Report from Dagstuhl Seminar 18361

Measuring the Complexity of Computational Content: From Combinatorial Problems to Analysis

Edited by

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Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 18361 "Measuring the Complexity of Computational Content: From Combinatorial Problems to Analysis". It includes abstracts of talks presented during the seminar and open problems that were discussed, as well as a bibliography on Weihrauch complexity that was started during the previous meeting (Dagstuhl seminar 15392) and that saw some significant growth in the meantime. The session "Solved problems" is dedicated to the solutions to some of the open questions raised in the previous meeting (Dagstuhl seminar 15392).

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Edited in cooperation with Marta Fiori Carones

1 Executive Summary

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Reducibilities such as many-one, Turing or polynomial-time reducibility have been an extraordinarily important tool in theoretical computer science from its very beginning. In recent years these reducibilites have been transferred to the continuous setting, where they allow us to classify computational problems on real numbers and other continuous data types.

In the late 1980s Weihrauch has introduced a reducibility that can be seen as an analogue of many-one reducibility for (multi-valued) functions on infinite data types. This reducibility,

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now called *Weihrauch reducibility*, was studied since the 1990s by Weihrauch's school of computable analysis and flourished recently when Gherardi and Marcone proposed this reducibility as a tool for a uniform approach to reverse analysis.

Reverse mathematics aims to classify theorems according to the axioms that are needed to prove these theorems in second-order arithmetic. This proof theoretic approach yields non-uniform classifications of the computational content of certain theorems. However, many of these classifications also have uniform content and Weihrauch complexity allows us to study this uniform computational content directly using methods of computability theory.

This perspective has motivated Dorais, Dzhafarov, Hirst, Mileti and Shafer, on the one hand, Hirschfeldt and Jockusch, on the other hand, to study combinatorial problems using this approach. This research has led to a number of further reducibilities (computable reducibility, generalized Weihrauch reducibility and others) that can be seen as non-uniform or less resource sensitive versions of Weihrauch reducibility. Using this toolbox of reducibilities one can now adjust the instruments exactly according to the degree of uniformity and resource sensitivity that one wants to capture.

A precursor seminar[1] that was also held at Dagstuhl has been instrumental in bringing together researchers from these different communities for the first time. This has created a common forum and fostered several research developments in this field. We believe that the current seminar was very successful in strengthening and deepening the collaborations between the involved communities. Ample time was left and successfully used for research in groups. A novelty of the current seminar was a special session at which solutions of open problems from the previous seminar were presented. To see that several of the major open problems of the previous meetings were solved in the meantime was inspiring and motivating! Some of the solutions involve new techniques with a wider applicability. Hopefully, we will see solutions to some of the open questions presented at the current seminar in the not too far future! Altogether, the seminar did proceed in a highly productive atmosphere, thanks to many excellent contributions from participants. Inspired by these contributions the organizers are planning to edit a special issue of the journal *Computability* dedicated to this seminar.

This report includes abstracts of many talks that were presented during the seminar, it includes a list of some of the open problems that were discussed, as well as a bibliography on Weihrauch complexity that was started during the previous meeting and that saw significant growth in the meantime. Altogether, this report reflects the extraordinary success of our seminar and we would like to use this opportunity to thank all participants for their valuable contributions and the Dagstuhl staff for their excellent support!

[1] 15392 Measuring the Complexity of Computational Content: Weihrauch Reducibility and Reverse Analysis, see https://doi.org/10.4230/DagRep.5.9.77

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3 Overview of Talks

3.1 Weihrauch goes Brouwerian

Vasco Brattka (Universität der Bundeswehr – München, DE) and
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© Vasco Brattka and Guido Gherardi
Main reference Vasco Brattka, Guido Gherardi: "Weihrauch goes Brouwerian", CoRR, Vol. abs/1809.00380, 2018.
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We prove that the Weihrauch lattice can be transformed into a Brouwer algebra by the consecutive application of two closure operators in the appropriate order: first completion and then parallelization. The closure operator of completion is a new closure operator that we introduce. It transforms any problem into a total problem on the completion of the respective types, where we allow any value outside of the original domain of the problem. This closure operator is of interest by itself, as it generates a total version of Weihrauch reducibility that is defined like the usual version of Weihrauch reducibility, but in terms of total realizers. From a logical perspective completion can be seen as a way to make problems independent of their premises. Alongside with the completion operator and total Weihrauch reducibility we need to study precomplete representations that are required to describe these concepts. In order to show that the parallelized total Weihrauch lattice forms a Brouwer algebra, we introduce a new multiplicative version of an implication. While the parallelized total Weihrauch lattice forms a Brouwer algebra with this implication, the total Weihrauch lattice fails to be a model of intuitionistic linear logic in two different ways. In order to pinpoint the algebraic reasons for this failure, we introduce the concept of a Weihrauch algebra that allows us to formulate the failure in precise and neat terms. Finally, we show that the Medvedev Brouwer algebra can be embedded into our Brouwer algebra, which also implies that the theory of our Brouwer algebra is Jankov logic.

