Packing Geometric Objects with Optimal Worst-Case Density

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Abstract

We motivate and visualize problems and methods for packing a set of objects into a given container, in particular a set of different-size circles or squares into a square or circular container. Questions of this type have attracted a considerable amount of attention and are known to be notoriously hard. We focus on a particularly simple criterion for deciding whether a set can be packed: comparing the total area A of all objects to the area C of the container. The *critical packing density* δ^* is the largest value A/C for which any set of area A can be packed into a container of area C. We describe algorithms that establish the critical density of squares in a square ($\delta^* = 0.5$), of circles in a square ($\delta^* = 0.5390\ldots$), regular octagons in a square ($\delta^* = 0.5685\ldots$), and circles in a circle ($\delta^* = 0.5$).

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1 Introduction

Deciding whether a set of disks can be packed into a given container is a fundamental geometric optimization problem that has attracted considerable attention. Disk packing also has numerous applications in engineering, science, operational research and everyday life, e.g., for packaging cylinders [1, 8], bundling tubes or cables [22, 24], or the cutting industry [23]. Further applications stem from forestry [23] and origami design [13].

Like many other packing problems, disk packing is typically quite difficult; what is more, the combinatorial hardness is compounded by the geometric complications of dealing with irrational coordinates that arise when packing circular objects. This is reflected by the limitations of provably optimal results for the optimal value for the smallest sufficient disk container (and hence, the densest such disk packing in a disk container), a problem that was discussed by Kravitz [12] in 1967: Even when the input consists of just 13 unit disks, the optimal value for the densest disk-in-disk packing was only established in 2003 [7], while the optimal value for 14 unit disks is still unproven. The enormous challenges of establishing densest disk packings are also illustrated by a long-standing open conjecture by Erdős and Oler from 1961 [19] regarding optimal packings of n unit disks into an equilateral triangle, which has only been proven up to $n = 15$. For other examples of mathematical work on densely packing relatively small numbers of identical disks, see [5, 6, 9, 16], and [10, 15, 20] for related experimental work. Many authors have considered heuristics for circle packing problems, see [11, 23] for overviews of numerous heuristics and optimization methods. The best known solutions for packing equal disks into squares, triangles and other shapes are continuously published on Specht's website `http://packomania.com` [21].

The related problem of packing square objects has also been studied for a long time. The decision problem whether it is possible to pack a given set of squares into the unit square was shown to be strongly NP-complete by Leung et al. [14], using a reduction from 3-Partition. Already in 1967, Moon and Moser [17] found a sufficient condition. They proved that it is possible to pack a set of squares into the unit square in a shelf-like manner if their combined area (i.e., the sum over all square areas) does not exceed $\frac{1}{2}$. At the same time, $\frac{1}{2}$ is the *largest upper area bound* one can hope for, because two squares larger than the quarter-squares shown in Fig. 1 cannot be packed. We call the ratio between the largest combined object area that can always be packed and the area of the container the problem's *critical packing density*.

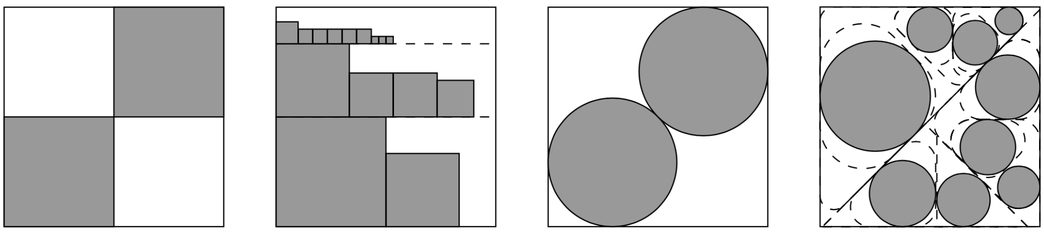

Figure 1 (1) An instance of critical packing density for squares in a square. (2) An example packing produced by Moon and Moser's Shelf Packing. (3) An instance of critical packing density for disks in a square. (4) An example packing produced by Morr's Split Packing.

The equivalent problem of establishing the critical packing density for disks in a square was posed by Demaine, Fekete, and Lang [2] and resolved relatively recently by Morr, Fekete and Scheffer [4, 18]. Making use of Split Packing, a recursive procedure for cutting the

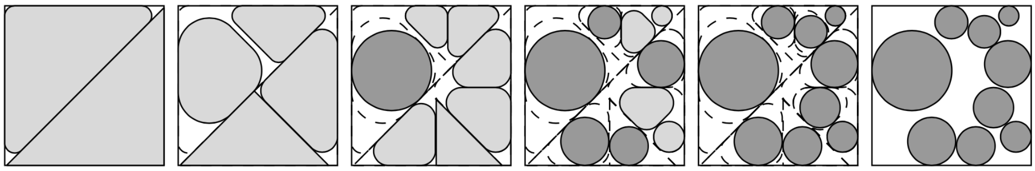

Figure 2 An example run of Split Packing.

