The Δ-calculus: Syntax and Types

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Abstract

We present the Δ-calculus, an explicitly typed λ-calculus with strong pairs, projections and explicit type coercions. The calculus can be parametrized with different intersection type theories $\mathcal{T}$, e.g. the Coppo-Dezani, the Coppo-Dezani-Sallé, the Coppo-Dezani-Venneri and the Barendregt-Coppo-Dezani ones, producing a family of Δ-calculi with related intersection typed systems. We prove the main properties like Church-Rosser, unicity of type, subject reduction, strong normalization, decidability of type checking and type reconstruction. We state the relationship between the intersection type assignment systems à la Curry and the corresponding intersection typed systems à la Church by means of an essence function translating an explicitly typed Δ-term into a pure λ-term one. We finally translate a Δ-term with type coercions into an equivalent one without them; the translation is proved to be coherent because its essence is the identity. The generic Δ-calculus can be parametrized to take into account other intersection type theories as the ones in the Barendregt et al. book.

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1 Introduction

Intersection type theories $\mathcal{T}$ were first introduced as a form of ad hoc polymorphism in (pure) λ-calculi à la Curry. The paper by Barendregt, Coppo, and Dezani [4] is a classic reference, while [5] is a definitive reference.

Intersection type assignment systems $\lambda^{\mathcal{T}}_{\cap}$ have been well known in the literature for almost 40 years for many reasons: among them, characterization of strongly normalizing λ-terms [5], λ-models [1], automatic type inference [28], type inhabitation [45, 40], type unification [18].

As intersection had its classical development for type assignment systems, many papers tried to find an explicitly typed λ-calculus à la Church corresponding to the original intersection type assignment systems à la Curry. The programming language Forsythe, by Reynolds [41], is probably the first reference, while Pierce's Ph.D. thesis [36] combines also unions, intersections and bounded polymorphism. In [49] intersection types were used as a foundation for typed intermediate languages for optimizing compilers for higher-order

polymorphic programming languages; implementations of typed programming language featuring intersection (and union) types can be found in SML-CIDRE [15] and in StardustML [19, 20].

Annotating pure λ-terms with intersection types is not simple: a classical example is the difficulty to decorate the bound variable of the explicitly typed polymorphic identity $\lambda x{:}?.x$ such that the type of the identity is $(\sigma \to \sigma) \cap (\tau \to \tau)$: previous attempts showed that the full power of the intersection type discipline can be easily lost.

In this paper, we define and prove the main properties of the Δ-calculus, a generic intersection typed system for an explicitly typed λ-calculus à la Church enriched with strong pairs, denoted by $\langle \Delta_1 , \Delta_2 \rangle$, projections, denoted by $pr_i\, \Delta$, and type coercions, denoted by Δ^σ.

A *strong pair* $\langle \Delta_1 , \Delta_2 \rangle$ is a special kind of cartesian product such that the two parts of a pair satisfy a given property $\mathcal{R}$ on their "essence", that is $\wr \Delta_1 \wr \mathcal{R} \wr \Delta_2 \wr$.

An *essence* $\wr \Delta \wr$ of a Δ-term is a pure λ-term obtained by erasing type decorations, projections and choosing one of the two elements inside a strong pair. As examples

$$\begin{aligned} \wr \langle \lambda x{:}\sigma \cap \tau.pr_2\, x \, , \lambda x{:}\sigma \cap \tau.pr_1\, x \rangle \wr &= \lambda x.x \\ \wr \lambda x{:}(\sigma \to \tau) \cap \sigma.(pr_1\, x)(pr_2\, x) \wr &= \lambda x.x\, x \\ \wr \lambda x{:}\sigma \cap (\tau \cap \rho).\langle \langle pr_1\, x \, , pr_1\, pr_2\, x \rangle \, , pr_2\, pr_2\, x \rangle \wr &= \lambda x.x \end{aligned}$$

and so on. Therefore, the essence of a Δ-term is its untyped skeleton: a strong pair $\langle \Delta_1 , \Delta_2 \rangle$ can be typechecked if and only if $\wr \Delta_1 \wr \mathcal{R} \wr \Delta_2 \wr$ is verified, otherwise the strong pair will be ill-typed. The essence also gives the exact mapping between a term and its typing à la Church and its corresponding term and type assignment à la Curry. Changing the parameters $\mathcal{T}$ and $\mathcal{R}$ results in defining a totally different intersection typed system.

A *type coercion* Δ^τ is a term of type τ whose type-decoration denotes an application of a subsumption rule to the term Δ of type σ such that $\sigma \leqslant_{\mathcal{T}} \tau$: if we omit type coercions, then we lose the uniqueness of type property.

For the purpose of this paper, we study the four well-known intersection type theories $\mathcal{T}$, namely Coppo-Dezani $\mathcal{T}_{\mathrm{CD}}$ [12], Coppo-Dezani-Sallé $\mathcal{T}_{\mathrm{CDS}}$ [13], Coppo-Dezani-Venneri $\mathcal{T}_{\mathrm{CDV}}$ [14] and Barendregt-Coppo-Dezani $\mathcal{T}_{\mathrm{BCD}}$ [4]. We will inspect the above type theories using three equivalence relations $\mathcal{R}$ on pure λ-terms, namely $\equiv, =_\beta$, and $=_{\beta\eta}$.

The combination of the above $\mathcal{T}$ and $\mathcal{R}$ allows to define ten meaningful typed systems for the Δ-calculus that can be pictorially displayed in a "Δ-chair" (see Definition 9). Following the same style as in the Barendrengt et al. book [5], the edges in the chair represent an inclusion relation over the set of derivable judgments.

Section 3 shows a number of typable examples in the systems presented in the Δ-chair: each example is provided with a corresponding type assignment derivation of its essence. Some historical examples of Pottinger [39], Hindley [25] and Ben-Yelles [6] are essentially re-decorated and inhabited (when possible) in the Δ-calculus. The aim of this section is both to make the reader comfortable with the different intersection typed systems, and to give a first intuition of the correspondence between Church-style and Curry-style calculi.

Section 4 proves the metatheory for all the systems in the Δ-chair: Church-Rosser, unicity of type, subject reduction, strong normalization, decidability of type checking and type reconstruction and studies the relations between intersection type assignment systems à la Curry and the corresponding intersection typed systems à la Church. Notions of soundness, completeness and isomorphism will relate type assignment and typed systems. We also show how to remove type coercions Δ^τ defining a translation function, denoted by $\|\_\|$, inspired by the one of Tannen et al. [8]: the intuition of the translation is that if Δ has type σ and $\sigma \leqslant_{\mathcal{T}} \tau$, then $\|\sigma \leqslant_{\mathcal{T}} \tau\|$ is a Δ-term of type $\sigma \to \tau$, $(\|\sigma \leqslant_{\mathcal{T}} \tau\|\ \|\Delta\|)$ has type τ and $\wr \|\sigma \leqslant_{\mathcal{T}} \tau\| \wr$ is the identity $\lambda x.x$.

1.1 λ-calculi with intersection types à la Church

Several calculi à la Church appeared in the literature: they capture the power of intersection types; we briefly review them.

The Forsythe programming language by Reynolds [41] annotates a λ-abstraction with types as in $\lambda x{:}\sigma_1|\cdots|\sigma_n.M$. However, there is no typed term whose type erasure is the combinator $\mathsf{K} \equiv \lambda x.\lambda y.x$, with the type $(\sigma \to \sigma \to \sigma) \cap (\tau \to \tau \to \tau)$.

Pierce [37] improves Forsythe by using a **for** construct to build ad hoc polymorphic typing, as in $\mathbf{for}\,\alpha \in \{\sigma,\tau\}.\lambda x{:}\alpha.\lambda y{:}\alpha.x$. However, there is no typed term whose type erasure is $\lambda x.\lambda y.\lambda z.(x\,y\,,x\,z)$, with the type
$((\sigma \to \rho) \cap (\tau \to \rho') \to \sigma \to \tau \to \rho \times \rho') \cap ((\sigma \to \sigma) \cap (\sigma \to \sigma) \to \sigma \to \sigma \to \sigma \times \sigma)$.

Freeman and Pfenning [21] introduced refinement types, that is types that allow *ad hoc* polymorphism for ML constructors. Intuitively, refinement types can be seen as subtypes of a standard type: the user first defines a type and then the refinement types of this type. The main motivation for these refinement types is to allow non-exhaustive pattern matching, which becomes exhaustive for a given refinement of the type of the argument. As an example, we can define a type `boolexp` for boolean expressions, with constructors `True`, `And`, `Not` and `Var`, and a refinement type `ground` for boolean expressions without variables, with the same constructors except `Var`: then, the constructor `True` has type $\texttt{boolexp} \cap \texttt{ground}$, the constructor `And` has type $(\texttt{boolexp} * \texttt{boolexp} \to \texttt{boolexp}) \cap (\texttt{ground} * \texttt{ground} \to \texttt{ground})$ and so on. However, intersection is meaningful only when using constructors.

Wells et al. [49] introduced λ^{CIL}, a typed intermediate λ-calculus for optimizing compilers for higher-order programming languages. The calculus features intersection, union and flow types, the latter being useful to optimize data representation. λ^{CIL} can faithfully encode an intersection type assignment derivation by introducing the concept of virtual tuple, i.e. a special kind of pair whose type erasure leads to exactly the same untyped λ-term. A parallel context and parallel substitution, similar to the notion of [29, 30], is defined to reduce expressions in parallel inside a virtual tuple. Subtyping is defined only on flow types and not on intersection types: this system can encode the $\lambda^{\mathrm{CD}}_{\cap}$ type assignment system.

Wells and Haak [50] introduced λ^{B}, a more compact typed calculus encoding of λ^{CIL}: in fact, by comparing Fig. 1 and Fig. 2 of [50] we can see that the set of typable terms with intersection types of λ^{CIL} and λ^{B} are the same. In that paper, virtual tuples are removed by introducing branching terms, typable with branching types, the latter representing intersection type schemes. Two operations on types and terms are defined, namely expand, expanding the branching shape of type annotations when a term is substituted into a new context, and select, to choose the correct branch in terms and types. As there are no virtual tuples, reductions do not need to be done in parallel. As in [49], the $\lambda^{\mathrm{CD}}_{\cap}$ type assignment system can be encoded.

Frisch et al. [22] designed a typed system with intersection, union, negation and recursive types. The authors inherit the usual problem of having a domain space $\mathcal{D}$ that contains all the terms and, at the same time, all the functions from $\mathcal{D}$ to $\mathcal{D}$. They prevent this by having an auxiliary domain space which is the disjoint union of $\mathcal{D}^2$ and $\mathcal{P}(\mathcal{D}^2)$. The authors interpret types as sets in a well-suited model where the set-inspired type constructs are interpreted as the corresponding to set-theoretical constructs. Moreover, the model manages higher-order functions in an elegant way. The subtyping relation is defined as a relation on the set-theoretical interpretation $[\![\_]\!]$ of the types. For instance, the problem $\sigma \cap \tau \leqslant \sigma$ will be interpreted as $[\![\sigma]\!] \cap [\![\tau]\!] \subseteq [\![\sigma]\!]$, where $\cap$ becomes the set intersection operator, and the decision program actually decides whether $([\![\sigma]\!] \cap [\![\tau]\!]) \cap \overline{[\![\sigma]\!]}$ is the empty set.

Bono et al. [7] introduced a relevant and strict parallel term constructor to build inhabitants of intersections and a simple call-by-value parallel reduction strategy. In that paper, an infinite number of constants $c^{\sigma\Rightarrow\tau}$ is applied to typed variables x^{σ} such that $c^{\sigma\Rightarrow\tau}\,x^{\sigma}$ is upcasted to type τ. The paper also uses a local renaming typing rule, which changes the type decoration in λ-abstractions, as well as coercions. Term synchronicity in the tuples is guaranteed by the typing rules. The calculus uses van Bakel's strict version [46] of the $\mathcal{T}_{\mathrm{CD}}$ intersection type theory.

