Criticality of Regular Formulas

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Abstract

We define the **criticality** of a boolean function $f : \{0,1\}^n \to \{0,1\}$ as the minimum real number $\lambda \geq 1$ such that

$$\mathbb{P}\left[\ \mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p) \geq t\ \right] \leq (p\lambda)^t$$

for all $p \in [0,1]$ and $t \in \mathbb{N}$, where $\boldsymbol{R}_p$ is the p-random restriction and $\mathsf{DT}_{\mathsf{depth}}$ is decision-tree depth. Criticality is a useful parameter: it implies an $O(2^{(1-\frac{1}{2\lambda})n})$ bound on the decision-tree size of f, as well as a $2^{-\Omega(k/\lambda)}$ bound on Fourier weight of f on coefficients of size $\geq k$.

In an unpublished manuscript [11], the author showed that a combination of Håstad's switching and multi-switching lemmas [5, 6] implies that AC^0 circuits of depth $d+1$ and size s have criticality at most $O(\log s)^d$. In the present paper, we establish a stronger $O(\frac{1}{d}\log s)^d$ bound for **regular formulas**: the class of AC^0 formulas in which all gates at any given depth have the same fan-in. This result is based on

(i) a novel switching lemma for bounded size (unbounded width) DNF formulas, and

(ii) an extension of (i) which analyzes a canonical decision tree associated with an entire depth-d formula.

As corollaries of our criticality bound, we obtain an improved #SAT algorithm and tight Linial-Mansour-Nisan Theorem for regular formulas, strengthening previous results for AC^0 circuits due to Impagliazzo, Matthews, Paturi [7] and Tal [17]. As a further corollary, we increase from $o(\frac{\log n}{\log\log n})$ to $o(\log n)$ the number of quantifier alternations for which the QBF-SAT (quantified boolean formula satisfiability) algorithm of Santhanam and Williams [14] beats exhaustive search.

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1 Introduction

For a boolean function f, we consider the random variable $\mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p)$ (the decision-tree depth of f under a p-random restriction) parameterized by $p \in [0,1]$. For every f, there is a sufficient small value of $p > 0$ such that $\mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p)$ satisfies an exponential tail bound. This "sufficiently small" is quantified by the following notion of *criticality*.

▶ **Definition 1.** *For $\lambda \in \mathbb{R}_{\geq 1}$, we say that a boolean function f is* λ-critical *if*

$$\mathbb{P}\left[\ \mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p) \geq t\ \right] \leq (p\lambda)^t$$

for all $p \in [0,1]$ and $t \in \mathbb{N}$. The criticality *of f is the minimum $\lambda \in \mathbb{R}_{\geq 1}$ for which f is λ-critical.*

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Criticality has been implicitly studied in previous works, although we are unaware if this parameter of boolean functions has been named before. Most notably, Håstad's switching lemma [5] is equivalent to the statement that every width-w CNF or DNF formula is $O(w)$-critical. Motivating our study of criticality is the observation that upper bounds on criticality imply upper bounds on decision-tree size.

▶ **Theorem 2.** *If $f : \{0,1\}^n \to \{0,1\}$ is λ-critical, then $\mathsf{DT}_{\mathsf{size}}(f)$ is at most $O(2^{(1-\frac{1}{2\lambda})n})$.*

In light of Theorem 2, it is reasonable to expect upper bounds on the criticality of a class of boolean functions to yield (randomized) #SAT algorithms.

In an unpublished manuscript [11], we observed that a combination of Håstad's switching lemma [5] and "multi-swithing lemma" [6] can be used to show that AC^0 circuits of depth $d+1$ and size s have criticality $O(\log s)^d$. Via Theorem 2, this implies an essentially tight upper bound on the decision-tree size of AC^0 circuits and yields a randomized #SAT algorithm with parameters matching that of Impagliazzo, Matthews and Paturi [7] for AC^0 circuits of super-linear size $n^{1+\Omega(1)}$. In the present paper, we improve these results by giving a quantitatively stronger upper bound on the criticality of regular AC^0 formulas, where *regular* means that all gates at the same height have equal fan-in.

▶ **Theorem 3.** *Regular AC^0 formulas of depth $d+1$ and size s have criticality $O(\frac{1}{d}\log s)^d$ (specifically, at most $60^d(\frac{1}{d}\ln s + 1)^d$).*

Theorem 3 unifies (and arguably simplifies) several of the main results on AC^0 circuits, including bounds on decision-tree size and the Fourier spectrum and #SAT algorithms. In addition, by obtaining quantitative stronger versions of these results for regular AC^0 formulas, we improve an algorithm of Santhanam and Williams [14] for satisfiability of quantified boolean formulas with bounded-many quantifier blocks.

1.1 Known bounds on criticality

The following bounds on criticality are immediate or known from previous work.

(1) If f is a boolean function which depends on n variables, then it is n-critical. This follows from

$$\mathbb{P}\left[\ \mathsf{DT}_{\mathsf{depth}}(f{\restriction}\boldsymbol{R}_p) \geq t\ \right] \leq \mathbb{P}\left[\ \mathbf{Bin}(n,p) \geq t\ \right] \leq p^t\binom{n}{t} \leq (pn)^t.$$

(2) If f has decision-tree depth k, then f is k-critical. This follows from the folklore bound: for all $t \geq 1$,

$$\mathbb{P}\left[\ \mathsf{DT}_{\mathsf{depth}}(f{\restriction}\boldsymbol{R}_p) \geq t\ \right] \leq 2^{t-1}p^t\binom{k}{t} \leq \frac{2^{t-1}}{t!}(pk)^t \leq (pk)^t.$$

(The first inequality is shown by induction on t.)

(3) Showing that (1) and (2) are tight: the n-variable parity function has criticality exactly $\lambda = n$. This follows from

$$np(1-p)^{n-1} = \mathbb{P}\left[\ \mathbf{Bin}(n,p) = 1\ \right] \leq \mathbb{P}\left[\ \mathsf{DT}_{\mathsf{depth}}(\mathrm{PARITY}_n{\restriction}\boldsymbol{R}_p) \geq 1\ \right] \leq p\lambda.$$

Therefore, $\lambda \geq n(1-p)^{n-1}$ for all $p \in (0,1]$, hence $\lambda \geq n$.

(4) Håstad's switching lemma [5] shows that every width-w CNF or DNF formula is $O(w)$-critical.

(5) An alternative switching lemma in [11] (included in Section 4 of this paper) shows that every size-m CNF or DNF formula is $O(\log m)$-critical.

(6) By a combination of Hastad's switching lemma [5] and multi-switching lemma [6], it is shown in [11] that every boolean function f computable by an AC^0 circuit of depth $d+1$ and size s is $O(\log s)^d$-critical. The switching lemma is used to show

$$\mathbb{P}\left[\, \mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p) \geq t \,\right] \leq \left(p \cdot O(\log s)^d\right)^t$$

for $t \leq \log s$, while the multi-switching lemma establishes this inequality for $t > \log s$.

1.2 Formulas vs. circuits

Every AC^0 circuit of depth $d+1$ and size s is equivalent to a regular AC^0 formula of depth $d+1$ and size at most s^d. Theorem 3 therefore implies the $O(\log s)^d$ criticality bound for AC^0 circuits.

The proof of our quantitatively stronger $O(\frac{1}{d}\log s)^d$ bound for regular AC^0 formulas is based on a novel "depth-d switching lemma". Previous switching lemmas analyze the so-called *canonical decision tree* of bounded-width depth-2 formula under a random restriction. In contrast, we analyze a certain canonical decision tree associated with a depth-d formula under a sequence of random restrictions. The bound we obtain is in terms of *top fan-in*, as opposed to *width* (i.e., bottom fan-in of a depth-2 formula).

While our proof of Theorem 3 relies on the assumption of regularity, we conjecture that the $O(\frac{1}{d}\log s)^d$ criticality bound applies to all AC^0 formulas. In this connection, let us mention that a previous result of the author [12] implies that every boolean function f computed by a (not necessarily regular) AC^0 formula of depth $d+1$ and size s satisfies

$$\mathbb{P}\left[\, \mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p) \geq t \,\right] \leq \left(p \cdot O(\tfrac{1}{d}\log s)^d\right)^t$$

for all $t \leq \log s$. The method of [12] applies Håstad's switching lemma to AC^0 formulas in a more efficient way. However, this method encounters the same $\log s$ barrier mentioned in §1.1(6). We do not know how to establish criticality for non-regular AC^0 formulas by proving the above inequality for $t > \log s$.

1.3 Outline of the paper

In Section 2 we state the definitions of AC^0 formulas, restrictions, decision trees, and present some key inequalities. Section 3 gives some results on criticality, including a proof of Theorem 2. In Section 4 we show that size-m DNF formulas have criticality $O(\log m)$ via a novel switching lemma argument. (This section and Appendix A are independent of the rest of the paper, but serve a warm-up for the more complicated switching lemmas that follow.) In Section 5, we introduce a *canonical decision tree* associated with an entire depth-d formula under a chain of restrictions. Sections 6 and 7 prove switching lemmas for this notion of canonical decision tree. Section 8 contains the proof of Theorem 3. Section 9 discusses satisfiability algorithms. The paper concludes with some open questions in Section 10.

2 Preliminaries

$\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \dots\}$. For $n \in \mathbb{N}$, $[n]$ is the set $\{1, \dots, n\}$ (in particular, $[0]$ is the empty set). $\ln(\cdot)$ is the natural logarithm and $\log(\cdot)$ is the base-2 logarithm. We consistently use boldface for random objects.

Throughout this paper, we fix an arbitrary set V whose elements we call *variable indices*. Without loss of generality, $V = [n]$; however, since the nature and number of variable indices plays no role in our switching lemma, we prefer to think of V as an abstract set. (The only time we assume $V = [n]$ is when speaking of boolean functions $f : \{0,1\}^n \to \{0,1\}$ in §3.)

▶ **Definition 4** (AC^0 formulas). *A* depth-0 formula *is a constant* 0 *or* 1 *or a literal* X_v *or* $\overline{X}_v$ *where* v *is a variable index. For* $d \geq 1$*, a* depth-d formula *is a syntactic object of the form* $\mathrm{OR}(F_1, \ldots, F_m)$ *or* $\mathrm{AND}(F_1, \ldots, F_m)$ *where* $m \geq 1$ *and* $F_1, \ldots, F_m$ *are depth* $d-1$ *formulas.*

We measure size *of a formula by the number of depth-*1 *subformulas. Formally,*

$$\mathrm{size}(F) := \begin{cases} 0 & \text{if } F \text{ has depth } 0, \\ 1 & \text{if } F \text{ has depth } 1, \\ \sum_{\ell=1}^{m} \mathrm{size}(F_\ell) & \text{if } F \text{ has depth} \geq 2 \text{ and is the OR or AND of } F_1, \ldots, F_m. \end{cases}$$

Up to a constant factor, size is equivalent to the number of gates in F*.*

The (syntactic) support *of a formula is the set of variable indices* v *such that the literal* X_v *or* $\overline{X}_v$ *occurs as a depth-*0 *subformula. Throughout this paper, all definitions and proofs* by induction *are, first, with respect to depth, and second, with respect to support size.*

If F *is a formula, we write* $F \equiv 0$ *(resp.* $F \equiv 1$*) if* F *computes the constant* 0 *function (resp. the constant* 1 *function).*

A depth-d *formula is* regular *if there exist integers* $m_2, \ldots, m_d \geq 1$ *such that, for all* $i \in \{2, \ldots, d\}$*, every depth* i *subformula has top fan-in* m_i*. Note that such a formula has size* $\prod_{i=2}^{d} m_i$*.*

▶ **Definition 5** (Restrictions and inputs). *A* restriction *is a partial function* ϱ *from* V *to* $\{0,1\}$*, viewed as a subset of* $V \times \{0,1\}$*, whose elements we denote by* $v \mapsto b$*. We write* $\mathrm{Dom}(\varrho)$ *for the domain of* ϱ*, and we write* $\mathrm{Stars}(\varrho)$ *for the set* $V \setminus \mathrm{Dom}(\varrho)$ *of "unrestricted" variable indices.*

An input *is a restriction with domain* V *(i.e., a total function from* V *to* $\{0,1\}$*, as opposed to a string in* $\{0,1\}^{|V|}$*).*

Two restrictions ϱ *and* σ *are* consistent *(we also say that* σ *is* ϱ-consistent*) if* $\varrho(v) = \sigma(v)$ *for all* $v \in \mathrm{Dom}(\varrho) \cap \mathrm{Dom}(\sigma)$*. In this case, the union* $\varrho \cup \sigma$ *is a restriction. We say that* σ *is a* refinement *of* ϱ *if* $\varrho \subseteq \sigma$ *(i.e.,* σ *extends* ϱ *by fixing additional variables).*

If F *is a formula and* ϱ *is a restriction, we denote by* $F\restriction_\varrho$ *the formula obtained from* F *by relabeling literals according to* ϱ *(we do not perform any simplification to* F*). Formally, we have the induction definition:*

$$F\restriction_\varrho = \begin{cases} F & \text{if } F \text{ is a constant or a literal } X_v \text{ or } \overline{X}_v \text{ where } v \in \mathrm{Stars}(\varrho), \\ 0 & \text{if } F \text{ is } X_v \text{ and } \varrho(v) = 0, \text{ or } F \text{ is } \overline{X}_v \text{ and } \varrho(v) = 1, \\ 1 & \text{if } F \text{ is } X_v \text{ and } \varrho(v) = 1, \text{ or } F \text{ is } \overline{X}_v \text{ and } \varrho(v) = 0, \\ \mathrm{OR/AND}(F_1\restriction_\varrho, \ldots, F_m\restriction_\varrho) & \text{if } F \text{ is } \mathrm{OR/AND}(F_1, \ldots, F_m). \end{cases}$$

Note that the support of $F\restriction_\varrho$ *equals the support of* F *minus the domain of* ϱ*.*

For $p \in [0,1]$*, the* p-random restriction $\boldsymbol{R}_p$ *is the random restriction which independent maps each variable index* v *to* 0 *or* 1 *with probability* $\frac{1-p}{2}$*, or leaves* v *unrestricted with probability* p*. For a restriction* ϱ*, a* p-random refinement *of* ϱ *is the random restriction* $\boldsymbol{R}_p$ *conditioned on being an extension of* ϱ *(i.e., conditioned on* $\varrho \subseteq \boldsymbol{R}_p$*).*

▶ **Definition 6** (Sequences and bitstrings). *For integers $s, t \in \mathbb{N}$ and arbitrary sequences $\alpha = \langle \alpha_1, \ldots, \alpha_s \rangle$ and $\beta = \langle \beta_1, \ldots, \beta_t \rangle$, we write $\alpha \circ \beta$ for the concatenated sequence $\langle \alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t \rangle$. The unique sequence of length 0 is denoted by $\langle\rangle$. (Note that $\langle\rangle$ is the identity with respect to concatenation.)*

We refer to sequences $a = \langle a_1, \ldots, a_s \rangle$ in the set $\{0,1\}^s$ as bitstrings. *(To avoid confusion, we regard inputs to formulas as total functions $V \to \{0,1\}$ rather than as ordered bitstrings in the set $\{0,1\}^{|V|}$.) For $t \geq s$, we write $\{0,1\}^t_s$ for the set of bitstrings $q = \langle q_1, \ldots, q_t \rangle \in \{0,1\}^t$ such that $q_1 + \cdots + q_t = s$. For bitstrings $a \in \{0,1\}^s$ and $b \in \{0,1\}^t$ and $q \in \{0,1\}^t_s$, we write $b \leftarrow^q a$ for the bitstring $\langle c_1, \ldots, c_t \rangle$ defined by*

$$c_j := \begin{cases} b_j & \text{if } q_j = 0, \\ a_i & \text{if } q_j = 1 \text{ and } q_1 + \cdots + q_{j-1} = i \text{ (i.e., } q_j \text{ is the } i^{th} \text{ 1-coordinate of } q). \end{cases}$$

That is, $b \leftarrow^q a$ overwrites b with a in the indices specified by q.