3.2 Effectivity and Reducibility with Ordinal Turing Machines

Merlin Carl (Universität Konstanz, DE)

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Main reference Merlin Carl: "Generalized Effective Reducibility", in Proc. of the Pursuit of the Universal – 12th Conference on Computability in Europe, CiE 2016, Paris, France, June 27 – July 1, 2016, Proceedings, Lecture Notes in Computer Science, Vol. 9709, pp. 225–233, Springer, 2016.
URL http://dx.doi.org/10.1007/978-3-319-40189-8_23

By taking Turing computability as its basic notion of effectivity, the study of Weihrauch reducibility is restricted to realms where objects are countable or can be encoded by countable objects. By replacing Turing machines with Koepke's Ordinal Turing Machines (OTMs), we obtain a notion of effective reducibility that applies to sets of arbitrary size. We can then ask for arbitrary Π_2-statements in the language of set theory whether they are effective or whether one is effectively reducible to the other. As a sample application, we consider several variants of the axiom of choice and see that the versions with systems of representations and choice functions are effectively equivalent, while the well-ordering principle is strictly stronger.

By taking OTMs as the underlying concept of effectivity, we can also reinterpret the realizability interpretation of intutionistic logic, thus obtaining a notion of effectivity for set-theoretical statements of arbitrary quantifier complexity. In this sense, the axioms of KP turn out to be effective, while the power set axiom and the axioms of replacement and separation are not.

3.3 Around finite basis results for topological embeddability between functions

Raphael Carroy (Universität Wien, AT)

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We say that a function f embeds (topologically) in a function g when there are two (topological) embeddings σ and τ satisfying $\tau \circ f = g \circ \sigma$. This quasi-order is a strengthening of the topological strong Weihrauch reducibility. In recent years, various subclasses of analytic functions were shown to admit a finite bases under topological embeddability, including non-σ-continuous functions (Solecki-Pawlikovski-Sabok) and non-Baire-class-one functions (in a joint work with Benjamin Miller). In an effort to understand if topological embeddability could be a well-quasi-order, which would mean that every subclass of functions admits a finite basis under embeddability, we recently proved a dichotomy for spaces of continuous functions with compact Polish 0-dimensional domains: embeddability is either analytic complete or a well-quasi-order.

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3.4 On the Solvability Complexity Index hierarchy, the computational spectral problem and computer assisted proofs

Matthew Colbrook (Cambridge University, GB)

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Joint work of Matthew Colbrook, Anders Hansen
Main reference Matthew J. Colbrook, Bogdan Roman and Anders C. Hansen: "How to compute spectra with error control", submitted, 2018.
Main reference Jonathan Ben-Artzi, Matthew J. Colbrook, Anders C. Hansen, Olavi Nevanlinna and Markus Seidel: " On the solvability complexity index hierarchy and towers of algorithms", 2018.

We will discuss the Solvability Complexity Index (SCI) hierarchy, which is a classification hierarchy for all types of problems in computational mathematics that allows for classifications determining the boundaries of what computers can achieve in scientific computing. The SCI hierarchy captures many key computational issues in the history of mathematics including the insolvability of the quintic, Smale's problem on the existence of iterative generally convergent algorithm for polynomial root finding [1] (and McMullen's solution [2]), the computational spectral problem [3], inverse problems, optimisation, PDEs etc., and also mathematical logic.

Perhaps surprisingly, many of the classifications in the SCI hierarchy do not depend on the model of computation used.

The SCI hierarchy allows for solving the long standing computational spectral problem, and reveals potential surprises. For example, the problem of computing spectra of compact operators, for which the method has been known for decades, is strictly harder than the problem of computing spectra of Schrödinger operators with bounded potentials, which has been open for more than half a century. We also provide an algorithm for the latter problem, thus finally resolving this issue [4]. Moreover, the SCI hierarchy helps classifying problems suitable for computer assisted proofs. In particular, undecidable or non-computable problems are used in computer assisted proofs, where the recent example of the resolution of Kepler's conjecture (Hilbert's 18th problem) is a striking phenomenon [5]. However, only certain classes of non-computable problems can be used in computer assisted proofs, and the SCI hierarchy helps detecting such classes. As we will discuss, the problems of computing spectra of compact operators and Schrödinger operators with bounded potentials are both non-computable, however, whereas the compact case is in general unsuitable for computer assisted proofs, the Schrödinger case is indeed suitable. We will also discuss exciting new algorithms for computing spectra with error control and provide some cutting edge numerical examples [6].

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6 Matthew J. Colbrook, Bogdan Roman and Anders C. Hansen. *How to compute spectra with error control*, submitted, (2018)

3.5 Some properties of the countable space S_0

Matthew de Brecht (Kyoto University, JP)

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In a generalization of Hurewicz's dichotomy theorem, we showed that a countably based co-analytic space is either quasi-Polish or else it contains a Π^0_2 subspace homeomorphic to one of four particular countable spaces (called S_2, S_1, S_D, and S_0). The spaces S_2 (the rationals), S_1 (the cofinite topology on the integers), and S_D (the Alexandrov topology on the natural numbers) are relatively well-known spaces and are often used as counter examples to various completeness properties (such as the Baire category theorem or sobriety).