1.2 Logics for intersection types

Proof-functional (or strong) logical connectives, introduced by Pottinger [39], take into account the shape of logical proofs, thus allowing for polymorphic features of proofs to be made explicit in formulæ. This differs from classical or intuitionistic connectives where the meaning of a compound formula is only dependent on the truth value or the provability of its subformulæ.

Pottinger was the first to consider the intersection $\cap$ as a proof-functional connective. He contrasted it to the intuitionistic connective $\wedge$ as follows: *"The intuitive meaning of $\cap$ can be explained by saying that to assert $A \cap B$ is to assert that one has a reason for asserting A which is also a reason for asserting B, while to assert $A \wedge B$ is to assert that one has a pair of reasons, the first of which is a reason for asserting A and the second of which is a reason for asserting B".*

A simple example of a logical theorem involving intuitionistic conjunction which does not hold for proof-functional conjunction is $(A \supset A) \wedge (A \supset B \supset A)$. Otherwise there would exist a term which behaves both as I and as K. Later, Lopez-Escobar [32] and Mints [33] investigated extensively logics featuring both proof-functional and intuitionistic connectives especially in the context of realizability interpretations.

It is not immediate to extend the judgments-as-types Curry-Howard paradigm to logics supporting proof-functional connectives. These connectives need to compare the shapes of derivations and do not just take into account their provability, i.e. the inhabitation of the corresponding type.

There are many proposals to find a suitable logics to fit intersection types; among them we cite [48, 42, 34, 10, 7, 38], and previous papers by the authors [16, 31, 43].

1.3 Raising the Δ-calculus to a Δ-framework.

Our long term goal is to build a prototype of a theorem prover based on the Δ-calculus and proof-functional logic. Recently [27], we have extended a subset of the generic Δ-calculus with other proof-functional operators like union types, relevant arrow types, together with dependent types as in the Edinburgh Logical Framework [24]: a preliminary implementation of a type checker appeared in [43] by the authors. In a nutshell:

Strong disjunction is a proof-functional connective that can be interpreted as the union type $\cup$ [16, 43]: it contrasts with the intuitionistic connective $\vee$. As Pottinger did for intersection, we could informally say that asserting $(A \cup B) \supset C$ corresponds to have a same reason for both $A \supset C$ and $B \supset C$.

A simple example of a logical theorem involving intuitionistic disjunction which does not hold for strong disjunction is $((A \supset B) \cup B) \supset A \supset B$. Otherwise there would exist a term which behaves both as I and as K.

Minimal type theory $\leqslant_{\min}$

(refl) $\sigma \leqslant \sigma$ (incl) $\sigma \cap \tau \leqslant \sigma, \sigma \cap \tau \leqslant \tau$

(glb) $\rho \leqslant \sigma, \rho \leqslant \tau \Rightarrow \rho \leqslant \sigma \cap \tau$ (trans) $\sigma \leqslant \tau, \tau \leqslant \rho \Rightarrow \sigma \leqslant \rho$

Axiom schemes

($\mathtt{U}_{top}$) $\sigma \leqslant \mathtt{U}$ ($\mathtt{U}_{\to}$) $\mathtt{U} \leqslant \sigma \to \mathtt{U}$

($\to\cap$) $(\sigma \to \tau) \cap (\sigma \to \rho) \leqslant \sigma \to (\tau \cap \rho)$

Rule scheme

($\to$) $\sigma_2 \leqslant \sigma_1, \tau_1 \leqslant \tau_2 \Rightarrow \sigma_1 \to \tau_1 \leqslant \sigma_2 \to \tau_2$

Figure 1 Minimal type theory $\leqslant_{\min}$, axioms and rule schemes (see Fig. 13.2 and 13.3 of [5]).

$$\frac{x{:}\sigma \in B}{B \vdash^{\mathcal{T}}_{\cap} x : \sigma}\ (ax) \qquad \frac{B, x{:}\sigma \vdash^{\mathcal{T}}_{\cap} M : \tau}{B \vdash^{\mathcal{T}}_{\cap} \lambda x.M : \sigma \to \tau}\ (\to I)$$

$$\frac{B \vdash^{\mathcal{T}}_{\cap} M : \sigma \quad B \vdash^{\mathcal{T}}_{\cap} M : \tau}{B \vdash^{\mathcal{T}}_{\cap} M : \sigma \cap \tau}\ (\cap I) \qquad \frac{B \vdash^{\mathcal{T}}_{\cap} M : \sigma \to \tau \quad B \vdash^{\mathcal{T}}_{\cap} N : \sigma}{B \vdash^{\mathcal{T}}_{\cap} M\,N : \tau}\ (\to E)$$

$$\frac{B \vdash^{\mathcal{T}}_{\cap} M : \sigma \cap \tau}{B \vdash^{\mathcal{T}}_{\cap} M : \sigma}\ (\cap E_1) \qquad \frac{B \vdash^{\mathcal{T}}_{\cap} M : \sigma \cap \tau}{B \vdash^{\mathcal{T}}_{\cap} M : \tau}\ (\cap E_2)$$

$$\frac{\mathtt{U} \in \mathbb{A}}{B \vdash^{\mathcal{T}}_{\cap} M : \mathtt{U}}\ (top) \qquad \frac{B \vdash^{\mathcal{T}}_{\cap} M : \sigma \quad \sigma \leqslant_{\mathcal{T}} \tau}{B \vdash^{\mathcal{T}}_{\cap} M : \tau}\ (\leqslant_{\mathcal{T}})$$

Figure 2 Generic intersection type assignment system $\lambda^{\mathcal{T}}_{\cap}$ (see Figure 13.8 of [5]). Although rules $(\cap E_i)$ are derivable with $\leqslant_{\min}$, we add them for clarity.

Strong (relevant) implication is yet another proof-functional connective that was interpreted in [2] as a relevant arrow type $\to_r$. As explained in [2], it can be viewed as a special case of implication whose related function space is the simplest one, namely the one containing only the identity function. Because the operators $\supset$ and $\to_r$ differ, $A \to_r B \to_r A$ is not derivable.

Dependent types, as introduced in the Edinburgh Logical Framework [24] by Harper et al., allows considering proofs as first-class citizens albeit differently with respect to proof-functional logics. The interaction of both dependent and proof-functional operators is intriguing: the former mentions proofs explicitly, while the latter mentions proofs implicitly. Their combination therefore opens up new possibilities of formal reasoning on proof-theoretic semantics.

2 Syntax, Reduction and Types

▶ **Definition 1** (Type atoms, type syntax, type theories and type assignment systems). *We briefly review some basic definition from Subsection 13.1 of [5], in order to define type assignment systems. The set of atoms, intersection types, intersection type theories and intersection type assignment systems are defined as follows:*

1. **(Atoms).** *Let $\mathbb{A}$ be a set of symbols which we will call type atoms, and let $\mathtt{U}$ be a special type atom denoting the universal type. In particular, we will use $\mathbb{A}_\infty = \{\mathtt{a}_i \mid i \in \mathbb{N}\}$ with $\mathtt{a}_i$ being different from $\mathtt{U}$ and $\mathbb{A}^{\mathtt{U}}_\infty = \mathbb{A}_\infty \cup \{\mathtt{U}\}$.*
2. **(Syntax).** *The syntax of intersection types, parametrized by $\mathbb{A}$, is: $\sigma ::= \mathbb{A} \mid \sigma \to \sigma \mid \sigma \cap \sigma$.*

Table 1 Type theories $\lambda^{\mathrm{CD}}_{\cap}$, $\lambda^{\mathrm{CDS}}_{\cap}$, $\lambda^{\mathrm{CDV}}_{\cap}$, and $\lambda^{\mathrm{BCD}}_{\cap}$. The ref. column refers to the original article these theories come from.

$\lambda^{\mathcal{T}}_{\cap}$	$\mathcal{T}$	$\mathbb{A}$	$\leqslant_{\min}$ plus	ref.
$\lambda^{\mathrm{CD}}_{\cap}$	$\mathcal{T}_{\mathrm{CD}}$	$\mathbb{A}_{\infty}$	–	[12]
$\lambda^{\mathrm{CDS}}_{\cap}$	$\mathcal{T}_{\mathrm{CDS}}$	$\mathbb{A}^{\mathtt{U}}_{\infty}$	$(\mathtt{U}_{top})$	[13]
$\lambda^{\mathrm{CDV}}_{\cap}$	$\mathcal{T}_{\mathrm{CDV}}$	$\mathbb{A}_{\infty}$	$(\to), (\to\cap)$	[14]
$\lambda^{\mathrm{BCD}}_{\cap}$	$\mathcal{T}_{\mathrm{BCD}}$	$\mathbb{A}^{\mathtt{U}}_{\infty}$	$(\to), (\to\cap), (\mathtt{U}_{top}), (\mathtt{U}\to)$	[4]

3. **(Intersection type theories $\mathcal{T}$).** *An intersection type theory $\mathcal{T}$ is a set of sentences of the form $\sigma \leqslant \tau$ satisfying at least the axioms and rules of the minimal type theory $\leqslant_{min}$ defined in Figure 1. The type theories $\mathcal{T}_{\mathrm{CD}}, \mathcal{T}_{\mathrm{CDV}}, \mathcal{T}_{\mathrm{CDS}}$, and $\mathcal{T}_{\mathrm{BCD}}$ are the smallest type theories over $\mathbb{A}$ satisfying the axioms and rules given in Table 1. We write $\mathcal{T}_1 \sqsubseteq \mathcal{T}_2$ if, for all σ, τ such that $\sigma \leqslant_{\mathcal{T}_1} \tau$, we have that $\sigma \leqslant_{\mathcal{T}_2} \tau$. In particular, $\mathcal{T}_{\mathrm{CD}} \sqsubseteq \mathcal{T}_{\mathrm{CDV}} \sqsubseteq \mathcal{T}_{\mathrm{BCD}}$ and $\mathcal{T}_{\mathrm{CD}} \sqsubseteq \mathcal{T}_{\mathrm{CDS}} \sqsubseteq \mathcal{T}_{\mathrm{BCD}}$. We will sometimes note, for instance,* BCD *instead of $\mathcal{T}_{\mathrm{BCD}}$.*
4. **(Intersection type assignment systems $\lambda^{\mathcal{T}}_{\cap}$).** *We define in Figure 2 an infinite collection of type assignment systems parametrized by a set of atoms $\mathbb{A}$ and a type theory $\mathcal{T}$. We name four particular type assignment systems in the table below, which is an excerpt from Figure 13.4 of [5]. $B \vdash^{\mathcal{T}}_{\cap} M : \sigma$ denotes a derivable type assignment judgment in the type assignment system $\lambda^{\mathcal{T}}_{\cap}$. Type checking is not decidable for $\lambda^{\mathrm{CD}}_{\cap}$, $\lambda^{\mathrm{CDV}}_{\cap}$, $\lambda^{\mathrm{CDS}}_{\cap}$, and $\lambda^{\mathrm{BCD}}_{\cap}$.*

2.1 The Δ-calculi

Intersection type assignment systems and Δ-calculi have in common their type syntax and intersection type theories. The generic syntax of the Δ-calculus is defined as follows.

▶ **Definition 2** (Generic Δ-calculus syntax).

$$\Delta \quad ::= \quad u_{\Delta} \mid x \mid \lambda x{:}\sigma.\Delta \mid \Delta\,\Delta \mid \langle \Delta\,, \Delta \rangle \mid pr_i\,\Delta \mid \Delta^{\sigma} \qquad\qquad i \in \{1,2\}$$

Intuitively, u_{Δ} ranges over an infinite set of constants, indexed with a particular Δ-term. Δ^{σ} denotes an explicit coercion of Δ to type σ. The expression $\langle \Delta\,, \Delta \rangle$ denotes a pair that, following the Lopez-Escobar jargon [32], we call "strong pair" with respective projections pr_1 and pr_2. Note that pr_i is not a term: if it were a term, then pr_i should have several types, or a parametric polymorphic type. But our calculi do not have parametric polymorphism, and they have unicity of typing.