▶ **Definition 7** (Ordered restrictions). *An* ordered restriction *is a sequence $\beta = \langle v_1 \mapsto b_1, \ldots, v_t \mapsto b_t \rangle$ where $t \in \mathbb{N}$ and each $v_i \mapsto b_i$ is an ordered pairs with $v_i \in V$ and $b_i \in \{0,1\}$ such that $v_1, \ldots, v_t$ are distinct. As a matter of notation, we sometimes identify β with its underlying (unordered) restriction $\{v_1 \mapsto b_1, \ldots, v_t \mapsto b_t\}$, for instance, by writing $\mathrm{Dom}(\beta)$ for $\{v_1, \ldots, v_t\}$ or $F|_\beta$ for $F|_{\{v_1 \mapsto b_1, \ldots, v_t \mapsto b_t\}}$.*

For an ordered restriction $\beta = \langle v_1 \mapsto b_1, \ldots, v_t \mapsto b_t \rangle$ and a set of variable indices $S \subseteq V$ and a bitstring $a = \langle a_1, \ldots, a_s \rangle \in \{0,1\}^{|\mathrm{Dom}(\beta) \cap S|}$, we write $\beta \leftarrow_S a$ for the ordered restriction $\langle v_1 \mapsto c_1, \ldots, v_t \mapsto c_t \rangle$ where

$$c_j := \begin{cases} b_j & \text{if } v_j \notin S, \\ a_i & \text{if } v_j \text{ is the } i^{th} \text{ variable of } \mathrm{Dom}(\beta) \cap S \text{ in the order given by } \beta. \end{cases}$$

In other words, $\langle c_1, \ldots, c_t \rangle = \langle b_1, \ldots, b_t \rangle \leftarrow^q a$ where $q \in \{0,1\}^t_s$ is the bitstring defined by $q_j = 1 :\Leftrightarrow v_j \in S$.

▶ **Definition 8** (Decision trees). *A* decision tree *is a finite rooted binary tree T in which each non-leaf is labeled by a variable index v and the two edges to its children are labeled by 0 and 1. We require the variable indices on any root-to-leaf branch are distinct; each root-to-leaf branch therefore corresponds to an ordered restriction. We measure* size *by the number of leaves and* depth *by the maximum number of non-leaves on a root-to-leaf path.*

We say that a decision tree T determines *a boolean function f if the restricted function $f|_\alpha$ is constant for each ordered restriction α corresponding to a branch of T. (We might also say that T "computes" f, if we regard T as having output values on leaves.) The* decision-tree size *(resp.* decision-tree depth*) of a boolean function f, denoted $\mathsf{DT}_{\mathsf{size}}(f)$ (resp. $\mathsf{DT}_{\mathsf{depth}}(f)$), is the minimum size (resp. depth) of a decision tree that determines f.*

Later on, it will be convenient to identify decision trees with the set of ordered restrictions corresponding to branches. From this perspective, a decision tree is a nonempty set T^* of ordered restrictions such that, for all $\langle v_1 \mapsto a_1, \ldots, v_s \mapsto a_s \rangle \in T^*$ and $i \in [s]$,

- *if $\langle v_1 \mapsto a_1, \ldots, v_{i-1} \mapsto a_{i-1}, v'_i \mapsto a'_i \rangle$ is an initial subsequence of any element of T^*, then $v'_i = v_i$,*
- *$\langle v_1 \mapsto a_1, \ldots, v_{i-1} \mapsto a_{i-1}, v_i \mapsto 1 - a_i \rangle$ is an initial subsequence of some element of T^*.*

2.1 Inequalities

▶ **Lemma 9.** *For every integer $s \geq 1$ and $\varepsilon \in (0,1]$,*

$$\sum_{t=s}^{\infty}(1-\varepsilon)^t\binom{t-1}{s-1} = \left(\frac{1-\varepsilon}{\varepsilon}\right)^s.$$

Proof. Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be independent Bernoulli(ε) random variables. Then

$$1 = \mathbb{P}\left[\sum_{i=1}^{\infty}\boldsymbol{X}_i \geq s\right] = \sum_{t=s}^{\infty}\mathbb{P}\left[\boldsymbol{X}_t = 1 \text{ and } \sum_{i=1}^{t-1}\boldsymbol{X}_i = s-1\right] = \sum_{t=s}^{\infty}\varepsilon^s(1-\varepsilon)^{t-s}\binom{t-1}{s-1}.$$

The identity follows by multiplying both sides by $((1-\varepsilon)/\varepsilon)^s$. ◀

The next inequality also comes up in the $\mathrm{AC}^0[\oplus]$ formula lower bound of Rossman and Srinivasan [13].

▶ **Lemma 10.** *For all real numbers $a, b, c \geq 0$,*

$$\left(\frac{a}{c}+1\right)^c(b+1) \leq \left(\frac{a+b}{c+1}+1\right)^{c+1}.$$

Proof. The lemma is trivial if $c = 0$ (under the convention that $(\frac{a}{0}+1)^0 = 1$), so assume $c > 0$. Let $f(a,b,c) := \mathrm{RHS} - \mathrm{LHS}$. Then

$$\frac{\partial}{\partial b}f(a,b,c) = \left(\frac{a+b}{c+1}+1\right)^c - \left(\frac{a}{c}+1\right)^c.$$

Note that this is an increasing function of b with a zero at $b = a/c$. Therefore, $f(a,b,c)$ is minimal at $b = a/c$ where it takes value $f(a, a/c, c) = 0$. ◀

As a corollary, we get:

▶ **Lemma 11.** *For all integers $d, m_1, \ldots, m_d \geq 1$,*

$$\prod_{i=1}^{d}(\ln m_i + 1) \leq \left(\frac{1}{d}\ln\left(\prod_{i=1}^{d} m_i\right) + 1\right)^d.$$

Proof. For each $j \in [d-1]$, Lemma 10 implies

$$\left(\frac{1}{j-1}\ln\left(\prod_{i=1}^{j-1} m_i\right) + 1\right)^{j-1}(\ln m_j + 1) \leq \left(\frac{1}{j}\ln\left(\prod_{i=1}^{j} m_i\right) + 1\right)^j.$$

The lemma follows from these $d-1$ inequalities. ◀

The final inequality of this section plays a key role in our switching lemma analysis.

▶ **Lemma 12.** *Let I be a finite set and let $\mu : I \to [0,1]$ be a function such that $\sum_{i\in I}\mu(i) \leq 1$. Then for every function $t : I \to \mathbb{R}_{\geq 1}$ and $s \in \mathbb{R}_{\geq 1}$,*

$$\sum_{i\in I}\left(\frac{t(i)}{s}\right)^s\mu(i) \leq \left(\frac{1}{s}\ln\left(\sum_{i\in I}e^{t(i)}\mu(i)\right) + 1\right)^s.$$

Proof. Let $\lambda := \sum_{i\in I} \mu(i)$ ($\in [0,1]$) and let $\nu(i) := \mu(i)/\lambda$ ($\geq \mu(i)$). We have

$$\sum_{i\in I} \left(\frac{t(i)}{s}\right)^s \mu(i) = \lambda \sum_{i\in I} \left(\frac{t(i)}{s}\right)^s \nu(i) \leq \lambda \sum_{i\in I} \left(\frac{1}{s}\ln(e^{t(i)}) + 1\right)^s \nu(i)$$
$$\leq \lambda \left(\frac{1}{s}\ln\left(\sum_{i\in I} e^{t(i)}\nu(i)\right) + 1\right)^s$$

by Jensen's inequality since $a \mapsto (\frac{1}{s}\ln(a)+1)^s$ is concave over $a \in \mathbb{R}_{\geq 1}$

$$= \lambda \left(\frac{1}{s}\ln\left(\sum_{i\in I} e^{t(i)}\mu(i)\right) + 1 - \frac{1}{s}\ln(\lambda)\right)^s$$
$$\leq \left(\frac{1}{s}\ln\left(\sum_{i\in I} e^{t(i)}\mu(i)\right) + 1\right)^s$$

since $\lambda \mapsto \lambda(a - \frac{1}{s}\ln(\lambda))^s$ is increasing over $\lambda \in (0,1]$ (hence maximal at $\lambda = 1$) for every $a \in \mathbb{R}_{\geq 1}$. ◀

In the special case $t(i) = \ln(1/\mu(i))$ and $s = 1$, we get the inequality $\sum_{i\in I} \mu(i)\ln(1/\mu(i)) \leq \ln|I| + 1$. For distributions μ, this is essentially the inequality $\mathbb{H}(\mu) \leq \log|\mathrm{Support}(\mu)|$ for Shannon entropy; when μ is a sub-distribution, this inequality requires $+ O(1)$ on the righthand side.

3 Implications of Criticality

Before presenting our main result on regular AC^0 formulas, we give some general results on λ-critical functions. We begin with the following upper bound on decision-tree size, which is slightly stronger than Theorem 2.

▶ **Proposition 13.** *If $f : \{0,1\}^n \to \{0,1\}$ is $\frac{1}{2\varepsilon}$-critical, then* $\mathsf{DT}_{\mathsf{size}}(f) \leq 2^{n-\varepsilon n-\sqrt{\varepsilon n}+\log(\varepsilon n)+O(1)}$.

Proof. We first note that the proposition is trivial if $\varepsilon > \frac{1}{2}$, since no non-constant boolean function has criticality < 1. If $n < 10$ or $\varepsilon < \frac{2}{n}$, then the bound $\mathsf{DT}_{\mathsf{size}}(f) \leq 2^{n-\varepsilon n-\sqrt{\varepsilon n}+\log(\varepsilon n)+O(1)}$ follows from the trivial bound $\mathsf{DT}_{\mathsf{size}}(f) \leq 2^n$ by choosing a large enough constant. We may therefore assume that $n \geq 10$ and $\varepsilon \in [\frac{2}{n}, \frac{1}{2}]$.

Let $p := \varepsilon - \frac{1}{n}$ and note that $p \in [\frac{1}{n}, \frac{1}{2}]$. We have

$$\mathbb{E}\left[\,\mathsf{DT}_{\mathsf{size}}(f\restriction \boldsymbol{R}_p)\,\right] \leq \mathbb{E}\left[\,2^{\mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p)}\,\right] = \sum_{t=0}^{\infty} 2^t\, \mathbb{P}\left[\,\mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p) = t\,\right]$$
$$\leq \sum_{t=0}^{\infty} 2^t \left(\frac{p}{2\varepsilon}\right)^t = \sum_{t=0}^{\infty} \left(1 - \frac{1}{\varepsilon n}\right)^t = \varepsilon n.$$

We next make use of the following facts:

- $\mathbb{P}\left[\,\mathbf{Bin}(n,p) \geq pn + \sqrt{pn+1}\,\right] > 0.05$
  (this bound holds for all $n \geq 10$ and $p \in [\frac{1}{n}, \frac{1}{2}]$, as can be shown using estimates in [19]),
- $\mathsf{DT}_{\mathsf{size}}(f) \leq \sum_{\varrho: S\to\{0,1\}} \mathsf{DT}_{\mathsf{size}}(f\restriction_{\varrho})$ for every set of variable indices $S \subseteq [n]$.

Let $\boldsymbol{S}$ be a p-random subset of $[n]$ (i.e., a uniform random subset of size $\mathbf{Bin}(n,p)$), and let $\boldsymbol{\varrho}$ be a uniform random function $[n] \setminus \boldsymbol{S} \to \{0,1\}$. Note that $\boldsymbol{\varrho}$ is a p-random restriction.

