Modal Embeddings and Calling Paradigms

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Abstract

We study the computational interpretation of the two standard modal embeddings, usually named after Girard and Gödel, of intuitionistic logic into IS4. As source system we take either the call-by-name (cbn) or the call-by-value (cbv) lambda-calculus with simple types. The target system can be taken to be the, arguably, simplest fragment of IS4, here recast as a very simple lambda-calculus equipped with an indeterminate lax monoidal comonad. A slight refinement of the target and of the embeddings shows that: the target is a calculus indifferent to the calling paradigms cbn/cbv, obeying a new paradigm that we baptize call-by-box (cbb), and enjoying standardization; and that Girard's (resp. Gödel's) embbedding is a translation of cbn (resp. cbv) lambda-calculus into this calculus, using a compilation technique we call protecting-by-a-box, enjoying the preservation and reflection properties known for cps translations - but in a stronger form that allows the extraction of standardization for cbn or cbv as consequence of standardization for cbb. The modal target and embeddings achieve thus an unification of call-by-name and call-by-value as call-by-box.

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1 Introduction

It is a fact reported in textbooks [16] that there are two main embeddings of intuitionistic logic into (intuitionistic) modal logic S4, the original one due to Gödel and a more recent one named after Girard. What is the computational meaning of this fact? In particular, why two? Similar questions concerning the embedding of intuitionistic logic into linear logic have been answered long ago: the $(!A \multimap B)$- and $!(A \multimap B)$-translations already introduced in the seminal paper [4] correspond to the two calling mechanisms of functional programming, call-by-name (cbn) and call-by-value (cbv), which are thus "explained in terms of logical translations, bringing them into the scope of the Curry-Howard isomorphism" [11]. Through these results in terms of linear logic we can glimpse what the computational explanation of the embeddings into modal logic is.

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The claim of the present paper is that it is desirable to give a direct analysis of the embeddings into modal logic S4. First, such analysis is more abstract, because it is done in terms of an indeterminate $\Box$-modality, and so the analysis applies to all the possible instantiations of the modality, including the ! modality of linear logic. Second, such analysis will be carried farther than what was done before with linear logic. Here is what we obtained:

First, the target of the modal embeddings can be defined, in a first moment, as the simplest fragment of intuitionistic S4 where the problem of closure under substitution is solved. This target can be presented as a very simple λ-calculus equipped with an indeterminate lax monoidal comonad. In a second moment, the target should be slightly refined into a λ-calculus with several noteworthy properties: (i) it follows a new calling mechanism, *call-by-box*; (ii) it is equipped with a notion of evaluation, *weak-and-external* reduction, which is *indifferent* to cbn and cbv; (iii) it enjoys a standardization theorem that makes explicit the contribution of evaluation to a notion of standard reduction. In the description of Plotkin [12], the target system is a calculus and a programming language, and the standardization theorem links both. We say all these ingredients turn call-by-box (cbb) into a new *calling paradigm*.

Second, the embeddings can be defined, as expected, on the cbn λ-calculus (in the case of Girard) and on Plotkin's cbv λ-calculus (in the case of Gödel). But, after a refinement of Gödel's embedding, both can be seen as having as target the above refined target. When this is done, the embeddings can be described as a compilation of cbn or cbv into cbb, following a new technique that we call *protecting-by-a-box*. This technique improves the old protecting-by-a-lambda, already discussed in [12], which only achieves the compilation of cbn into cbv. *Au contraire*, protecting-by-a-box is capable of compiling both cbn and cbv into the indifferent paradigm cbb. In addition, the refined embeddings enjoy properties of preservation and reflection at the levels of reduction and evaluation, and also standard reduction. So all the *translation, simulation, and indifference properties* of cps-translations [12] hold of the refined embeddings, and so they can be offered as an improvement of protecting-by-a-lambda alternative to cps-translations.

Third, the indifference property of the cbb target, which at first only means that reduction or evaluation in the full target captures cbn (resp. cbv) reduction or evaluation when restricted to the image of Girard's (resp. Gödel's) embedding, actually goes much further: all the translation, simulation, and indifference properties cooperate to show that the standardization theorems for the cbn and cbv λ-calculi can be extracted from the standardization theorem of the cbb target. Because of all this, we feel entitled to say that the cbb target, together with the refined modal embeddings, achieve a *modal unification of call-by-name and call-by-value*

Plan of the paper. Section 2 recalls the cbn (i.e. ordinary) λ-calculus $\lambda_{\mathtt{n}}$ and Plotkin's cbv λ-calculus $\lambda_{\mathtt{v}}$. Section 3 recasts the modal embeddings as maps from $\lambda_{\mathtt{n}}$ or $\lambda_{\mathtt{v}}$ into a simple, modal target language $\lambda_{\Box}$. Section 4 motivates and introduces the refined target $\lambda_{\mathtt{b}}$ and proves that it enjoys standardization. Section 5 introduces the refined embeddings and proves their properties. Section 6 briefly shows how to instantiate our results with two $\Box$-modalities. Section 7 concludes.

2 Background

The source calculi of the modal translations we will study in this paper are the call-by-name and call-by value λ-calculi. In this section we fix notation, terminology and several definitions regarding these calculi, including what we mean by a "calling paradigm", and by "indifference property", and how we define standard reduction.

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A_1 \vdash M : A_2}{\Gamma \vdash \lambda x.M : A_1 \supset A_2} \qquad \frac{\Gamma \vdash M : A_1 \supset A_2 \quad \Gamma \vdash N : A_1}{\Gamma \vdash MN : A_2}$$

Figure 1 (Shared) typing rules of source calculi $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$.

The modal translations will be defined on untyped source calculi, but at the same time we will develop simply-typed versions of the source calculi and translations. The source calculi are based on the set of λ-terms, given by

$$M, N, P, Q ::= x \mid \lambda x.M \mid MN$$

A value is a term of the form x or $\lambda x.M$. Values are ranged over by V, W.

We consider two reduction rules

$$(\lambda x.M)N \to [N/x]M \qquad (\beta_{\mathtt{n}}) \qquad\qquad (\lambda x.M)V \to [V/x]M \qquad (\beta_{\mathtt{v}})$$

As usual, $\to_{\beta_{\mathtt{n}}}$ (resp. $\to_{\beta_{\mathtt{v}}}$) denotes the compatible closure of $\beta_{\mathtt{n}}$ (resp. $\beta_{\mathtt{v}}$). Compatible closure is the closure under the term formers for λ-abstraction and application, *i.e.* closure under the rules:

$$\frac{M \to M'}{MN \to M'N}\ (\mu) \qquad \frac{N \to N'}{MN \to MN'}\ (\nu) \qquad \frac{M \to M'}{\lambda x.M \to \lambda x.M'}\ (\xi)$$

When we equip the λ-terms with $\to_{\beta_{\mathtt{n}}}$ we obtain the ordinary λ-calculus, or **call-by-name λ-calculus** here denoted $\lambda_{\mathtt{n}}$; when we equip the λ-terms with $\to_{\beta_{\mathtt{v}}}$ we obtain Plotkin's **cbv λ-calculus** [12], here denoted $\lambda_{\mathtt{v}}$.

We will develop in parallel the typed version of these calculi. Here, types are given by:

$$A, A' ::= X \mid A \supset A'$$

Let Γ range over sets of type assignments $x : A$ with all x distinct. The typing system derives sequents of the form $\Gamma \vdash M : A$. The typing rules are given in Fig. 1. Logically, this is a presentation of intuitionistic implicational logic.

We define sub-relations of $\to_{\beta_{\mathtt{n}}}$ and $\to_{\beta_{\mathtt{v}}}$. To this end, we need the closure rule $\nu_<$, the restriction of ν above where M is V. Then we define: $\to_{\mathtt{w}}$ as $\beta_{\mathtt{n}}$ closed under μ and ν; $\to_{\mathtt{n}}$ as $\beta_{\mathtt{n}}$ closed under μ; $\to_{\mathtt{v}}$ as $\beta_{\mathtt{v}}$ closed under μ and $\nu_<$.

In $\to_{\mathtt{w}}$, reduction under λ's is forbidden, it is in this sense that $\to_{\mathtt{w}}^*$ is called **weak reduction**. In $\to_{\mathtt{w}}$, reduction in applications can occur both in function position or argument position, in any order. Relations $\to_{\mathtt{n}}$ and $\to_{\mathtt{v}}$ are two ways of restricting $\to_{\mathtt{w}}$ to get a deterministic relation (a partial function). The effect of $\nu_<$ is to force reduction in function position first (as reduction in arguments can only occur when the function position term is already a value). This option we call *left-first* and convey the idea with the symbol $<$. Notice $\nu_<$ has to be combined with $\beta_{\mathtt{v}}$: closing $\beta_{\mathtt{n}}$ under μ and $\nu_<$ does not give a deterministic relation (since a $\beta_{\mathtt{n}}$ redex with a non-value argument can reduce in two ways in this relation). We call $\to_{\mathtt{n}}^*$ and $\to_{\mathtt{v}}^*$ respectively **call-by-name evaluation** and **call-by-value evaluation**. Weak reduction and cbn evaluation make sense in $\lambda_{\mathtt{n}}$ while cbv evaluation makes sense in $\lambda_{\mathtt{v}}$.