The essence function $\wr\_\wr$ is an erasing function mapping typed Δ-terms into pure λ-terms. It is defined as follows.

▶ **Definition 3** (Essence function).

$$\wr x \wr \stackrel{def}{=} x \qquad \wr \Delta^{\sigma} \wr \stackrel{def}{=} \wr \Delta \wr \qquad \wr u_{\Delta} \wr \stackrel{def}{=} \wr \Delta \wr$$

$$\wr \lambda x{:}\sigma.\Delta \wr \stackrel{def}{=} \lambda x.\wr \Delta \wr \qquad \wr \Delta_1\,\Delta_2 \wr \stackrel{def}{=} \wr \Delta_1 \wr \wr \Delta_2 \wr$$

$$\wr \langle \Delta_1\,, \Delta_2 \rangle \wr \stackrel{def}{=} \wr \Delta_1 \wr \qquad \wr pr_i\,\Delta \wr \stackrel{def}{=} \wr \Delta \wr \qquad i \in \{1,2\}$$

One could argue that the choice of $\wr \langle \Delta_1\,, \Delta_2 \rangle \wr \stackrel{def}{=} \wr \Delta_1 \wr$ is arbitrary and could have been replaced with $\wr \langle \Delta_1\,, \Delta_2 \rangle \wr \stackrel{def}{=} \wr \Delta_2 \wr$. However, the typing rules will ensure that, if $\langle \Delta_1\,, \Delta_2 \rangle$

is typable, then, for some suitable equivalence relation $\mathcal{R}$, we have that $\wr\Delta_1\wr\ \mathcal{R}\ \wr\Delta_2\wr$. Thus, strong pairs can be viewed as constrained cartesian products.

The generic reduction semantics reduces terms of the Δ-calculus as follows.

▶ **Definition 4** (Generic reduction semantics). *Syntactical equality is denoted by* $\equiv$.

1. **(Substitution).** *Substitution on Δ-terms is defined as usual, with the additional rules:*

$$u_{\Delta_1}[\Delta_2/x] \stackrel{def}{=} u_{(\Delta_1[\Delta_2/x])} \qquad \text{and} \qquad \Delta_1^{\sigma}[\Delta_2/x] \stackrel{def}{=} (\Delta_1[\Delta_2/x])^{\sigma}$$

2. **(One-step reduction).** *We define two notions of reduction:*

$$\begin{array}{llll} (\lambda x{:}\sigma.\Delta_1)\,\Delta_2 & \longrightarrow_{\beta} & \Delta_1[\Delta_2/x] & (\beta) \\ pr_i\,\langle\Delta_1\,,\Delta_2\rangle & \longrightarrow_{pr_i} & \Delta_i \quad i\in\{1,2\} & (pr_i\,) \end{array}$$

Observe that $(\lambda x{:}\sigma.\Delta_1)^{\sigma}\,\Delta_2$ is not a redex, because the λ-abstraction is coerced. The contextual closure is defined as usual except for reductions inside the index of u_{Δ} that are forbidden (even though substitutions are propagated). We write $\longrightarrow_{\beta pr_i}$ for the contextual closure of the (β) and $(pr_i\,)$ notions of reduction. We also define a synchronous contextual closure, which is like the usual contextual closure except for the strong pairs, as defined in point (3)*. Synchronous contextual closure of the notions of reduction generates the reduction relation $\longrightarrow^{\parallel}_{\beta pr_i}$.*

3. **(Synchronous closure of $\longrightarrow^{\parallel}$).** *Synchronous closure is defined on the strong pairs with the following constraint:*

$$\frac{\Delta_1 \longrightarrow^{\parallel} \Delta_1' \quad \Delta_2 \longrightarrow^{\parallel} \Delta_2' \quad \wr\Delta_1'\wr \equiv \wr\Delta_2'\wr}{\langle\Delta_1\,,\Delta_2\rangle \longrightarrow^{\parallel} \langle\Delta_1'\,,\Delta_2'\rangle}\ (Clos^{\parallel})$$

Note that we reduce in the two components of the strong pair;

4. **(Multistep reduction).** *We write $\twoheadrightarrow_{\beta pr_i}$ (resp. $\twoheadrightarrow^{\parallel}_{\beta pr_i}$) as the reflexive and transitive closure of $\longrightarrow_{\beta pr_i}$ (resp. $\longrightarrow^{\parallel}_{\beta pr_i}$);*
5. **(Congruence).** *We write $=_{\beta pr_i}$ as the congruence relation generated by $\twoheadrightarrow_{\beta pr_i}$.*

We mostly consider βpr_i-reductions, thus to ease the notation we omit the subscript in βpr_i-reductions.

Note that η-reduction can be defined as in the untyped λ-calculus. The next definition introduces a notion of synchronization inside strong pairs. This notion is also valid in untyped calculi.

▶ **Definition 5** (Synchronization). *A Δ-term is synchronous if and only if, for all its subterms of the shape $\langle\Delta_1\,,\Delta_2\rangle$, we have that $\wr\Delta_1\wr \equiv \wr\Delta_2\wr$.*

It is easy to verify that $\longrightarrow^{\parallel}$ preserves synchronization, while this is not the case for $\longrightarrow$. The next definition introduces a generic intersection typed system for the Δ-calculus that is parametrizable by suitable equivalence relations on pure λ-terms $\mathcal{R}$ and type theories $\mathcal{T}$.

▶ **Definition 6** (Generic intersection typed system). *The generic intersection typed system is defined in Figure 3. We denote by $\Delta^{\mathcal{T}}_{\mathcal{R}}$ a particular typed system with the type theory $\mathcal{T}$ and under an equivalence relation $\mathcal{R}$ and by $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$ a corresponding typing judgment.*

The typing rules are intuitive for a calculus à la Church except rules $(\cap I)$, (top) and $(\leqslant_{\mathcal{T}})$. The typing rule for a strong pair $(\cap I)$ is similar to the typing rule for a cartesian product, except for the side-condition $\wr\Delta_1\wr\ \mathcal{R}\ \wr\Delta_2\wr$, forcing the two parts of the strong pair to have essences compatible under $\mathcal{R}$, thus making a strong pair a special case of a cartesian pair.

$$\frac{\mathtt{U} \in \mathbb{A}}{B \vdash^{\mathcal{T}}_{\mathcal{R}} u_\Delta : \mathtt{U}}\ (top) \qquad \frac{x{:}\sigma \in B}{B \vdash^{\mathcal{T}}_{\mathcal{R}} x : \sigma}\ (ax) \qquad \frac{B, x{:}\sigma \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \tau}{B \vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma.\Delta : \sigma \to \tau}\ (\to I)$$

$$\frac{B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1 : \sigma \quad B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_2 : \tau \quad \wr \Delta_1 \wr\, \mathcal{R}\, \wr \Delta_2 \wr}{B \vdash^{\mathcal{T}}_{\mathcal{R}} \langle \Delta_1 , \Delta_2 \rangle : \sigma \cap \tau}\ (\cap I) \qquad \frac{B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1 : \sigma \to \tau \quad B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_2 : \sigma}{B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1\, \Delta_2 : \tau}\ (\to E)$$

$$\frac{B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma \cap \tau}{B \vdash^{\mathcal{T}}_{\mathcal{R}} pr_1\, \Delta : \sigma}\ (\cap E_1) \qquad \frac{B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma \cap \tau}{B \vdash^{\mathcal{T}}_{\mathcal{R}} pr_2\, \Delta : \tau}\ (\cap E_2) \qquad \frac{B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma \quad \sigma \leqslant_{\mathcal{T}} \tau}{B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta^{\tau} : \tau}\ (\leqslant_{\mathcal{T}})$$

Figure 3 Generic intersection typed system $\Delta^{\mathcal{T}}_{\mathcal{R}}$.

For instance, $\langle \lambda x{:}\sigma.\lambda y{:}\tau.x\ , \lambda x{:}\sigma.x\rangle$ is not typable in $\Delta^{\mathcal{T}}_{\equiv}$; $\langle (\lambda x{:}\sigma.x)\, y\ , y\rangle$ is not typable in $\Delta^{\mathcal{T}}_{\equiv}$ but it is in $\Delta^{\mathcal{T}}_{=_\beta}$; $\langle x\ , \lambda y{:}\sigma.((\lambda z{:}\tau.z)\, x)\, y\rangle$ is not typable in $\Delta^{\mathcal{T}}_{\equiv}$ nor $\Delta^{\mathcal{T}}_{=_\beta}$ but it is in $\Delta^{\mathcal{T}}_{=_{\beta\eta}}$. In the typing rule (top), the subscript Δ in u_Δ is not necessarily typable so $\wr u_\Delta \wr$ can easily be any arbitrary λ-term. The typing rule $(\leqslant_{\mathcal{T}})$ allows to change the type of a Δ-term from σ to τ if $\sigma \leqslant_{\mathcal{T}} \tau$: the term in the conclusion must record this change with an explicit type coercion $\_^{\tau}$, producing the new term Δ^{τ}: explicit type coercions are important to keep the unicity of typing derivations.

The next definition introduces a partial order over equivalence relations on pure λ-terms and an inclusion over typed systems as follows.

▶ **Definition 7** ($\mathcal{R}$ and $\sqsubseteq$).

1. *Let $\mathcal{R} \in \{\equiv, =_\beta, =_{\beta\eta}\}$. $\mathcal{R}_1 \sqsubseteq \mathcal{R}_2$ if, for all pure λ-terms M, N such that $M\ \mathcal{R}_1\ N$, we have that $M\ \mathcal{R}_2\ N$;*
2. *$\Delta^{\mathcal{T}_1}_{\mathcal{R}_1} \sqsubseteq \Delta^{\mathcal{T}_2}_{\mathcal{R}_2}$ if, whenever $B \vdash^{\mathcal{T}_1}_{\mathcal{R}_1} \Delta : \sigma$, we have $B \vdash^{\mathcal{T}_2}_{\mathcal{R}_2} \Delta : \sigma$.*

▶ **Lemma 8.**

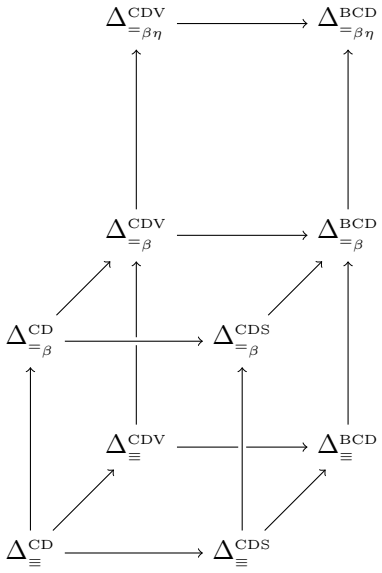
1. *$\Delta^{\mathrm{CD}}_{\mathcal{R}} \sqsubseteq \Delta^{\mathrm{CDS}}_{\mathcal{R}} \sqsubseteq \Delta^{\mathrm{BCD}}_{\mathcal{R}}$ and $\Delta^{\mathrm{CD}}_{\mathcal{R}} \sqsubseteq \Delta^{\mathrm{CDV}}_{\mathcal{R}} \sqsubseteq \Delta^{\mathrm{BCD}}_{\mathcal{R}}$;*
2. *$\Delta^{\mathcal{T}_1}_{\mathcal{R}_1} \sqsubseteq \Delta^{\mathcal{T}_2}_{\mathcal{R}_2}$ if $\mathcal{T}_1 \sqsubseteq \mathcal{T}_2$ and $\mathcal{R}_1 \sqsubseteq \mathcal{R}_2$.*

2.2 The Δ-chair

The next definition classifies ten typed systems for the Δ-calculus: some of them already appeared (sometime with a different notation) in the literature by the present authors.

