq 9 + \ell$. Now let S be obtained from S' by removing all vertices of $(S' \cap U) \setminus \{p, q, x, y\}$, adding vertex x if $S' \cap \{x, y\} = \emptyset$, and adding vertex y if $S' \cap \{p, q\} = \emptyset$. Then $|S| \leq |S'| - 9$ since we remove $9 + \ell$ vertices and add ℓ new ones. Since S contains at least one vertex from $\{p, q\}$ and at least one vertex from $\{x, y\}$, it dominates the two triangles in G . Since it contains a superset of the terminal vertices that S' contains, the remaining vertices of the graph are dominated as before. Hence S is a dominating set in G and $\text{OPT}_{DS}(G) \leq \text{OPT}_{DS}(G') - 9$. ◀

Using the material so far, we can prove that the transformation operation in Figure 2 increases the size of an optimal dominating set by exactly 48.

► **Lemma 16.** *Let $\{a, b\}$ and $\{c, d\}$ be two disjoint edges of a graph G . Let G' be the graph obtained by replacing these two edges as in Figure 2. Then $\text{OPT}_{DS}(G') = \text{OPT}_{DS}(G) + 48$.*

Proof. The transformation depicted in Figure 2 can be broken down into six steps: transform $\{a, b\}$ into a double-path structure, transform $\{c, d\}$ into a double-path structure, and perform four operations in which crossing triangles are replaced by gadgets. By Lemma 10, the two double-path insertions increase the size of a minimum dominating set by exactly $2 \cdot 6$. By Lemma 15, the four steps in which crossing triangles are eliminated increase the size of a minimum domination set by exactly $4 \cdot 9$. Hence $\text{OPT}_{DS}(G') = \text{OPT}_{DS}(G) + 12 + 36$. ◀

Using Lemma 16 we easily obtain the following.

► **Lemma 17.** *There is a useful crossover gadget for DOMINATING SET.*

Proof. The gadget that is inserted to replace two edges $\{a, b\}$ and $\{c, d\}$ in the procedure of Figure 2 is planar and has its terminals u, u', v, v' on the outer face in the appropriate cyclic ordering. Since Lemma 16 shows that the replacement increases the size of a minimum dominating set by exactly 48, it follows that the structure serves as a useful crossover gadget for DOMINATING SET as per Definition 5. ◀

Theorem 2 now follows by combining Lemma 17 with the planarization argument of Theorem 6 and the lower bound for the nonplanar case given by Theorem 9.

► **Theorem 2.** *Assuming SETH, there is no $\varepsilon > 0$ such that DOMINATING SET on a planar graph G given along with a linear layout of cutwidth k can be solved in time $\mathcal{O}^*((3 - \varepsilon)^k)$.*

Proof. Suppose DOMINATING SET on a planar graph with a given linear layout of cutwidth k can be solved in $\mathcal{O}^*((3 - \varepsilon)^k)$ time for some $\varepsilon > 0$, by an algorithm called A . Then DOMINATING SET on a nonplanar graph with a given layout of cutwidth k can be solved in time $\mathcal{O}^*((3 - \varepsilon)^k)$ by reducing it to a planar graph with a linear layout of cutwidth $k + \mathcal{O}(1)$ (using Theorem 6 and the existence of a useful crossover gadget; this blows up the graph size by at most a polynomial factor) and then running A . By Theorem 9, this contradicts SETH. ◀

5 Conclusion

In this work we have investigated whether SETH-based lower bounds for solving problems on graphs of bounded treewidth also apply for (1) planar graphs and (2) graphs of bounded cutwidth. To answer these questions, we showed that the graph parameter cutwidth can be preserved when reducing to a planar instance using suitably restricted crossover gadgets.

For both problems considered in this work, the runtime lower bound for solving the problem on graphs of bounded cutwidth continues to hold for *planar* graphs of bounded cutwidth. Hence planarity seems to offer no algorithmic advantage when working with graphs of bounded cutwidth. Moreover, for both INDEPENDENT SET and DOMINATING SET the runtime lower bound for the treewidth parameterization also applies for cutwidth.

Future work may explore other combinatorial problems on graphs of bounded cutwidth. For example, what is the optimal running time for FEEDBACK VERTEX SET, ODD CYCLE TRANSVERSAL, or HAMILTONIAN CYCLE on graphs of bounded cutwidth? What is the complexity of the cutwidth parameterization of these problems on planar graphs? For the GRAPH q -COLORING problem, these questions are answered in a recent manuscript by an overlapping set of authors [21]: planarity offers no advantage, but the parameterization by cutwidth k can be solved in time $\mathcal{O}^*(2^k)$ for all q , sharply contrasting that the treewidth parameterization cannot be solved in time $\mathcal{O}^*((q - \varepsilon)^k)$ under SETH.

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