The **standardization** theorem for $\lambda_{\mathtt{n}}$ says that $M \to_{\beta_{\mathtt{n}}}^* N$ iff M reduces in a *standard* way to N; it states the completeness of that standard way of reducing. The specification of the standard way of reducing can be made by characterizing what reduction sequences are accepted as standard [1], or by axiomatizing the relation that M reduces in a standard

$$\frac{}{x \Rightarrow_{\mathtt{n}} x}\ VAR \quad \frac{M \Rightarrow_{\mathtt{n}} N}{\lambda x.M \Rightarrow_{\mathtt{n}} \lambda x.N}\ ABS \quad \frac{M \Rightarrow_{\mathtt{n}} M' \quad N \Rightarrow_{\mathtt{n}} N'}{MN \Rightarrow_{\mathtt{n}} M'N'}\ APL$$

$$\frac{M \to_{\mathtt{n}}^* \lambda x.M' \quad [N/x]M' \Rightarrow_{\mathtt{n}} P}{MN \Rightarrow_{\mathtt{n}} P}\ RDX$$

Figure 2 Standard reduction in $\lambda_{\mathtt{n}}$.

$$\frac{}{x \Rightarrow_{\mathtt{v}} x}\ VAR \quad \frac{M \Rightarrow_{\mathtt{v}} N}{\lambda x.M \Rightarrow_{\mathtt{v}} \lambda x.N}\ ABS \quad \frac{M \Rightarrow_{\mathtt{v}} M' \quad N \Rightarrow_{\mathtt{v}} N'}{MN \Rightarrow_{\mathtt{v}} M'N'}\ APL$$

$$\frac{M \to_{\mathtt{v}}^* \lambda x.M' \quad N \to_{\mathtt{v}}^* V \quad [V/x]M' \Rightarrow_{\mathtt{v}} P}{MN \Rightarrow_{\mathtt{v}} P}\ RDX$$

Figure 3 Standard reduction in $\lambda_{\mathtt{v}}$.

way to N [8]. In Fig. 2 we give one such axiomatization (it thus is in the spirit of [8], but notice no use is made of the vector notation), with the relation denoted $M \Rightarrow_{\mathtt{n}} N$. It is straightforward to see that $\Rightarrow_{\mathtt{n}}$ is contained in $\to_{\beta_{\mathtt{n}}}^*$, and from such a proof one extracts a notion of **standard reduction sequence**: it starts with cbn evaluation (corresponding to applications of rule RDX) of the given application MN, until one decides to freeze the outer construct and do reduction inside the subexpressions (corresponding to application of the other rules). The inclusion of $\to_{\beta_{\mathtt{n}}}^*$ in $\Rightarrow_{\mathtt{n}}$ is the real content of the standardization theorem, and will be obtained later as a particular case of a more general, unifying result.

We also give a definition of the relation $M \Rightarrow_{\mathtt{v}} N$ (M reduces in a standard way to N in $\lambda_{\mathtt{v}}$), again not by characterizing standard reduction sequences [12], rather by the inductive definition in Fig. 3. These rules determine a similar notion of standard reduction sequence: cbv evaluation of the given application MN, until one decides to freeze the outer construct and do reduction inside the subexpressions. The standardization theorem for $\lambda_{\mathtt{v}}$ will also be obtained later as a particular case of the same more general, unifying result.

Call-by-name and call-by-value are **calling paradigms**, in the sense that each comprises: a variant of the β-rule, specifying how functions are called, and generating a notion of deterministic evaluation and a notion of full reduction. According to [12], the evaluation relation can be understood as a programming language, and the full reduction (more precisely, the related notion of equality) can be understood as the corresponding calculus. The standardization theorem shows how evaluation can be used in a specific, but complete, way of reducing, and thereby links the programming language and the calculus.

Call-by-name and call-by-value are related by cps-translations, one from cbn to cbv, another the other way around [12]. The maps link full reduction or evaluation in the source system to full reduction or evaluation in the target system, respectively, and these are the **translation** and **simulation** properties of the maps [12]. But the target of the cps-translations is the subset of λ-terms given by the grammar

$$M, N ::= V \mid MV \qquad V ::= x \mid \lambda x.M$$

This is such a restricted set of terms that we cannot observe any difference between $\beta_{\mathtt{n}}$- and $\beta_{\mathtt{v}}$-reduction, and that all three relations $\to_{\mathtt{w}}$, $\to_{\mathtt{n}}$, and $\to_{\mathtt{v}}$ collapse to the same relation, viz. $\beta_{\mathtt{v}}$ closed under μ – a kind of intersection between $\to_{\mathtt{n}}$ and $\to_{\mathtt{v}}$. So, in this subset of terms

$$\frac{}{\Gamma, x : B \vdash x : B} \qquad \frac{\Gamma, x : B \vdash M : A}{\Gamma \vdash \lambda x.M : B \supset A} \qquad \frac{\Gamma \vdash M : B \supset A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{box}(M) : \Box A} \qquad \frac{\Gamma \vdash M : \Box A}{\Gamma \vdash \varepsilon(M) : A}$$

Figure 4 Typing rules of the modal target calculus $\lambda_\Box$.

we cannot observe any difference between cbn and cbv evaluation; and weak reduction is also deterministic and coincides with those. This is the **indifference property** of the image of cps-translations.

Modal embeddings, we will see, also provide translations and simulations of cbn and cbv into a shared target enjoying an indifference property and following a new calling paradigm.

3 Modal embeddings

In this section, we recast the two modal embeddings of intuitionistic logic into intuitionistic modal logic S4 due to Girard and Gödel [16] as translations from $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ into a very simple λ-calculus, named $\lambda_\Box$. This system corresponds to a fragment of intuitionistic S4, but will prove to be strong enough to interpret call-by-name and call-by-value. We will recall the well-known properties of preservation of reduction by the embeddings in considerable detail, since this will be useful to motivate the refinements in the following sections.

3.1 Modal target calculus $\lambda_\Box$

We will develop in parallel the untyped and typed versions of the comonadic language $\lambda_\Box$. The terms are given by:

$$M, N, P, Q ::= x \mid \lambda x.M \mid MN \mid \mathsf{box}(M) \mid \varepsilon(M)$$

On this set we define two reduction rules:

$$(\lambda x.M)N \to [N/x]M \qquad (\beta_\supset) \qquad\qquad \varepsilon(\mathsf{box}(M)) \to M \qquad (\beta_\Box)$$

Here $[N/x]M$ denotes ordinary substitution.

As usual, for $R \in \{\beta_\supset, \beta_\Box\}$ or $R = \beta_\supset \cup \beta_\Box$, $\to_R$ denotes the closure of R under all term formers. In other words, $\to_R$ denotes the closure of R under five rules: rules μ, ν and ξ, allowing reduction under application or abstraction, plus two rules allowing reduction under box and ε. Instead of $\to_{\beta_\supset \cup \beta_\Box}$ normally we only write $\to$; so, in this case, $\to^*$ (resp. $\to^+$) stands for the reflexive-transitive (resp. transitive) closure of $\to_{\beta_\supset \cup \beta_\Box}$.

Typing helps grasping this term language. Types are given by:

$$A ::= X \mid B \supset A \mid B \qquad\qquad B ::= \Box A$$

Contexts Γ are sets of declarations $x : B$ where each x is declared at most once. The typing system derives sequents $\Gamma \vdash M : A$, and the typing rules are in Fig. 4.

The simple-minded $\Box$ introduction rule is appropriate because of the restriction of contexts to boxed types (for arbitrary contexts this rule is unsound for intuitionistic S4). This restriction, in turn, dictates the restriction of left-hand-sides of implications to boxed types.

$$
\begin{array}{rclrcl}
x^\sharp_U & = & x & (\mathsf{box}(M))^\sharp_U & = & \mathsf{cobind}(x_1, \ldots, x_n, x_1 \ldots x_n . M^\sharp_U) \\
(\lambda x.\, M)^\sharp_U & = & \lambda x. M^\sharp_{U,x} & & & (x_1, \ldots, x_n = U, \text{ all } x_i \text{ distinct}) \\
(MN)^\sharp_U & = & M^\sharp_U N^\sharp_U & (\varepsilon(M))^\sharp_U & = & \varepsilon(M^\sharp_U)
\end{array}
$$

Figure 5 Scoped term translation from $\lambda_\Box$ to term calculus of **IS4**.

The $\beta_\Box$ reduction rule corresponds to the rule for contracting $\Box$ introduction/elimination detours.

Substitution enjoys the usual admissible typing rule, but with a restriction to boxed types imposed by the definition of contexts:

$$
\frac{\Gamma \vdash N : B \quad \Gamma, x : B \vdash M : A}{\Gamma \vdash [N/x]M : A} \tag{1}
$$

The proof uses admissibility of weakening in the case $M = \lambda y.M'$. The latter follows by an immediate induction, but crucially depends on the restriction of contexts to boxed types.

▶ **Proposition 1** (Subject reduction of $\lambda_\Box$). *In $\lambda_\Box$, if $M \to N$ and $\Gamma \vdash M : A$, then $\Gamma \vdash N : A$.*

Proof. By induction on $\to$. The base case relative to $\beta_\supset$ uses the typing rule for substitution (1). The base case relative to $\beta_\Box$ is immediate by inversion of the typing rules for ε and box. ◀

3.2 Comparison of $\lambda_\Box$ with IS4

The difficulty of formulating a satisfactory natural deduction system for the modal logic S4 is well known since Prawitz [13]. The issue is not at the level of provability, but rather at the level of normalization: to guarantee that the system is closed under substitution [2]. Bierman and de Paiva [2], in their natural deduction system **IS4** for full intuitionistic S4 (recalled in Appendix A), need to work with the general introduction rule for the modality, which brings certain complications. In contrast, in $\lambda_\Box$, we adopt the naive introduction rule for the modality, but have to operate with the restricted implications and sequents that ensure closure under substitution.

Therefore, on the level of logic alone, $\lambda_\Box$ is a fragment of Bierman and de Paiva's **IS4**. As a term calculus, $\lambda_\Box$ is a fragment of Bierman and de Paiva's term calculus, which is a calculus for a cartesian closed category equipped with a lax monoidal comonad. Fig. 5 gives a translation of scoped $\lambda_\Box$ terms to scoped **IS4** terms. U ranges over sets of variables; $M^\sharp_U$ is well-defined if $\mathrm{FV}(M) \subseteq U$. We write $|\Gamma|$ for the set $\{x \mid x : A \in \Gamma\}$. This translation not only preserves typing and reduction steps of $\lambda_\Box$ terms, but also *reflects* typing and reduction steps of the **IS4** terms in its image, thus isolating a fragment of **IS4** isomorphic to $\lambda_\Box$:

▶ **Proposition 2** (Preservation and reflection of typing and reduction from $\lambda_\Box$ to IS4).
1. *For all Γ, A in $\lambda_\Box$, $\Gamma \vdash M : A$ in $\lambda_\Box$ iff $\Gamma \vdash M^\sharp_{|\Gamma|} : A$ in the term calculus for* **IS4**.
2. *For all U s. t. $\mathrm{FV}(M) \subseteq U$, $M \to N$ in $\lambda_\Box$ iff $M^\sharp_U \to N^\sharp_U$ in the term calculus for* **IS4**.