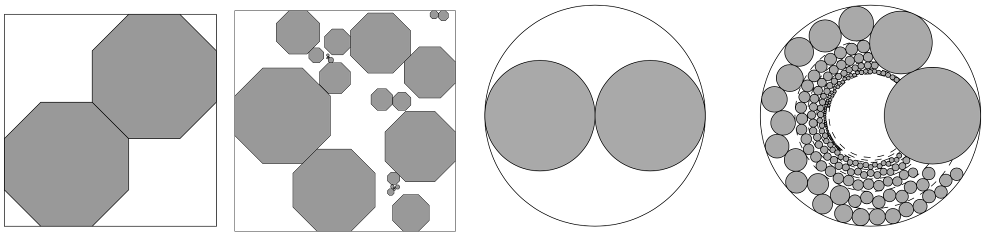

Figure 3 (1) An instance of critical packing density for regular octagons in a square. (2) An example packing produced by Split Packing. (3) An instance of critical packing density for circles in a circles. (4) An example packing produced by Boundary/Ring Packing of Fekete, Keldenich, and Scheffer.

container into *hats*, i.e., triangular pieces with rounded corners, they proved that the critical packing density of disks in a square is $\frac{\pi}{3+2\sqrt{2}} \approx 0.539$. As illustrated in Fig. 3 (1) and (2), Split Packing can be generalized to also achieve critical packing density for other shapes such as regular octagons.

The analogous question of establishing the critical packing density for disks in a disk has just been resolved by Fekete, Keldenich and Scheffer [3]. As illustrated in Fig. 3 (3) and (4), it is 0.5.

2 Squares in a Square: Shelf Packing

The basic idea for Shelf Packing (see Fig. 1 (2)) is to greedily insert the items in order of decreasing size, starting in the lower left corner. Items are placed from left to right with aligned bottoms, until an item no longer fits to the right of its predecessor; when that happens, an item is placed on top of the leftmost item in the previous row. Assuming that this fails to place the last item completely inside the container yields an estimate for the packed area that exceeds 1/2; this involves relatively simple algebraic transformations, as shown in the video.

3 Circles in a Square: Split Packing

The key idea for Split Packing is to recursively subdivide both the set of items and the container in a fashion that yields a similar, balanced split for both. A schematic overview is shown in Fig. 4: After sorting the circles by decreasing size (Step 1), the set of items is greedily split into two subsets, such that the difference between their total area does not exceed the area of the last subset (Step 2); the area split is also applied to the container along a diagonal cut (Step 3). A finer point is to notice that a guaranteed minimum size of circles in a subset makes it sufficient to consider triangular subcontainers with rounded corners (“hats”), as the acute corners will not be required to accommodate bigger circles; as a consequence, asymmetric diagonal cuts leave a bigger piece that can still be treated as such a hat, allowing recursion. This is carried out recursively on both subsets (Steps 4 and 5), until a subset consists of a single circle, which is then simply packed into its corresponding triangle (Step 6). The resulting subdivision and packing are shown in Steps 7 and 8. In the end, each subcontainer achieves a packing density of at least $\frac{\pi}{3+2\sqrt{2}} \approx 0.539$, establishing the critical bound; see Fig. 2 for an example run.

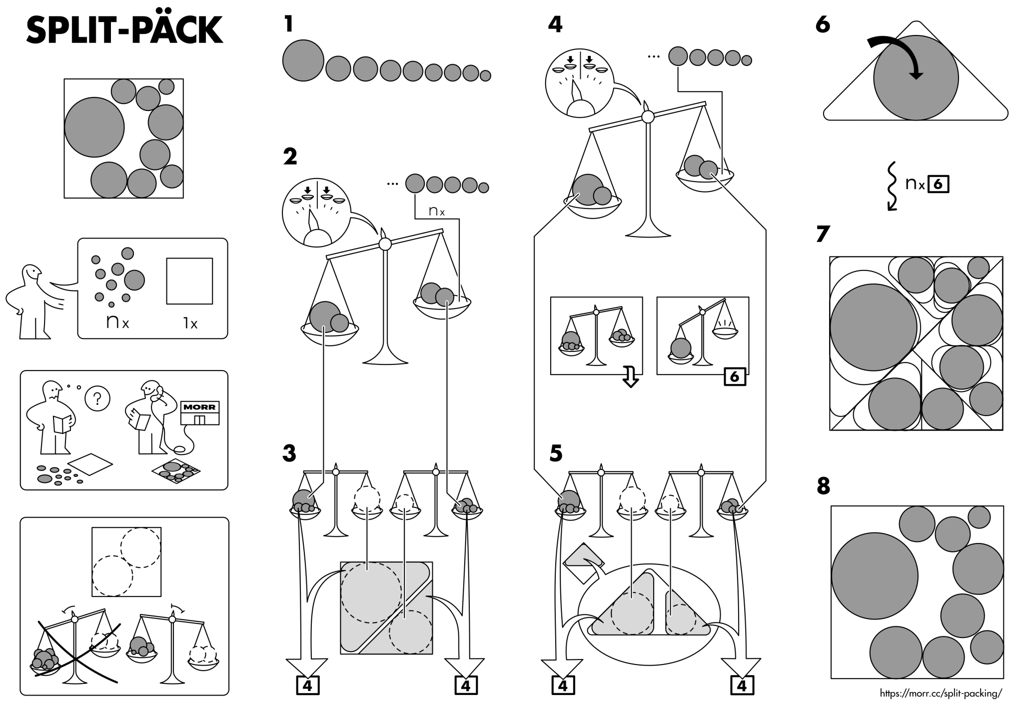

Figure 4 A schematic illustration of Split Packing.