For any $c > 0$, we have

$$\begin{aligned}
&\mathbb{1}[\, \mathsf{DT}_{\mathsf{size}}(f) > c2^{n-\varepsilon n} \,] \\
&\quad \le \mathop{\mathbb{P}}_{\boldsymbol{S}}\left[\, 2^{n-|\boldsymbol{S}|} \mathop{\mathbb{E}}_{\boldsymbol{\varrho}}[\, \mathsf{DT}_{\mathsf{size}}(f\restriction_{\boldsymbol{\varrho}}) \,] > c2^{n-\varepsilon n} \,\right] \\
&\quad \le \mathop{\mathbb{P}}_{\boldsymbol{S}}\left[\, \left(|\boldsymbol{S}| < pn + \sqrt{pn+1} \right) \text{ or } \left(\mathop{\mathbb{E}}_{\boldsymbol{\varrho}}[\, \mathsf{DT}_{\mathsf{size}}(f\restriction_{\boldsymbol{\varrho}}) \,] > c2^{pn+\sqrt{pn+1}-\varepsilon n} \right) \,\right] \\
&\quad \le \mathop{\mathbb{P}}_{\boldsymbol{S}}[\, |\boldsymbol{S}| < pn + \sqrt{pn+1} \,] + \mathop{\mathbb{P}}_{\boldsymbol{S}}\left[\, \mathop{\mathbb{E}}_{\boldsymbol{\varrho}}[\, \mathsf{DT}_{\mathsf{size}}(f\restriction_{\boldsymbol{\varrho}}) \,] > c2^{\sqrt{\varepsilon n}-1} \,\right] \\
&\quad \le \mathbb{P}[\, \mathbf{Bin}(n,p) < pn + \sqrt{pn+1} \,] + \frac{2}{c2^{\sqrt{\varepsilon n}}} \mathbb{E}[\, \mathsf{DT}_{\mathsf{size}}(f\restriction_{\boldsymbol{R}_p}) \,] \\
&\quad < 0.95 + \frac{2\varepsilon n}{c2^{\sqrt{\varepsilon n}}}.
\end{aligned}$$

Setting $c := 40\varepsilon n / 2^{\sqrt{\varepsilon n}}$, we have $\mathbb{1}[\, \mathsf{DT}_{\mathsf{size}}(f) > c2^{n-\varepsilon n} \,] < 1$. We conclude that

$$\mathsf{DT}_{\mathsf{size}}(f) \le c2^{n-\varepsilon n} = 40 \cdot 2^{n-\varepsilon n-\sqrt{\varepsilon n}+\log(\varepsilon n)}. \qquad \blacktriangleleft$$

The following theorem (which includes Theorem 2) lists several consequences of criticality, which follow from Proposition 13 as well as results of Linial, Mansour and Nisan [9] and Tal [17] relating the Fourier spectrum of a boolean function to its degree under a p-random restriction.

▶ **Theorem 14** (Implications of criticality). *If $f : \{0,1\}^n \to \{0,1\}$ is λ-critical, then*

(1) $\mathsf{DT}_{\mathsf{size}}(f) \le O(2^{(1-\frac{1}{2\lambda})n})$,

(2) *f agrees with* PARITY_n *on at most* $\frac{1}{2} + O(2^{-n/2\lambda})$ *fraction of inputs,*

(3) $\mathbb{P}[\, \mathsf{deg}(f\restriction_{\boldsymbol{R}_p}) \ge t \,] \le (p\lambda)^t$ *for all p and t,*

(4) $\sum_{S \subseteq [n] : |S| \ge k} \widehat{f}(S)^2 \le 2e \cdot e^{-k/\lambda}$ *for all k,*

(5) $\sum_{S \subseteq [n] : |S| = k} |\widehat{f}(S)| \le O(\lambda)^k$ *for all k.*

Proof. (1) follows immediately from Proposition 13. Property (2) is a consequence of (1). Property (3) follows from the definition of criticality and the fact that $\mathsf{deg}(\cdot) \le \mathsf{DT}_{\mathsf{depth}}(\cdot)$. Linial, Mansour and Nisan [9] showed that (3) ⇒ (4). Tal [17] showed that (4) ⇒ (5) (and moreover that (4) ⇒ (3), i.e., properties (3) and (4) are equivalent up to constant in the $O(\cdot)$). ◀

We conclude this section by observing that any exponential tail bound on $\mathsf{DT}_{\mathsf{depth}}(f\restriction_{\boldsymbol{R}_q})$ implies an upper bound on criticality.

▶ **Proposition 15.** *Let f be a boolean function, let $q, \varepsilon \in (0,1]$, and suppose $\mathbb{P}[\, \mathsf{DT}_{\mathsf{depth}}(f\restriction_{\boldsymbol{R}_q}) = t \,] \le (1-\varepsilon)^t$ for all $t \in \mathbb{N}$. Then f is $\frac{2}{\varepsilon q}$-critical.*

Proof. Let $0 \le p \le q$, let $\boldsymbol{\varrho}_1$ be a q-random restriction over the variables of f, and let $\boldsymbol{\varrho}_2$ be a p/q-random restriction over $\mathrm{Stars}(\boldsymbol{\varrho}_1)$. Then using §1.1(2) and Lemma 9, we have

$$
\begin{aligned}
&\mathbb{P}\left[\, \mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_p) \geq t \,\right] \\
&= \mathop{\mathbb{E}}_{\boldsymbol{\varrho}_1}\left[\, \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_2}\left[\, \mathsf{DT}_{\mathsf{depth}}((f\restriction_{\boldsymbol{\varrho}_1})\restriction_{\boldsymbol{\varrho}_2}) \geq t \,\right] \,\right] \\
&= \sum_{k=t}^{\infty} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1}\left[\, \mathsf{DT}_{\mathsf{depth}}(f\restriction_{\boldsymbol{\varrho}_1}) = k \,\right] \mathop{\mathbb{E}}_{\boldsymbol{\varrho}_1}\left[\, \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_2}\left[\, \mathsf{DT}_{\mathsf{depth}}((f\restriction_{\boldsymbol{\varrho}_1})\restriction_{\boldsymbol{\varrho}_2}) \geq t \,\right] \,\middle|\, \mathsf{DT}_{\mathsf{depth}}(f\restriction_{\boldsymbol{\varrho}_1}) = k \,\right] \\
&\leq \sum_{k=t}^{\infty} \mathbb{P}\left[\, \mathsf{DT}_{\mathsf{depth}}(f\restriction \boldsymbol{R}_q) = k \,\right] \max_{g\,:\,\mathsf{DT}_{\mathsf{depth}}(g)=k} \mathbb{P}\left[\, \mathsf{DT}_{\mathsf{depth}}(g\restriction \boldsymbol{R}_{p/q}) \geq t \,\right] \\
&\leq \sum_{k=t}^{\infty} (1-\varepsilon)^k \left(\frac{2p}{q}\right)^t \binom{k}{t} = \left(\frac{2p}{q}\right)^t \frac{(1-\varepsilon)^{t-1}}{\varepsilon^t} \leq \left(\frac{2p}{\varepsilon q}\right)^t .
\end{aligned}
$$

◀

4 Criticality of Size-m DNF Formulas

In this section, we show that size-m (unbounded width) DNF formulas have criticality $O(\log m)$ via a novel switching lemma. This switching lemma is based on convexity (it uses the inequality in Lemma 12). As a simple illustration of the underlying idea, in Appendix A we present a simple entropy argument showing that size-m DNF formulas have average sensitivity $O(\log m)$.

The switching lemma for DNF formulas in this section serves as a warm-up for more complicated switching lemmas for (sequences of) depth-d formulas in Sections 6 and 7. Those switching lemmas analyze a different construction of canonical decision trees. (Our result for DNF formulas is technically distinct from the depth-2 case of our depth-d switching lemma.)

Let us now fix a DNF formula $F = \mathrm{OR}(F_1, \ldots, F_m)$ where each term F_ℓ is an AND of literals. We identify each F_ℓ with an ordered restriction $\beta_\ell = \langle v_1 \mapsto b_1, \ldots, v_t \mapsto b_t \rangle$ corresponding to its unique minimal satisfying assignment, and we let $V_\ell = \mathrm{Dom}(\beta_\ell) = \{v_1, \ldots, v_t\}$. We say that a restriction ϱ *satisfies* F_ℓ if $\beta_\ell \subseteq \varrho$, and we say that ϱ *falsifies* F_ℓ if there exists $v \in V_\ell \cap \mathrm{Dom}(\varrho)$ such that $\beta_\ell(v) \neq \beta_\ell(\varrho)$.

For restrictions ϱ, we define the *canonical decision tree* $\mathcal{CDT}(F, \varrho)$ inductively as follows:

- If ϱ satisfies F_ℓ for any $\ell \in [m]$, or if ϱ falsifies F_ℓ for every $\ell \in [m]$, then $\mathcal{CDT}(F)$ is the trivial decision tree $\{\langle\rangle\}$.
- Otherwise, let $\ell \in [m]$ be the unique index such that ϱ falsifies $F_1, \ldots, F_{\ell-1}$ but not F_ℓ. Let $Q := V_\ell \cap \mathrm{Stars}(\varrho)$ and note that $|Q| \geq 1$. In this case, $\mathcal{CDT}(F, \varrho)$ queries all variables in Q, receives answers $\alpha : Q \to \{0,1\}$, and then proceeds as the decision tree $\mathcal{CDT}(F, \varrho \cup \alpha)$.

  Formally, if $\beta_\ell = \langle v_1 \mapsto b_1, \ldots, v_t \mapsto b_t \rangle$ and $Q = \{v_{i_1}, \ldots, v_{i_s}\}$ where $1 \leq i_1 < \cdots < i_s \leq t$, then we have

$$
\begin{aligned}
\mathcal{CDT}(F, \varrho \cup \alpha) := \{(v_{i_1} \mapsto a_1, \ldots, v_{i_s} \mapsto a_s) \circ \beta :& \\
a \in \{0,1\}^s,\ \beta \in \mathcal{CDT}(F, \varrho \cup \{v_{i_1} \mapsto a_1, \ldots, v_{i_s} \mapsto a_s\})\}&.
\end{aligned}
$$

Note that the decision tree $\mathcal{CDT}(F, \varrho)$ determines the function computed by $F\restriction_\varrho$.

▶ **Lemma 16.** *Suppose $\mathcal{CDT}(F, \varrho)$ has depth $s \geq 1$. Then there exist*

- *integers $r \in [s]$ and $s_1, \ldots, s_r \geq 1$ with $s_1 + \cdots + s_r = s$,*
- *integers $1 \leq \ell_1 < \cdots < \ell_r \leq m$,*
- *sets $Q_i \subseteq V_{\ell_i} \setminus (V_{\ell_1} \cup \cdots \cup V_{\ell_{i-1}})$ with $|Q_i| = s_i$ and restrictions $\alpha_i, \sigma_i : Q_i \to \{0,1\}$ for each $i \in [r]$*

such that, for all $i \in [r]$,

(i) $\varrho \cup \alpha_1 \cup \cdots \cup \alpha_{i-1}$ *falsifies* $F_{\ell'}$ *for all* $1 \le \ell' < \ell_i$,

(ii) $\varrho \cup \alpha_1 \cup \cdots \cup \alpha_{i-1} \cup \sigma_i$ *satisfies* F_{ℓ_i},

(iii) $Q_i = (V_{\ell_i} \setminus (V_{\ell_1} \cup \cdots \cup V_{\ell_{i-1}})) \cap \mathrm{Stars}(\varrho)$.

Proof. Straightforward from unpacking the inductive definition of $\mathcal{CDT}(F, \varrho)$. ◀

▶ **Lemma 17.** *Let* $F = \mathrm{OR}(F_1, \ldots, F_m)$ *be a DNF formula and let* $\boldsymbol{\varrho}$ *be a p-random restriction. Then*

$$\mathbb{P}[\ \mathcal{CDT}(F, \boldsymbol{\varrho}) \text{ has depth } s\ ] \le (8ep\log(em))^s.$$

Proof. By Lemma 16, we have

$$\begin{aligned}
&\mathop{\mathbb{P}}_{\boldsymbol{\varrho}}[\ \mathcal{CDT}(F, \boldsymbol{\varrho}) \text{ has depth } s\ ] \\
&\quad\le \sum_{r, s_1, \ldots, s_r, \ell_1, \ldots, \ell_r, Q_1, \ldots, Q_r, \alpha_1, \ldots, \alpha_r, \sigma_1, \ldots, \sigma_r} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}}[\ \text{(i),(ii),(iii) for all } i \in [r]\ ] \\
&\quad\le 2^s \max_{r, \vec{s}} \sum_{\vec{\ell}, \vec{Q}, \vec{\alpha}, \vec{\sigma}} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}}[\ \text{(i),(ii),(iii) for all } i \in [r]\ ].
\end{aligned}$$

The second inequality uses the fact that there are at most 2^s possibilities for data $(r, s_1, \ldots, s_r)$.

Let $\boldsymbol{x} : V \to \{0,1\}$ be a uniform random completion of $\boldsymbol{\varrho}$. For any restriction γ, let $\boldsymbol{x}^\gamma$ be the input where $\boldsymbol{x}^\gamma(v)$ equals $\gamma(v)$ if $v \in \mathrm{Dom}(\gamma)$ and $x(v)$ otherwise. For any $r, \vec{s}, \vec{\ell}, \vec{Q}, \vec{\alpha}, \vec{\sigma}$, note that

$$\begin{aligned}
\mathop{\mathbb{P}}_{\boldsymbol{\varrho}}[\ \text{(i),(ii),(iii) for all } i \in [r]\ ] &= 2^s \mathop{\mathbb{P}}_{\boldsymbol{\varrho}, \boldsymbol{x}}[\ \text{(i),(ii),(iii) for all } i \in [r] \text{ and } \sigma_1 \cup \cdots \cup \sigma_r \subseteq \boldsymbol{x}\ ] \\
&= 2^s \mathop{\mathbb{P}}_{\boldsymbol{\varrho}, \boldsymbol{x}}[\ \text{(i'),(ii'),(iii') for all } i \in [r]\ ] \\
&= (2p)^s (1-p)^{|V_1 \cup \cdots \cup V_{\ell_r}| - s} \mathop{\mathbb{P}}_{\boldsymbol{x}}[\ \text{(i'),(ii') for all } i \in [r]\ ] \\
&\le (2p)^s \mathop{\mathbb{P}}_{\boldsymbol{x}}[\ \text{(i'),(ii') for all } i \in [r]\ ]
\end{aligned}$$

where

(i') $\boldsymbol{x}^{\alpha_1 \cup \cdots \cup \alpha_{i-1}}$ falsifies $F_{\ell'}$ and $1 \le \ell' < \ell_i$,

(ii') $\boldsymbol{x}^{\alpha_1 \cup \cdots \cup \alpha_{i-1}}$ satisfies F_{ℓ_i},

(iii') $Q_i = (V_{\ell_i} \setminus (V_{\ell_1} \cup \cdots \cup V_{\ell_{i-1}})) \cap \mathrm{Stars}(\boldsymbol{\varrho})$.