▶ **Proposition 3** (Conservativity of IS4 over $\lambda_\Box$). *For all Γ, A in $\lambda_\Box$, if $\Gamma \vdash M : A$ in the term calculus for* **IS4**, *then there exists N such that $\Gamma \vdash N : A$ in $\lambda_\Box$.*

See the appendix for proofs.[1]

[1] In the case box(M) of the translation in Fig. 5, it is important to “rebind” all variables of M to

$$\begin{array}{rcl} X^\circ & = & X \\ (A_1 \supset A_2)^\circ & = & \Box A_1^\circ \supset A_2^\circ \end{array} \qquad \begin{array}{rcl} x^\circ & = & \varepsilon(x) \\ (\lambda x.M)^\circ & = & \lambda x.M^\circ \\ (MN)^\circ & = & M^\circ \mathsf{box}(N^\circ) \end{array}$$

Figure 6 Translation from λ_n to $\lambda_\Box$ ("Girard's translation").

3.3 Modal embeddings $(\cdot)^\circ$ and $(\cdot)^*$

The two modal translations are presented as maps from λ-terms to $\lambda_\Box$-terms. The original, and easier to grasp, motivation is logical, so the mapping of types and type preservation by the translations is presented right away. We also detail the preservation of reduction steps, which has been observed many times in many contexts [5, 11, 3]. Later, we will strengthen these results.

The translation from λ_n to $\lambda_\Box$ is in Fig. 6. On the level of types (i.e. the underlying logic) it is inspired by translation of intuitionistic logic into linear logic [4], based on the characteristic decomposition of intuitionistic implication into linear implication and the ! modality.[2]

In the following, $\Box\Gamma^\circ = x_1 : \Box A_1^\circ, \ldots, x_n : \Box A_n^\circ$ when $\Gamma = x_1 : A_1, \ldots, x_n : A_n$.

▶ **Proposition 4** (Preservation and reflection of typing by Girard's translation). *$\Gamma \vdash M : A$ in λ_n iff $\Box\Gamma^\circ \vdash M^\circ : A^\circ$ in $\lambda_\Box$.*

Proof. In each direction, routine induction on the given typing derivation. For example, in the "if" direction: the case $M = x$ follows by the axiom of λ_n, because the hypothesis implies $x : A$ is in Γ; the case $M = \lambda x.N$ follows because the hypothesis implies, for some A_1, A_2, $A = A_1 \supset A_2$ and $\Box\Gamma^\circ, x : \Box A_1^\circ \vdash N^\circ : A_2^\circ$ (through the immediate subderivation of the given typing derivation), so the IH applies and the $\supset$ introduction typing rule of λ_n can be used to conclude. ◀

▶ **Lemma 5.** *$[\mathsf{box}(N^\circ)/x]M^\circ \to^*_{\beta_\Box} ([N/x]M)^\circ$.*

Proof. By induction on M. The critical case $M = x$ reads: $LHS = \varepsilon(\mathsf{box}(N^\circ)) \to_{\beta_\Box} N^\circ = RHS$. ◀

▶ **Proposition 6** (Preservation of reduction by Girard's translation). *If $M \to_{\beta_\mathsf{n}} N$ in λ_n, then $M^\circ \to^+ N^\circ$ in $\lambda_\Box$.*

Proof. By induction on $M \to_{\beta_\mathsf{n}} N$. The base case uses the previous lemma:

$$((\lambda x.M)N)^\circ = (\lambda x.M^\circ)\mathsf{box}(N^\circ) \to_{\beta_\supset} [\mathsf{box}(N^\circ)/x]M^\circ \to^*_{\beta_\Box} ([N/x]M)^\circ \ . \tag{2}$$

◀

The translation from λ_v to $\lambda_\Box$ is in Fig. 7. A^* is denoted $A^\Box$ in [16], where it is defined directly by recursion on A: $X^\Box = \Box X$ and $(A_1 \supset A_2)^\Box = \Box(A_1^\Box \supset A_2^\Box)$. These are two styles for defining the same translation of types. The style we adopted is the same of Gödel's 1933 paper, while the alternative style was proposed by McKinsey-Tarski [15].

match the typing rule of cobind in **IS4** (introduction rule of $\Box$). Contrary to $\lambda_\Box$, contexts of **IS4** may contain unboxed formulas. The cobind typing rule is essentially a combination of the box typing rule (introduction rule of $\Box$ in $\lambda_\Box$), relying on a fully boxed context $x_1 : B_1, ..., x_n : B_n$, with a multicut.

[2] We could call Girard's translation the $(\Box A \supset B)$-translation.

$$
\begin{array}{rcl}
A^* & = & \Box A^\bullet \\
 & & \\
X^\bullet & = & X \\
(A_1 \supset A_2)^\bullet & = & \Box A_1^\bullet \supset \Box A_2^\bullet
\end{array}
\qquad
\begin{array}{rcl}
V^* & = & \mathsf{box}(V^\bullet) \\
(MN)^* & = & \varepsilon(M^*)N^* \\
x^\bullet & = & \varepsilon(x) \\
(\lambda x.M)^\bullet & = & \lambda x.M^*
\end{array}
$$

Figure 7 Translation from λ_v to $\lambda_\Box$ ("Gödel's translation").

At the term level, the translation is organized in two levels: there is a translation of terms M^* and a translation of values $V^\bullet$. A simpler translation with $x^* = x$ was possible, but the adopted version (with an η-expansion in the translation of variables) allows a uniform translation of values as terms: $V^* = \mathsf{box}(V^\bullet)$. It is sound to $\eta_\Box$-expand a variable of type A^* because A^* is a boxed type.[3]

In the following, $\Gamma^* = x_1 : A_1^*, \ldots, x_n : A_n^*$ when $\Gamma = x_1 : A_1, \ldots, x_n : A_n$.

▶ **Proposition 7** (Preservation and reflection of typing by Gödel's translation)**.**

1. $\Gamma \vdash M : A$ *in* λ_v *iff* $\Gamma^* \vdash M^* : A^*$ *in* $\lambda_\Box$*.*
2. $\Gamma \vdash V : A$ *in* λ_v *iff* $\Gamma^* \vdash V^\bullet : A^\bullet$ *in* $\lambda_\Box$*.*

Proof. In each direction, the two statements are proved by mutual induction on the given typing derivation. ◀

▶ **Lemma 8.**

1. $[\mathsf{box}(V^\bullet)/x]M^* \to^*_{\beta_\Box} ([V/x]M)^*$.
2. $[\mathsf{box}(V^\bullet)/x]W^\bullet \to^*_{\beta_\Box} ([V/x]W)^\bullet$.

Proof. By simultaneous induction on M and W. The critical case $W = x$ reads: $LHS = \varepsilon(\mathsf{box}(V^\bullet)) \to_{\beta_\Box} V^\bullet = RHS$. ◀

▶ **Proposition 9** (Preservation of reduction by Gödel's translation)**.** *If* $M \to_{\beta_\mathsf{v}} N$ *in* λ_v*, then* $M^* \to^+ N^*$ *in* $\lambda_\Box$*.*

Proof. By induction on $M \to_{\beta_\mathsf{v}} N$. The base case reads:

$$
\begin{array}{rcl}
((\lambda x.M)V)^* & = & \varepsilon(\mathsf{box}(\lambda x.M^*))\mathsf{box}(V^\bullet) \\
 & \to_{\beta_\Box} & (\lambda x.M^*)\mathsf{box}(V^\bullet) \to_{\beta_\supset} [\mathsf{box}(V^\bullet)/x]M^* \to^*_{\beta_\Box} ([V/x]M)^*
\end{array}
\tag{3}
$$

where the last reduction is justified by the previous lemma. ◀

▶ **Example 10.** Reflection of reduction along the translation from λ_v to $\lambda_\Box$ fails. Let $P := (\lambda x.xM)N$ (with $x \notin FV(M)$) and $Q := NM$. Then $P \to_{\beta_\mathsf{v}} Q$ does *not* hold in general. But $P^* \to^+ Q^*$ does hold, since:

$$
P^* = \varepsilon(\mathsf{box}(\lambda x.\varepsilon(\mathsf{box}(\varepsilon(x)))M^*))N^* \to^2_{\beta_\Box} (\lambda x.\varepsilon(x)M^*)N^* \to_{\beta_\supset} \varepsilon(N^*)M^* = Q^* \ .
$$

[3] We could call Gödel's translation the $(\Box A \supset \Box B)$-translation, and call McKinsey-Tarski translation the $(\Box(A \supset B))$-translation. There is again a connection with translations of intuitionistic logic into linear logic. The "call-by-value" translation of intuitionistic logic into linear logic is sometimes called the $!(A \multimap B)$-translation or the $(!A \multimap !B)$-translation. The former is found in the original paper by Girard [4]; the second is briefly mentioned by Lafont [7] in the discussion of the translation of λ-calculus, and used in [11]. In the translation of λ-terms into linear logic proofs and proof nets, Mackie [9] does the η-expansion of variables in the $(!A \multimap !B)$-translation and does not do it in the $!(A \multimap B)$-translation.

This example is adapted from one given in [14], where the same remark is made about the translation of λ_{v} into a linear λ-calculus (yet another, similar example is given in [11]). The path followed in [14] is to grow the source calculus from Plotkin's λ_{v} to Moggi's computational λ-calculus in order to derive more reductions. In the next section we follow a different path and shrink the target calculus.

4 Refined modal target calculus λ_{b}

In order to give a deeper analysis of the modal embeddings, and refine the results of the previous section, we will identify in this section a sublanguage λ_{b} of $\lambda_{\Box}$. In the next section, we give refined versions of the embeddings whose images lie in λ_{b}. In this section, we start with a motivation for the refinements. Next we define λ_{b} and prove some properties, notably a standardization theorem.

4.1 Motivation

We start by analyzing whether Proposition 6 and 9 could be stated as equivalences, so that we would also have reflection of reduction. By inspection of calculations (2) and (3), one recognizes an obstacle in the uncontrolled proliferation of $\beta_{\Box}$-reduction steps in the reduction between images. We regard $\beta_{\Box}$-reduction steps as **administrative** [12], and seek to hide them somehow. In the cps-literature, administrative steps are hidden by performing them at compile time by an optimized version of the translation [12]. Here, we will also use such an idea, but combined with another one: a refined definition of the target system will also avoid many administrative steps.

Inspecting (2) again one sees that administrative steps in the image of Girard's map come from Lemma 5. But notice that a substitution is always triggered with a box (a term of the form $\mathsf{box}(N)$) as the actual parameter; since in the target of Girard's map variables are always wrapped with ε, every actual replacement generates an administrative redex. A solution is to pass, not the box, but the contents of the box (the box is open), which will replace, not x, but $\varepsilon(x)$. The adoption of this special β-rule is a simple trick that eliminates all the administrative steps in the image of Girard's map.