In this talk we will look more closely at the space S_0 (finite sequences of natural numbers with a very weak topology), which is less well-known. Although S_0 does has some nice completeness properties (it is sober and every closed subset is a Baire space), we will show that it also resembles the space of rationals in several ways.

3.6 Ishihara's Boundedness Principle BD-N and below

Hannes Diener (University of Canterbury – Christchurch, NZ)

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The aim of constructive reverse mathematics (CRM) is to classify theorems and principles over intuitionistic logic. The resulting hierarchy in many parts resembles parts of (Simpson style) reverse mathematics and parts of the Weihrauch lattice.

BD-N is one of the weakest principles that is of interest in CRM. It was introduced in the 1990ies by Hajime Ishihara to find a "logical" counterpart to the analytical statement that all sequentially continuous functions defined on a separable metric space are point-wise continuous. That characterisation makes it seem like quite a straightforward principle, however, from 2010 onward, there have been a number of statements identified that are all implied by BD-N, but that surprisingly lie strictly below it. Furthermore there is little understanding between how this statements interact.

This talk tries to present these ideas and hopefully initiate some discussion on whether this situation is reflected in the Weihrauch lattice or some variation thereof.

3.7 Some results in higher levels of the Weihrauch lattice

Jun Le Goh (Cornell University, US)

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We present some results regarding higher levels of the Weihrauch lattice. We show that comparability of well-orderings is Weihrauch equivalent to its weak version, answering a question of Marcone. The proof proceeds via the ATR-like problem of producing the jump hierarchy on a given well-ordering. We also formulate a "two-sided" version of ATR: given a linear ordering L and a set of natural numbers A, produce either a jump hierarchy on L which starts with A, or an infinite L-descending sequence. We show that this problem is closely related to Koenig's duality theorem about countable bipartite graphs.

3.8 Trees Describing Topological Weihrauch Degrees of Multivalued Functions

Peter Hertling (Universität der Bundeswehr – München, DE)

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We suggest definitions of continuous strong Weihrauch reducibility and of continuous Weihrauch reducibility on the set of functions mapping a subset of the Baire space to some quasi-order. Then we present descriptions of the corresponding topological strong Weihrauch degrees and of the topological Weihrauch degrees of Δ^0_2 measurable functions mapping the Baire space to some better-quasi-order, by suitable trees and forests and suitable reducibility relations on forests. We also consider Wadge degrees. Furthermore, we show that this leads to a similar description of the Wadge degrees, the topological strong Weihrauch degrees and the topological Weihrauch degrees of multivalued functions defined on a subset of a countably based T_0-space with range in a finite discrete space.

3.9 Leaf management

Jeffry L. Hirst (Appalachian State University – Boone, US)

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Joint work of Caleb Davis, Jeffry Hirst, Jake Pardo
Main reference Caleb Davis, Jeffry Hirst, Jake Pardo, Tim Ransom: "Reverse mathematics and colorings of hypergraphs", Archive for Mathematical Logic, November, 2018.
URL https://doi.org/10.1007/s00153-018-0654-z

We demonstrate a process for transforming trees into trees with sets of leaf nodes. This process can be used to eliminate bootstrapping in certain reverse mathematics arguments, and may prove useful in calibrating Weihrauch strength of some statements. This talk includes joint work with Caleb Davis and Jake Pardo.

3.10 Degrees of randomized computability (Informal talk)

Rupert Hölzl (Bundeswehr University Munich, DE)

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Joint work of Rupert Hölzl, Christopher P. Porter

In this survey we discuss work of Levin and V'yugin on collections of sequences that are non-negligible in the sense that they can be computed by a probabilistic algorithm with high probability. More precisely, Levin and V'yugin introduced an ordering on collections of sequences that are closed under Turing equivalence. Roughly speaking, given two such collections $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is less than $\mathcal{B}$ in this ordering if $\mathcal{A} \setminus \mathcal{B}$ is negligible. The degree structure associated with this ordering, the *Levin-V'yugin degrees* (or *LV-degrees*) can be shown to be a Boolean algebra, and in fact a measure algebra.

We demonstrate the interactions of this work with recent results in computability theory and algorithmic randomness: First, we recall the definition of the Levin-V'yugin algebra and identify connections between its properties and classical properties from computability theory. In particular, we apply results on the interactions between notions of randomness and Turing reducibility to establish new facts about specific LV-degrees, such as the LV-degree of the collection of 1-generic sequences, that of the collection of sequences of hyperimmune degree, and those collections corresponding to various notions of effective randomness. Next, we provide a detailed explanation of a complex technique developed by V'yugin that allows the construction of semi-measures into which computability-theoretic properties can be encoded. We provide examples of the uses of this technique by explicating and extending V'yugin's results about the LV-degrees of the collection of Martin-Löf random sequences and the collection of sequences of DNC degree, as well as results concerning atoms of the LV-degrees.