Letting

$$\mu(\vec{\ell}, \vec{Q}, \vec{\alpha}) := \mathop{\mathbb{P}}_{\boldsymbol{x}}[\ \text{(i'),(ii') for all } i \in [r]\ ],$$

we have

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}}[\ \mathcal{CDT}(F, \boldsymbol{\varrho}) \text{ has depth } t\ ] \le (4p)^s \max_{r, \vec{s}} \sum_{\vec{\ell}, \vec{Q}, \vec{\alpha}} \mu(\vec{\ell}, \vec{Q}, \vec{\alpha}).$$

We next observe that, given any $\ell_1, \ldots, \ell_i$, there are $2^{s_i} \binom{|V_{\ell_i} \setminus (V_{\ell_1} \cup \cdots \cup V_{\ell_{i-1}})|}{s_i}$ choices

for (Q_i, α_i). Therefore,

$$\sum_{\vec{\ell},\vec{Q},\vec{\alpha}} \mu(\vec{\ell},\vec{Q},\vec{\alpha}) \le 2^s \sum_{\ell_1} \max_{Q_1,\alpha_1} \binom{|V_{\ell_1}|}{s_1} \sum_{\ell_2} \max_{Q_2,\alpha_2} \binom{|V_{\ell_2} \setminus V_{\ell_1}|}{s_2} \cdots$$
$$\cdots \sum_{\ell_r} \max_{Q_r,\alpha_r} \binom{|V_{\ell_2} \setminus (V_{\ell_1} \cup \cdots \cup V_{\ell_{r-1}})|}{s_r} \mu(\vec{\ell},\vec{Q},\vec{\alpha})$$
$$\le 2^s \sum_{\ell_1} \max_{Q_1,\alpha_1} \sum_{\ell_2} \max_{Q_2,\alpha_2} \sum_{\ell_r} \max_{Q_r,\alpha_r} \binom{|V_{\ell_1} \cup \cdots \cup V_{\ell_r}|}{s} \mu(\vec{\ell},\vec{Q},\vec{\alpha}).$$

We replace each $\sum \max$ with $\max \sum$ as follows. Let $Q_i^\star$ and $\alpha_i^\star$ range over functions $Q_i^\star(\ell_1,\dots,\ell_i) \in \binom{V_{\ell_i} \setminus (V_{\ell_1} \cup \cdots \cup V_{\ell_{i-1}})}{s_i}$ and $\alpha_i^\star(\ell_1,\dots,\ell_i) : Q_i^\star(\ell_1,\dots,\ell_i) \to \{0,1\}$, and let $\mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star)$ be short for

$$\mu(\langle \ell_1,\dots,\ell_r\rangle, \langle Q_1^\star(\ell_1),\dots,Q_r^\star(\ell_1,\dots,\ell_r)\rangle, \langle \alpha_1^\star(\ell_1),\dots,\alpha_r^\star(\ell_1,\dots,\ell_r)\rangle).$$

This allows us replace each $\sum_{\ell_1,\dots,\ell_i} \max_{Q_i,\alpha_i}$ with $\max_{Q_i^\star,\alpha_i^\star} \sum_{\ell_1,\dots,\ell_i}$ to obtain

$$\sum_{\vec{\ell},\vec{Q},\vec{\alpha}} \mu(\vec{\ell},\vec{Q},\vec{\alpha}) \le 2^s \max_{\vec{Q}^\star,\vec{\alpha}^\star} \sum_{\vec{\ell}} \binom{|V_{\ell_1} \cup \cdots \cup V_{\ell_r}|}{s} \mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star).$$

A key observation is that, for any given $\vec{Q}^\star$ and $\vec{\alpha}^\star$, we have $\sum_{\vec{\ell}} \mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star) \le 1$. To see why, note that each input x determines at most one sequence $\vec{\ell} = \langle \ell_1,\dots,\ell_r\rangle$ such that (i') and (ii') hold for all $i \in [r]$, that is, $x^{\alpha_1^\star(\ell_1) \cup \cdots \cup \alpha_{i-1}^\star(\ell_1,\dots,\ell_{i-1})}$ satisfies F_{ℓ_i} and falsifies $F_{\ell'}$ for all $\ell' < \ell_i$. Therefore, the events (over random $\boldsymbol{x}$) defining probabilities $\mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star)$ are mutually exclusive. We now have the following bound, using Lemma 12 for the last inequality:

$$\sum_{\vec{\ell}} \binom{|V_{\ell_1} \cup \cdots \cup V_{\ell_r}|}{s} \mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star) \le \sum_{\vec{\ell}} \left(\frac{e|V_{\ell_1} \cup \cdots \cup V_{\ell_r}|}{s}\right)^s \mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star)$$
$$= \left(\frac{e}{\ln(2)}\right)^s \sum_{\vec{\ell}} \left(\frac{\ln(2^{|V_{\ell_1} \cup \cdots \cup V_{\ell_r}|})}{s}\right)^s \mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star)$$
$$\le \left(\frac{e}{\ln(2)}\right)^s \left(\frac{1}{s} \ln\left(\sum_{\vec{\ell}} 2^{|V_{\ell_1} \cup \cdots \cup V_{\ell_r}|} \mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star)\right) + 1\right)^s.$$

A second key observation is that, for any $\vec{\ell},\vec{Q},\vec{\alpha}$, we have

$$\mu(\vec{\ell},\vec{Q},\vec{\alpha}) = \mathbb{P}[\text{ (i'),(ii') for all } i \in [r]\ ]$$
$$\le \mathbb{P}[\text{ (ii') for all } i \in [r]\ ]$$
$$= \begin{cases} (1/2)^{|V_{\ell_1} \cup \cdots \cup V_{\ell_r}|} & \text{if } \bigwedge_{i \in [r]} F_{\ell_i}\restriction_{\alpha_i \cup \cdots \cup \alpha_{i-1}} \text{ is satisfiable,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{\vec{\ell}} 2^{|V_{\ell_1} \cup \cdots \cup V_{\ell_r}|} \mu(\vec{\ell},\vec{Q}^\star,\vec{\alpha}^\star) \le \sum_{\vec{\ell}} 1 \le \binom{m}{r} \le m^r \le m^s.$$

Putting the pieces together, we conclude

$$\mathbb{P}_{\boldsymbol{\varrho}}[\ \mathcal{CDT}(F,\boldsymbol{\varrho}) \text{ has depth } s\ ] \le \left(\frac{8ep}{\ln(2)}\right)^s \left(\frac{1}{s}\ln(m^s) + 1\right)^s = \left(8ep\log(em)\right)^s. \qquad \blacktriangleleft$$

▶ **Corollary 18.** *Every size-m DNF formula has criticality at most* $16e\log(em)$.

Proof. Let F be a size-m DNF formula. Without loss of generality, let $t \geq 1$ and $0 < p \leq (16e\log(em))^{-1}$. Then

$$\begin{aligned}\mathop{\mathbb{P}}_{\boldsymbol{\varrho}}[\ \mathsf{DT}_{\mathsf{depth}}(F\restriction_{\boldsymbol{\varrho}}) \geq t\ ] &\leq \sum_{s=t}^{\infty} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}}[\ \mathcal{CDT}(F,\boldsymbol{\varrho}) \text{ has depth } s\ ] \\ &\leq \sum_{s=t}^{\infty} \Big(8ep\log(em)\Big)^{s} \leq \Big(16ep\log(em)\Big)^{t}.\end{aligned}$$

◀

5 The Canonical Decision Tree of a Depth-d Formula

In this section, we define the canonical decision tree of a depth-d formula F under a chain of restrictions $\varrho_1 \subseteq \cdots \subseteq \varrho_d$, denoted $\mathcal{T}^*_{\varrho_1,\ldots,\varrho_d}(F)$. The primary definition, however, is of a richer object $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$.

▶ **Definition 19.** *For every $d \in \mathbb{N}$ and chain of restrictions $\varrho_1 \subseteq \cdots \subseteq \varrho_d$ and depth-d formula F, we define a set of ordered restrictions $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ as follows. In the base case $d = 0$, let*

$$\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F) := \begin{cases}\{\langle\rangle\} & \text{if } F \text{ is a constant } 0 \text{ or } 1,\\ \{\langle v \mapsto 0\rangle, \langle v \mapsto 1\rangle\} & \text{if } F \text{ is a literal } X_v \text{ or } \overline{X}_v.\end{cases}$$

For $d \geq 1$, the definition is inductive. Suppose $F = \mathrm{OR}(F_1,\ldots,F_m)$ where each F_ℓ is a depth $d-1$ formula. Assume that $T_{\varrho_1,\ldots,\varrho_{d-1}}(F_\ell)$ is defined for all $\ell \in [m]$ and that $T_{\varrho_1,\ldots,\varrho_d}(F\restriction_\gamma)$ is defined for every restriction γ whose domain includes at least one variable index in the support of F. We consider three cases:

- **(i)** *If $F_1\restriction_{\varrho_d} \equiv \cdots \equiv F_m\restriction_{\varrho_d} \equiv 0$, then $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F) := \{\langle\rangle\}$.*
- **(ii)** *If $F_1\restriction_{\varrho_d} \equiv \cdots \equiv F_{\ell-1}\restriction_{\varrho_d} \equiv 0$ and $F_\ell\restriction_{\varrho_d} \equiv 1$, then $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F) := \{\langle\rangle\}$.*
- **(iii)** *If $F_1\restriction_{\varrho_d} \equiv \cdots \equiv F_{\ell-1}\restriction_{\varrho_d} \equiv 0$ and $F_\ell\restriction_{\varrho_d}$ computes a non-constant function, then $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ is the set of ordered restrictions $\langle v_1 \mapsto a_1, \ldots, v_u \mapsto a_u\rangle$ of length $u \geq 1$ such that there exist $t \in [u]$ and $b \in \{0,1\}^t$ satisfying*
  - *$\langle v_1 \mapsto b_1, \ldots, v_t \mapsto b_t\rangle \in \mathcal{T}_{\varrho_1,\ldots,\varrho_{d-1}}(F_\ell)$,*
  - *$\langle v_{t+1} \mapsto a_{t+1}, \ldots, v_u \mapsto a_u\rangle \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F\restriction_{\{v_1\mapsto a_1,\ldots,v_t\mapsto a_t\}})$, and*
  - *for all $i \in [t]$, if $v_i \in \mathrm{Dom}(\varrho_d)$, then $b_i = a_i = \varrho_d(v_i)$; and if $v_i \in \mathrm{Stars}(\varrho_d)$, then*

$$b_i = \begin{cases} a_i & \text{if } (F_\ell\restriction_{\varrho_d})\restriction_{\{v_1\mapsto b_1,\ldots,v_{i-1}\mapsto b_{i-1},\, v_i\mapsto a_i\}} \not\equiv 0,\\ 1-a_i & \text{if } (F_\ell\restriction_{\varrho_d})\restriction_{\{v_1\mapsto b_1,\ldots,v_{i-1}\mapsto b_{i-1},\, v_i\mapsto a_i\}} \equiv 0.\end{cases}$$

Finally, $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ is defined in the same way if $F = \mathrm{AND}(F_1,\ldots,F_m)$, but with the roles 0 and 1 exchanged.

▶ **Lemma 20.** *$\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ is nonempty and every $\alpha = \langle v_1 \mapsto a_1, \ldots, v_u \mapsto a_u\rangle \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ satisfies:*

- **(a)** *α is consistent with ϱ_d (i.e., for all $i \in [u]$, if $v_i \in \mathrm{Dom}(\varrho_d)$, then $a_i = \varrho_d(v_i)$),*
- **(b)** *the support of F contains $\mathrm{Dom}(\alpha)$ (i.e., for all $i \in [u]$, the literal X_{v_i} or $\overline{X}_{v_i}$ occurs as a depth-0 subformula of F),*
- **(c)** *for all $i \in [u]$, if $v_i \in \mathrm{Stars}(\varrho_d)$, then $\langle v_1 \mapsto a_1, \ldots, v_{i-1} \mapsto a_{i-1}, v_i \mapsto 1 - a_i\rangle$ is an initial subsequence of some element of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$,*

(d) *for all $i \in [u]$ and every variable index v'_i and bit $a'_i \in \{0,1\}$, if $\langle v_1 \mapsto a_1, \ldots, v_{i-1} \mapsto a_{i-1}, v'_i \mapsto a'_i\rangle$ is an initial subsequence of any element of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$, then $v'_i = v_i$,*

(e) *the function computed by $F\restriction_{\varrho_d \cup \alpha}$ is constant (i.e., $F\restriction_{\varrho_d\cup\alpha} \equiv 0$ or $F\restriction_{\varrho_d\cup\alpha} \equiv 1$).*

Proof. Though the proof is straightforward from Definition 19, we include full details. Note that the lemma is trivial when $d = 0$ as well as in cases (i) and (ii) when $d \geq 1$. So we assume that $F = \mathrm{OR}(F_1, \ldots, F_m)$ falls under case (iii), as witnessed by $\ell \in [m]$. By the induction hypothesis, we may assume that the lemma holds with respect to F_ℓ as well as $F\restriction_\gamma$ for every restriction γ whose domain includes at least one variable index in the support of F.

We first establish that $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ is nonempty. By the induction hypothesis, $\mathcal{T}_{\varrho_1,\ldots,\varrho_{d-1}}(F_\ell)$ is nonempty. Since $F_\ell\restriction_{\varrho_d}$ is non-constant and $F_\ell\restriction_{\varrho_d\cup\beta}$ is constant for every $\beta \in \mathcal{T}_{\varrho_1,\ldots,\varrho_{d-1}}(F_\ell)$, there exists $\beta \in \mathcal{T}_{\varrho_1,\ldots,\varrho_{d-1}}(F_\ell)$ such that $F_\ell\restriction_{\varrho_d\cup\beta} \equiv 1$. Let $\beta = \langle v_1 \mapsto b_1, \ldots, v_t \mapsto b_t\rangle$ and note that $t \geq 1$. For all $i \in [t]$, we have $(F_\ell\restriction_{\varrho_d})\restriction_{\{v_1\mapsto b_1,\ldots,v_i\mapsto b_i\}} \not\equiv 0$, since this is the same formula as $F_\ell\restriction_{\varrho_d\cup\{v_1\mapsto b_1,\ldots,v_i\mapsto b_i\}}$ by ϱ_d-consistency of β. By the definition of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ in case (iii), it follows that $\beta\circ\gamma \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ for every $\gamma \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F\restriction_\beta)$. Nonemptiness of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ therefore follows from nonemptiness of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F\restriction_\beta)$, which we know by the induction hypothesis applied to $F\restriction_\beta$ (noting that $\mathrm{Dom}(\beta)$ contains a variable index in the support of F, namely v_1, which is in the support of F_ℓ by the induction hypothesis applied to F_ℓ).

