Properties of Minimal-Perimeter Polyominoes

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Abstract

In this video, we survey some results concerning polyominoes, which are sets of connected cells on the square lattice, and specifically, minimal-perimeter polyominoes, that are polyominoes with the minimal-perimeter from all polyominoes of the same size.

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1 Introduction

In this video, we discuss the combinatorial and geometric properties of minimal-perimeter polyominoes. A polyomino is a set of edge-connected squares on the square lattice. The size of a polyomino is the number of squares composing it. The problem of counting polyominoes dates back to the 1950s, when it was studied in parallel in the fields of combinatorics [7] and statistical physics [6]. The number of polyominoes of area n is denoted in the literature by $A(n)$. No formula for $A(n)$ is known as of today, but there exist a few algorithms for computing values of $A(n)$, such as those of Redelmeier [12] and Jensen [8]. Using Jensen's algorithm, values of $A(n)$ were computed up to $n = 56$. The existence of the *growth constant* of $A(n)$, namely, $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$, was shown by Klarner [9]. More than 30 years later, Madras [11] showed that $\lim_{n\to\infty} A(n+1)/A(n)$ exists and, thus, is equal to λ. The best known lower and upper bounds on λ are 4.0025 [5] and 4.6496 [10], respectively. The currently best *estimate* of λ is 4.0625696 ± 0.0000005 [8].

The perimeter of a polyomino is the set of empty squares adjacent to the polyomino. The number of polyominoes with a given size and perimeter size was studied in several papers. Recently, Asinowski et al. [2] provided formulae for the number of polyominoes of a given area and perimeter that is close to the maximum possible perimeter.

In contrast to polyominoes with large perimeter size, a minimal-perimeter polyomino is a polyomino with the minimum possible perimeter for its size. Minimal-perimeter polyominoes were studied by Altshular et al. [1] and by Sieben [13], providing formulae for the minimum perimeter size possible for a given size of a polyomino. Recently, we provided some combinatorial results about the sets of minimal-perimeter polyominoes [3, 4]. In this video, we show some of these results.

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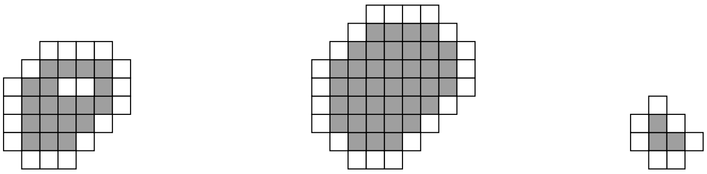

(a) Q (b) $I(Q)$ (c) $D(Q)$

Figure 1 A polyomino Q and the respective inflated and deflated polyominoes. The gray cells represent polyomino cells, while the white cells represent perimeter cells.

2 Results

As mentioned in the introduction, a polyomino, denoted by Q, is a set of edge-connected cells. The perimeter of a polyomino, denoted by $\mathcal{P}(Q)$, is the set of empty cells adjacent to the polyomino.

In this video, we present several results about the properties of minimal-perimter polyominoes, namely, polyominoes with the minimum number of perimeter cells for their size. A key concept for these results is the inflation operation. Formally, the inflation of a polyomino Q, denoted by $I(Q)$, is defined as $I(Q) = Q \cup \mathcal{P}(Q)$, or in words, inflating a polyomino is the operation of expanding a polyomino by its perimeter. Similarly, the deflation of a polyomino Q, denoted by $D(Q)$, is the removal of the cells adjacent to some perimeter cells. Formally, we define the border of Q, denoted by $\mathcal{B}(Q)$ as the set of cells adjacent to a perimeter cell. Similarly to inflation, the deflated polyomino is defined as $D(Q) = Q \backslash \mathcal{B}(Q)$. Those concepts are depicted in Figure 1.

2.1 Inflating Minimal-Perimeter Polyominoes

Using the above concepts, we present in the video the following theorems [3].

▶ **Theorem 1.** *[3, Thm. 3] If Q is a minimal-perimeter polyomino, then $I(Q)$ is a minimal-perimeter polyomino as well.*

▶ **Theorem 2.** *[3] Let Q be a polyomino for which $D(Q)$ is a minimal-perimeter polyomino. Then, for any other minimal-perimeter polyomino Q' of the same size as Q, we have that $D(Q')$ is a minimal-perimeter polyomino as well.*

Combining these results, we obtain the following theorem.

▶ **Theorem 3.** *[3, Thm. 4] Let M_n be the set of minimal-perimeter polyominoes of size n. Then, for $n \geq 3$, we have that $|M_n| = \left|M_{n+\epsilon(n)}\right|$, where $\epsilon(n)$ is the perimeter size of a minimal-perimeter polyomino of size n.*

Theorem 3 is demonstrated in Figure 2. Theorem 3 is the main result shown in [3]; we refer the reader to the full paper for more details.