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3.11 Average-case polynomial-time computability of the three-body problem

Akitoshi Kawamura (Kyushu University, JP)

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Joint work of Akitoshi Kawamura, Holger Thies, Martin Ziegler
Main reference Akitoshi Kawamura, Holger Thies, Martin Ziegler: "Average-Case Polynomial-Time Computability of Hamiltonian Dynamics", in Proc. of the 43rd International Symposium on Mathematical Foundations of Computer Science, MFCS 2018, August 27-31, 2018, Liverpool, UK, LIPIcs, Vol. 117, pp. 30:1–30:17, Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, 2018.
URL http://dx.doi.org/10.4230/LIPIcs.MFCS.2018.30

We apply average-case complexity theory to physical problems modeled by continuous-time dynamical systems. The computational complexity when simulating such systems for a bounded time-frame mainly stems from trajectories coming close to complex singularities of the system. We show that if for most initial values the trajectories do not come close to singularities the simulation can be done in polynomial time on average. For Hamiltonian systems we relate this to the volume of "almost singularities" in phase space and give some general criteria to show that a Hamiltonian system can be simulated efficiently on average. As an application we show that the planar circular-restricted three-body problem is average-case polynomial-time computable.

3.12 Weihrauch reducibility for some third order principles

Takayuki Kihara (Nagoya University, JP)

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In order to examine the degrees of difficulty of separation principles on topological spaces, we introduce Weihrauch reducibilty for some third order principles. For instance, in terms of third order continuous Weihrauch reducibility, we show that (1) LLPO is not reducible to the closed separation principle on a separable metrizable space; (2) the open separation principle on a non-discrete second-countable Hausdorff space is equivalent to the uniform-LPO (the map on 1 returning the Kleene's type 2 object 2E) which is strictly stronger than lim; and (3) the coanalytic separation principle on a Polish space is located strictly between (some versions of) the Borel choice and the analytic choice.

3.13 Cohesiveness in the Tree Ramsey Theorem for Pairs

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Joint work of C. T. Chong, Wei Li, Wei Wang, Yue Yang
Main reference C. T. Chong, Wei Li, Wei Wang, Yue Yang: "On the strength of Ramsey's theorem for trees", preprint.

In this talk, we present a version of cohesiveness in the setting of the tree Ramsey Theorem. We prove that the cohesiveness for trees is Pi^1_1 conservative over $P\Sigma_1 + B\Sigma_2$. It is a joint work with C. T. Chong, Lu Liu and Yue Yang.

3.14 Using a Weihrauch degree finitely many times

Arno Pauly (Swansea University, GB)

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The closure operator $^\diamond$ introduced in [3] captures the idea of using a Weihrauch degree finitely many times, without any requirements on a priorily bounding the number of uses:

▶ **Definition 1.** $f^\diamond$ has instances
- A register machine program M using f as a primitive operation (could be non-deterministic!)
- An input x for M on which M halts

and provides $M(x)$ as solutions.

It is intimately linked to the generalized Weihrauch reducibility by Hirschfeldt and Jockusch:

▶ **Observation 1.** $f \leq_W g^\diamond$ iff $f \leq_{gW} g$.

The following example (jww Kazuto Yoshimura) shows that it does not even have to hold that the number of oracles uses depends on the input – it can depend on intermediate results to the oracel calls:

▶ **Example 2.** Let $(q_i)_{i\in\omega}$ be strongly Turing-incomparable. Define F by $F(0^\omega) = \{iq_i \mid i \in \omega\}$, $F(q_{i+1}) = q_i$. Then $q_0 \leq_W F^\diamond$, but we have no bounds for the *run-time*.

Various classifications or stability results for $^\diamond$ have been proven. We shall list some of those:

▶ **Theorem 3.**
- $\mathrm{LPO}^\diamond \equiv_W \mathrm{C}_\mathbb{N}$ *(Neumann & Pauly [3])*
- $\mathrm{C}^\diamond_{\{0,1\}^\omega} \equiv_W \mathrm{C}_{\{0,1\}^\omega}$, $\mathrm{C}^\diamond_\mathbb{R} \equiv_W \mathrm{C}_\mathbb{R}$, $\mathrm{C}^\diamond_{\omega^\omega} \equiv_W \mathrm{C}_{\omega^\omega}$
- $\mathrm{Sort}^\diamond \equiv_W \Pi^0_2\mathrm{C}_\mathbb{N}$ *(Gassner, P. & Steinberg)*
- $(\Sigma^0_\alpha\mathrm{LPO})^\diamond \equiv_W \Pi^0_\alpha\mathrm{C}_\mathbb{N}$ *(Brattka, Gherardi, Hölzl, Nobrega & P.)*
- $\mathrm{C}_{\{0,1\}^\omega,\sharp<\infty} \equiv_W \mathrm{C}^\diamond_{\{0,1\}^\omega<\infty}$ *(Pauly & Tsuiki [1])*
- $\mathrm{C}^\diamond_{\{0,1\}^\omega,\sharp\leq 2} \equiv_W \coprod_{n\in\mathbb{N}} \mathrm{C}_{\{0,1\}^\omega,\sharp\leq n} \equiv_W \mathrm{C}^*_{\{0,1\}^\omega,\sharp\leq 2}$ *(Pauly & Tsuiki [1])*
- $\mathrm{C}^\diamond_{\{0,1\}^\omega,\sharp=2} \equiv_W \coprod_{n\in\mathbb{N}} \mathrm{C}_{\{0,1\}^\omega,\sharp=n} \equiv_W \mathrm{C}^*_{\{0,1\}^\omega,\sharp=2}$ *(Pauly & Tsuiki [1])*