▶ **Definition 9** (Δ-chair). *Ten typed systems $\Delta^{\mathcal{T}}_{\mathcal{R}}$ can be drawn pictorially in a Δ-chair, where the arrows represent an inclusion relation. $\Delta^{\mathrm{CD}}_{\equiv}$ corresponds roughly to [29, 30] (in the expression $M@\Delta$, M is the essence of Δ) and in its intersection part to [43]; $\Delta^{\mathrm{CDS}}_{\equiv}$ corresponds roughly in its intersection part to [17], $\Delta^{\mathrm{BCD}}_{\equiv}$ corresponds in its intersection part to [31], $\Delta^{\mathrm{CD}}_{=_{\beta\eta}}$ corresponds in its intersection part to [16]. The other typed systems are basically new. The main properties of these systems are:*

1. *All the $\Delta^{\mathcal{T}}_{\equiv}$ systems enjoys the synchronous subject reduction property, the other systems also enjoy ordinary subject reduction (Theorem 29);*
2. *All the systems strongly normalize (Theorem 33);*
3. *All the systems correspond to the to original type assignment systems except $\Delta^{\mathrm{CD}}_{=_\beta}$, $\Delta^{\mathrm{CDV}}_{=_\beta}$, $\Delta^{\mathrm{CDV}}_{=_{\beta\eta}}$ and $\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$ (Theorem 35);*
4. *Type checking and type reconstruction are decidable for all the systems, except $\Delta^{\mathrm{CDS}}_{=_\beta}$, $\Delta^{\mathrm{BCD}}_{=_\beta}$, and $\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$ (Theorem 37).*



3 Examples

This section shows examples of typed derivations $\Delta^{\tau}_{\mathcal{R}}$ and highlights the corresponding type assignment judgment in $\lambda^{\tau}_{\cap}$ they correspond to, in the sense that we have a derivation $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$ and another derivation $B \vdash^{\mathcal{T}}_{\cap} \wr \Delta \wr : \sigma$. The correspondence between intersection typed systems $\Delta^{\tau}_{\mathcal{R}}$ and intersection type assignment $\lambda^{\tau}_{\cap}$ will be defined in Subsection 5.1.

▶ **Example 10** (Polymorphic identity)**.** In all of the intersection type assignment systems $\lambda^{\tau}_{\cap}$ we can derive $\vdash^{\mathcal{T}}_{\cap} \lambda x.x : (\sigma \to \sigma) \cap (\tau \to \tau)$. A corresponding Δ-term is: $\langle \lambda x{:}\sigma.x \, , \lambda x{:}\tau.x \rangle$ that can be typed in all of the typed systems of the Δ-chair as follows

$$\dfrac{\dfrac{x{:}\sigma \vdash^{\mathcal{T}}_{\mathcal{R}} x : \sigma}{\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma.x : \sigma \to \sigma} \quad \dfrac{x{:}\tau \vdash^{\mathcal{T}}_{\mathcal{R}} x : \tau}{\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\tau.x : \tau \to \tau} \quad \lambda x.x \; \mathcal{R} \; \lambda x.x}{\vdash^{\mathcal{T}}_{\mathcal{R}} \langle \lambda x{:}\sigma.x \, , \lambda x{:}\tau.x \rangle : (\sigma \to \sigma) \cap (\tau \to \tau)}$$

▶ **Example 11** (Auto application)**.** In all of the intersection type assignment systems we can derive $\vdash^{\mathcal{T}}_{\cap} \lambda x.x\,x : ((\sigma \to \tau) \cap \sigma) \to \tau$. A corresponding Δ-term is: $\lambda x{:}(\sigma \to \tau) \cap \sigma.(pr_1\, x)(pr_2\, x)$ that can be typed in all of the typed systems of the Δ-chair as follows

$$\dfrac{\dfrac{\dfrac{x{:}(\sigma \to \tau) \cap \sigma \vdash^{\mathcal{T}}_{\mathcal{R}} x : (\sigma \to \tau) \cap \sigma}{x{:}(\sigma \to \tau) \cap \sigma \vdash^{\mathcal{T}}_{\mathcal{R}} pr_1\, x : \sigma \to \tau} \quad \dfrac{x{:}(\sigma \to \tau) \cap \sigma \vdash^{\mathcal{T}}_{\mathcal{R}} x : (\sigma \to \tau) \cap \sigma}{x{:}(\sigma \to \tau) \cap \sigma \vdash^{\mathcal{T}}_{\mathcal{R}} pr_2\, x : \sigma}}{x{:}(\sigma \to \tau) \cap \sigma \vdash^{\mathcal{T}}_{\mathcal{R}} (pr_1\, x)(pr_2\, x) : \tau}}{\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}(\sigma \to \tau) \cap \sigma.(pr_1\, x)(pr_2\, x) : (\sigma \to \tau) \cap \sigma \to \tau}$$

▶ **Example 12** (Some examples in $\Delta^{\mathrm{CDS}}_{\mathcal{R}}$)**.** In $\lambda^{\mathrm{CDS}}_{\cap}$ we can derive $\vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\cap} (\lambda x.\lambda y.x) : \sigma \to \mathtt{U} \to \sigma$, and using this type assignment, we can derive $z{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\cap} (\lambda x.\lambda y.x)\, z\, z : \sigma$. A corresponding

Δ-term is: $(\lambda x{:}\sigma.\lambda y{:}\mathtt{U}.x)\, z\, z^{\mathtt{U}}$ that can be typed in $\Delta^{\mathrm{CDS}}_{\mathcal{R}}$ as follows

$$\dfrac{\dfrac{\dfrac{z{:}\sigma, x{:}\sigma, y{:}\mathtt{U} \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} x : \sigma}{z{:}\sigma, x{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} \lambda y{:}\mathtt{U}.x : \mathtt{U} \to \sigma}}{z{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} \lambda x{:}\sigma.\lambda y{:}\mathtt{U}.x : \sigma \to \mathtt{U} \to \sigma} \quad z{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} z : \sigma}{z{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} (\lambda x{:}\sigma.\lambda y{:}\mathtt{U}.x)\, z : \mathtt{U} \to \sigma} \qquad \dfrac{z{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} z : \sigma \quad \sigma \leqslant_{\mathcal{T}_{\mathrm{CDS}}} \mathtt{U}}{z{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} z^{\mathtt{U}} : \mathtt{U}}}{z{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} (\lambda x{:}\sigma.\lambda y{:}\mathtt{U}.x)\, z\, z^{\mathtt{U}} : \sigma}$$

As another example, we can also derive $\vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\cap} \lambda x.x : \sigma \to \sigma \cap \mathtt{U}$. A corresponding Δ-term is: $\lambda x{:}\sigma.\langle x\, , x^{\mathtt{U}}\rangle$ that can be typed in $\Delta^{\mathrm{CDS}}_{\mathcal{R}}$ as follows

$$\dfrac{\dfrac{x{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} x : \sigma \quad \sigma \leqslant_{\mathcal{T}_{\mathrm{CDS}}} \mathtt{U}}{x{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} x^{\mathtt{U}} : \mathtt{U}} \qquad x \mathcal{R} x}{x{:}\sigma \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} \langle x\, , x^{\mathtt{U}}\rangle : \sigma \cap \mathtt{U}}}{\vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} \lambda x{:}\sigma.\langle x\, , x^{\mathtt{U}}\rangle : \sigma \to \sigma \cap \mathtt{U}}$$

▶ **Example 13** (An example in $\Delta^{\mathrm{CDV}}_{\mathcal{R}}$)**.** In $\lambda^{\mathrm{CDV}}_{\cap}$ we can prove the commutativity of intersection, i.e. $\vdash^{\mathcal{T}_{\mathrm{CDV}}}_{\cap} \lambda x.x : \sigma \cap \tau \to \tau \cap \sigma$. A corresponding Δ-term is: $\langle \lambda x{:}\sigma \cap \tau.pr_2\, x\, , \lambda x{:}\sigma \cap \tau.pr_1\, x\rangle^{(\sigma\cap\tau)\to(\tau\cap\sigma)}$ that can be typed in $\Delta^{\mathrm{CDV}}_{\mathcal{R}}$ as follows

$$\dfrac{\dfrac{\dfrac{x{:}\sigma \cap \tau \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} x : \sigma \cap \tau}{x{:}\sigma \cap \tau \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} pr_2\, x : \tau}}{\vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} \lambda x{:}\sigma \cap \tau.pr_2\, x : (\sigma \cap \tau) \to \tau} \quad \dfrac{\dfrac{x{:}\sigma \cap \tau \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} x : \sigma \cap \tau}{x{:}\sigma \cap \tau \vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} pr_1\, x : \sigma}}{\vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} \lambda x{:}\sigma \cap \tau.pr_1\, x : (\sigma \cap \tau) \to \sigma} \quad \lambda x.x \mathcal{R} \lambda x.x}{\vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} \langle \lambda x{:}\sigma \cap \tau.pr_2\, x\, , \lambda x{:}\sigma \cap \tau.pr_1\, x\rangle : ((\sigma \cap \tau) \to \tau) \cap ((\sigma \cap \tau) \to \sigma)} *}{\vdash^{\mathcal{T}_{\mathrm{CDS}}}_{\mathcal{R}} \langle \lambda x{:}\sigma \cap \tau.pr_2\, x\, , \lambda x{:}\sigma \cap \tau.pr_1\, x\rangle^{(\sigma\cap\tau)\to(\tau\cap\sigma)} : (\sigma \cap \tau) \to (\tau \cap \sigma)}$$

where $*$ is $((\sigma \cap \tau) \to \tau) \cap ((\sigma \cap \tau) \to \sigma) \leqslant_{\mathcal{T}_{\mathrm{CDV}}} (\sigma \cap \tau) \to (\tau \cap \sigma)$.