Now consider any $\alpha = \langle v_1 \mapsto a_1, \ldots, v_u \mapsto a_u\rangle \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$, as witnessed by some $t \in [u]$ and $b \in \{0,1\}^t$ in definition of case (iii). Let $\gamma := \{v_1 \mapsto a_1, \ldots, v_t \mapsto a_t\}$. We establish properties (a)–(e) in order.

(a): Suppose $i \in [u]$ and $v_i \in \mathrm{Dom}(\varrho_d)$. If $i \in \{1,\ldots,t\}$, then $a_i = \varrho_d(v_i)$ by definition of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ in case (iii). Otherwise, if $i \in \{t+1,\ldots,u\}$, then $a_i = \varrho_d(v_i)$ by property (a) with respect to $F\restriction_\gamma$.

(b): From the induction hypothesis, we know that $v_1, \ldots, v_t$ are in the support of F_ℓ (hence also the support of F) and that $v_{t+1}, \ldots, v_t$ are in the support of $F\restriction_\gamma$ (hence also the support of F).

(c): First note that $(F_\ell\restriction_{\varrho_d})\restriction_{\{v_1\mapsto b_1,\ldots,v_{i-1}\mapsto b_{i-1}\}} \not\equiv 0$ for all $i \in [t]$, as easily shown by induction on i. It then follows from the definition of case (iii) that for all $i \in [t]$, if $v_i \in \mathrm{Stars}(\varrho_d)$, then $\langle v_1 \mapsto a_1, \ldots, v_{i-1} \mapsto a_{i-1}, v_i \mapsto 1 - a_i\rangle$ is an initial subsequence of some element of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$. The same conclusion for all $i \in \{t+1,\ldots,u\}$ follows from property (c) with respect to $F\restriction_\gamma$.

(d): If $i \in [t]$ and $\langle v_1 \mapsto a_1, \ldots, v_{i-1} \mapsto a_{i-1}, v'_i \mapsto a'_i\rangle$ is an initial subsequence of an element of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$, then by definition of case (iii), $\langle v_1 \mapsto b_1, \ldots, v_{i-1} \mapsto b_{i-1}, v'_i \mapsto a'_i\rangle$ is initial subsequence of an element of $\mathcal{T}_{\varrho_1,\ldots,\varrho_{d-1}}(F_\ell)$ and therefore $v'_i = v_i$. For $i \in \{t+1,\ldots,u\}$, the conclusion follows from property (d) with respect to $F\restriction_\gamma$.

(e): Since $\langle v_{t+1} \mapsto a_{t+1}, \ldots, v_u \mapsto a_u\rangle \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F\restriction_\gamma)$, the formula $(F\restriction_\gamma)\restriction_{\varrho_d\cup\{v_{t+1}\mapsto a_{t+1},\ldots,v_u\mapsto a_u\}}$, which is the same formula as $F\restriction_{\varrho_d\cup\alpha}$ by ϱ_d-consistency of α, computes a constant function by property (e) with respect to $F\restriction_\gamma$. ◀

▶ **Definition 21.** *For $\alpha = \langle v_1 \mapsto a_1, \ldots, v_u \mapsto a_u\rangle \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$, let α^* denote the subsequence*

$$\alpha^* := \langle v_{i_1} \mapsto a_{i_1}, \ldots, v_{i_s} \mapsto a_{i_s}\rangle$$

for the unique $1 \leq i_1 < \cdots < i_s \leq u$ such that $\{v_{i_1}, \ldots, v_{i_s}\} = \mathrm{Dom}(\alpha) \cap \mathrm{Stars}(\varrho_d)$. Let

$$\mathcal{T}^*_{\varrho_1,\ldots,\varrho_d}(F) := \{\alpha^* : \alpha \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)\}.$$

▶ **Lemma 22.** *$\mathcal{T}^*_{\varrho_1,\ldots,\varrho_d}(F)$ is the set of branches of a decision tree determining $F\restriction_{\varrho_d}$. Moreover, each element of $\mathcal{T}^*_{\varrho_1,\ldots,\varrho_d}(F)$ is a subsequence of a unique element of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$.*

Proof. Straightforward from Lemma 20. ◀

▶ **Definition 23.** *We call $\mathcal{T}^*_{\varrho_1,\ldots,\varrho_d}(F)$ the* canonical decision tree of $F\restriction_{\varrho_d}$ under $\varrho_1,\ldots,\varrho_d$. *For a bitstring $a \in \{0,1\}^s$ and an ordered restriction α, we write "$\mathcal{T}^{(a)}_{\varrho_1,\ldots,\varrho_d}(F) = \alpha$" if $\alpha \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ and there exist variable indices $v_1,\ldots,v_s$ such that $\alpha^* = \langle v_1 \mapsto a_1, \ldots, v_s \mapsto a_s\rangle$. We say that "$\mathcal{T}^{(a)}_{\varrho_1,\ldots,\varrho_d}(F)$ exists" if $\mathcal{T}^{(a)}_{\varrho_1,\ldots,\varrho_d}(F) = \alpha$ for any $\alpha \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$.*

Note that, by Lemma 22, if $\mathcal{T}^{(a)}_{\varrho_1,\ldots,\varrho_d}(F)$ exists then $\mathcal{T}^{(a)}_{\varrho_1,\ldots,\varrho_d}(F) = \alpha$ for a unique $\alpha \in \mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ (justifying our use of the equality symbol). We may regard $\mathcal{T}^{(\cdot)}_{\varrho_1,\ldots,\varrho_d}(F)$ as a partial function from bitstrings to elements of the set $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$.

6 Depth-d Switching Lemma

In this section, we consider a depth-d formula $F = \mathrm{OR}(F_1,\ldots,F_m)$ and study the branches of $\mathcal{T}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d}(F)$ where $\boldsymbol{\varrho}_1 \subseteq \cdots \subseteq \boldsymbol{\varrho}_d$ is a chain of random restrictions where $\boldsymbol{\varrho}_d$ is a p-random refinement of $\boldsymbol{\varrho}_{d-1}$ and formulas $F_1,\ldots,F_m$ satisfy a certain hypothesis with respect to $\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}$. This allows us to bound the probability that $\mathcal{T}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d}(F)$ has an a-branch for any string $a \in \{0,1\}^s$. We refer to the main result of this section, Proposition 25, as the "depth-d switching lemma" since it analyzes the canonical decision tree of F in a similar manner as Håstad's switching lemma analyzes the canonical decision tree of a CNF or DNF formula.

Proposition 25 is in fact a special case of the slightly more general Proposition 27 ("serial depth-d switching lemma"), which we prove in the next section. The proofs are essentially the same, but with Proposition 25 we have fewer indices to keep track of. The next lemma unpacks the recursive definition of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ to obtain a more explicit characterization of its branches. This lemma associates $\mathcal{T}^{(a)}_{\varrho_1,\ldots,\varrho_d}(F)$, whenever this exists, with certain data $(r,\vec{s},\vec{\ell},\vec{t},\vec{b},\vec{q})$.

▶ **Lemma 24** (Unpacking $\mathcal{T}^{(a)}_{\varrho_1,\ldots,\varrho_d}(F)$)**.** *Let $F = \mathrm{OR}(F_1,\ldots,F_m)$ be a depth-d formula, let $\varrho_1 \subseteq \cdots \subseteq \varrho_d$ be restrictions, let $s \geq 1$, and let $a \in \{0,1\}^s$. If $\mathcal{T}^{(a)}_{\varrho_1,\ldots,\varrho_d}(F)$ exists, then there exist*

- *integers $r \in [s]$ and $s_1,\ldots,s_r \geq 1$ with $s_1 + \cdots + s_r = s$,*
- *integers $1 \leq \ell_1 < \cdots < \ell_r \leq m$,*
- *integers $t_i \geq s_i$ and bitstrings $b_i \in \{0,1\}^{t_i}$ and $q_i \in \{0,1\}^{t_i}_{s_i}$ for each $i \in [r]$*

with the property that there exist unique ordered restrictions $\beta_1,\ldots,\beta_r$ such that, for all $i \in [r]$,

- **(i)** *$(F_{\ell'}\restriction_{\gamma_i})\restriction_{\varrho_d} \equiv 0$ for all $1 \leq \ell' < \ell_i$,*
- **(ii)** *$(F_{\ell_i}\restriction_{\gamma_i})\restriction_{\varrho_d} \not\equiv 0$,*
- **(iii)** *$\mathcal{T}^{(b_i)}_{\varrho_1,\ldots,\varrho_{d-1}}(F_{\ell_i}\restriction_{\gamma_i}) = \beta_i$,*
- **(iv)** *β_i is ϱ_d-consistent and $(F_{\ell_i}\restriction_{\gamma_i})\restriction_{\varrho_{d-1}\cup\beta_i} \equiv 1$,*
- **(v)** *"q_i identifies $\mathrm{Stars}(\varrho_d)$ within $\mathrm{Dom}(\beta_i) \cap \mathrm{Stars}(\varrho_{d-1})$" in the following sense: for all $j \in [t_i]$,*

$$q_{i,j} = 1 \iff \mathrm{Stars}(\varrho_d) \text{ contains the } j^{th} \text{ variable of } \mathrm{Dom}(\beta_i) \cap \mathrm{Stars}(\varrho_{d-1}) \text{ in the order given by } \beta_i.$$

where
$$\gamma_i := (\beta_1 \circ \cdots \circ \beta_{i-1}) \leftarrow_{\mathrm{Stars}(\varrho_{d-1})} c_i,$$
$$c_i := (\, b_1 \circ \cdots \circ b_{i-1}\,) \leftarrow^{q_1 \circ \cdots \circ q_{i-1}} \langle a_1, \ldots, a_{s_1+\cdots+s_{i-1}} \rangle.$$

(Note that conditions (iii) and (v) imply that $|\mathrm{Dom}(\beta_i) \cap \mathrm{Stars}(\varrho_{d-1})| = t_i$ and $|\mathrm{Dom}(\beta_i) \cap \mathrm{Stars}(\varrho_d)| = s_i$ and $\gamma_i = (\beta_1 \circ \cdots \circ \beta_{i-1}) \leftarrow_{\mathrm{Stars}(\varrho_d)} \langle a_1, \ldots, a_{s_1+\cdots+s_{i-1}} \rangle$.)

Proof. Straightforward from Definition 19. ◀

▶ **Proposition 25** ("Depth-d switching lemma"). *Let $F = \mathrm{OR}(F_1, \ldots, F_m)$ be a depth-d formula. Suppose $\boldsymbol{\varrho}_1 \subseteq \cdots \subseteq \boldsymbol{\varrho}_{d-1}$ are random restrictions such that the following holds: for all integers $r \geq 1$ and $1 \leq \ell_1 < \cdots < \ell_r \leq m$ and $t_1, \ldots, t_r \geq 1$ and bitstrings $b_1, \ldots, b_r, c_1, \ldots, c_r$ where $b_i \in \{0,1\}^{t_i}$ and $c_i \in \{0,1\}^{t_1+\cdots+t_{i-1}}$,*

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}} \left[\exists \beta_1, \ldots, \beta_r \bigwedge_{i \in [r]} \left(\mathcal{T}^{(b_i)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}}(F_{\ell_i}\restriction_{(\beta_1 \circ \cdots \circ \beta_{i-1}) \leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_{d-1})} c_i}) = \beta_i \right) \right] \leq \left(\frac{1}{2e}\right)^{t_1+\cdots+t_r}.$$

Then for every integer $s \geq 1$ and bitstring $a \in \{0,1\}^s$ and $p \in [0,1]$, letting $\boldsymbol{\varrho}_d$ be a p-random refinement of $\boldsymbol{\varrho}_{d-1}$, we have

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \left[\mathcal{T}^{(a)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d}(F) \text{ exists } \right] \leq \big(4ep(\ln m + 1)\big)^s.$$

Proof. By Lemma 24 and a union bound,

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \left[\mathcal{T}^{(a)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d}(F) \text{ exists } \right]$$
$$\leq \sum_{\substack{r,s_1,\ldots,s_r \\ \ell_1,\ldots,\ell_r \\ t_1,\ldots,t_r \\ b_1,\ldots,b_r \\ q_1,\ldots,q_r}} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \left[\begin{array}{l} \exists \beta_1, \ldots, \beta_r \text{ such that, for all } i \in [r], \\ \quad \text{(i)} \ (F_{\ell'}\restriction_{\gamma_i})\restriction_{\boldsymbol{\varrho}_d} \equiv 0 \text{ for all } 1 \leq \ell' < \ell_i \\ \quad \text{(ii)} \ (F_{\ell_i}\restriction_{\gamma_i})\restriction_{\boldsymbol{\varrho}_d} \not\equiv 0 \\ \quad \text{(iii)} \ \mathcal{T}^{(b_i)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}}(F_{\ell_i}\restriction_{\gamma_i}) = \beta_i \\ \quad \text{(iv)} \ \beta_i \text{ is } \boldsymbol{\varrho}_d\text{-consistent and } (F_{\ell_i}\restriction_{\gamma_i})\restriction_{\boldsymbol{\varrho}_{d-1} \cup \beta_i} \equiv 1 \\ \quad \text{(v)} \ q_i \text{ identifies } \mathrm{Stars}(\boldsymbol{\varrho}_d) \text{ within } \mathrm{Dom}(\beta_i) \cap \mathrm{Stars}(\boldsymbol{\varrho}_{d-1}) \\ \text{where } \gamma_i := (\beta_1 \circ \cdots \circ \beta_{i-1}) \leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_{d-1})} c_i, \\ \qquad c_i := (\, b_1 \circ \cdots \circ b_{i-1}\,) \leftarrow^{q_1 \circ \cdots \circ q_{i-1}} \langle a_1, \ldots, a_{s_1+\cdots+s_{i-1}} \rangle \end{array} \right]$$
$$\leq 2^s \max_{r,s_1,\ldots,s_r} \sum_{\substack{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r, \\ b_1,\ldots,b_r,q_1,\ldots,q_r}} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \left[\exists \beta_1, \ldots, \beta_r \text{ such that (i)–(v) for all } i \in [r] \right].$$

Henceforth, we fix $r, s_1, \ldots, s_r$ and bound the sum over $\vec{\ell}, \vec{t}, \vec{b}, \vec{q}$.