For the last three items, we recall:

▶ **Definition 4** (Le Roux & P. [2]; Tsuiki & P. [1])**.** Let $\mathrm{C}_{\{0,1\}^\omega,\sharp=n}$, $\mathrm{C}_{\{0,1\}^\omega,\sharp\leq n}$, $\mathrm{C}_{\{0,1\}^\omega,\sharp<\infty}$ be closed choice on 2^ω restricted to sets of cardinality n, at most n, or finite.

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3.15 Overt choice on CoPolish spaces

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Choice principles are cornerstones in the Weihrauch lattice, as many important Weihrauch degrees are characterised by a choice problem. Overt choice means the computational task of picking a point in a closed set given by positive information. From Computable Analysis we know that overt choice is computable on computable Polish spaces.

We show that overt choice is discontinuous on CoPolish spaces like the vector space of polynomials or the space of tempered distributions. The discontinuity is caused by the fact that these spaces are not Frechet-Urysohn spaces. There is a minimal non-Frechet-Urysohn CoPolish space $Smin$ which embeds as a closed subspace into every other such space. Overt choice on $Smin$ turns out to be Weihrauch equivalent to LPO.

On the positive side, we show that overt-compact choice on CoPolish spaces is continuous. It is even computable, if the CoPolish space meets some reasonable effectivity conditions. Finally we present a Choice Elimination Theorem for compact choice on CoPolish spaces.

3.16 Q-Wadge degrees as free structures

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Based on ideas, notions and results of P Hertling, J. Duparc and V. Selivanov, T. Kihara and A. Montalban have recently characterized up to isomorphism the structure W_Q of Wadge degrees of Borel Q-partitions of the Baire space, for every countable better quasiorder Q. The characterization is in terms of the so called h-quasiorder on suitably iterated Q-labeled countable well founded forests. Since the corresponding precise definitions are rather long and technical, we attempt here to find a clear shorter characterization.

To achieve this goal, we formulate some easy axioms for a theory T in a language expanding the language of sigma-semilattices. Then we show that many initial segments of W_Q (including W_Q itself) are (reducts of) free structures of suitable subtheories of T. Informally, in this way we obtain a kind of axiomatizations for the initial segments of W_Q.

3.17 Polynomial-time Weihrauch reductions

Florian Steinberg (INRIA Sophia Antipolis, FR)

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Main reference Florian Steinberg: "Computational Complexity Theory for Advanced Function Spaces in Analysis", PhD thesis 2017.

The complexity of operators on the real functions has been a topic of interest for some time (see [1]). However, until fairly recently, complexity theoretical considerations on continuous strutures where limited by the framework. While complexity theory for function on the the real numbers worked reasonably well, many function spaces were known to be "to broad" to be captured. Thus, complexity considerations about operators were confined to be point-wise. Nontheless, interesting results were proven in this setting: For instance that the integration operator preserves the class of polynomial-time computable functions if and only if FP = #P.

This changed, when in 2012 Kawamura and Cook introduced a framework for complexity theory for operators from analysis that allowed for a uniform treatment of operators on real functions by relying on type-two complexity theory. The added uniformity requirement often removes the dependence of results on separation results about complexity classes. For instance, within Kawamura and Cooks framework, it is possible to prove that the integration operator is not polynomial-time computable. In his PhD Thesis and subsequent work, Kawamura introduced a corresponding notion of reducibility and provides some examples of uniformizations. This reducibility is a polynomial-time version of Weihrauch reducibility and can be used to gain further insight in the properties of the operators that are related to separation of complexity classes.

We give a short introduction to the framework of Kawamura and Cook and an overview over what is known about polynomial-time Weihrauch reducibility so far. It turns out that there are some interesting differences to non resource-restricted Weihrauch reducibility. For instance, strong Weihrauch reducibility may fail not only for information theoretic reasons but also because the operator to reduce to forgets about the sizes of instances. For illustration we take a closer look at a uniformization of one of Friedmann and Ko's results about integration of real functions that was part of the authors PhD project.

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3.18 Proof-theoretic characterization of Weihrauch reducibility

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First, we discuss some counterexamples to the theorems of the article [2] by Rutger Kuyper about the characterization of Weihrauch reducibility in RCA_0.