▶ **Example 14** (Another polymorphic identity in $\Delta^{\mathcal{T}}_{=_\beta}$)**.** In all the $\Delta^{\mathcal{T}}_{=_\beta}$ you can type the following Δ-term: $\langle \lambda x{:}\sigma.x\, , (\lambda x{:}\tau{\to}\tau.x)\, (\lambda x{:}\tau.x)\rangle$. The typing derivation is thus

$$\dfrac{\dfrac{x{:}\sigma \vdash^{\mathcal{T}}_{=_\beta} x : \sigma}{\vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\sigma.x : \sigma \to \sigma} \quad \dfrac{\dfrac{x{:}\tau \to \tau \vdash^{\mathcal{T}}_{=_\beta} x : \tau \to \tau}{\vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\tau{\to}\tau.x : (\tau \to \tau) \to (\tau \to \tau)} \quad \dfrac{x{:}\tau \vdash^{\mathcal{T}}_{=_\beta} x : \tau}{\vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\tau.x : \tau \to \tau}}{\vdash^{\mathcal{T}}_{=_\beta} (\lambda x{:}\tau{\to}\tau.x)\, (\lambda x{:}\tau.x) : \tau \to \tau \quad \lambda x.x =_\beta (\lambda x.x)\, (\lambda x.x)}}{\vdash^{\mathcal{T}}_{=_\beta} \langle \lambda x{:}\sigma.x\, , (\lambda x{:}\tau{\to}\tau.x)\, (\lambda x{:}\tau.x)\rangle : (\sigma \to \sigma) \cap (\tau \to \tau)}$$

▶ **Example 15** (Two examples in $\Delta^{\mathrm{BCD}}_{\equiv}$ and $\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$)**.** In $\lambda^{\mathrm{BCD}}_{\cap}$ we can can type any term, including the non-terminating term $\Omega \stackrel{def}{=} (\lambda x.x\, x)\, (\lambda x.x\, x)$. More precisely, we have $\vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\cap} \Omega : \mathtt{U}$. A corresponding Δ-term whose essence is Ω is $(\lambda x{:}\mathtt{U}.x^{\mathtt{U}\to\mathtt{U}}\, x)\, (\lambda x{:}\mathtt{U}.x^{\mathtt{U}\to\mathtt{U}}\, x)^{\mathtt{U}}$ that can be typed in $\Delta^{\mathrm{BCD}}_{\mathcal{R}}$ as follows

$$\dfrac{\dfrac{*}{\vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\mathcal{R}} \lambda x{:}\mathtt{U}.x^{\mathtt{U}\to\mathtt{U}}\, x : \mathtt{U} \to \mathtt{U}} \quad \dfrac{\dfrac{*}{\vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\mathcal{R}} \lambda x{:}\mathtt{U}.x^{\mathtt{U}\to\mathtt{U}}\, x : \mathtt{U} \to \mathtt{U}} \quad \mathtt{U} \to \mathtt{U} \leqslant_{\mathcal{T}_{\mathrm{BCD}}} \mathtt{U}}{\vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\mathcal{R}} (\lambda x{:}\mathtt{U}.x^{\mathtt{U}\to\mathtt{U}}\, x)^{\mathtt{U}} : \mathtt{U}}}{\vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\mathcal{R}} (\lambda x{:}\mathtt{U}.x^{\mathtt{U}\to\mathtt{U}}\, x)\, (\lambda x{:}\mathtt{U}.x^{\mathtt{U}\to\mathtt{U}}\, x)^{\mathtt{U}} : \mathtt{U}}$$

where $*$ is

$$\dfrac{\dfrac{x{:}\mathtt{U} \vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\mathcal{R}} x : \mathtt{U} \quad \mathtt{U} \leqslant_{\mathcal{T}_{\mathrm{BCD}}} \mathtt{U} \to \mathtt{U}}{x{:}\mathtt{U} \vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\mathcal{R}} x^{\mathtt{U}\to\mathtt{U}} : \mathtt{U} \to \mathtt{U}} \qquad x{:}\mathtt{U} \vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\mathcal{R}} x : \mathtt{U}}{x{:}\mathtt{U} \vdash^{\mathcal{T}_{\mathrm{BCD}}}_{\mathcal{R}} x^{\mathtt{U}\to\mathtt{U}}\, x : \mathtt{U}}$$

In $\lambda^{\text{BCD}}_{\cap}$ we can type $x{:}\mathtt{U} \to \mathtt{U} \vdash^{\mathcal{T}_{\text{BCD}}}_{\cap} x : (\mathtt{U} \to \mathtt{U}) \cap (\sigma \to \mathtt{U})$. A corresponding Δ-term whose essence is x is $\langle x\,, \lambda y{:}\sigma.x\,y^{\mathtt{U}}\rangle$ that can be typed in $\Delta^{\text{BCD}}_{=_{\beta\eta}}$ as follows

$$\dfrac{\dfrac{\dfrac{}{x{:}\mathtt{U}\to\mathtt{U},\, y{:}\sigma \vdash^{\mathcal{T}_{\text{BCD}}}_{=_{\beta\eta}} x : \mathtt{U}\to\mathtt{U}} \qquad \dfrac{\dfrac{}{x{:}\mathtt{U}\to\mathtt{U},\, y{:}\sigma \vdash^{\mathcal{T}_{\text{BCD}}}_{=_{\beta\eta}} y : \sigma} \quad \sigma \leqslant \mathtt{U}}{x{:}\mathtt{U}\to\mathtt{U},\, y{:}\sigma \vdash^{\mathcal{T}_{\text{BCD}}}_{=_{\beta\eta}} y^{\mathtt{U}} : \mathtt{U}}}{x{:}\mathtt{U}\to\mathtt{U},\, y{:}\sigma \vdash^{\mathcal{T}_{\text{BCD}}}_{=_{\beta\eta}} x\,y^{\mathtt{U}} : \mathtt{U}}}{x{:}\mathtt{U}\to\mathtt{U} \vdash^{\mathcal{T}_{\text{BCD}}}_{=_{\beta\eta}} x : \mathtt{U}\to\mathtt{U} \quad x{:}\mathtt{U}\to\mathtt{U} \vdash^{\mathcal{T}_{\text{BCD}}}_{=_{\beta\eta}} \lambda y{:}\sigma.x\,y^{\mathtt{U}} : \sigma\to\mathtt{U}} \qquad x =_{\beta\eta} \lambda y.x\,y}{x{:}\mathtt{U}\to\mathtt{U} \vdash^{\mathcal{T}_{\text{BCD}}}_{=_{\beta\eta}} \langle x\,, \lambda y{:}\sigma.x\,y^{\mathtt{U}}\rangle : (\mathtt{U}\to\mathtt{U}) \cap (\sigma\to\mathtt{U})}$$

Note that the $=_{\beta\eta}$ condition has an interesting loophole, as it is well known that $\lambda^{\text{BCD}}_{\cap}$ does not enjoy $=_{\eta}$ conversion property. Theorem 35 will show that we can construct a Δ-term which does not correspond to any $\lambda^{\text{BCD}}_{\cap}$ derivation.

▶ **Example 16** (Pottinger). The following examples can be typed in all the type theories of the Δ-chair (we also display the corresponding pure λ-terms typable in $\lambda^{\mathcal{T}}_{\cap}$). These are encodings from the examples à la Curry given by Pottinger in [39].

$\lambda x.\lambda y.x\,y$
$\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}(\sigma\to\tau)\cap(\sigma\to\rho).\lambda y{:}\sigma.\langle (pr_1\,x)\,y)\,, (pr_2\,x)\,y\rangle : (\sigma\to\tau)\cap(\sigma\to\rho)\to\sigma\to\tau\cap\rho$

$\lambda x.\lambda y.x\,y$
$\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma\to\tau\cap\rho.\langle\lambda y{:}\sigma.pr_1\,(x\,y)\,, \lambda y{:}\sigma.pr_2\,(x\,y)\rangle : (\sigma\to\tau\cap\rho)\to(\sigma\to\tau)\cap(\sigma\to\rho)$

$\lambda x.\lambda y.x\,y$
$\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma\to\rho.\lambda y{:}\sigma\cap\tau.x\,(pr_1\,y) : (\sigma\to\rho)\to\sigma\cap\tau\to\rho$

$\lambda x.\lambda y.x$
$\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma\cap\tau.\lambda y{:}\sigma.pr_2\,x : \sigma\cap\tau\to\sigma\to\tau$

$\lambda x.\lambda y.x\,y\,y$
$\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma\to\tau\to\rho.\lambda y{:}\sigma\cap\tau.x\,(pr_1\,y)\,(pr_2\,y) : (\sigma\to\tau\to\rho)\to\sigma\cap\tau\to\rho$

$\lambda x.x$
$\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma\cap\tau.pr_1\,x : \sigma\cap\tau\to\sigma$

$\lambda x.x$
$\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma.\langle x\,, x\rangle : \sigma\to\sigma\cap\sigma$

$\lambda x.x$
$\vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma\cap(\tau\cap\rho).\langle\langle pr_1\,x\,, pr_1\,pr_2\,x\rangle\,, pr_2\,pr_2\,x\rangle : \sigma\cap(\tau\cap\rho)\to(\sigma\cap\tau)\cap\rho$

In the same paper, Pottinger lists some types that cannot be inhabited by any intersection type assignment ($\not\vdash^{\mathcal{T}}_{\cap}$) in an empty context, namely: $\sigma\to(\sigma\cap\tau)$, and $(\sigma\to\tau)\to(\sigma\to\rho)\to\sigma\to\tau\cap\rho$, and $((\sigma\cap\tau)\to\rho)\to\sigma\to\tau\to\rho$. It is not difficult to verify that the above types cannot be inhabited by any of the typed systems of the Δ-chair because of the failure of the essence condition in the strong pair type rule.

▶ **Example 17** (Intersection is not the conjunction operator). This counter-example is from the corresponding counter-example à la Curry given by Hindley [26] and Ben-Yelles [6]. The intersection type $(\sigma\to\sigma)\cap((\sigma\to\tau\to\rho)\to(\sigma\to\tau)\to\sigma\to\rho)$ where the left part of the intersection corresponds to the type for the combinator I and the right part for the combinator S cannot be assigned to a pure λ-term. Analogously, the same intersection type cannot be assigned to any Δ-term.

3.1 On synchronization and subject reduction

For the typed systems $\Delta^{\mathcal{T}}_{\equiv}$, strong pairs have an intrinsic notion of synchronization: some redexes need to be reduced in a synchronous fashion unless we want to create meaningless Δ-terms that cannot be typed. Consider the Δ-term $\langle(\lambda x{:}\sigma.x)\, y\, , (\lambda x{:}\sigma.x)\, y\rangle$: if we use the $\longrightarrow$ reduction relation, then the following reduction paths are legal:

$$\langle(\lambda x{:}\sigma.x)\, y\, , (\lambda x{:}\sigma.x)\, y\rangle \begin{array}{l} \nearrow^{\beta}\ \langle(\lambda x{:}\sigma.x)\, y\, , y\rangle \searrow_{\beta} \\ \searrow_{\beta}\ \langle y\, , (\lambda x{:}\sigma.x)\, y\rangle \nearrow^{\beta} \end{array} \langle y\, , y\rangle$$

More precisely, the first and second redexes are rewritten asynchronously, thus they cannot be typed in any typed system $\Delta^{\mathcal{T}}_{\equiv}$, because we fail to check the left and the right part of the strong pair to be the same: the $\longrightarrow^{\parallel}$ reduction relation prevents this loophole and allows to type all redexes. In summary, $\longrightarrow^{\parallel}$ can be thought of as the natural reduction relation for the typed systems $\Delta^{\mathcal{T}}_{\equiv}$.

4 Metatheory of $\Delta^{\mathcal{T}}_{\mathcal{R}}$

For lack of space all proofs are omitted: the interested reader can find more technical details in the related full version at `https://arxiv.org/abs/1803.09660`.

4.1 General properties

Unless specified, all properties applies to the intersection typed systems $\Delta^{\mathcal{T}}_{\mathcal{R}}$. There are several elegant ways to prove the Church-Rosser property: among others we could use the historical method of Tait and Martin-Löf (see Definition 3.2.3 of [3]), or the van Oostrom and van Raamsdonk technique [47], or the Takahashi parallel reduction technique [44]. Parallel reduction semantics extends Definition 4 and it is inductively defined as follows.