Let $\boldsymbol{x}$ be a uniform random completion of $\boldsymbol{\varrho}_d$. For each choice of $\vec{\ell}, \vec{t}, \vec{b}, \vec{q}$, we have the following key sequence of (in)equalities, which we state below and justify afterwards:

$$
\begin{aligned}
&\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d}\Big[\ \exists \beta_1,\ldots,\beta_r \text{ such that (i)–(v) for all } i\in[r]\ \Big] \\
&\qquad = 2^s \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\Big[\ \exists \beta_1,\ldots,\beta_r \text{ such that (i)–(v) and } \beta_i\subseteq \boldsymbol{x} \text{ for all } i\in[r]\ \Big] \\
&\qquad \le 2^s \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\left[\begin{array}{l}
\exists \beta_1,\ldots,\beta_r \text{ such that, for all } i\in[r], \\
\quad \text{(i')}\ \ (F_{\ell'}\restriction_{\gamma_i})(\boldsymbol{x}) = 0 \text{ for all } 1\le \ell' < \ell_i \\
\quad \text{(ii')}\ \ (F_{\ell_i}\restriction_{\gamma_i})(\boldsymbol{x}) = 1 \\
\quad \text{(iii')}\ \ \mathcal{T}^{(b_i)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}}(F_{\ell_i}\restriction_{\gamma_i}) = \beta_i \\
\quad \text{(iv')}\ \ \beta_i \subseteq \boldsymbol{x} \\
\quad \text{(v')}\ \ q_i \text{ identifies } \mathrm{Stars}(\boldsymbol{\varrho}_d) \text{ within } \mathrm{Dom}(\beta_i)\cap \mathrm{Stars}(\boldsymbol{\varrho}_{d-1}) \\
\text{where } \gamma_i := (\beta_1\circ\cdots\circ\beta_{i-1}) \leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_{d-1})} c_i, \\
\qquad\qquad c_i := (\,b_1\circ\cdots\circ b_{i-1}\,) \leftarrow^{q_1\circ\cdots\circ q_{i-1}} \langle a_1,\ldots,a_{s_1+\cdots+s_{i-1}}\rangle
\end{array}\right] \\
&\qquad = (2p)^s(1-p)^{(t_1+\cdots+t_r-s)}\mu(\vec{\ell},\vec{t},\vec{b},\vec{q}) \\
&\qquad \le (2p)^s\mu(\vec{\ell},\vec{t},\vec{b},\vec{q})
\end{aligned}
$$

where

$$
\mu(\vec{\ell},\vec{t},\vec{b},\vec{q}) := \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\Big[\ \exists \beta_1,\ldots,\beta_r \text{ such that (i')–(iv') for all } i\in[r]\ \Big].
$$

The first equality follows from the independence of conditions (i)–(v) (which only depend on $\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d$) and the event that $(\beta_1\cup\cdots\cup\beta_r)\subseteq(\boldsymbol{x}\setminus\varrho_d)$ for any fixed ϱ_d in the support of $\boldsymbol{\varrho}_d$ (this event has probability 2^{-s} since $|\mathrm{Dom}(\beta_i)\cap\mathrm{Stars}(\varrho_d)| = s_i$ for each $i\in[r]$). The subsequent inequality follows from the observation that conditions (i)–(v) together with $(\beta_1\cup\cdots\cup\beta_r)\subseteq\boldsymbol{x}$ imply conditions (i')–(v'). The next equality follows from the independence of conditions (i')–(iv') and condition (v'). To see this, consider the following alternative way of generating $\boldsymbol{\varrho}_d$ and $\boldsymbol{x}$ given $\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}$: first generate $\boldsymbol{x}$ as a uniform random completion of $\boldsymbol{\varrho}_{d-1}$ (rather than of $\boldsymbol{\varrho}_d$), then obtain $\boldsymbol{\varrho}_d$ from $\boldsymbol{x}$ by randomly removing each pair $v\mapsto\boldsymbol{x}_v$ with $v\in\mathrm{Stars}(\boldsymbol{\varrho}_{d-1})$ independently with probability $1-p$. The independence of conditions (i')–(iv') and condition (v') is now seen by observing that the former only depends on $\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}$ and $\boldsymbol{x}$, while the latter only depends on $\boldsymbol{\varrho}_d$ (for any fixed $\varrho_1,\ldots,\varrho_{d-1},x$ in the support of $\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1},\boldsymbol{x}$). The probability of the latter event is precisely $p^s(1-p)^{(t_1+\cdots+t_r-s)}$ since for each $i\in[r]$, the set $\mathrm{Dom}(\beta_i)\cap\mathrm{Stars}(\varrho_{d-1})$ contains t_i variables, of which $\mathrm{Stars}(\boldsymbol{\varrho}_d)$ is required to include exactly the s_i variables specified by q_i.

Combining the above inequalities, we have

$$
\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d}\big[\ \mathcal{T}^{(a)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d}(F) \text{ exists}\ \big] \le (4p)^s \max_{r,s_1,\ldots,s_r}\ \sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r,q_1,\ldots,q_r} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}).
$$

We next turn to bounding both the individual probabilities $\mu(\vec{\ell},\vec{t},\vec{b},\vec{q})$ and their sum

$\sum_{\vec{\ell},\vec{t},\vec{b},\vec{q}} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q})$. Ignoring conditions (i') and (ii'), we have

$$\begin{aligned}
&\mu(\vec{\ell},\vec{t},\vec{b},\vec{q}) \\
&\leq \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\left[\ \exists \beta_1,\ldots,\beta_r \text{ such that (iii') and (iv') for all } i\in[r]\ \right] \\
&= \left(\frac{1}{2}\right)^{t_1+\cdots+t_r} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}}\left[\ \exists \beta_1,\ldots,\beta_r \text{ such that (iii') for all } i\in[r]\ \right] \\
&= \left(\frac{1}{2}\right)^{t_1+\cdots+t_r} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}}\left[\begin{array}{l}\exists \beta_1,\ldots,\beta_r \text{ such that, for all } i\in[r],\\ \mathcal{T}^{(b_{h,i})}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}}(F_{\ell_i}\restriction_{(\beta_1\circ\cdots\circ\beta_{i-1})\leftarrow \mathrm{Stars}(\boldsymbol{\varrho}_{d-1})c_i}) = \beta_i \\ \text{where } c_i := (b_1\circ\cdots\circ b_{i-1}) \leftarrow^{q_1\circ\cdots\circ q_{i-1}} \langle a_1,\ldots,a_{s_1+\cdots+s_{i-1}}\rangle\end{array}\right] \\
&\leq \left(\frac{1}{4e}\right)^{t_1+\cdots+t_r}.
\end{aligned}$$

The first equality follows from independence of conditions (iii') and (iv'). The second equality is a restatement of condition (iii'). The last inequality uses the hypothesis of the theorem concerning formulas $F_1,\ldots,F_m$.

We next bound the sum $\sum_{\vec{\ell},\vec{t},\vec{b},\vec{q}} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q})$. We start out by observing that

$$\sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r,q_1,\ldots,q_r} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\left[\ \exists \beta_1,\ldots,\beta_r \text{ such that (i')–(v') for all } i\in[r]\ \right] \leq 1,$$

since these events are mutually exclusive. To see why, consider any $\varrho_1,\ldots,\varrho_d,x$ in the support of $\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d,\boldsymbol{x}$ and notice that there is a unique process of uniquely determining $\vec{\ell},\vec{t},\vec{b},\vec{q}$ (if any exist) such that conditions (i')–(v') hold. First, we find the unique ℓ_1 (if any exists) such that $F_{\ell_1}(x)=1$ and $F_{\ell'}(x)=0$ for all $1\leq \ell'<\ell_1$ (note that $\gamma_1=\langle\rangle$). Next, let β_1 be the unique branch of $\mathcal{T}_{\varrho_1,\ldots,\varrho_{d-1}}(F_{\ell_1})$ consistent with x, let b_i be the sequence of answers to the queried variable indices on this branch, and let t_i be the length of b_1. If $F_{\ell_1}(x)=0$ or $|\mathrm{Dom}(\beta_1)\cap\mathrm{Stars}(\varrho_d)|\neq s_1$, then the process fails; otherwise, let $q_1\in\{0,1\}^{t_1}_{s_1}$ be the unique bitstring that identifies $\mathrm{Stars}(\varrho_d)$ within $\mathrm{Dom}(\beta_1)\cap\mathrm{Stars}(\varrho_{d-1})$. Having uniquely determined ℓ_1,t_1,b_1,q_1, the process continues by finding the unique ℓ_2 (if any exists) such that $(F_{\ell_2}\restriction_{\gamma_2})(x)=1$ and $(F_{\ell'}\restriction_{\gamma_2})(x)=0$ for all $1\leq\ell'<\ell_1$ (note that γ_2 is completed determined by previous data β_1,b_1,q_1). Continuing in this manner, we find unique t_2,b_2,q_2, etc.

Note that condition (v') uniquely determines bitstrings $q_1,\ldots,q_s$. This condition is omitted from the events in probabilities $\mu(\vec{\ell},\vec{t},\vec{b},\vec{q})$, which therefore are not mutually exclusive as the choice of $q_i\in\{0,1\}^{t_i}_{s_i}$ is now free. However, we can restore mutual exclusivity as follows. For each $i\in[r]$, let $q^\star_i$ range over functions associating each sequence of partial data $(\ell_1,t_1,b_1,\ldots,\ell_i,t_i,b_i)$ with an element $q^\star_i(\ell_1,t_1,b_1,\ldots,\ell_i,t_i,b_i)\in\{0,1\}^{t_i}_{s_i}$. Let

$$\mu(\vec{\ell},\vec{t},\vec{b},\vec{q^\star}) := \mu\Big(\vec{\ell},\vec{t},\vec{b},\langle q^\star_1(\ell_1,t_1,b_1),\ldots,q^\star_r(\ell_1,t_1,b_1,\ldots,\ell_r,t_r,b_r)\rangle\Big).$$

For any choice of $q^\star_1,\ldots,q^\star_r$, the events in probabilities $\mu(\vec{\ell},\vec{t},\vec{b},\vec{q^\star})$ are mutually exclusive over $\vec{\ell},\vec{t},\vec{b}$. (It is a subtle but important point that $q^\star_i$ is a function of $(\ell_1,t_1,b_1,\ldots,\ell_i,t_i,b_i)$ independent of any "future" data ℓ_j,t_j,b_j for $j>i$.) Therefore,

$$\sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q^\star}) \leq 1.$$

We now have the bound

$$\begin{aligned}
&\sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r,q_1,\ldots,q_r} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}) \\
&\qquad \le \sum_{\ell_1,t_1,b_1} \binom{t_1}{s_1} \max_{q_1} \cdots \sum_{\ell_r,t_r,b_r} \binom{t_r}{s_r} \max_{q_r} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}) \\
&\qquad = \max_{q_1^\star,\ldots,q_r^\star} \sum_{\ell_1,t_1,b_1} \binom{t_1}{s_1} \cdots \sum_{\ell_r,t_r,b_r} \binom{t_r}{s_r} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}^\star) \\
&\qquad \le \max_{q_1^\star,\ldots,q_r^\star} \sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r} \binom{t_1+\cdots+t_r}{s} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}^\star) \\
&\qquad \le \max_{q_1^\star,\ldots,q_r^\star} \sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r} e^s \left(\frac{t_1+\cdots+t_r}{s}\right)^s \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}^\star) \\
&\qquad \le \max_{q_1^\star,\ldots,q_r^\star} e^s \left(\frac{1}{s} \ln\left(\sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r} e^{(t_1+\cdots+t_r)} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}^\star)\right) + 1\right)^s
\end{aligned}$$

where the final inequality is by Lemma 12.

We next have $\sum_{\vec{\ell},\vec{t},\vec{b}} e^{(t_1+\cdots+t_r)} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}^\star) \le m^s$ as follows:

$$\begin{aligned}
\sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r} e^{(t_1+\cdots+t_r)} \mu(\vec{\ell},\vec{t},\vec{b},\vec{q}^\star) &\le \sum_{\ell_1,\ldots,\ell_r,t_1,\ldots,t_r,b_1,\ldots,b_r} \left(\frac{1}{4}\right)^{t_1+\cdots+t_r} \\
&\le \sum_{\ell_1,\ldots,\ell_r} \sum_{t=r}^{\infty} \left(\frac{1}{4}\right)^t \sum_{t_1,\ldots,t_r,b_1,\ldots,b_r \,:\, t_1+\cdots+t_r=t} 1 \\
&= \sum_{\ell_1,\ldots,\ell_r} \sum_{t=r}^{\infty} \left(\frac{1}{2}\right)^t \binom{t-1}{r-1} \\
&= \sum_{\ell_1,\ldots,\ell_r} 1 = \binom{m}{r} \le m^r \le m^s.
\end{aligned}$$

The first inequality uses our bound $\mu(\vec{\ell},\vec{t},\vec{b},\vec{q}) \le (1/8e)^{(t_1+\cdots+t_r)}$ (which holds for any $q_1,\ldots,q_r$ including $q_1^\star(\ell_1,t_1,b_1),\ldots,q_r^\star(\ell_1,t_1,b_1,\ldots,\ell_r,t_r,b_r)$). The first equality is due to the fact that there are $\binom{t-1}{r-1}$ choices for integers $t_1,\ldots,t_r \ge 1$ such that $t_1+\cdots+t_r = t$, and there are 2^{t_i} choices for each bitstring $b_i \in \{0,1\}^{t_i}$. The second equality uses Lemma 9. Finally, we use the fact that $r \le s$ since $s_1,\ldots,s_r \ge 1$ are integers such that $s_1+\cdots+s_r = s$.