Secondly, we present some results of our own: Affine logic is a refinement of classical logic that restricts contraction. We define affine Peano arithmetic in all finite types in order to characterize different formalizations of Weihrauch reducibility for different classes of total problems. We do this by combining a variation of Gödel's Dialectica interpretation for classical affine logic due to Masaru Shirahata [3], a functional interpretation by Benno van den Berg, Eyvind Briseid, and Pavol Safarik for nonstandard arithmetic [1], and a hereditarily defined notion of computability for higher types derived from associates.

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3.19 Computable planar curves intersect in a computable point

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Main reference Klaus Weihrauch: "Computable planar paths intersect in a computable point", CoRR, arXiv:1708.07460v2, 2017.
URL https://arxiv.org/abs/1708.07460

Consider two paths $f, g : [0; 1] \to [0; 1]^2$ in the unit square such that $f(0) = (0, 0)$, $f(1) = (1, 1)$, $g(0) = (0, 1)$ and $g(1) = (1, 0)$. By continuity of f and g there is a point of intersection. We prove that there is a computable point of intersection if f and g are computable.

The article has been accepted by the journal "Computability" and will appear soon.

4 Solved questions

4.1 Joins in the strong Weihrauch degrees

Damir D. Dzhafarov (University of Connecticut – Storrs, US)

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The Weihrauch degrees and strong Weihrauch degrees are partially ordered structures representing degrees of unsolvability of various mathematical problems. Their study has been widely applied in computable analysis, complexity theory, and more recently, also in computable combinatorics. We answer an open question about the algebraic structure of the strong Weihrauch degrees, by exhibiting a join operation that turns these degrees into a lattice. Previously, the strong Weihrauch degrees were only known to form a lower semi-lattice. We then show that unlike the Weihrauch degrees, which are known to form a distributive lattice, the lattice of strong Weihrauch degrees is not distributive. Therefore, the two structures are not isomorphic.

4.2 Separating products of Weihrauch degrees

Takayuki Kihara (Nagoya University, JP)

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Joint work of Takayuki Kihara, Arno Pauly
Main reference Takayuki Kihara, Arno Pauly: "Dividing by Zero – How Bad Is It, Really?", in Proc. of the 41st International Symposium on Mathematical Foundations of Computer Science, MFCS 2016, August 22-26, 2016 – Kraków, Poland, LIPIcs, Vol. 58, pp. 58:1–58:14, Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, 2016.
URL http://dx.doi.org/10.4230/LIPIcs.MFCS.2016.58
Main reference Takayuki Kihara, Arno Pauly: "Finite choice, convex choice and sorting". Preprint.

We show that the compositional product of LLPO and AoUC is not Weihrauch reducible to finite parallelization of AoUC [1], and the the compositional product of IVT and AoUC is not Weihrauch reducible to any finite dimensional convex choice [2]. This solves two open problems raised at a recent Dagstuhl meeting 15392 on Weihrauch reducibility.

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4.3 ATR_0 in the Weihrauch lattice

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Joint work of Takayuki Kihara, Alberto Marcone, Arne Pauly

This is a survey on the progress made since the previous Dagstuhl workshop on the study within the Weihrauch lattice of problems arising from statement lying at the upper levels of the reverse mathematics hierarchy. In particular, we consider statements equivalent, or closely related, to ATR_0, such as various set-existence axioms, comparability of well-orders, the perfect tree theorem, and open determinacy. The Weihrauch degrees appearing in this research include Unique Choice and Choice on Baire space.

The results will be included in a joint paper with Takayuki Kihara and Arno Pauly.

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1 Takayuki Kihara, Alberto Marcone, and Arno Pauly. *Searching for an analogue of* ATR_0 *in the Weihrauch lattice*. In preparation.

4.4 RT^2_2 compared to the product of SRT^2_2 and COH

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Joint work of Damir D. Dzhafarov, Jun Le Goh, Denis R. Hirschfeldt, Ludovic Patey, Arno Pauly
Main reference Damir D. Dzhafarov, Jun Le Goh, Denis R. Hirschfeldt, Ludovic Patey, Arno Pauly: "Ramsey's theorem and pro ducts in the Weihrauch degrees", CoRR (2018), arXiv:1804.10968
URL https://arxiv.org/abs/1804.10968

Ramsey's theorem for pairs and two colors (RT^2_2) asserts that every 2-coloring of $[\mathbb{N}]^2$ admits an infinite monochromatic set. RT^2_2 can be decomposed into a stable version (SRT^2_2) and the cohesiveness principle (COH). From the viewpoint of Weihrauch reducibility, RT^2_2 is a consequence of the compositional product of SRT^2_2 and COH and implies their coproduct. In a previous Dagstuhl seminar, it was asked which reversals hold.

In this talk, we present a complete overview of the question and show that none of the reversal holds. In particular, we prove that the cartesian product of SRT^2_2 and COH is not Weihrauch reducible to RT^2_2.

This is a joint work with Damir Dzhafarov, Jun Le Goh, Denis Hirschfeldt and Arno Pauly.