2.2 Counting Minimal-Perimeter Polyominoes

The bijection shown in Theorem 3 provides the number of minimal-perimeter polyominoes for any size which is obtained by inflating a set of minimal-perimeter polyominoes of some smaller size n, for which we know the cardinality of M_n. However, there are sizes of polyominoes, for which the set of minimal-perimeter polyominoes is not created by inflating the polyominoes

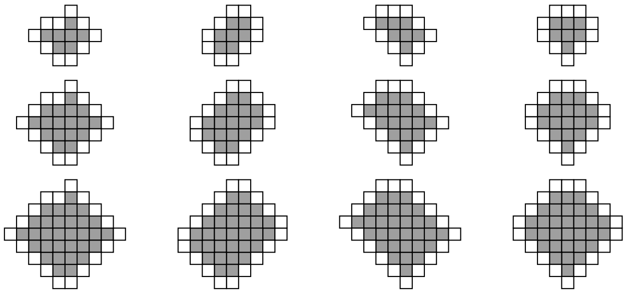

Figure 2 A demonstration of Theorem 3. The first row contains all the minimal-perimeter polyominoes of size 7, the second row contains all the minimal-perimeter polyominoes of size 17, and the last row contains all the minimal-perimeter polyominoes of size 31. Notice that the polyominoes in each row are the inflated versions of the respective polyominoes in the previous row.

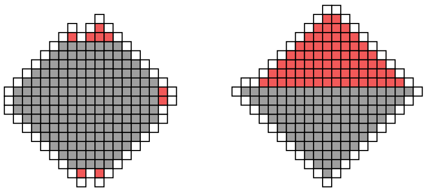

Figure 3 An example of the decomposition of two minimal-perimeter polyominoes into their the body (gray) and caps (red).

in another set of minimal-perimeter polyominoes. The properties of those sizes are discussed in [4]. However, due to time constraints, we focus in the video only on the question of how many minimal-perimeter polyominoes there are for a given size.

In order to count the minimal-perimeter polyominoes of a given size, we show that any minimal-perimeter polyomino can be decomposed into an octogonal shaped polyomino, with the same perimeter as the original polyomino (referred to as the "body" of the polyomino), and "caps," which are added on top of the octagon edges and do not change the perimeter. This decomposition is depicted in Figure 3. Using this decomposition, we were able to devise an efficient algorithm for counting minimal-perimeter polyominoes of a given size. Data we obtained are shown in Figure 4. For more information, we refer the reader to [4].

3 Future Work

Several questions in this area are yet to be answered. First, how do those results generalize to other lattices, such as polyiamonds on the triangular lattice, polyhexes on the hexagonal lattice, and polycubes on the cubical lattice in higher dimensions?

Second, the results shown in Figure 4 raise some questions about the behavior of the upper envelope of the sequence $|M_n|$. It seems that this sequence is exponential, and if it is, we would like to achieve good bounds on its growth rate.

4 The Video

The video depicts the above results. It was created mostly using Powtoon. The 3-dimensional animation was created using Maya Autodesk, and the graph animation was created programatically using the D3.js library.

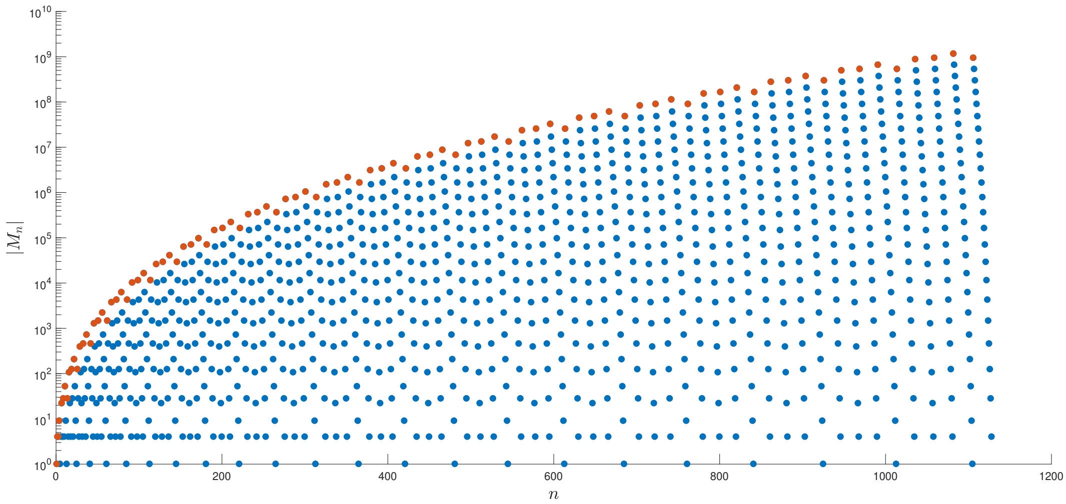

Figure 4 Values of $|M_n|$. The values which are not created by the inflation operation are shown in red.

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