How about Gödel's map? Inspecting calculation (3) one sees again that the $\beta_{\supset}$-redex has a box as argument, and that the subsequent administrative steps are justified by the lemma that shows how substitution commutes with the translation, namely Lemma 8. Again, a special β-reduction step could open the box before executing the substitution. But the $\beta_{\supset}$-reduction step has a preliminary administrative step, and not all occurrences of $\varepsilon(M)$ in the image of the map have the form $\varepsilon(x)$. The latter two problems have a common solution. In the translation of application in Fig. 7, one should put $(\lambda z.\varepsilon(z)N^*)M^*$. The preliminary administrative step in (3) will become a special β-reduction step, and $\varepsilon(x)$ will suffice in the images of the translation as a monolithic term form instead of x and $\varepsilon(M)$.

4.2 Definition of λ_{b} and relationship with $\lambda_{\Box}$

We call our refined modal calculus λ_{b}. Its terms are given by the grammar:

$$M, N, P, Q, T ::= \varepsilon(x) \mid \lambda x.M \mid MN \mid \mathsf{box}(N)$$

Note that variables x and ε are amalgamated, and constrained to the construction $\varepsilon(x)$. **Values** V are terms of the form $\varepsilon(x)$ or $\lambda x.M$. **Boxes** are terms of the form $\mathsf{box}(N)$, ranged over by B.

Types and sequents of $\lambda_{\mathtt{b}}$ are as for $\lambda_\Box$. Recall that in implications the antecedent must be a boxed type, and accordingly types in contexts must be boxed. The typing rules of $\lambda_{\mathtt{b}}$ are as for $\lambda_\Box$, except that the typing rule for $\varepsilon(x)$ corresponds to the obvious combination of the typing rules of $\lambda_\Box$ for variables and for ε, that reads as follows:

$$\frac{}{\Gamma, x : \Box A \vdash \varepsilon(x) : A}$$

Immediately: for any $M \in \lambda_{\mathtt{b}}$, $\Gamma \vdash M : A$ in $\lambda_{\mathtt{b}}$ iff $\Gamma \vdash M : A$ in $\lambda_\Box$.

The unique reduction rule of $\lambda_{\mathtt{b}}$ is:

$$(\lambda x.M)\mathsf{box}(N) \to [N/\varepsilon(x)]M \qquad (\beta_{\mathtt{b}})$$

where $[N/\varepsilon(x)]M$ is defined by recursion on M, and all clauses are homomorphic, except for the critical clauses:

$$[N/\varepsilon(x)]\varepsilon(x) = N \qquad\qquad [N/\varepsilon(x)]\varepsilon(y) = \varepsilon(y) \quad (x \neq y) \ .$$

As usual, $\to_{\beta_{\mathtt{b}}}$ denotes the compatible closure of $\beta_{\mathtt{b}}$, *i.e.* the closure of $\beta_{\mathtt{b}}$ under all term formers of $\lambda_{\mathtt{b}}$.

The $\beta_{\mathtt{b}}$-rule is a package of reduction steps of $\lambda_\Box$. In fact, for $M, N \in \lambda_{\mathtt{b}}$:

$$(\lambda x.M)\mathsf{box}(N) \to_{\beta_\supset} [\mathsf{box}(N)/x]M \to^*_{\beta_\Box} [N/\varepsilon(x)]M$$

where $[\mathsf{box}(N)/x]M \to^*_{\beta_\Box} [N/\varepsilon(x)]M$ is easily established by induction on M.

Rule $\beta_{\mathtt{b}}$ only fires when the argument is a box. For this reason we speak of **call-by-box**. In the typed setting, and since function types are always of the form $B \supset A$, arguments are always of boxed types – nevertheless, arguments are not necessarily boxes. We will prove call-by-box (abbreviated cbb) to be a calling paradigm, in the sense of Section 2, by defining evaluation and standard reduction and proving standardization.

We define sub-relations of $\to_{\beta_{\mathtt{b}}}$. To this end consider the closure rules:

$$\frac{M \to M'}{MN \to M'N}\ (\mu) \quad \frac{M \to M'}{M\mathsf{B} \to M'\mathsf{B}}\ (\mu_>) \quad \frac{N \to N'}{MN \to MN'}\ (\nu) \quad \frac{N \to N'}{VN \to VN'}\ (\nu_<)$$

Then: $\to_{\mathtt{we}}$ is inductively defined by $\beta_{\mathtt{b}}$ and μ and ν; $\to_{\mathtt{we}>}$ is inductively defined by $\beta_{\mathtt{b}}$ and $\mu_>$ and ν; $\to_{\mathtt{we}<}$ is inductively defined by $\beta_{\mathtt{b}}$, μ and $\nu_<$.

Notice: we always close the same β-rule (hence a single calling paradigm is at stake). Relation $\to_{\mathtt{we}}$ is called **weak** (because values do not reduce) and **external** - because boxes do not reduce. Relation $\to^*_{\mathtt{we}}$ is called **call-by-box evaluation**. Here, evaluation is taken in a relaxed sense, since the relation $\to_{\mathtt{we}}$ is non-deterministic: the cbb "evaluation" of a given application MN consists of the interleaved cbb "evaluation" of M and N in any order until a $\beta_{\mathtt{b}}$-redex emerges at root position. We may turn $\to_{\mathtt{we}}$ into a deterministic relation, by imposing either the left-first ($<$) or right-first ($>$) order of reduction in applications.[4] Cbn (resp. cbv) evaluation on $\lambda_{\mathtt{b}}$ will be defined later, as a restriction of $\to^*_{\mathtt{we}>}$ (resp. $\to^*_{\mathtt{we}<}$).

4.3 Properties of $\lambda_{\mathtt{b}}$

The main property of $\lambda_{\mathtt{b}}$ is how it unifies call-by-name and call-by-value, and will be seen in Section 5. Here we chose to show subject reduction, because it is a sanity check for a modal calculus, and standardization, because we want to promote call-by-box to a calling paradigm.

[4] Recall how the combination of rules $\beta_{\mathtt{n}}$, μ and $\nu_<$ on λ-terms failed to produce a deterministic relation.

$$\frac{}{\varepsilon(x) \Rightarrow_{\mathsf{b}} \varepsilon(x)}\ VAR \quad \frac{M \Rightarrow_{\mathsf{b}} N}{\lambda x.M \Rightarrow_{\mathsf{b}} \lambda x.N}\ ABS \quad \frac{M \Rightarrow_{\mathsf{b}} M' \quad N \Rightarrow_{\mathsf{b}} N'}{MN \Rightarrow_{\mathsf{b}} M'N'}\ APL \quad \frac{M \Rightarrow_{\mathsf{b}} N}{\mathsf{box}(M) \Rightarrow_{\mathsf{b}} \mathsf{box}(N)}\ BOX$$

$$\frac{M \to_{\mathtt{we}}^{*} \lambda x.M' \quad N \to_{\mathtt{we}}^{*} \mathsf{box}(N') \quad [N'/\varepsilon(x)]M' \Rightarrow_{\mathsf{b}} P}{MN \Rightarrow_{\mathsf{b}} P}\ RDX$$

Figure 8 Standard reduction of λ_{b}.

$$\frac{}{M \Rightarrow_{\mathsf{b}} M}\ (1) \qquad \frac{}{(\lambda x.M)\mathsf{box}(N) \Rightarrow_{\mathsf{b}} [N/\varepsilon(x)]M}\ (2) \qquad \frac{M \Rightarrow_{\mathsf{b}} M' \quad N \Rightarrow_{\mathsf{b}} N'}{[N/\varepsilon(x)]M \Rightarrow_{\mathsf{b}} [N'/\varepsilon(x)]M'}\ (3)$$

$$\frac{M \to_{\mathtt{we}} N \Rightarrow_{\mathsf{b}} P}{M \Rightarrow_{\mathsf{b}} P}\ (4) \qquad \frac{M \to_{\mathtt{we}}^{*} N \Rightarrow_{\mathsf{b}} P}{M \Rightarrow_{\mathsf{b}} P}\ (5)$$

$$\frac{M \Rightarrow_{\mathsf{b}} \lambda x.M' \quad N \Rightarrow_{\mathsf{b}} \mathsf{box}(N')}{MN \Rightarrow_{\mathsf{b}} [N'/\varepsilon(x)]M'}\ (6) \qquad \frac{M \Rightarrow_{\mathsf{b}} (\lambda x.M')\mathsf{box}(N')}{M \Rightarrow_{\mathsf{b}} [N'/\varepsilon(x)]M'}\ (7) \qquad \frac{M \Rightarrow_{\mathsf{b}} N \to_{\beta_{\mathsf{b}}} P}{M \Rightarrow_{\mathsf{b}} P}\ (8)$$

Figure 9 Admissible rules of λ_{b}.

▶ **Proposition 11** (Subject reduction of λ_{b}). *In λ_{b}, if $M \to_{\beta_{\mathsf{b}}} N$ and $\Gamma \vdash M : A$, then $\Gamma \vdash N : A$.*

Proof. This is an immediate consequence of subject reduction for $\lambda_{\Box}$ (Prop. 1), and the facts (i) $\Gamma \vdash M : A$ in λ_{b} iff $\Gamma \vdash M : A$ in $\lambda_{\Box}$, and (ii) $M \to_{\beta_{\mathsf{b}}} N$ implies $M \to_{\beta}^{*} N$ in $\lambda_{\Box}$. ◀

Fig. 8 gives an inductive definition of the relation "M reduces in a standard way to N in λ_{b}", denoted $M \Rightarrow_{\mathsf{b}} N$.

▶ **Theorem 12** (Standardization of λ_{b}). *In λ_{b}, $M \to_{\beta_{\mathsf{b}}}^{*} N$ iff $M \Rightarrow_{\mathsf{b}} N$.*

Proof. Our proof is inspired in Loader's proof for λ-calculus [8], but notice no use of vector notation is made, and the definition of $\Rightarrow_{\mathsf{b}}$ makes explicit the contribution of evaluation.