4.5 Grouping principle

Keita Yokoyama (JAIST – Ishikawa, JP) and Ludovic Patey (University Claude Bernard – Lyon, FR)

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Main reference Ludovic Patey and Keita Yokoyama: "The proof-theoretic strength of Ramsey's theorem for pairs and two colors, Advances in Mathematics", 330:1034–1070, 2018.
URL https://doi.org/10.1016/j.aim.2018.03.035

Grouping principle is a technical combinatorial statement which is a direct consequence of Ramsey's theorem. In the previous seminar (Dagstuhl seminar 15392), Yokoyama posed a question "what is the reverse mathematical strength of the grouping principle for pairs and two colors?" Patey answered this quesiton by showing that any computable instance of the stable version of the grouping principle for pairs admits has a low solution.

5 Open problems

5.1 Density and minimality properties of the Weihrauch lattice

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This open problem is related to lattice theoretic properties of the Weihrauch lattice and its variants. These questions apply to the Weihrauch lattice itself, to the strong Weihrauch lattice, to the parallelized Weihrauch lattice, the parallelized total Weihrauch lattice and other variants:

1. What can be said about density properties of the corresponding lattice?
2. Are there regions where the lattice is dense and others where it is not? Can those be classified?
3. Are there minimal pairs or atoms?

Basically nothing is known about the answers to such questions!

5.2 Ramsey's theorem: products versus colors

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We consider Ramsey's theorem for a fixed cardinality n and k colors. It is easy to see that the m–fold product of Ramsey's theorem for k colors is strongly Weihrauch reducible to a single instance with k^m colors (all for the fixed cardinality n) [31, Corollary 3.18 (1)]. This means that colors can make up for products. Does the converse hold true, i.e., can products make up for colors? More precisely, is there a number m for each k, such that Ramsey's theorem for k colors is Weihrauch reducible to the m–fold product of Ramsey's theorem for only 2 colors (all for the fixed cardinality n)? (See also [31, Question 3.22].) The answer is yes for cardinality $n = 1$ [31, Proposition 3.23], but not known for higher cardinalities $n \geq 2$.

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5.3 Weihrauch strength of countable well-orderings

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What is the Weihrauch strength of various statements about countable well-orderings? In the reverse mathematics setting, they tend to clump into two groups, one at the ACA_0 level and the other at ATR_0. Do they separate in the Weihrauch hierarchy?

Possibly useful resources include the survey of ordinal arithmetic in Reverse Mathematics 2001 [1] and Sierpinski's text, Cardinal and Ordinal Numbers [2]. Also see the related work by Jun Le Goh and by Alberto Marcone and his affiliates.
A small subproject: Examine statements related to indecomposable ordinals.

- Weak comparability of indecomposable well-orderings.
- If α is well-ordered, then ω^α is well-ordered. (Consider the contrapositive to formulate this as a Weihrauch problem.)
- If α is indecomposable, then there is a β such that $\alpha = \omega^\beta$. (Here, the equality could indicate weak comparability or strong comparability.)

In the subproject, the reverse mathematical analysis of the statements has already been completed. One could also select a previously unanalyzed statement from Sierpinski [2] and do both the reverse mathematical analysis and the Weihrauch analysis.

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5.4 Some questions around Weihrauch counterparts of ATR

Takayuki Kihara (Nagoya University, JP)

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Main reference Paul-Elliot Anglès D'Auriac and Takayuki Kihara: "A comparison of various analytic choice principles", Preprint.

Goh introduced the two-sided version ATR_2 of arithmetical transfinite recursion, and Anglès D'Auriac and Kihara [1] introduced its variant $ATR_{2'}$ which is shown to be arithmetically Weihrauch equivalent to the Σ^1_1-choice on Cantor space.

Q1. Is $ATR_{2'}$ arithmetically Weihrauch equivalent to ATR_2?

Anglès D'Auriac and Kihara [1] showed that the Σ^1_1-choice on Baire space is not Weihrauch reducible to the parallelization of the Σ^1_1-choice on the natural numbers.

Q2. Is the Σ^1_1-choice on Baire space hyperarithmetically Weihrauch reducible to the parallelization of the Σ^1_1-choice on the natural numbers?

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5.5 Two open questions from Dagstuhl Seminar 18361

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These two questions concern the Weihrauch degrees of problems in algebra. The first concerns vector spaces. The elements of a countable vector space over $\mathbb{Q}$ can be identified with elements of $\mathbb{N}$, so that the elementary diagram can be encoded canonically as an element of $2^{\mathbb{N}}$. We can use this representation to ask about the degrees of problems in linear algebra. For example, the problem of producing a basis for a countable vector space over $\mathbb{Q}$ has Weihrauch degree $\widehat{\mathsf{LPO}}$, and in the setting of reverse mathematics the analogous principle of second order arithmetic is equivalent to ACA_0 over RCA_0. The first question relates to the problem of finding a proper finite dimensional subspace of a countable vector space over $\mathbb{Q}$.