The key insight (a balanced, recursive split into subsets can be used analogously to the container by a diagonal cut) can also be applied to other kinds of objects: See Fig. 3 for the case of regular octagons, for which the critical packing density is $8(5\sqrt{2}-7) \approx 0.5685\ldots$. Similar results can be obtained for other shapes and triangular containers, see the paper [4] for details.

4 Circles in a Circle: Boundary/Ring Packing

For the scenario of packing circles into a circle, the curved container boundary requires a different approach. The algorithm of Fekete, Keldenich and Scheffer [3] applies subroutines. The circles we want to pack into a circle container $\mathcal{C}$ are considered in decreasing order of radius. In the following, we sketch the algorithm of Fekete, Keldenich and Scheffer; see the full paper [3] for details.

First we check whether the two largest circles allow a recursive packing of the remaining circles, see Figure 5 (top-left). If we cannot apply the recursion, two subroutines are used: *Ring Packing* and *Boundary Packing*, see Figures 5 (top-middle) and (top-right).

Ring Packing packs circles into a ring R, alternating between touching the inner and outer boundary of the ring. If the current circle and the previously packed circle could pass each other in R, Ring Packing partitions the ring into two rings and continues by packing into these two rings. Ring Packing continues until the current circle cannot be packed into any open ring, see Figure 5 (bottom).

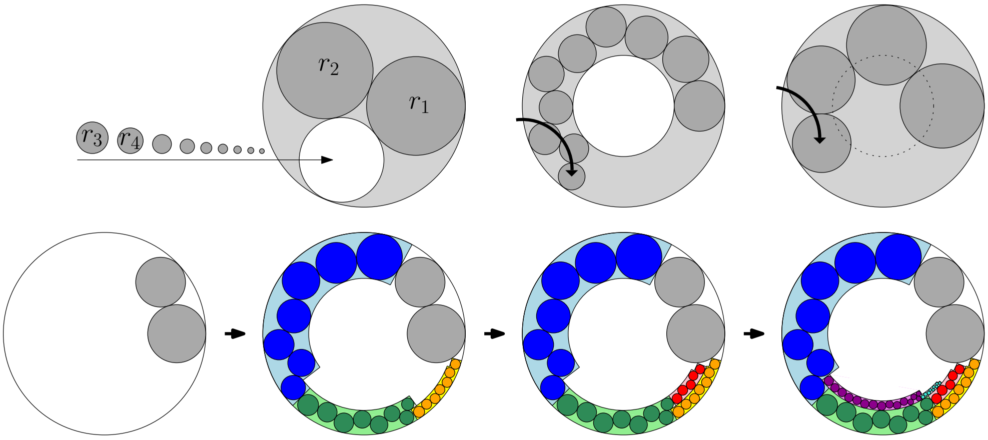

Figure 5 The approach of Fekete, Keldenich and Scheffer [3]: (Top, left to right) Recursion, Ring Packing and Boundary Packing. (Bottom) A stepwise run of the algorithm.

Boundary Packing packs the input circles into the container, touching its boundary and the previously packed circle until the radius of the current circle is below a given threshold.

Using a combination of manual and automated case checking, the analysis ensures a packing density of at least 0.5; see Figure 6 for an illustration and the full paper for details.

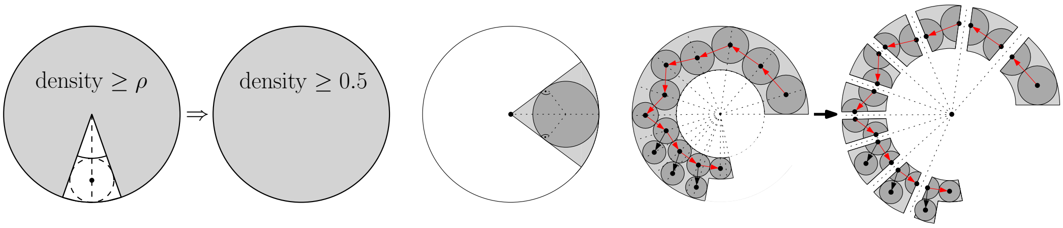

Figure 6 The analytic approach for ensuring a packing density of at least 0.5.

5 The Video

The video opens with a number of practical illustrations for packing problems, followed by a sketch of NP-hardness proofs and additional geometric difficulties when packing circles. After introducing the concept of critical packing density, we sketch the algorithmic proof by Moon and Moser for the critical packing density of squares in a square. This is followed by an illustrated description of Morr's Split Packing algorithm for circles in a square, along with extensions to other shapes and containers. Finally, we sketch a current result on the critical packing density of circles in a circle.

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