The "if" direction is a very simple induction on $M \Rightarrow_{\mathsf{b}} N$ and just uses the facts that $\to_{\beta_{\mathsf{b}}}^{*}$ is reflexive, transitive and compatible, and that $\to_{\mathtt{we}}^{*} \subseteq \to_{\beta_{\mathsf{b}}}^{*}$.

The "only if" direction is proved by establishing the admissibility of the rules (1) to (8) in Fig. 9. Once this is done, the proof of the "only if" implication is by induction on $M \to_{\beta_{\mathsf{b}}}^{*} N$, and follows immediately from rules (1) and (8).

The proof of (1) is an easy induction on M. Then (2) follows from RDX and (1). The proof of (3) is by induction on $M \Rightarrow_{\mathsf{b}} M'$. The case RDX requires the substitution lemma for λ_{b}'s substitution, plus the following property of $\to_{\mathtt{we}}$: if $M \to_{\mathtt{we}} M'$ then $[N/\varepsilon(x)]M \to_{\mathtt{we}} [N/\varepsilon(x)]M'$. Rule (5) follows easily from (4), and the latter is proved by induction on $M \to_{\mathtt{we}} N$. Rule (6) is proved by induction on $M \Rightarrow_{\mathsf{b}} \lambda x.M'$ with subinduction on $N \Rightarrow_{\mathsf{b}} \mathsf{box}(N')$. Use is made of (3) and (5). Then, (7) follows easily from (6) by induction on $M \Rightarrow_{\mathsf{b}} (\lambda x.M')\mathsf{box}(N')$. Finally, (8) is proved by induction on $M \Rightarrow_{\mathsf{b}} N$, and uses (7). ◀

From the proof of the "if" implication of this theorem, one extracts a notion of **standard reduction sequence**: it starts with cbb evaluation (corresponding to applications of rule RDX) of the given application MN, until one decides to freeze the outer construct and do reduction inside the subexpressions (corresponding using the other rules in Fig. 8).

5 Refined embeddings into $\lambda_{\mathtt{b}}$

Continuing to implement the refinements motivated at the beginning of Section 4, we now introduce variants of Girard's and Gödel's translations from, respectively, $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ into the refined target calculus $\lambda_{\mathtt{b}}$. The hope was to improve Propositions 6 and 9, achieving reflection, in addition to preservation, of reduction by the maps.

In fact, we will obtain much more, as preservation and reflection will work at the levels of reduction, evaluation, and standard reduction – see Theorems 14 and 18 below. The results at the level of reduction and evaluation correspond to the **translation property** and **simulation property** of the maps, in the terminology of cps-translations [12]. The image of each map enjoys its own **indifference property**, and these two properties together are the indifference property of $\lambda_{\mathtt{b}}$. What is new compared to cps-translations is that the simulation and indifference properties cooperate to accomplish the results at the level of standard reduction; and the latter, together with the standardization for $\lambda_{\mathtt{b}}$, and the results at the level of reduction allow the extraction of the standardization for $\lambda_{\mathtt{n}}$ or $\lambda_{\mathtt{v}}$ as corollaries.

Translation of types stays unchanged both for the refined Girard's and Gödel's translations. At the level of terms, Girard's translation stays the same, but Gödel's translation will suffer a slight refinement, as promised at the beginning of Section 4. We reuse the symbols $(\cdot)^{\circ}$, $(\cdot)^{*}$ and $(\cdot)^{\bullet}$.

5.1 Refined Girard's embedding

We start with Girard's translation. Fig. 6 can be read *ipsis verbis* as defining a translation from $\lambda_{\mathtt{n}}$ to $\lambda_{\mathtt{b}}$. The refinement comes, not from the translation, but from the refined functioning of the target system. Preservation and reflection of typing, as stated in Prop. 4 for $\lambda_{\Box}$ for the original Girard's translation, holds in the same way for $\lambda_{\mathtt{b}}$ for the refined translation.

The image of the term translation is the subset of $\lambda_{\mathtt{b}}$ terms given by the grammar

$$M, N ::= \varepsilon(x) \mid \lambda x.M \mid M\,\mathsf{box}(N) \tag{4}$$

Let us call this subset **Girard's image**.

Due to the restricted form of arguments in applications, relations $\to_{\mathtt{we}}$ and $\to_{\mathtt{we>}}$ collapse on Girard's image to the same relation, one which can alternatively be defined as $\beta_{\mathtt{b}}$ closed under μ. This property of Girard's image we call its **indifference property**, by analogy with our account of the indifference property on λ-terms given at the end of Section 2. This single deterministic relation on Girard's image is denoted $\to_{\mathtt{n}}$. By **call-by-name evaluation** on $\lambda_{\mathtt{b}}$ we mean $\to_{\mathtt{n}}^{*}$. The terminology is justified by Theorem 14 below.

▶ **Lemma 13.** $[N^{\circ}/\varepsilon(x)]M^{\circ} = ([N/x]M)^{\circ}$.

▶ **Theorem 14** (Properties of refined Girard's translation)**.**
1. *(Preservation and reflection of reduction)* $M \to_{\beta_{\mathtt{n}}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^{\circ} \to_{\beta_{\mathtt{b}}} N^{\circ}$ *in* $\lambda_{\mathtt{b}}$*.*
2. *(Preservation and reflection of evaluation)* $M \to_{\mathtt{n}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^{\circ} \to_{\mathtt{n}} N^{\circ}$ *in* $\lambda_{\mathtt{b}}$*.*
3. *(Preservation and reflection of standard reduction)* $M \Rightarrow_{\mathtt{n}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^{\circ} \Rightarrow_{\mathtt{b}} N^{\circ}$ *in* $\lambda_{\mathtt{b}}$*.*

Proof.

Proof of 1. The "only if" half is proved by induction on $M \to_{\beta_{\mathtt{n}}} N$. The base case uses Lemma 13: $((\lambda x.M)N)^{\circ} = (\lambda x.M^{\circ})\mathsf{box}(N^{\circ}) \to_{\beta_{\mathtt{b}}} [N^{\circ}/\varepsilon(x)]M^{\circ} = ([N/x]M)^{\circ}$. The "if" half follows from this fact: the image of $(\cdot)^{\circ}$ is closed under reduction and any reduction between images is an image of a source reduction. Symbolically: if $M^\circ \to_{\beta_\mathtt{b}} N'$, then there is a λ-term N such that $N^\circ = N'$ and $M \to_{\beta_\mathtt{n}} N$. The proof is by induction on the λ-term M.

Proof of 2. By inspection of the proof of 1.

Proof of 3. The "only if" implication is proved by induction on $M \Rightarrow_\mathtt{n} N$. The case RDX uses item 2 of this theorem. To prove the "if" direction, one proves something stronger: if $M^\circ \Rightarrow_\mathtt{b} Q$ in $\lambda_\mathtt{b}$ then there is $N \in \lambda_\mathtt{n}$ such that $N^\circ = Q$ and $M \Rightarrow_\mathtt{n} N$ in $\lambda_\mathtt{n}$. The proof is by induction on $M^\circ \Rightarrow_\mathtt{b} Q$. The case RDX, besides Lemma 13, uses a strong form of the indifference property of Girard's image: $M^\circ \to_\mathtt{we} Q$ iff $M^\circ \to_\mathtt{n} Q$; and a strong form of item 2 of the present theorem: if $M^\circ \to_\mathtt{n} Q$ then there is $N \in \lambda_\mathtt{n}$ such that $N^\circ = Q$ and $M \to_\mathtt{n} N$ in $\lambda_\mathtt{n}$. ◀

▶ **Corollary 15** (Standardization of $\lambda_\mathtt{n}$). *In $\lambda_\mathtt{n}$, $M \to^*_{\beta_\mathtt{n}} N$ iff $M \Rightarrow_\mathtt{n} N$.*

Proof. Follows from Thm. 12 (standardization for $\lambda_\mathtt{b}$), and parts 1 and 3 of Thm. 14. ◀

5.2 Refined Gödel's embedding

Now we turn to Gödel's translation. We will make use of the abbreviation

$$\mathsf{raise}(M) := \lambda z.\varepsilon(z)M$$

We immediately remark the following derived rules:

$$\frac{\Gamma \vdash M : B}{\Gamma \vdash \mathsf{raise}(M) : (\Box(B \supset B')) \supset B'} \qquad \frac{N \to^*_\mathtt{we} \mathsf{box}(N') \quad M \to^*_\mathtt{we} \mathsf{box}(\lambda x.M')}{\mathsf{raise}(N)M \to^*_\mathtt{we} [N'/\varepsilon(x)]M'} \tag{5}$$

The latter is proved by the following reduction sequence:

$$\mathsf{raise}(N)M \to^*_\mathtt{we} \mathsf{raise}(N)\mathsf{box}(\lambda x.M') \to_\mathtt{we} (\lambda x.M')N \to^*_\mathtt{we} (\lambda x.M')\mathsf{box}(N') \to_\mathtt{we} [N'/\varepsilon(x)]M' \tag{6}$$

Gödel's term translation introduced before in Fig. 7 is refined thus:

$$V^* = \mathsf{box}(V^\bullet) \qquad (MN)^* = \mathsf{raise}(N^*)M^* \qquad x^\bullet = \varepsilon(x) \qquad (\lambda x.M)^\bullet = \lambda x.M^*$$

As said, the translation of types remains unchanged, and Proposition 7 about preservation and reflection of typing holds again, now for $\lambda_\mathtt{b}$.

The image of the translation is contained in the subset of $\lambda_\mathtt{b}$ terms given by the grammar

$$M ::= \mathsf{box}(V) \mid VM \qquad\qquad V ::= \varepsilon(x) \mid \lambda x.M \tag{7}$$

For the exact image, the V in VM should be constrained to $\mathsf{raise}(M')$. But the subset (7) has the advantage of being closed under $\to_{\beta_\mathtt{b}}$. Let us allow ourselves the abuse of calling (7) **Gödel's image**.[5]

Due to the restricted form of the term in function position in applications, relations $\to_\mathtt{we}$ and $\to_{\mathtt{we}<}$ collapse on Gödel's image to the same relation, one which can alternatively

[5] Gödel's image can be obtained in another way. Notice Girard's image is the set of $\lambda_\mathtt{b}$-terms where the following is valid: a term is a box iff it is the argument term of an application. Likewise, we might want to characterize Gödel's image as the set of $\lambda_\mathtt{b}$-terms where the following is valid: a term is a value iff it is the function term of an application. But the latter has to be complemented with the imposition that the contents of boxes are values (otherwise, closure under reduction would not be guaranteed).

be defined as β_{b} closed under ν. This property of Gödel's image we call its **indifference property**, again by analogy with our account of the indifference property on λ-terms given at the end of Section 2, but also in analogy with what happens in Girard's image. This single deterministic relation on Gödel's image is denoted $\to_{\mathsf{v}}$. By **call-by-value evaluation** on λ_{b} we mean $\to_{\mathsf{v}}^*$. The terminology is justified by Theorem 18 below.