▶ **Definition 18** (Parallel reduction semantics).

$$\begin{array}{rcll}
x & \Longrightarrow & x & \text{and} \qquad u_{\Delta} \Longrightarrow u_{\Delta} \\
\Delta^{\sigma} & \Longrightarrow & (\Delta')^{\sigma} & \text{if } \Delta \Longrightarrow \Delta' \\
\Delta_1\, \Delta_2 & \Longrightarrow & \Delta'_1\, \Delta'_2 & \text{if } \Delta_1 \Longrightarrow \Delta'_1 \text{ and } \Delta_2 \Longrightarrow \Delta'_2 \\
\lambda x{:}\sigma.\Delta & \Longrightarrow & \lambda x{:}\sigma.\Delta' & \text{if } \Delta \Longrightarrow \Delta' \\
(\lambda x{:}\sigma.\Delta_1)\, \Delta_2 & \Longrightarrow & \Delta'_1[\Delta'_2/x] & \text{if } \Delta_1 \Longrightarrow \Delta'_1 \text{ and } \Delta_2 \Longrightarrow \Delta'_2 \\
\langle \Delta_1\, , \Delta_2\rangle & \Longrightarrow & \langle \Delta'_1\, , \Delta'_2\rangle & \text{if } \Delta_1 \Longrightarrow \Delta'_1 \text{ and } \Delta_2 \Longrightarrow \Delta'_2 \\
pr_i\, \Delta & \Longrightarrow & pr_i\, \Delta' & \text{if } \Delta \Longrightarrow \Delta' \quad \text{and } i \in \{1,2\} \\
pr_i\, \langle \Delta_1\, , \Delta_2\rangle & \Longrightarrow & \Delta'_i & \text{if } \Delta_i \Longrightarrow \Delta'_i \text{ and } i \in \{1,2\}
\end{array}$$

Intuitively, $\Delta \Longrightarrow \Delta'$ means that Δ' is obtained from Δ by simultaneous contraction of some βpr_i-redexes possibly overlapping each other. Church-Rosser can be achieved by proving a stronger statement, namely $\Delta \Longrightarrow \Delta'$ implies $\Delta' \Longrightarrow \Delta^*$, where Δ^* is a Δ-term determined by Δ and independent from Δ'. The above statement is satisfied by the term Δ^* which is, in turn, obtained from Δ by contracting all the redexes existing in Δ simultaneously.

▶ **Definition 19** (The map $\_^*$).

$$
\begin{array}{rcl@{\qquad}rcl}
x^* & \stackrel{def}{=} & x & u_\Delta^* & \stackrel{def}{=} & u_\Delta \\
(\Delta^\sigma)^* & \stackrel{def}{=} & (\Delta^*)^\sigma & \langle \Delta_1 \,, \Delta_2 \rangle^* & \stackrel{def}{=} & \langle \Delta_1^* \,, \Delta_2^* \rangle \\
(\lambda x{:}\sigma.\Delta)^* & \stackrel{def}{=} & \lambda x{:}\sigma.\Delta^* & (\lambda x{:}\sigma.\Delta_1)\,\Delta_2^* & \stackrel{def}{=} & \Delta_1^*[\Delta_2^*/x] \\
(\Delta_1\,\Delta_2)^* & \stackrel{def}{=} & \Delta_1^*\,\Delta_2^* & \multicolumn{3}{c}{\textit{if } \Delta_1\,\Delta_2 \textit{ is not a } \beta\textit{-redex}} \\
(pr_i\,\langle \Delta_1 \,, \Delta_2 \rangle)^* & \stackrel{def}{=} & \Delta_i^* & \multicolumn{3}{c}{i \in \{1,2\}} \\
(pr_i\,\Delta)^* & \stackrel{def}{=} & pr_i\,\Delta^* & \multicolumn{3}{c}{\textit{if } \Delta \textit{ is not a strong pair}}
\end{array}
$$

The next technical lemma will be useful in showing that Church-Rosser for $\longrightarrow\!\!\!\!\!\rightarrow$ can be inherited from Church-Rosser for $\Longrightarrow$.

▶ **Lemma 20.**

1. *If* $\Delta_1 \longrightarrow \Delta_1'$, *then* $\Delta_1 \Longrightarrow \Delta_1'$;
2. *if* $\Delta_1 \Longrightarrow \Delta_1'$, *then* $\Delta_1 \longrightarrow\!\!\!\!\!\rightarrow \Delta_1'$;
3. *if* $\Delta_1 \Longrightarrow \Delta_1'$ *and* $\Delta_2 \Longrightarrow \Delta_2'$, *then* $\Delta_1[\Delta_2/x] \Longrightarrow \Delta_1'[\Delta_2'/x]$;
4. $\Delta_1 \Longrightarrow \Delta_1^*$.

Now we have to prove the Church-Rosser property for the parallel reduction.

▶ **Lemma 21** (Confluence property for $\Longrightarrow$). *If* $\Delta \Longrightarrow \Delta'$, *then* $\Delta' \Longrightarrow \Delta^*$.

The Church-Rosser property follows.

▶ **Theorem 22** (Confluence). *If* $\Delta_1 \longrightarrow\!\!\!\!\!\rightarrow \Delta_2$ *and* $\Delta_1 \longrightarrow\!\!\!\!\!\rightarrow \Delta_3$, *then there exists* Δ_4 *such that* $\Delta_2 \longrightarrow\!\!\!\!\!\rightarrow \Delta_4$ *and* $\Delta_3 \longrightarrow\!\!\!\!\!\rightarrow \Delta_4$.

At this point, a small remark about η-reduction in the Δ-calculus could be useful. It is well-known by Nederpelt [35] that Church-Rosser for $\beta\eta$-reduction in the λ-calculus à la Church is just false; Geuvers in [23] proved the Church-Rosser for $\beta\eta$-reduction for well-typed terms in Pure Type Systems; the same result can be easily proved for the Δ-calculus.

The next lemma says that all type derivations for Δ have an unique type.

▶ **Lemma 23** (Unicity of typing). *If* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, *then* σ *is unique.*

The next lemma proves inversion properties on typable Δ-terms.

▶ **Lemma 24** (Generation).

1. *If* $B \vdash^{\mathcal{T}}_{\mathcal{R}} x : \sigma$, *then* $x{:}\sigma \in B$;
2. *if* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \lambda x{:}\sigma.\Delta : \rho$, *then* $\rho \equiv \sigma \to \tau$ *for some* τ *and* $B, x{:}\sigma \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \tau$;
3. *if* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1\,\Delta_2 : \tau$, *then there is* σ *such that* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1 : \sigma \to \tau$ *and* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_2 : \sigma$;
4. *if* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \langle \Delta_1 \,, \Delta_2 \rangle : \rho$, *then there is* σ, τ *such that* $\rho \equiv \sigma \cap \tau$ *and* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1 : \sigma$ *and* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_2 : \tau$ *and* $\wr\,\Delta_1\,\wr\;\mathcal{R}\;\wr\,\Delta_2\,\wr$;
5. *if* $B \vdash^{\mathcal{T}}_{\mathcal{R}} pr_1\,\Delta : \sigma$, *then there is* τ *such that* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma \cap \tau$;
6. *if* $B \vdash^{\mathcal{T}}_{\mathcal{R}} pr_2\,\Delta : \tau$, *then there is* σ *such that* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma \cap \tau$;
7. *if* $B \vdash^{\mathcal{T}}_{\mathcal{R}} u_\Delta : \sigma$, *then* $\sigma \equiv \mathtt{U}$;
8. *if* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta^\tau : \rho$, *then* $\rho \equiv \tau$ *and there is* σ *such that* $\sigma \leqslant_{\mathcal{T}} \tau$ *and* $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$.

The next lemma says that all subterms of a typable Δ-term are typable too.

▶ **Lemma 25** (Subterms typability). *If $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, and Δ' is a subterm of Δ, then there exists B' and τ such that $B' \supseteq B$ and $B' \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta' : \tau$.*

As expected, the weakening and strengthening properties on contexts are verified.

▶ **Lemma 26** (Free-variable properties).
1. *If $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, and $B' \supseteq B$, then $B' \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$;*
2. *if $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, then $\mathrm{FV}(\Delta) \subseteq Dom(B)$;*
3. *if $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, $B' \subseteq B$ and $\mathrm{FV}(\Delta) \subseteq Dom(B')$, then $B' \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$.*

The next lemma also says that essence is closed under substitution.

▶ **Lemma 27** (Substitution).
1. *$\wr \Delta_1[\Delta_2/x] \wr \equiv \wr \Delta_1 \wr[\wr \Delta_2 \wr/x]$;*
2. *If $B, x{:}\sigma \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1 : \tau$ and $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_2 : \sigma$, then $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1[\Delta_2/x] : \tau$.*

In order to prove subject reduction, we need to prove that reducing Δ-terms preserve the side-condition $\wr \Delta_1 \wr \mathcal{R} \wr \Delta_2 \wr$ when typing the strong pair $\langle \Delta_1 , \Delta_2 \rangle$. We prove this in the following lemma.

▶ **Lemma 28** (Essence reduction).
1. *If $B \vdash^{\mathcal{T}}_{\equiv} \Delta_1 : \sigma$ and $\Delta_1 \longrightarrow \Delta_2$, then $\wr \Delta_1 \wr =_\beta \wr \Delta_2 \wr$;*
2. *for $\mathcal{R} \in \{=_\beta, =_{\beta\eta}\}$, if $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1 : \sigma$ and $\Delta_1 \longrightarrow \Delta_2$, then $\wr \Delta_1 \wr \mathcal{R} \wr \Delta_2 \wr$;*
3. *if $B \vdash^{\mathcal{T}}_{=_{\beta\eta}} \Delta_1 : \sigma$ and $\Delta_1 \longrightarrow_\eta \Delta_2$, then $\wr \Delta_1 \wr =_\eta \wr \Delta_2 \wr$.*

The next theorem states that all the $\Delta^{\mathcal{T}}_{\equiv}$ typed systems preserve synchronous βpr_i-reduction, and all the $\Delta^{\mathcal{T}}_{=_\beta}$ and $\Delta^{\mathcal{T}}_{=_{\beta\eta}}$ typed systems preserve βpr_i-reduction.

▶ **Theorem 29** (Subject reduction for βpr_i).
1. *If $B \vdash^{\mathcal{T}}_{\equiv} \Delta_1 : \sigma$ and $\Delta_1 \longrightarrow^{\parallel} \Delta_2$, then $B \vdash^{\mathcal{T}}_{\equiv} \Delta_2 : \sigma$;*
2. *for $\mathcal{R} \in \{=_\beta, =_{\beta\eta}\}$, if $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_1 : \sigma$ and $\Delta_1 \longrightarrow \Delta_2$, then $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta_2 : \sigma$.*

The next theorem states that some of the typed systems on the back of the Δ-chair preserve η-reduction.

▶ **Theorem 30** (Subject reduction for η for $\mathcal{T}_{\mathrm{CDV}}, \mathcal{T}_{\mathrm{BCD}}$). *Let $\mathcal{T} \in \{\mathcal{T}_{\mathrm{CDV}}, \mathcal{T}_{\mathrm{BCD}}\}$. If $B \vdash^{\mathcal{T}}_{=_{\beta\eta}} \Delta_1 : \sigma$ and $\Delta_1 \longrightarrow_\eta \Delta_2$, then $B \vdash^{\mathcal{T}}_{=_{\beta\eta}} \Delta_2 : \sigma$.*

4.2 Strong normalization

The idea of the strong normalization proof is to embed typable terms of the Δ-calculus into Church-style terms of a target system, which is the simply-typed λ-calculus with pairs, in a structure-preserving way (and forgetting all the essence side-conditions). The translation is sufficiently faithful so as to preserve the number of reductions, and so strong normalization for the Δ-calculus follows from strong normalization for simply-typed λ-calculus with pairs. A similar technique has been used in [24] to prove the strong normalization property of LF and in [9] to prove the strong normalization property of a subset of $\lambda^{\mathrm{CD}}_{\cap}$.

The target system has one atomic type called $\circ$, a special constant term $u_\circ$ of type $\circ$ and an infinite number of constants c_σ of type σ for any type of the target system. We denote by $B \vdash_\times M : \sigma$ a typing judgment in the target system.

▶ **Definition 31** (Forgetful mapping).