Problem: Let $P\colon \subseteq 2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ be the partial multifuction that, given the atomic diagram of an infinite dimensional vector space over $\mathbb{Q}$, returns the characteristic function of a finite dimensional nonzero subspace of the vector space. What is the Weihrauch degree of P?

Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [1] proved that the principle of second order arithmetic analogous to P is equivalent to ACA_0 over RCA_0. Their proof has an interesting nonuniformity, as it relies on the ability to choose a basis for the finite dimensional subspace. It follows from their results that $\mathsf{WKL} \leq_W P \leq_W \widehat{\mathsf{LPO}}$. We suspect $P \equiv_W \widehat{\mathsf{LPO}}$, but a new proof method seems to be needed.

The second problem comes from group theory. It is a classical fact that every group with more than 2 elements has a nontrivial automorphism. We represent countably infinite groups by identifying their set of elements with $\mathbb{N}$, so that their elementary diagrams can be viewed as elements of $2^{\mathbb{N}}$. There is no loss of generality in assuming the identity element is identified with $0 \in \mathbb{N}$.

Problem: Let $A\colon \subseteq 2^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be the partial multifunction that, given the atomic diagram of a countably infinite group, produces a nontrivial automorphism of the group. What is the Weihrauch degree of A?

The known upper bound is $A \leq_W \mathsf{LPO} \times \mathsf{LPO}$. The two particular questions that LPO is used to answer are whether the group is abelian and whether every element has order 2. In particular, every computable countably infinite group has a computable nontrivial automorphism. If the Weihrauch degree of A is nontrivial, this provides another example of the importance of weak choice principles.

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5.6 Characterizing the diamond-operator

Arno Pauly (Swansea University, GB)

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The $^\diamond$-operator in the Weihrauch lattice captures the idea of making finitely many calls to an oracle available, without any a priori known bound on the number of calls. See the abstract "Using a Weihrauch degree finitely many times" abstract for details. It is clear that if $f \equiv_W f^\diamond$, then $1 \leq_W f$ and $f \equiv_W f \star f$. Our question is whether the converse holds:

Does $1 \leq_W f$ and $f \equiv_W f \star f$ imply $f \equiv_W f^\diamond$?

During the seminar, Linda Brown Westrick obtained a positive answer to this question.

5.7 Compact Hausdorff spaces are regular

Arno Pauly (Swansea University, GB)

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It is a well-known result from topology that compact Hausdorff spaces are regular. The traditional proof proceeds as follows: We are given $x \in \mathbf{X}$ and $A \in \mathcal{A}(\mathbf{X})$ with $x \notin A$. For each $y \in A$ there are disjoint opens $U_y \ni x$ and $V_y \ni y$, since $\mathbf{X}$ is Hausdorff. Consider the open cover $A \subseteq \bigcup_{y \in A} V_y$. By compactness of $\mathbf{X}$, there exists some finite $I \subseteq A$ such that already $A \subseteq \bigcup_{y \in I} V_y$. Now $\bigcup_{y \in I} V_y$ and $\bigcap_{y \in I} U_y$ are disjoint open sets separating x and A.

In computable topology, however, this argument does not go through. In order to obtain$\bigcup_{y \in A} V_y$ as an open set, we would require A as an overt set, not merely as a closed set. Restricted to countably-based spaces, a different approach was shown to work in [1]. Here, we ask whether the statement holds in general:

Is every computably Hausdorff computably compact represented spaces already computably regular?

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5.8 Characterization of overt choice on maximal CoPolish spaces

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It is known that there exist maximal CoPolish spaces X in the sense that any other CoPolish space is homeomorphic to a closed subspace of X. A CoPolish space is defined to be the direct limit of an increasing sequence of compact metric spaces. One example of a maximal CoPolish space is the Hilbert space l_2 equipped with the sequentialization of the weak* topology on l_2. Overt choice is the problem of picking a point in a closed subset given with positive information.

Question: Characterize the Weihrauch degree of overt choice $V(l_2)$ on l_2.

Note that overt choice on any CoPolish space is continuously Weihrauch reducible to $V(l_2)$ due to the maximality property.

5.9 Minimal continuous Weihrauch degrees

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Let $f \neq 0$ be any multifunction, where 0 denotes the nowhere defined problem.

Question: Does there exist a multifunction $g \neq 0$ such that g is strictly below f in the continuous Weihrauch lattice?

5.10 When can one step function Weihrauch compute another?

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Joint work of Adam Day, Rod Downey, Takayuki Kihara, Linda Westrick

Let $\leq$ denote the lexicographic order on Cantor space. For $A \in 2^\omega$, define the step function $s_A : 2^\omega \to 2$ to be the characteristic function of $\{X \in 2^\omega : A \leq X\}$.

Question: Characterize the pairs (A, B) for which $s_A \leq_W s_B$.

The little that is known about this is strange. If B is computable and s_B is discontinuous, then $s_A \leq_W s_B$ if and only if A is left-c.e. But if B is not computable and $s_A \leq_W s_B$, then A and B are Turing equivalent.

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For an always up-to-date version of this bibliography see http://cca-net.de/publications/weibib.php.

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