▶ **Lemma 16.** $[V^\bullet/\varepsilon(x)]M^* = ([V/x]M)^*$ *and* $[V^\bullet/\varepsilon(x)]W^\bullet = ([V/x]W)^\bullet$.

The refined Gödel embedding improves Proposition 9: as we will see below in items 1 and 3 of Theorem 18, we get rid of the proliferation of administrative steps, and even obtain reflection for full evaluation (evaluation until a value is output). However, the following example shows that reflection of (standard) reduction still does not hold in general.

▶ **Example 17.** Let us return to Example 10. Recall $P = (\lambda x.xM)N$ (with $x \notin FV(M)$), $Q = NM$, and $P \to_{\beta_{\mathsf{v}}} Q$ does not hold in general. With refined Gödel's translation, it is still the case that $P^* \to_{\beta_{\mathsf{b}}}^+ Q^*$ does hold in λ_{b}, since:

$$P^* = \mathsf{raise}(N^*)\mathsf{box}(\lambda x.\mathsf{raise}(M^*)x^*) \to_{\beta_{\mathsf{b}}} (\lambda x.\mathsf{raise}(M^*)x^*)N^* \to_{\beta_{\mathsf{b}}} (\lambda x.\varepsilon(x)M^*)N^* = Q^* \ .$$

Some other refinements are needed, namely the identification of sub-relations in λ_{b} which allow reflection of (standard) reduction. Again, this is in the spirit of shrinking the target.

First, we introduce a new β rule

$$\mathsf{raise}(\mathsf{box}(N))\mathsf{box}(\lambda x.P) \to [N/\varepsilon(x)]P \qquad (\beta_{\mathsf{b2}}) \ ,$$

corresponding to a sequence of two β_{b}-reduction steps:

$$\mathsf{raise}(\mathsf{box}(N))\mathsf{box}(\lambda x.P) \to_{\beta_{\mathsf{b}}} (\lambda x.P)\mathsf{box}(N) \to_{\beta_{\mathsf{b}}} [N/\varepsilon(x)]P \ . \tag{8}$$

Second, we define $\Rightarrow_{\mathsf{b2}}$, a sub-relation of $\Rightarrow_{\mathsf{b}}$ in λ_{b}: in Fig. 8, replace RDX by:

$$\frac{N \to_{\mathtt{we}}^* \mathsf{box}(N') \quad M \to_{\mathtt{we}}^* \mathsf{box}(\lambda x.M') \quad [N'/\varepsilon(x)]M' \Rightarrow_{\mathsf{b2}} Q}{\mathsf{raise}(N)M \Rightarrow_{\mathsf{b2}} Q}\ RDX2$$

This is a derivable rule of $\Rightarrow_{\mathsf{b}}$: it is illuminating to see how this rule corresponds to two applications of RDX where, in each of these, the first premiss follows by reflexivity and all the action happens in the second premiss.[6]

In addition, $\Rightarrow_{\mathsf{b2}}$ determines a notion of "standard" reduction sequence that, we now argue, is standard in the official sense derived from Theorem 12 and defined right after its proof in Section 4. Rule $RDX2$ determines that the initial segment of a "standard" reduction sequence does the parallel cbb evaluation of the components of the given $\mathsf{raise}(N)M$ until a β_{b2}-redex $\mathsf{raise}(\mathsf{box}(N'))\mathsf{box}(\lambda x.M')$ emerges and is immediately reduced. Now this initial segment, which is not strictly standard (because the evaluation of N happens inside $\mathsf{raise}(\cdot)$ which is a λ), has a corresponding standard reduction sequence, namely the sequence (6).

[6] For the purpose of studying Girard's map, the following particular case of RDX, where the action happens in the *first* premiss, would have sufficed:

$$\frac{M \to_{\mathtt{we}}^* \mathsf{box}(\lambda x.M') \quad [N/\varepsilon(x)]M' \Rightarrow_{\mathsf{b}} Q}{M\mathsf{box}(N) \Rightarrow_{\mathsf{b}} Q}$$

▶ **Theorem 18** (Properties of refined Gödel's translation)**.**

1. *(Preservation of reduction) If* $M \to_{\beta_{\mathtt{v}}} N$ *in* $\lambda_{\mathtt{v}}$ *then* $M^* \to^2_{\beta_{\mathtt{b}}} N^*$ *in* $\lambda_{\mathtt{b}}$*.*
2. *(Preservation and reflection of reduction)* $M \to_{\beta_{\mathtt{v}}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to_{\beta_{\mathtt{b2}}} N^*$ *in* $\lambda_{\mathtt{b}}$*.*
3. *(Preservation and reflection of evaluation)* $M \to^*_{\mathtt{v}} V$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to^*_{\mathtt{v}} V^*$ *in* $\lambda_{\mathtt{b}}$*.*
4. *(Preservation of standard reduction) If* $M \Rightarrow_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$ *then* $M^* \Rightarrow_{\mathtt{b}} N^*$ *in* $\lambda_{\mathtt{b}}$*.*
5. *(Preservation and reflection of standard red.)* $M \Rightarrow_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \Rightarrow_{\mathtt{b2}} N^*$ *in* $\lambda_{\mathtt{b}}$*.*

Proof.

Proof of 1. Follows from the "only if" half of item 2.

Proof of 2. The "only if" half is proved by induction on $M \to_{\beta_{\mathtt{v}}} N$. The "if" half is a consequence of two facts: (i) injectivity of $(\cdot)^*$; (ii) the image of $(\cdot)^*$ (resp. $(\cdot)^\bullet$) is closed for $\beta_{\mathtt{b2}}$-reduction and any $\beta_{\mathtt{b2}}$-reduction between images is an image of a source reduction. More precisely, the second fact is the conjunction of: (a) If $M^* \to_{\beta_{\mathtt{b2}}} P$, then there is a λ-term Q such that $Q^* = P$ and $M \to_{\beta_{\mathtt{v}}} Q$; (b) If $V^\bullet \to_{\beta_{\mathtt{b2}}} N$, then there is a λ-calculus value W such that $W^\bullet = N$ and $V \to_{\beta_{\mathtt{v}}} W$. This is proved by simultaneous induction on M and V.

Proof of 3. The result follows with the help of the following two facts:

Fact 1. If $M \to_{\mathtt{v}} N$ in $\lambda_{\mathtt{v}}$, then there exists P such that $M^* {\to_{\mathtt{v}}}^+ P$ and $N^* {\to_{\beta_{\mathtt{a}}}}^* P$ in $\lambda_{\mathtt{b}}$, where $\to_{\beta_{\mathtt{a}}}$ is *administrative* 1-step-reduction defined by closure under μ and ν of the rule

$$\mathsf{raise}(N)\mathsf{box}(M) \ \to \ MN \quad (\beta_{\mathtt{a}}) \ ,$$

which is the $\beta_{\mathtt{b}}$-step $(\lambda z.\varepsilon(z)N)\mathsf{box}(M) \ \to_{\beta_{\mathtt{b}}} \ MN$ (for $z \notin \mathrm{FV}(N)$).[7] Notice $N^* {\to_{\beta_{\mathtt{a}}}}^* P$ implies $N^* {\to_{\mathtt{v}}}^* P$.

Fact 2. If $M^* \to^*_{\mathtt{v}} Q$ in $\lambda_{\mathtt{b}}$, then: (i) if Q is a box, then $Q = V^*$, for some value V, and $M \to^*_{\mathtt{v}} V$ in $\lambda_{\mathtt{v}}$; and (ii) if $Q = N^*$, for some N, then $M \to^*_{\mathtt{v}} N$ in $\lambda_{\mathtt{v}}$.

The "if" part of item 3 actually holds when V is replaced by an arbitrary term N, as stated in part (ii) of Fact 2. The "only if" part of item 3 follows by induction on the length of the reduction sequence. Suppose $M \to_{\mathtt{v}} M_0 \to^*_{\mathtt{v}} V$. By Fact 1 above, there exists P s.t. $M^* \to^*_{\mathtt{v}} P$ and $M_0^* \to^*_{\mathtt{v}} P$. By IH, $M_0^* \to^*_{\mathtt{v}} V^*$. Hence, since reduction sequences starting at M_0^* are deterministic, $P \to^*_{\mathtt{v}} V^*$ or $V^* \to^*_{\mathtt{v}} P$. As the latter is equivalent to $V^* = P$ (because V^* cannot reduce), it follows $M^* \to^*_{\mathtt{v}} P \to^*_{\mathtt{v}} V^*$, as wanted.

The proof of Fact 1 above is by induction on $M \to_{\mathtt{v}} N$. In the base case, one actually proves $M^* \to_{\mathtt{v}} N^*$, using Lemma 16. In the step case where $M = VM_0 \to_{\mathtt{v}} VN_0 = N$, because $M_0 \to_{\mathtt{v}} N_0$, we find the need for the administrative reductions from N^*.

The proof of Fact 2 above is by induction on the length of $M^* \to^*_{\mathtt{v}} Q$.