- *On intersection types.*

$$|a_i| \stackrel{def}{=} \circ \quad \forall a_i \in \mathbb{A} \qquad \text{and} \qquad |\sigma \cap \tau| \stackrel{def}{=} |\sigma| \times |\tau| \qquad \text{and} \qquad |\sigma \to \tau| \stackrel{def}{=} |\sigma| \to |\tau|$$

- *On Δ-terms.*

$$\begin{array}{rcl@{\qquad}rcl} |x|_B & \stackrel{def}{=} & x & |u_\Delta|_B & \stackrel{def}{=} & u_\circ \\ |\lambda x{:}\sigma.\Delta|_B & \stackrel{def}{=} & \lambda x.|\Delta|_{B,x:\sigma} & |\Delta_1\,\Delta_2|_B & \stackrel{def}{=} & |\Delta_1|_B\,|\Delta_2|_B \\ |\langle \Delta_1\,, \Delta_2\rangle|_B & \stackrel{def}{=} & (|\Delta_1|_B\,, |\Delta_2|_B) & |pr_i\,\Delta|_B & \stackrel{def}{=} & pr_i\,|\Delta|_B \\ |\Delta^\tau|_B & \stackrel{def}{=} & c_{|\sigma|\to|\tau|}\,|\Delta|_B \quad \textit{if } B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma & & & \end{array}$$

- *The map can be easily extended to basis B.*

The following technical lemma states some properties of the forgetful function.

▶ **Lemma 32.**

1. *If $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, then $|\Delta|_B$ is defined, and, for all $B' \supseteq B$, $|\Delta|_B \equiv |\Delta|_{B'}$;*
2. *$|\Delta_1[\Delta_2/x]|_B \equiv |\Delta_1|_B[|\Delta_2|_B/x]$;*
3. *If $\Delta_1 \longrightarrow \Delta_2$, then $|\Delta_1|_B \longrightarrow |\Delta_2|_B$;*
4. *If $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$ then $|B| \vdash_\times |\Delta|_B : |\sigma|$.*

Strong normalization follows easily from the above lemmas.

▶ **Theorem 33** (Strong normalization). *If $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, then Δ is strongly normalizing.*

5 Typed systems à la Church vs. type assignment systems à la Curry

5.1 Relation between type assignment systems $\lambda^{\mathcal{T}}_{\cap}$ and typed systems $\Delta^{\mathcal{T}}_{\mathcal{R}}$

It is interesting to state some relations between type assignment systems à la Church and typed systems à la Curry. An interesting property is the one of isomorphism, namely the fact that whenever we assign a type σ to a pure λ-term M, the same type can be assigned to a Δ-term such that the essence of Δ is M. Conversely, for every assignment of σ to a Δ-term, a valid type assignment judgment of the same type for the essence of Δ can be derived. Soundness, completeness and isomorphism between intersection typed systems for the Δ-calculus and the corresponding intersection type assignment systems for the λ-calculus are defined as follows.

▶ **Definition 34** (Soundness, completeness and isomorphism). *Let $\Delta^{\mathcal{T}}_{\mathcal{R}}$ and $\lambda^{\mathcal{T}}_{\cap}$.*

1. *(Soundness, $\Delta^{\mathcal{T}}_{\mathcal{R}} \lhd \lambda^{\mathcal{T}}_{\cap}$). $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$ implies $B \vdash^{\mathcal{T}}_{\cap} \wr\,\Delta\,\wr : \sigma$;*
2. *(Completeness, $\Delta^{\mathcal{T}}_{\mathcal{R}} \rhd \lambda^{\mathcal{T}}_{\cap}$). $B \vdash^{\mathcal{T}}_{\cap} M : \sigma$ implies there exists Δ such that $M \equiv \wr\,\Delta\,\wr$ and $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$;*
3. *(Isomorphism, $\Delta^{\mathcal{T}}_{\mathcal{R}} \sim \lambda^{\mathcal{T}}_{\cap}$). $\Delta^{\mathcal{T}}_{\mathcal{R}} \rhd \lambda^{\mathcal{T}}_{\cap}$ and $\Delta^{\mathcal{T}}_{\mathcal{R}} \lhd \lambda^{\mathcal{T}}_{\cap}$.*

▶ **Theorem 35** (Soundness, completeness and isomorphism). *The following properties (left of Figure 4) between Δ-calculi and type assignment systems $\lambda^{\mathcal{T}}_{\cap}$ can be verified.*

The next theorem characterizes the class of strongly normalizing Δ-terms.

$\Delta^{\mathcal{T}}_{\mathcal{R}}$	$\Delta^{\mathcal{T}}_{\mathcal{R}} \lhd \lambda^{\mathcal{T}}_{\cap}$	$\Delta^{\mathcal{T}}_{\mathcal{R}} \rhd \lambda^{\mathcal{T}}_{\cap}$
$\Delta^{\mathrm{CD}}_{\equiv}$	$\surd$	$\surd$
$\Delta^{\mathrm{CDV}}_{\equiv}$	$\surd$	$\surd$
$\Delta^{\mathrm{CDS}}_{\equiv}$	$\surd$	$\surd$
$\Delta^{\mathrm{BCD}}_{\equiv}$	$\surd$	$\surd$
$\Delta^{\mathrm{CD}}_{=_\beta}$	$\times$	$\surd$
$\Delta^{\mathrm{CDV}}_{=_\beta}$	$\times$	$\surd$
$\Delta^{\mathrm{CDS}}_{=_\beta}$	$\surd$	$\surd$
$\Delta^{\mathrm{BCD}}_{=_\beta}$	$\surd$	$\surd$
$\Delta^{\mathrm{CDV}}_{=_{\beta\eta}}$	$\times$	$\surd$
$\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$	$\times$	$\surd$

$\Delta^{\mathcal{T}}_{\mathcal{R}}$	TC/TR
$\Delta^{\mathrm{CD}}_{\equiv}$	$\surd$
$\Delta^{\mathrm{CDV}}_{\equiv}$	$\surd$
$\Delta^{\mathrm{CDS}}_{\equiv}$	$\surd$
$\Delta^{\mathrm{BCD}}_{\equiv}$	$\surd$
$\Delta^{\mathrm{CD}}_{=_\beta}$	$\surd$
$\Delta^{\mathrm{CDV}}_{=_\beta}$	$\surd$
$\Delta^{\mathrm{CDS}}_{=_\beta}$	$\times$
$\Delta^{\mathrm{BCD}}_{=_\beta}$	$\times$
$\Delta^{\mathrm{CDV}}_{=_{\beta\eta}}$	$\surd$
$\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$	$\times$

Source	Target
$\Delta^{\mathrm{CD}}_{\equiv}$	$\Delta^{\mathrm{CD}}_{=_\beta}$
$\Delta^{\mathrm{CDV}}_{\equiv}$	$\Delta^{\mathrm{CDV}}_{=_{\beta\eta}}$
$\Delta^{\mathrm{CDS}}_{\equiv}$	$\Delta^{\mathrm{CDS}}_{=_\beta}$
$\Delta^{\mathrm{BCD}}_{\equiv}$	$\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$
$\Delta^{\mathrm{CD}}_{=_\beta}$	$\Delta^{\mathrm{CD}}_{=_\beta}$
$\Delta^{\mathrm{CDV}}_{=_\beta}$	$\Delta^{\mathrm{CDV}}_{=_{\beta\eta}}$
$\Delta^{\mathrm{CDS}}_{=_\beta}$	$\Delta^{\mathrm{CDS}}_{=_\beta}$
$\Delta^{\mathrm{BCD}}_{=_\beta}$	$\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$
$\Delta^{\mathrm{CDV}}_{=_{\beta\eta}}$	$\Delta^{\mathrm{CDV}}_{=_{\beta\eta}}$
$\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$	$\Delta^{\mathrm{BCD}}_{=_{\beta\eta}}$

Figure 4 On the left: Soundness, completeness, isomorphism. On the center: type checking/reconstruction. On the right: source and target languages of the translation.

▶ **Theorem 36** (Characterization). *Every strongly normalizing λ-term can be type-annotated so as to be the essence of a typable Δ-term.*

We can finally state decidability of type checking (TC) and type reconstruction (TR).

▶ **Theorem 37** (Decidability of type checking and type reconstruction). *Figure 4 (in the center) list decidability of type checking and type reconstruction.*

5.2 Subtyping and explicit coercions

The typing rule ($\leqslant_{\mathcal{T}}$) in the general typed system introduces type coercions: once a type coercion is introduced, it cannot be eliminated, so *de facto* freezing a Δ-term inside an explicit coercion. Tannen et al. [8] showed a translation of a judgment derivation from a "Source" system with subtyping (Cardelli's Fun [11]) into an "equivalent" judgment derivation in a "Target" system without subtyping (Girard system F with records and recursion). In the same spirit, we present a translation that removes all explicit coercions. Intuitively, the translation proceeds as follows: every derivation ending with rule ($\leqslant_{\mathcal{T}}$) is translated into the following (coercion-free) derivation, i.e.

$$\frac{B \vdash^{\mathcal{T}}_{\mathcal{R}'} \|\sigma \leqslant_{\mathcal{T}} \tau\| : \sigma \to \tau \quad B \vdash^{\mathcal{T}}_{\mathcal{R}'} \|\Delta\|_B : \sigma}{B \vdash^{\mathcal{T}}_{\mathcal{R}'} \|\sigma \leqslant_{\mathcal{T}} \tau\| \; \|\Delta\|_B : \tau} \; (\to E)$$

where $\mathcal{R}'$ is a suitable relation such that $\mathcal{R} \sqsubseteq \mathcal{R}'$, $\|-\|$ is a suitable translation function defined later in Definition 39. Note that changing of the type theory is necessary to guarantee well-typedness in the translation of strong pairs. Summarizing, we provide a type preserving translation of a Δ-term into a coercion-free Δ-term such that $\wr \Delta \wr =_{\beta\eta} \wr \Delta' \wr$. The following example illustrates some trivial compilations of axioms and rule schemes of Figure 1.

▶ **Example 38** (Translation of axioms and rule schemes of Figure 1).
(refl) the judgment $x{:}\sigma \vdash^{\mathcal{T}}_{\mathcal{R}} \langle x\,, x^{\sigma}\rangle : \sigma \cap \sigma$ is translated to a coercion-free judgment $x{:}\sigma \vdash^{\mathcal{T}}_{=_\beta} \langle x\,, (\lambda y{:}\sigma.y)\, x\rangle : \sigma \cap \sigma$;
(incl) the judgment $x{:}\sigma \cap \tau \vdash^{\mathcal{T}}_{\mathcal{R}} \langle x\,, x^{\tau}\rangle : (\sigma \cap \tau) \cap \tau$ is translated to a coercion-free judgment $x{:}\sigma \cap \tau \vdash^{\mathcal{T}}_{=_\beta} \langle x\,, (\lambda y{:}\sigma \cap \tau.pr_2\, y)\, x\rangle : (\sigma \cap \tau) \cap \tau$;