Putting together these inequalities, we get the desired bound

$$\Pr_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \left[\, \mathcal{T}^{(a)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d}(F) \text{ exists } \right] \le \left(4ep(\ln m + 1)\right)^s. \qquad \blacktriangleleft$$

7 Serial Depth-d Switching Lemma

We would like to prove Theorem 3 (our upper bound on the criticality of regular AC^0 formulas) by applying Proposition 25 ("depth-d switching lemma") to each layer of a regular AC^0 formula. Unfortunately, there is a mismatch between the *hypothesis* and the *conclusion* of Proposition 25: the hypothesis applies to a sequence of depth $d-1$ formulas, while the conclusion applies to single depth-d formula (and cannot therefore serve as the hypothesis for a depth $d+1$ formula).

In this section, we prove an extension of Proposition 25 which we require for Theorem 3. We call this result, Proposition 27, the "serial depth-d switching lemma", since it explores the canonical decision trees of a sequence of depth-d formulas $F_1, \dots, F_k$ in order. For integers $s_1, \dots, s_k \geq 1$ and bitstrings $a_h \in \{0,1\}^{s_h}$ $(h \in [k])$, we would like to bound the event that $\mathcal{T}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d}(F_1)$ has an a_1-branch, call it α_1, and that $\mathcal{T}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d}(F_2{\restriction}_{\alpha_1})$ has an a_2-branch α_2, etc., and finally that $\mathcal{T}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d}(F_d{\restriction}_{\alpha_1\circ\cdots\circ\alpha_{k-1}})$ has an a_k-branch α_k. However, in order for the conclusion of Proposition 27 to match the hypothesis, we need to consider a more general event where, instead of considering $\mathcal{T}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d}(F_h{\restriction}_{\alpha_1\circ\cdots\circ\alpha_{h-1}})$ in the hth stage, we instead apply the restriction $(\alpha_1 \circ \cdots \circ \alpha_{h-1}) \leftarrow_{\mathrm{Stars}(\varrho_d)} c_h$ (overwriting $\alpha_1 \circ \cdots \circ \alpha_{h-1}$ on all previously queried variables) where $c_h \in \{0,1\}^{s_1+\cdots+s_{h-1}}$ is an arbitrary bitstring. Although this makes notation in Proposition 27 slightly more cumbersome, the probabilistic main argument is nearly identical to Proposition 25.

▶ **Notation 1.** In what follows, we will consider integers $k \geq 1$ and $r_1, \dots, r_k \geq 1$ and various indexed families $\vec{w} = \{w_{h,i}\}_{h\in[k],i\in[r_h]}$. It is often convenient to regard $\vec{w}$ as a sequence of length $r_1 + \cdots + r_k$:

$$\vec{w} = \langle w_{1,1}, \dots, w_{1,r_1}, \dots\dots, w_{h,1}, \dots, w_{h,r_k}\rangle.$$

For $h \in [k]$ and $i \in [r_k + 1]$, notation "$w_{1,1}, \dots, w_{h,i-1}$" shall refer to the initial subsequence of length $r_1 + \cdots + r_{h-1} + i - 1$:

$$\langle w_{1,1}, \dots, w_{h,i-1}\rangle = \langle w_{1,1}, \dots, w_{1,r_1}, \dots\dots, w_{h-1,1}, \dots, w_{h-1,r_{h-1}}, w_{h,i}, \dots, w_{h,i-1}\rangle.$$

For example, if $w_{h,i}$ are integers (or bitstrings, ordered restrictions, etc.), we will write "$w_{1,1} + \cdots + w_{h,i-1}$" (or "$w_{1,1} \circ \cdots \circ w_{h,i-1}$") for the sum (or composition) of the first $r_1 + \cdots + r_{h-1} + i - 1$ elements of $\vec{w}$.

The following lemma plays the same role in Proposition 27 as Lemma 24 does in Proposition 25.

▶ **Lemma 26.** *Let $F_1, \dots, F_k$ be depth-d formulas where $F_h = \mathrm{OR}(F_{h,1}, \dots, F_{h,m})$ for each $h \in [k]$. Let $\varrho_1 \subseteq \cdots \subseteq \varrho_d$ be restrictions. Let $s_1, \dots, s_k \geq 1$ and let $a_h \in \{0,1\}^{s_h}$ and $c_h \in \{0,1\}^{s_1+\cdots+s_{h-1}}$ for each $h \in [k]$. Suppose there exist ordered restrictions $\alpha_1, \dots, \alpha_k$ such that, for all $h \in [k]$,*

$$\mathcal{T}^{(a_h)}_{\varrho_1,\dots,\varrho_d}(F_h{\restriction}_{(\alpha_1\circ\cdots\circ\alpha_{h-1})\leftarrow_{\mathrm{Stars}(\varrho_d)}c_h}) = \alpha_h.$$

Then there exist

- *integers $r_h \in [s_h]$ and $s_{h,1}, \dots, s_{h,r_h} \geq 1$ with $s_{h,1} + \cdots + s_{h,r_h} = s_h$ for each $h \in [k]$,*
- *integers $1 \leq \ell_{h,1} < \cdots < \ell_{h,r_h} \leq m$ for each $h \in [k]$,*
- *integers $t_{h,i} \geq s_{h,i}$ and bitstrings $b_{h,i} \in \{0,1\}^{t_{h,i}}$ and $q_{h,i} \in \binom{\{0,1\}^{t_{h,i}}}{s_{h,i}}$ for each $h \in [k]$ and $i \in [r_h]$*

with the property that there exist unique ordered restrictions $\beta_{1,1}, \dots, \beta_{h,r_h}$ such that, for all $h \in [k]$ and $i \in [r_h]$,

- **(i)** $(F_{h,\ell'}{\restriction}_{\gamma_{h,i}}){\restriction}_{\varrho_d} \equiv 0$ *for all* $1 \leq \ell' < \ell_{h,i}$,
- **(ii)** $(F_{h,\ell_{h,i}}{\restriction}_{\gamma_{h,i}}){\restriction}_{\varrho_d} \not\equiv 0$,
- **(iii)** $\mathcal{T}^{(b_{h,i})}_{\varrho_1,\dots,\varrho_{d-1}}(F_{h,\ell_{h,i}}{\restriction}_{\gamma_{h,i}}) = \beta_{h,i}$,
- **(iv)** $\beta_{h,i}$ *is ϱ_d-consistent and* $(F_{h,\ell_{h,i}}{\restriction}_{\gamma_{h,i}}){\restriction}_{\varrho_{d-1}\cup\beta_{h,i}} \equiv 1$,
- **(v)** *"$q_{h,i}$ identifies* $\mathrm{Stars}(\varrho_d)$ *within* $\mathrm{Dom}(\beta_{h,i}) \cap \mathrm{Stars}(\varrho_{d-1})$*" in the following sense: for all $j \in [t_{h,i}]$, we have $q_{h,i,j} = 1 \iff \mathrm{Stars}(\varrho_d)$ contains the j^{th} variable of $\mathrm{Dom}(\beta_{h,i}) \cap \mathrm{Stars}(\varrho_{d-1})$ in the order given by $\beta_{h,i}$.*

where

$$\gamma_{h,i} := (\beta_{1,1} \circ \cdots \circ \beta_{h,i-1}) \leftarrow_{\mathrm{Stars}(\varrho_{d-1})} d_{h,i},$$
$$d_{h,i} := (\, b_{1,1} \circ \cdots \circ b_{h,i-1}\,) \leftarrow^{q_{1,1} \circ \cdots \circ q_{h,i-1}} (c_h \circ \langle a_{h,1}, \ldots, a_{h,s_{h,1}+\cdots+s_{h,i-1}} \rangle).$$

(Note that conditions (iii) and (v) imply that $|\mathrm{Dom}(\beta_{h,i}) \cap \mathrm{Stars}(\varrho_{d-1})| = t_{h,i}$ and $|\mathrm{Dom}(\beta_{h,i}) \cap \mathrm{Stars}(\varrho_d)| = s_{h,i}$ and $\gamma_{h,i} = (\beta_{1,1} \circ \cdots \circ \beta_{h,i-1}) \leftarrow_{\mathrm{Stars}(\varrho_d)} (c_h \circ \langle a_{h,1}, \ldots, a_{h,s_{h,1}+\cdots+s_{h,i-1}} \rangle)$.)

Proof. Straightforward from Definition 19. ◀

▶ **Proposition 27** ("Serial depth-d switching lemma"). *Let $k, m \geq 1$, let $F_1, \ldots, F_k$ be depth-d formulas where $F_h = \mathrm{OR}(F_{h,1}, \ldots, F_{h,m})$ for each $h \in [k]$. Suppose $\boldsymbol{\varrho}_1 \subseteq \cdots \subseteq \boldsymbol{\varrho}_{d-1}$ are random restrictions such that, for all $r_1, \ldots, r_k \geq 1$ and $1 \leq \ell_{h,1} < \cdots < \ell_{h,r_h} \leq m$ and $t_{h,i} \geq 1$ and $b_{h,i} \in \{0,1\}^{t_{h,i}}$ and $d_{h,i} \in \{0,1\}^{t_{1,1}+\cdots+t_{h,i-1}}$ ($h \in [k]$ and $i \in [r_h]$),*

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}} \Big[\, \exists \beta_{1,1}, \ldots, \beta_{k,r_k} \bigwedge_{\substack{h \in [k] \\ i \in [r_h]}} \Big(\mathcal{T}^{(b_{h,i})}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}} (F_{h,\ell_{h,i}} \restriction_{(\beta_{1,1} \circ \cdots \circ \beta_{h,i-1}) \leftarrow_{\mathrm{Stars}(\varrho_{d-1})} d_{h,i}}) = \beta_{h,i} \Big) \,\Big] \leq \Big(\frac{1}{2e}\Big)^{t_{1,1}+\cdots+t_{k,r_k}}.$$

Then for all integers $s_1, \ldots, s_k \geq 1$ and bitstrings $a_h \in \{0,1\}^{s_h}$ and $c_h \in \{0,1\}^{s_1+\cdots+s_{h-1}}$ ($h \in [k]$) and $p \in [0,1]$, letting $\boldsymbol{\varrho}_d$ be a p-random refinement of $\boldsymbol{\varrho}_{d-1}$, we have

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \Big[\, \exists \alpha_1, \ldots, \alpha_k \bigwedge_{h \in [k]} \Big(\mathcal{T}^{(a_h)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} (F_h \restriction_{(\alpha_1 \circ \cdots \circ \alpha_{h-1}) \leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_d)} c_h}) = \alpha_h \Big) \,\Big] \leq \Big(4ep(\ln m + 1)\Big)^{s_1+\cdots+s_k}.$$

Note that Proposition 25 is precisely the special case $k = 1$ of Proposition 27. The following proof closely parallels the proof of Proposition 25 (with a few more indices to keep track of). We omit the justifications of certain (in)equalities that would be redundant.

Proof. Fix $s_1, \ldots, s_k$ and $a_1, \ldots, a_k$ and $c_1, \ldots, c_k$ and p, and let $s := s_1 + \cdots + s_k$.

Let $\vec{r} = \{r_h\}$ and $\vec{s} = \{s_{h,i}\}$ and $\vec{\ell} = \{\ell_{h,i}\}$ and $\vec{t} = \{t_{h,i}\}$ and $\vec{b} = \{b_{h,i}\}$ and $\vec{q} = \{q_{h,i}\}$ (where $h \in [k]$ and $[i] \in [r_h]$) range over data satisfying the bullet items of Lemma 26. We have

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \Big[\, \exists \alpha_1, \ldots, \alpha_k \bigwedge_{h \in [k]} \Big(\mathcal{T}^{(a_h)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} (F_h \restriction_{(\alpha_1 \circ \cdots \circ \alpha_{h-1}) \leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_d)} c_h}) = \alpha_h \Big) \,\Big]$$

$$\leq \sum_{\vec{r},\vec{s},\vec{\ell},\vec{t},\vec{b},\vec{q}} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \left[\begin{array}{l} \exists \beta_{1,1}, \ldots, \beta_{k,r_k} \text{ such that, for all } h \in [k] \text{ and } i \in [r_h], \\ \quad \text{(i)} \;\; (F_{h,\ell'} \restriction_{\gamma_{h,i}}) \restriction_{\boldsymbol{\varrho}_d} \equiv 0 \text{ for all } 1 \leq \ell' < \ell_{h,i} \\ \quad \text{(ii)} \;\; (F_{h,\ell_{h,i}} \restriction_{\gamma_{h,i}}) \restriction_{\boldsymbol{\varrho}_d} \not\equiv 0 \\ \quad \text{(iii)} \;\; \mathcal{T}^{(b_{h,i})}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d-1}} (F_{h,\ell_{h,i}} \restriction_{\gamma_{h,i}}) = \beta_{h,i} \\ \quad \text{(iv)} \;\; \beta_{h,i} \text{ is } \boldsymbol{\varrho}_d\text{-consistent and } (F_{h,\ell_{h,i}} \restriction_{\gamma_{h,i}}) \restriction_{\boldsymbol{\varrho}_{d-1} \cup \beta_{h,i}} \equiv 1 \\ \quad \text{(v)} \;\; q_{h,i} \text{ identifies } \mathrm{Stars}(\boldsymbol{\varrho}_d) \text{ within } \mathrm{Dom}(\beta_{h,i}) \cap \mathrm{Stars}(\boldsymbol{\varrho}_{d-1}) \\ \text{where } \gamma_{h,i} := (\beta_{1,1} \circ \cdots \circ \beta_{h,i-1}) \leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_{d-1})} d_{h,i}, \\ \qquad\quad d_{h,i} := (\, b_{1,1} \circ \cdots \circ b_{h,i-1}\,) \leftarrow^{q_{1,1} \circ \cdots \circ q_{h,i-1}} \\ \qquad\qquad\qquad\qquad (c_h \circ \langle a_{h,1}, \ldots, a_{h,s_{h,1}+\cdots+s_{h,i-1}} \rangle) \end{array} \right]$$

$$\leq 2^s \max_{\vec{r},\vec{s}} \sum_{\vec{\ell},\vec{t},\vec{b},\vec{q}} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_d} \Big[\, \exists \beta_{1,1}, \ldots, \beta_{k,r_k} \text{ such that (i)–(v) for all } h \in [k] \text{ and } i \in [r_h] \,\Big].$$