Proof of 4. By induction on $M \Rightarrow_{\mathtt{v}} N$. We spell out the case RDX. Suppose

$$\frac{M_1 \to^*_{\mathtt{v}} \lambda x.M_1' \quad M_2 \to^*_{\mathtt{v}} V \quad [V/x]M_1' \Rightarrow_{\mathtt{v}} N}{M_1 M_2 \Rightarrow_{\mathtt{v}} N}\ RDX$$

We want $\mathsf{raise}(M_2^*)M_1^* \Rightarrow_{\mathtt{b}} N^*$. By IH and Lemma 16, $([V/x]M_1')^* = [V^\bullet/\varepsilon(x)]M_1'^* \Rightarrow_{\mathtt{b}} N^*$. From $M_1 \to^*_{\mathtt{v}} \lambda x.M_1'$ we get, by item 3 of the current theorem, $M_1^* \to^*_{\mathtt{v}} (\lambda x.M_1')^* = \mathsf{box}(\lambda x.M_1'^*)$. By the indifference property, $M_1^* \to^*_{\mathtt{we}} \mathsf{box}(\lambda x.M_1'^*)$. Similarly, from $M_2 \to^*_{\mathtt{v}} V$ we get, by item 3 of the current theorem, $M_2^* \to^*_{\mathtt{v}} V^* = \mathsf{box}(V^\bullet)$. By the indifference property, $M_2^* \to^*_{\mathtt{we}} \mathsf{box}(V^\bullet)$. We conclude with two applications of RDX:

$$\frac{\lambda z.\varepsilon(z)M_2^* \to^*_{\mathtt{we}} \lambda z.\varepsilon(z)M_2^* \quad M_1^* \to^*_{\mathtt{we}} \mathsf{box}(\lambda x.M_1'^*) \quad (*)}{(\lambda z.\varepsilon(z)M_2^*)M_1^* \Rightarrow_{\mathtt{b}} N^*}\ RDX$$

[7] The first step in the sequence (8) is administrative in this sense.

where $(*)$ is

$$\frac{\lambda x.M_1'^* \to_{\mathtt{we}}^* \lambda x.M_1'^* \quad M_2^* \to_{\mathtt{we}}^* \mathsf{box}(V^\bullet) \quad [V^\bullet/\varepsilon(x)]M_1'^* \Rightarrow_{\mathtt{b}} N^*}{(\lambda x.M_1'^*)M_2^* \Rightarrow_{\mathtt{b}} N^*}\ RDX$$

Proof of 5. The "only if" direction is proved by changing a case in the proof of item 4 of the current theorem, namely the case RDX spelled out above. Indeed, the two applications of RDX that conclude the proof may be replaced by a single application of $RDX2$. For the "if" direction, we prove: if $M^* \Rightarrow_{\mathtt{b2}} Q$ then there is $N \in \lambda_{\mathtt{v}}$ such that $N^* = Q$ and $M \Rightarrow_{\mathtt{v}} N$. The proof is by induction on $M^* \Rightarrow_{\mathtt{b2}} Q$, and it heavily relies again on item 3 (simulation property) and the indifference property. ◀

Before extracting standardization of $\lambda_{\mathtt{v}}$ as corollary, we need the following addendum to the standardization theorem for $\lambda_{\mathtt{b}}$ (Theorem 12), concerning $\lambda_{\mathtt{b}}$-terms of the form M^*:

▶ **Theorem 19** (Addendum to standardization for $\lambda_{\mathtt{b}}$). *In $\lambda_{\mathtt{b}}$, if $M^* \to_{\beta_{\mathtt{b2}}}^* N^*$ then $M^* \Rightarrow_{\mathtt{b2}} N^*$.*

Proof. The proof has the same structure as that of Theorem 12. There is a single catch, because $\Rightarrow_{\mathtt{b2}}$ is not closed under prefixing a *single* $\to_{\mathtt{we}}$-step. So the rule for $\Rightarrow_{\mathtt{b2}}$ corresponding to rule (5) in Fig. 9 cannot be stated with $\to_{\mathtt{we}}^*$, it is stated with another special binary relation $P \twoheadrightarrow_{\mathtt{we2}} P'$, inductively defined over arbitrary $\lambda_{\mathtt{b}}$-terms by closing under μ and ν the following base rule:

$$\frac{Q \to_{\mathtt{we}}^* \mathsf{box}(V) \quad P \to_{\mathtt{we}}^* \mathsf{box}(\lambda y.P')}{\mathsf{raise}(Q)P \twoheadrightarrow_{\mathtt{we2}} [V/\varepsilon(y)]P'}$$

Notice $\twoheadrightarrow_{\mathtt{we2}} \subseteq \to_{\mathtt{we}}^*$. The second derived rule in (5) shows this for the base rule. ◀

▶ **Corollary 20** (Standardization for $\lambda_{\mathtt{v}}$). *In $\lambda_{\mathtt{v}}$, $M \to_{\beta_{\mathtt{v}}}^* N$ iff $M \Rightarrow_{\mathtt{v}} N$.*

Proof. The easy "if" implication follows by induction on $M \Rightarrow_{\mathtt{v}} N$. The hard "only if" implication goes via standardization for $\lambda_{\mathtt{b}}$. Suppose $M \to_{\beta_{\mathtt{v}}}^* N$. By item 2 of Theorem 18, we have $M^* \to_{\beta_{\mathtt{b2}}} N^*$. By Theorem 19, we obtain $M^* \Rightarrow_{\mathtt{b2}} N^*$ in $\lambda_{\mathtt{b}}$. Fortunately the addendum provided a statement with $\Rightarrow_{\mathtt{b2}}$ rather than $\Rightarrow_{\mathtt{b}}$, because reflection of standard reduction only works for $\Rightarrow_{\mathtt{b2}}$: item 5 of Theorem 18 concludes $M \Rightarrow_{\mathtt{v}} N$. ◀

6 Instantiations

Here we briefly illustrate two instantiations of the indeterminate comonad of $\lambda_{\mathtt{b}}$ with concrete comonads, namely the *trivial comonad* $\top \supset (\cdot)$ and the comonad ! of linear logic.

To provide a target for the trivial comonad instantiation, we add to the $\lambda_{\mathtt{v}}$-calculus a type $\top$ and a term $\star$, which we consider as a value of type $\top$. We name $\beta_{\mathtt{triv}}$ this particular case of $\beta_{\mathtt{v}}$: $(\lambda d.M)\star \to M$, with $d \notin \mathrm{FV}(M)$.

The *trivial instantiation* is defined in Fig. 10. Under this instantiation, a $\beta_{\mathtt{b}}$-reduction step in $\lambda_{\mathtt{b}}$ is simulated in this target by a $\beta_{\mathtt{v}}$-reduction step followed by a $\beta_{\mathtt{triv}}$-reduction sequence.

Composing the modal embeddings with the trivial instantiation we obtain embeddings into the considered extension of $\lambda_{\mathtt{v}}$. The resulting composition in the case of Girard's embedding is the following map $\mathcal{T} : \lambda_{\mathtt{n}} \to \lambda_{\mathtt{v}}$ (in the third clause, d is a dummy variable):

$$\mathcal{T}(x) = x\,\star \qquad \mathcal{T}(\lambda x.M) = \lambda x.\mathcal{T}(M) \qquad \mathcal{T}(MN) = \mathcal{T}(M)(\lambda d.\mathcal{T}(N))$$

$$
\begin{array}{rcl}
\mathsf{t}(X) & = & X \\
\mathsf{t}(B \supset A) & = & \mathsf{t}(B) \supset \mathsf{t}(A) \\
\mathsf{t}(\Box A) & = & \top \supset \mathsf{t}(A)
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{t}(\varepsilon(x)) & = & x\star \\
\mathsf{t}(\lambda x.M) & = & \lambda x.\mathsf{t}(M) \\
\mathsf{t}(MN) & = & \mathsf{t}(M)\mathsf{t}(N) \\
\mathsf{t}(\mathsf{box}(M)) & = & \lambda d.\mathsf{t}(M) \quad (d \notin \mathrm{FV}(M))
\end{array}
$$

Figure 10 Trivial instantiation of λ_{b}.

$$
\begin{array}{rcl}
\ell(X) & = & X \\
\ell(B \supset A) & = & \ell(B) \multimap \ell(A) \\
\ell(\Box A) & = & !\ell(A)
\end{array}
\qquad
\begin{array}{rcl}
\ell(\varepsilon(x)) & = & x \\
\ell(\lambda x.M) & = & \lambda y.\mathsf{let}\,!x = y\,\mathsf{in}\,\ell(M) \\
\ell(MN) & = & \ell(M)\ell(N) \\
\ell(\mathsf{box}(M)) & = & !\ell(M)
\end{array}
$$

Figure 11 Linear instantiation of λ_{b}.

This map preserves and reflects reduction, and is found in [5], where is described as "thunk introduction implemented in Λ", using the "protecting by a λ" technique [12].

Now we turn to the second instantiation of the comonad. The *linear instantiation* of types and terms of λ_{b} into the linear λ-calculus Lin of [11] is given in Figure 11. (Recall implications of λ_{b} have a boxed type in the antecedent.)

The image of the linear instantiation may be equipped with

$$(\lambda y.\mathsf{let}\,!x = y\,\mathsf{in}\,M)(!N) \to [N/x]M \qquad (\beta_\ell)$$

which amalgamates these two reduction steps:

$$(\lambda y.\mathsf{let}\,!x = y\,\mathsf{in}\,M)(!N) \to_{\beta_{\multimap}} \mathsf{let}\,!x = !N\,\mathsf{in}\,M \to_{\beta_!} [N/x]$$

A fragment of Lin is thus identified, and the linear instantiation becomes an isomorphism between λ_{b} and the fragment, with β_{b} corresponding to β_ℓ. We refrain from giving more details here.

Composing the modal embeddings with the linear instantiation we obtain the embeddings shown in Fig. 12. The two compositions are translations of λ_{n} and λ_{v} into the linear λ-calculus Lin which preserve reduction. These embeddings should be compared with those in [11]. The composition with Girard's embedding (the left column in Fig. 12) gives the cbn translation in *op. cit.*, whereas the composition with Gödel's embedding (the right column in Fig. 12) gives the cbv translation in *op. cit.* except for a refinement in the clause for application: in *op. cit.* the translation is $(MN)^* = (\mathsf{let}\,!z = M^*\,\mathsf{in}\,z)N^*$.

7 Final remarks

Our conclusion is that the refined modal embeddings achieve an unification of call-by-name and call-by-value by means of the calling paradigm call-by-box. With hindsight, it is obvious that call-by-box should be enough to interpret both cbn and cbv. Call-by-box comprises these high-level ideas: boxes are distinct from values and they are *not* values; in function applications, expressions in function position reduce to values, expressions in argument position reduce to boxes; functions are only called with boxes; evaluation is weak (values do not reduce) and external (boxes do not reduce). Call-by-box can thus be the target of a compilation technique which we call **protecting-by-a-box** and that works both for cbn

$$
\begin{array}{rcl}
x^{\circ\ell} & = & x \\
(\lambda x.M)^{\circ\ell} & = & \lambda y.\mathsf{let}\,!x = y\,\mathsf{in}\,M^{\circ\ell} \\
(MN)^{\circ\ell} & = & M^{\circ\ell}(!N^{\circ\ell})
\end{array}
\qquad
\begin{array}{rcl}
V^{*\ell} & = & !V^{\bullet\ell} \\
(MN)^{*\ell} & = & (\lambda y.\mathsf{let}\,!z = y\,\mathsf{in}\,zN^{*\ell})M^{*\ell} \\
x^{\bullet\ell} & = & x \\
(\lambda x.M)^{\bullet\ell} & = & \lambda y.\mathsf{let}\,!x = y\,\mathsf{in}\,M^{*\ell}
\end{array}
$$

Figure 12 Compositions of the modal embeddings and the linear instantiation.

and cbv: cbn is obtained by protecting arguments with boxes (the old idea of freezing the argument to delay its evaluation); cbv is obtained by restricting boxes to boxed values (hence functions are called with values only) and always having values wrapped as boxes (enabling the calling of functions with values).