(glb) the judgment $x{:}\sigma \vdash^{\mathcal{T}}_{\mathcal{R}} \langle x\,, x^{\sigma\cap\sigma}\rangle : \sigma \cap (\sigma \cap \sigma)$ is translated to a coercion-free judgment $x{:}\sigma \vdash^{\mathcal{T}}_{=_\beta} \langle x\,, (\lambda y{:}\sigma.\langle y\,, y\rangle)\, x\rangle : \sigma \cap (\sigma \cap \sigma)$;
($\mathtt{U}_{top}$) the judgment $x{:}\sigma \vdash^{\mathcal{T}}_{\mathcal{R}} \langle x\,, x^{\mathtt{U}}\rangle : \sigma \cap \mathtt{U}$ is translated to a coercion-free judgment $x{:}\sigma \vdash^{\mathcal{T}}_{=_\beta} \langle x\,, (\lambda y{:}\sigma.u_y)\, x\rangle : \sigma \cap \mathtt{U}$;
($\mathtt{U}_{\to}$) the judgment $x{:}\mathtt{U} \vdash^{\mathcal{T}}_{\mathcal{R}} \langle x\,, x^{\sigma\to\mathtt{U}}\rangle : \mathtt{U} \cap (\sigma \to \mathtt{U})$ is translated to a coercion-free judgment $x{:}\mathtt{U} \vdash^{\mathcal{T}}_{=_{\beta\eta}} \langle x\,, (\lambda f{:}\mathtt{U}.\lambda y{:}\sigma.u_{(f\,y)})\, x\rangle : \mathtt{U} \cap (\sigma \to \mathtt{U})$;
($\to\cap$) the judgment $x{:}(\sigma\to\tau)\cap(\sigma\to\rho) \vdash^{\mathcal{T}}_{\mathcal{R}} x^{\sigma\to\tau\cap\rho} : \sigma \to \tau\cap\rho$ is translated to a coercion-free judgment $x{:}(\sigma\to\tau)\cap(\sigma\to\rho) \vdash^{\mathcal{T}}_{=_{\beta\eta}} (\lambda f{:}(\sigma\to\tau)\cap(\sigma\to\rho).\lambda y{:}\sigma.\langle (pr_1\, f)\, y\,, (pr_2\, f)\, y\rangle)\, x : \sigma \to \tau\cap\rho$;
($\to$) the judgment $x{:}\sigma\to\tau\cap\rho \vdash^{\mathcal{T}}_{\mathcal{R}} \langle x\,, x^{\sigma\cap\rho\to\tau}\rangle : (\sigma\to\tau\cap\rho)\cap(\sigma\cap\rho\to\tau)$ is translated to a coercion-free judgment $x{:}\sigma\to\tau\cap\rho \vdash^{\mathcal{T}}_{=_{\beta\eta}} \langle x\,, (\lambda f{:}\sigma\to\tau\cap\rho.\lambda y{:}\sigma\cap\rho.pr_1\,(f\,(pr_1\,y)))\, x\rangle : (\sigma\to\tau\cap\rho)\cap(\sigma\cap\rho\to\tau)$;
(trans) the judgment $x{:}\sigma \vdash^{\mathcal{T}}_{\mathcal{R}} \langle x\,, (x^{\mathtt{U}})^{\sigma\to\mathtt{U}}\rangle : \sigma \cap (\sigma \to \mathtt{U})$ is translated to a coercion-free judgment $x{:}\sigma \vdash^{\mathcal{T}}_{=_{\beta\eta}} \langle x\,, (\lambda f{:}\mathtt{U}.\lambda y{:}\sigma.u_{(f\,y)})\,((\lambda y{:}\sigma.u_y)\, x)\rangle : \sigma \cap (\sigma \to \mathtt{U})$.

The next definition introduces two maps translating subtype judgments into explicit coercions functions and Δ-terms into coercion-free Δ-terms.

▶ **Definition 39** (Translations $\|-\|$ and $\|-\|_B$).

1. *The minimal type theory $\leqslant_{min}$ and the extra axioms and schemes are translated as follows.*

$$\begin{array}{lrcl}
(\text{refl}) & \|\sigma \leqslant_{\mathcal{T}} \sigma\| & \stackrel{def}{=} & \vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\sigma.x : \sigma \to \sigma \\
(\text{incl}_1) & \|\sigma\cap\tau \leqslant_{\mathcal{T}} \sigma\| & \stackrel{def}{=} & \vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\sigma\cap\tau.pr_1\, x : \sigma\cap\tau \to \sigma \\
(\text{incl}_2) & \|\sigma\cap\tau \leqslant_{\mathcal{T}} \tau\| & \stackrel{def}{=} & \vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\sigma\cap\tau.pr_2\, x : \sigma\cap\tau \to \tau \\
(\text{glb}) & \left\|\dfrac{\rho \leqslant_{\mathcal{T}} \sigma \quad \rho \leqslant_{\mathcal{T}} \tau}{\rho \leqslant_{\mathcal{T}} \sigma\cap\tau}\right\| & \stackrel{def}{=} & \vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\rho.\langle \|\rho \leqslant_{\mathcal{T}} \sigma\|\, x\,, \|\rho \leqslant_{\mathcal{T}} \tau\|\, x\rangle : \rho \to \sigma\cap\tau \\
(\text{trans}) & \left\|\dfrac{\sigma \leqslant_{\mathcal{T}} \tau \quad \tau \leqslant_{\mathcal{T}} \rho}{\sigma \leqslant_{\mathcal{T}} \rho}\right\| & \stackrel{def}{=} & \vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\sigma.\, \|\tau \leqslant_{\mathcal{T}} \rho\|\, (\|\sigma \leqslant_{\mathcal{T}} \tau\|\, x) : \sigma \to \rho \\
(\mathtt{U}_{top}) & \|\sigma \leqslant_{\mathcal{T}} \mathtt{U}\| & \stackrel{def}{=} & \vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\sigma.u_x : \sigma \to \mathtt{U}
\end{array}$$

$$(\mathtt{U}_{\to}) \qquad \|\mathtt{U} \leqslant_{\mathcal{T}} \sigma \to \mathtt{U}\| \stackrel{def}{=} \vdash^{\mathcal{T}}_{=_{\beta\eta}} \lambda f{:}\mathtt{U}.\lambda x{:}\sigma.u_{(f\,x)} : \mathtt{U} \to (\sigma \to \mathtt{U})$$

Let $\xi_1 \stackrel{def}{=} (\sigma\to\tau)\cap(\sigma\to\rho)$ and $\xi_2 \stackrel{def}{=} \sigma \to \tau\cap\rho$

$$(\to\cap) \qquad \|\xi_1 \leqslant_{\mathcal{T}} \xi_2\| \stackrel{def}{=} \vdash^{\mathcal{T}}_{=_{\beta\eta}} \lambda f{:}\xi_1.\lambda x{:}\sigma.\langle (pr_1\, f)\, x\,, (pr_2\, f)\, x\rangle : \xi_1 \to \xi_2$$

Let $\xi_1 \stackrel{def}{=} \sigma_1 \to \tau_1$ and $\xi_2 \stackrel{def}{=} \sigma_2 \to \tau_2$

$$(\to) \quad \left\|\dfrac{\sigma_2 \leqslant_{\mathcal{T}} \sigma_1 \quad \tau_1 \leqslant_{\mathcal{T}} \tau_2}{\sigma_1 \to \tau_1 \leqslant_{\mathcal{T}} \sigma_2 \to \tau_2}\right\| \stackrel{def}{=} \vdash^{\mathcal{T}}_{=_{\beta\eta}} \lambda f{:}\xi_1.\lambda x{:}\sigma_2.\, \|\tau_1 \leqslant_{\mathcal{T}} \tau_2\|\, (f\, (\|\sigma_2 \leqslant_{\mathcal{T}} \sigma_1\|\, x)) : \xi_1 \to \xi_2$$

2. *The translation $\|-\|_B$ is defined on Δ as follows.*

$$\begin{array}{rclcrcl}
\|u_\Delta\|_B & \stackrel{def}{=} & u_{\|\Delta\|_B} & & \|x\|_B & \stackrel{def}{=} & x \\
\|\lambda x{:}\sigma.\Delta\|_B & \stackrel{def}{=} & \lambda x{:}\sigma.\, \|\Delta\|_{B,x:\sigma} & & \|\Delta_1\,\Delta_2\|_B & \stackrel{def}{=} & \|\Delta_1\|_B\, \|\Delta_2\|_B \\
\|\langle\Delta_1\,, \Delta_2\rangle\|_B & \stackrel{def}{=} & \langle \|\Delta_1\|_B\,, \|\Delta_2\|_B\rangle & & \|pr_i\,\Delta\|_B & \stackrel{def}{=} & pr_i\, \|\Delta\|_B \qquad i \in \{1,2\} \\
\|\Delta^\tau\|_B & \stackrel{def}{=} & \|\sigma \leqslant_{\mathcal{T}} \tau\|\, \|\Delta\|_B & & \multicolumn{3}{l}{\textit{if } B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma.}
\end{array}$$

By looking at the above translation functions we can see that if $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, then $\|\Delta\|_B$ is defined and it is coercion-free.

The following key lemma states that a coercion function is always typable in $\Delta^{\mathcal{T}}_{=_{\beta\eta}}$, that it is essentially the identity and that, without using the rule schemes $(\to\cap)$, $(\mathtt{U}_\to)$, and $(\to)$ the translation can even be derivable in $\Delta^{\mathcal{T}}_{=_\beta}$.

▶ **Lemma 40** (Essence of a coercion is an identity)**.**
1. *If $\sigma \leqslant_{\mathcal{T}} \tau$, then $\vdash^{\mathcal{T}}_{=_{\beta\eta}} \|\sigma \leqslant_{\mathcal{T}} \tau\| : \sigma \to \tau$ and $\wr\, \|\sigma \leqslant_{\mathcal{T}} \tau\| \,\wr =_{\beta\eta} \lambda x.x$;*
2. *If $\sigma \leqslant_{\mathcal{T}} \tau$ without using the rule schemes $(\to\cap)$, $(\mathtt{U}_\to)$, and $(\to)$, then $\vdash^{\mathcal{T}}_{=_\beta} \|\sigma \leqslant_{\mathcal{T}} \tau\| : \sigma \to \tau$ and $\wr\, \|\sigma \leqslant_{\mathcal{T}} \tau\| \,\wr =_\beta \lambda x.x$.*

Proof. The proofs proceed in both parts by induction on the derivation of $\sigma \leqslant_{\mathcal{T}} \tau$. For instance, in case of (glb), we can verify that $\vdash^{\mathcal{T}}_{=_\beta} \lambda x{:}\rho.\langle \|\rho \leqslant_{\mathcal{T}} \sigma\|\, x\,, \|\rho \leqslant_{\mathcal{T}} \tau\|\, x\rangle : \rho \to \sigma \cap \tau$ using the induction hypotheses that $\|\rho \leqslant_{\mathcal{T}} \sigma\|$ (resp. $\|\rho \leqslant_{\mathcal{T}} \tau\|$) has type $\rho \to \sigma$ (resp. $\rho \to \tau$) and has an essence convertible to $\lambda x.x$. ◀

We can now prove the coherence of the translation as follows.

▶ **Theorem 41** (Coherence)**.** *If $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, then $B \vdash^{\mathcal{T}}_{\mathcal{R}'} \|\Delta\|_B : \sigma$ and $\wr\, \|\Delta\|_B \,\wr\, \mathcal{R}' \,\wr\, \Delta \,\wr$, where $\Delta^{\mathcal{T}}_{\mathcal{R}}$ and $\Delta^{\mathcal{T}}_{\mathcal{R}'}$ are respectively the source and target intersection typed systems given in Figure 4 (right part).*

Proof. By induction on the derivation. We illustrate the most important case, namely when the last type rule is $(\leqslant_{\mathcal{T}})$. In this case $\|\Delta^\tau\|_B$ is translated to $\|\sigma \leqslant_{\mathcal{T}} \tau\|\ \|\Delta\|_B$. By induction hypothesis we have that $B \vdash^{\mathcal{T}}_{\mathcal{R}} \Delta : \sigma$, and by Lemma 40 we have that $B \vdash^{\mathcal{T}}_{\mathcal{R}'} \|\sigma \leqslant_{\mathcal{T}} \tau\| : \sigma \to \tau$; therefore $B \vdash^{\mathcal{T}}_{\mathcal{R}'} \|\Delta^\tau\|_B : \tau$. Moreover, we know that $\wr\, \|\sigma \leqslant_{\mathcal{T}} \tau\| \,\wr\, \mathcal{R}'\ \lambda x.x$, and this gives $\wr\, \|\Delta^\tau\|_B \,\wr\, \mathcal{R}' \,\wr\, \|\Delta\|_B \,\wr$. Again by induction hypothesis we have that $\wr\, \|\Delta\|_B \,\wr\, \mathcal{R}' \,\wr\, \Delta \,\wr$, and this gives the thesis $\wr\, \|\Delta^\tau\|_B \,\wr\, \mathcal{R}' \,\wr\, \Delta^\tau \,\wr$. ◀

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