Notice this is a story with a cbn side and a cbv side. The cbn side is an abstraction of the protecting-by-a-λ technique, as shown through the trivial instantiation. The cbn side is also what the literature offers. Hatcliff and Danvy [5, 6] formalized an abstract version of protecting-by-a-λ as the thunk-introduction map from cbn λ-calculus into the λ-calculus with thunks (Λ_τ), and a separate variant of the thunk-introduction map directly into the λ-calculus. The former corresponds to our Girard's embedding, while the latter corresponds to the composition $\mathcal{T}$ of Section 6. But even here our results offer some improvements. First, we observed that $\mathcal{T}$ is connected to Girard's embedding through the trivial instantiation. Second, the precise formulation of the target system of Girard's embedding is important. Our care in formulating $\lambda_\Box$ so that it is closed under substitution and enjoys subject reduction is not matched in the treatment of typed Λ_τ [6]. And then we went further from $\lambda_\Box$ to λ_{b}, and this alone improved the properties of Girard's embedding as witnessed in Theorem 14.

The connection between calling paradigms and embeddings of intuitionistic logic into linear logic [4, 7, 9, 10] has its full treatment in [11]. Regarding cbn and cbv, our results match the results of [11], but in a more abstract and simpler setting - in fact we go further, since we treat proper standardization, and that is a key ingredient in our claim that cbn and cbv are unified by cbb. In obtaining results through embeddings into modal logic similar to those through embedding into linear logic, one sees that already modality, without the need for linearity, brings the calling mechanisms into the scope of the Curry-Howard isomorphism.

Inspired by linear logic, the bang-calculus [3] was recently proposed as a "generalization" of cbn and cbv. The part of *op. cit.* not concerned with denotational semantics compares with the initial part of the present paper, up to Section 3, where we dealt with unrefined target and embeddings (but notice the conceptual difference: for us, boxes are not values); all the work that comes after Section 3, about the refined target and embeddings, and which is the core of our contribution, is beyond the scope of [3].

As to future work, we would like to deepen the study of instantiations, both by considering other instantiations, and by investigating whether one obtains, through the composition of instantiations with embeddings, not only known maps, but also their properties.

References

1 H.P. Barendregt. *The Lambda Calculus*. North-Holland, 1984.

2 G. M. Bierman and V. C. V. de Paiva. On an intuitionistic modal logic. *Studia Logica*, 65:383–416, 2000.

3 T. Ehrhard and G. Guerrieri. The bang calculus: an untyped lambda-calculus generalizing call-by-name and call-by-value. In *Proc. of PPDP '16*, pages 174–187. ACM, 2016.

4 J.-Y. Girard. Linear logic. *Theor. Comput. Sci.*, 50(1):1–102, 1987.

5 J. Hatcliff and O. Danvy. Thunks and the λ-calculus. *J. Funct. Program.*, 7(3):303–319, 1997.

6 J. Hatcliff and O. Danvy. Thunks and the λ-Calculus (Extended Version). Technical Report BRICS RS-97-7, DIKU, 1997.

7 Y. Lafont. From proof-nets to interaction nets. In J. Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, pages 225–247. Cambridge Univ. Press, 1995.

8 Ralph Loader. Notes on simply typed lambda calculus. Technical Report ECS-LFCS-98-381, LFCS, Univ. of Edinburgh, 1998.

9 I. Mackie. *The Geometry of Implementation.* PhD thesis, Imperial College, 1994.

10 I. Mackie. Encoding strategies in the lambda calculus with interaction nets. In *Revised Selected Papers from IFL 2005*, volume 4015 of *Lect. Notes in Comput. Sci.*, pages 19–36. Springer, 2006.

11 J. Maraist, M. Odersky, D. N. Turner, and P. Wadler. Call-by-name, call-by-value, call-by-need and the linear lambda calculus. *Theor. Comput. Sci.*, 228(1-2):175–210, 1999.

12 G. Plotkin. Call-by-name, call-by-value and the λ-calculus. *Theor. Comput. Sci.*, 1:125–159, 1975.

13 D. Prawitz. *Natural Deduction.* Almquist and Wiksell, Stockholm, 1965.

14 A. Sabry and P. Wadler. A reflection on call-by-value. *ACM Trans. Program. Lang. Syst.*, 19(6):916–941, 1997.

15 A. Troelstra. Introductory note to 1933f. In *Kurt Gödel Collected Works*, pages 296–299. Oxford Univ. Press, 1986.

16 A. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge Univ. Press, 2000.

A Natural deduction system for the logic IS4

The formulae of the logic **IS4** are given by the grammar

$$
\begin{array}{lcl}
A & ::= & X \mid A \supset A \mid B \\
B & ::= & \Box A
\end{array}
$$

Formulae of the form B are called boxed formulae. Note that the antecedent of an implication need not be boxed.

Sequents of the natural deduction system are $\Gamma \vdash A$ where the antecedent Γ is a multiset of formulae, again not necessarily boxed. The proof rules are

$$
\frac{}{\Gamma, A \vdash A} \quad \frac{\Gamma, A_1 \vdash A_2}{\Gamma \vdash A_1 \supset A_2} \quad \frac{\Gamma \vdash A_1 \supset A_2 \quad \Gamma \vdash A_1}{\Gamma \vdash A_2}
$$

$$
\frac{\Gamma \vdash B_1 \quad \Gamma \vdash B_n \quad B_1, \ldots, B_n \vdash A}{\Gamma \vdash \Box A} \quad \frac{\Gamma \vdash \Box A}{\Gamma \vdash A}
$$

Note that the $B_1, \ldots, B_n$ in the $\Box$ introduction rule are boxed formulae.

In the corresponding term calculus, terms are given by the grammar

$$
M, N ::= x \mid \lambda x.M \mid MN \mid \mathsf{cobind}(M_1, \ldots, M_n, x_1 \ldots x_n.N) \mid \varepsilon(M)
$$

where for the term form $\mathsf{cobind}(M_1, \ldots, M_n, x_1 \ldots x_n.N)$ it is required that $\mathrm{FV}(N) \subseteq \{x_1, \ldots, x_n\}$.

Typing judgments are $\Gamma \vdash M : A$ where the antecedent Γ is a set of type assignments $x : A$ with all x distinct. The typing rules are

$$
\frac{}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma, x : A_1 \vdash M : A_2}{\Gamma \vdash \lambda x.M : A_1 \supset A_2} \quad \frac{\Gamma \vdash M : A_1 \supset A_2 \quad \Gamma \vdash N : A_1}{\Gamma \vdash MN : A_2}
$$

$$
\frac{\Gamma \vdash M_1 : B_1 \quad \Gamma \vdash M_n : B_n \quad x_1 : B_1, \ldots, x_n : B_n \vdash N : A}{\Gamma \vdash \mathsf{cobind}(M_1, \ldots, M_n, x_1 \ldots x_n.N) : \Box A} \quad \frac{\Gamma \vdash M : \Box A}{\Gamma \vdash \varepsilon(M) : A}
$$

The reduction relation between terms is given by the axioms

$$\frac{}{(\lambda x.M)N \to [N/x]M}\ \beta_{\supset}$$

$$\frac{}{\varepsilon(\mathsf{cobind}(M_1,\ldots,M_n,x_1\ldots x_n.N)) \to [M_1/x_1,\ldots,M_n/x_n]N}\ \beta_{\Box}$$

together with the rules of compatible closure.

In the categorical semantics of the natural deduction system of **IS4** in terms of a cartesian closed category with a lax monoidal comonad, ε corresponds to the counit of the comonad and cobind to a combination of the comultiplication and the lax monoidality laws.

▶ **Proposition 2.**
1. *For all* Γ, A *in* $\lambda_{\Box}$*,* $\Gamma \vdash M : A$ *in* $\lambda_{\Box}$ *iff* $\Gamma \vdash M^{\sharp}_{|\Gamma|} : A$ *in the term calculus for* **IS4***.*
2. *For all* U *s. t.* $\mathrm{FV}(M) \subseteq U$*,* $M \to N$ *in* $\lambda_{\Box}$ *iff* $M^{\sharp}_{U} \to N^{\sharp}_{U}$ *in the term calculus for* **IS4***.*

Proof.
1. The "only if" direction is proved by induction on the given $\lambda_{\Box}$ typing derivation. The case for the typing rule of box goes through in **IS4** using the $\Box$ introduction rule, where the first n premises are axioms (one for each of the n formulas in the context). The "if" direction is by induction on M. In the case $M = \mathsf{box}(N)$, assuming $\Gamma = x_1 : B_1, \ldots, x_n : B_n$, $M^{\sharp}_{|\Gamma|} = \mathsf{cobind}(x_1,\ldots,x_n,x_1\ldots x_n.N^{\sharp}_{|\Gamma|})$ and the hypothesis implies $A = \Box A_0$ and $\Gamma \vdash N^{\sharp}_{|\Gamma|} : A_0$ in $\lambda_{\Box}$ (for some A_0), so the IH and the $\Box$ introduction rule of $\lambda_{\Box}$ can be used to conclude.
2. Routine induction on the given reduction derivation in both directions. ◀

▶ **Proposition 3.** *For all* Γ, A *in* $\lambda_{\Box}$*, if* $\Gamma \vdash M : A$ *in the term calculus for* **IS4***, then there exists* N *such that* $\Gamma \vdash N : A$ *in* $\lambda_{\Box}$*.*

Proof. By induction on the given **IS4** typing derivation. The case for the $\Box$ introduction rule needs admissibility of both weakening and the typing rule for substitution. ◀