Letting $\boldsymbol{x}$ be a uniform random completion of $\boldsymbol{\varrho}_d$, we have

$$\begin{aligned}
&\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d}\Big[\ \exists \beta_{1,1},\dots,\beta_{k,r_k} \text{ such that (i)–(v) for all } h\in[k] \text{ and } i\in[r_h]\ \Big]\\
&= 2^s \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\Big[\ \exists \beta_{1,1},\dots,\beta_{k,r_k} \text{ such that (i)–(v) and } \beta_{h,i}\subseteq\boldsymbol{x} \text{ for all } h\in[k],\, i\in[r_h]\ \Big]\\
&\le 2^s \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\left[\begin{array}{l}
\exists \beta_{1,1},\dots,\beta_{k,r_k} \text{ such that, for all } h\in[k] \text{ and } i\in[r_h],\\
\quad \text{(i')}\ \ (F_{h,\ell'}\restriction_{\gamma_{h,i}})(\boldsymbol{x})=0 \text{ for all } 1\le\ell'<\ell_{h,i}\\
\quad \text{(ii')}\ \ (F_{h,\ell_{h,i}}\restriction_{\gamma_{h,i}})\restriction(\boldsymbol{x})=1\\
\quad \text{(iii')}\ \ \mathcal{T}^{(b_{h,i})}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_{d-1}}(F_{h,\ell_{h,i}}\restriction_{\gamma_{h,i}})=\beta_{h,i}\\
\quad \text{(iv')}\ \ \beta_{h,i}\subseteq\boldsymbol{x}\\
\quad \text{(v')}\ \ q_{h,i} \text{ identifies } \mathrm{Stars}(\boldsymbol{\varrho}_d) \text{ within } \mathrm{Dom}(\beta_{h,i})\cap\mathrm{Stars}(\boldsymbol{\varrho}_{d-1})\\
\text{where } \gamma_{h,i}:=(\beta_{1,1}\circ\cdots\circ\beta_{h,i-1})\leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_{d-1})} d_{h,i},\\
\qquad d_{h,i}:=(\,b_{1,1}\circ\cdots\circ b_{h,i-1}\,)\leftarrow^{q_{1,1}\circ\cdots\circ q_{h,i-1}}\\
\qquad\qquad (c_h\circ\langle a_{h,1},\dots,a_{h,s_{h,1}+\cdots+s_{h,i-1}}\rangle)
\end{array}\right]\\
&= (2p)^s(1-p)^{(t_{1,1}+\cdots+t_{k,r}-s)}\mu(\vec{\ell},\vec{t},\vec{b},\vec{q})\\
&\le (2p)^s\mu(\vec{\ell},\vec{t},\vec{b},\vec{q})
\end{aligned}$$

where

$$\mu(\vec{\ell},\vec{t},\vec{b},\vec{q}):=\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\Big[\ \exists \beta_{1,1},\dots,\beta_{k,r_k} \text{ such that (i')–(iv') for all } h\in[k] \text{ and } i\in[r_h]\ \Big].$$

Ignoring conditions (i') and (ii'), we have

$$\begin{aligned}
&\mu(\vec{\ell},\vec{t},\vec{b},\vec{q})\\
&\le \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_d,\boldsymbol{x}}\Big[\ \exists \beta_{1,1},\dots,\beta_{k,r_k} \text{ such that (iii') and (iv') for all } h\in[k] \text{ and } i\in[r_h]\ \Big]\\
&= \left(\frac{1}{2}\right)^{t_{1,1}+\cdots+t_{k,r_k}} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_{d-1}}\Big[\ \exists \beta_{1,1},\dots,\beta_{k,r_k} \text{ such that (iii') for all } h\in[k] \text{ and } i\in[r_h]\ \Big]\\
&= \left(\frac{1}{2}\right)^{t_{1,1}+\cdots+t_{k,r_k}} \mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_{d-1}}\left[\begin{array}{l}
\exists \beta_{1,1},\dots,\beta_{k,r_k} \text{ such that, for all } h\in[k] \text{ and } i\in[r_h],\\
\mathcal{T}^{(b_{h,i})}_{\boldsymbol{\varrho}_1,\dots,\boldsymbol{\varrho}_{d-1}}(F_{h,\ell_{h,i}}\restriction_{(\beta_{1,1}\circ\cdots\circ\beta_{h,i-1})\leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_{d-1})}d_{h,i}})=\beta_{h,i} \text{ where}\\
d_{h,i}:=(\,b_{1,1}\circ\cdots\circ b_{h,i-1}\,)\leftarrow^{q_{1,1}\circ\cdots\circ q_{h,i-1}}\\
\qquad\qquad (c_h\circ\langle a_{h,1},\dots,a_{h,s_{h,1}+\cdots+s_{h,i-1}}\rangle)
\end{array}\right]\\
&\le \left(\frac{1}{4e}\right)^{t_{1,1}+\cdots+t_{k,r_k}}.
\end{aligned}$$

For each $h\in[k]$ and $i\in[r_h]$, let $q^\star_{h,i}$ range over functions associating each sequence $(\ell_{1,1},t_{1,1},b_{1,1},\dots,\ell_{h,i},t_{h,i},b_{h,i})$ with an element $q^\star_{h,i}(\ell_{1,1},t_{1,1},b_{1,1},\dots,\ell_{h,i},t_{h,i},b_{h,i})\in\{0,1\}^{t_{h,i}}_{s_{h,i}}$. Let

$$\mu(\vec{\ell},\vec{t},\vec{b},\vec{q}^\star):=\mu\Big(\vec{\ell},\vec{t},\vec{b},\langle q^\star_{1,1}(\ell_{1,1},t_{1,1},b_{1,1}),\dots,q^\star_{k,r_k}(\ell_{1,1},t_{1,1},b_{1,1},\dots,\ell_{k,r_k},t_{k,r_k},b_{k,r_k})\rangle\Big).$$

For every choice of $q^\star_{1,1},\dots,q^\star_{k,r_k}$, we have $\sum_{\vec\ell,\vec t,\vec b}\mu(\vec\ell,\vec t,\vec b,\vec{q^\star})\le 1$. Therefore, using Lemma 9,

$$\begin{aligned}\sum_{\vec\ell,\vec t,\vec b,\vec q}\mu(\vec\ell,\vec t,\vec b,\vec q) &\le \sum_{\ell_{1,1},t_{1,1},b_{1,1}}\binom{t_{1,1}}{s_{1,1}}\max_{q_{1,1}}\cdots\sum_{\ell_{k,r_k},t_{k,r_k},b_{k,r_k}}\binom{t_{k,r_k}}{s_{k,r_k}}\max_{q_{k,r_k}}\mu(\vec\ell,\vec t,\vec b,\vec q)\\ &= \max_{\vec{q^\star}}\sum_{\ell_{1,1},t_{1,1},b_{1,1}}\binom{t_{1,1}}{s_{1,1}}\cdots\sum_{\ell_{k,r_k},t_{k,r_k},b_{k,r_k}}\binom{t_{k,r_k}}{s_{k,r_k}}\mu(\vec\ell,\vec t,\vec b,\vec{q^\star})\\ &\le \max_{\vec{q^\star}}\sum_{\vec\ell,\vec t,\vec b}\binom{t_{1,1}+\cdots+t_{k,r_k}}{s}\mu(\vec\ell,\vec t,\vec b,\vec{q^\star})\\ &\le \max_{\vec{q^\star}} e^s\left(\frac1s\ln\left(\sum_{\vec\ell,\vec t,\vec b}e^{(t_{1,1}+\cdots+t_{k,r_k})}\mu(\vec\ell,\vec t,\vec b,\vec{q^\star})\right)+1\right)^s.\end{aligned}$$

Finally, we have

$$\begin{aligned}\sum_{\vec\ell,\vec t,\vec b}e^{(t_{1,1}+\cdots+t_{k,r_k})}\mu(\vec\ell,\vec t,\vec b,\vec{q^\star}) &\le \sum_{\vec\ell,\vec t,\vec b}\left(\frac14\right)^{t_{1,1}+\cdots+t_{k,r_k}}\\ &\le \sum_{\vec\ell}\sum_{t=r_1+\cdots+r_k}^{\infty}\left(\frac14\right)^t\sum_{\vec t,\vec b\colon t_{1,1}+\cdots+t_{k,r_k}=t}1\\ &= \sum_{\vec\ell}\sum_{t=r_1+\cdots+r_k}^{\infty}\left(\frac12\right)^t\binom{t-1}{r_1+\cdots+r_k-1}\\ &= \sum_{\vec\ell}1=\binom{m}{r_1}\cdots\binom{m}{r_k}\le m^{(r_1+\cdots+r_k)}\le m^s.\end{aligned}$$

Putting together these inequalities, we get the desired bound

$$\mathop{\mathbb{P}}_{\boldsymbol\varrho_1,\dots,\boldsymbol\varrho_d}\left[\ \exists\alpha_1,\dots,\alpha_k\bigwedge_{h\in[k]}\left(\mathcal{T}^{(a_h)}_{\boldsymbol\varrho_1,\dots,\boldsymbol\varrho_d}(F_h\restriction_{(\alpha_1\circ\cdots\circ\alpha_{h-1})\leftarrow\mathrm{Stars}(\boldsymbol\varrho_d)c_h})=\alpha_h\right)\ \right]\le\left(4ep(\ln m+1)\right)^s.\ \blacktriangleleft$$

8 Proof of Theorem 3

The following lemma is required for the analysis of depth-1 subformulas in the proof of Theorem 3.

▶ **Lemma 28.** *Let $\boldsymbol\varrho$ be a p-random restriction. Let $F_1,\dots,F_k$ be depth-1 formulas, let $s_1,\dots,s_k\ge1$, and let $a_h\in\{0,1\}^{s_h}$ and $c_h\in\{0,1\}^{s_1+\cdots+s_{h-1}}$ for each $h\in[k]$. Then*

$$\mathop{\mathbb{P}}_{\boldsymbol\varrho}\left[\ \exists\alpha_1,\dots,\alpha_k\bigwedge_{h\in[k]}\left(\mathcal{T}^{(a_h)}_{\boldsymbol\varrho}(F_h\restriction_{(\alpha_1\circ\cdots\circ\alpha_{h-1})\leftarrow\mathrm{Stars}(\boldsymbol\varrho)c_h})=\alpha_h\right)\ \right]\le(2p)^{s_1+\cdots+s_k}.$$

Proof. Assume $F_1\restriction_{\boldsymbol\varrho}$ computes a non-constant function, since otherwise $\mathcal{T}^{(a_1)}_{\boldsymbol\varrho}(F_1)$ does not exist (note that $\alpha_1=\langle\rangle$). Let V_1 be the set of variable indices occurring in literals of F_1. Observe that $\mathcal{T}^{(a_1)}_{\boldsymbol\varrho}(F_1)$ exists if, and only if, $|V_1\cap\mathrm{Stars}(\boldsymbol\varrho)|=s_1$ and $\boldsymbol\varrho$ gives the unique assignment to variables in $V_1\cap\mathrm{Dom}(\boldsymbol\varrho)$ such that $F_1\restriction_{\boldsymbol\varrho}$ is non-constant. This happens with probability $\binom{|V_1|}{s_1}p^{s_1}(\frac{1-p}{2})^{|V_1|-s_1}$, which is at most $(2p)^{s_1}$. Also note that this event only depends $\boldsymbol\varrho|_{V_1}$ (i.e., the partial function $\boldsymbol\varrho$ restricted to $\mathrm{Dom}(\boldsymbol\varrho)\cap V_1$). If this event holds, then we have $\mathcal{T}^{(a_1)}_{\boldsymbol\varrho}(F_1)=\alpha_1$ for the unique ordered restriction α_1 with $\mathrm{Dom}(\alpha_1)=V_1\cap\mathrm{Stars}(\boldsymbol\varrho)$ whose values are given by a_1.

Next let $F_2' := F_2\restriction_{\alpha_1 \leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho})} c_1}$ and assume that $F_2'\restriction_{\boldsymbol{\varrho}}$ computes a non-constant function, since otherwise $\mathcal{T}_{\boldsymbol{\varrho}}^{(a_2)}(F_2')$ does not exist. Let V_2 be the set of variable indices occurring in literals of F_2', and note that $V_1 \cap V_2 = \emptyset$. Observe that $\mathcal{T}_{\boldsymbol{\varrho}}^{(a_2)}(F_2')$ exists if, and only if, $|V_2 \cap \mathrm{Stars}(\boldsymbol{\varrho})| = s_2$ and $\boldsymbol{\varrho}$ gives the unique assignment to variables in $V_2 \cap \mathrm{Dom}(\boldsymbol{\varrho})$ such that $F_2'\restriction_{\boldsymbol{\varrho}}$ is non-constant. Conditioned on the value of $\boldsymbol{\varrho}|_{V_1}$, this second event happens with probability $\binom{|V_2|}{s_2} p^{s_2} (\frac{1-p}{2})^{|V_2|-s_2}$, which is at most $(2p)^{s_2}$. Moreover, this second event only depends on $\boldsymbol{\varrho}|_{V_2}$.

Continuing in this manner, we conclude that the event in question holds with probability at most $(2p)^{s_1+\cdots+s_k}$. ◀

Proof of Theorem 3. Let F be a regular formula of depth $d+1$ and size s. For $i \in \{2,\ldots,d+1\}$, let m_i be the top fan-in of depth-i subformulas of F. Let

$$\lambda := \prod_{i=2}^{d+1} 8e^2(\ln m_i + 1).$$

Note that $s = m_2 \cdots m_{d+1}$ and therefore by Lemma 11

$$\lambda \leq (8e^2)^d \left(\frac{1}{d}\ln s + 1\right)^d.$$

We claim that F is λ-critical. To see this, consider any $p \in [0, \frac{1}{\lambda}]$. Let $\boldsymbol{\varrho}_1 \subseteq \cdots \subseteq \boldsymbol{\varrho}_{d+1}$ be a sequence of random restrictions where

- $\boldsymbol{\varrho}_1$ is a $\dfrac{1}{4e}$-random restriction,
- $\boldsymbol{\varrho}_i$ is a $\dfrac{1}{8e^2(\ln m_i + 1)}$-random refinement of $\boldsymbol{\varrho}_i$ for each $i \in \{2,\ldots,d\}$,
- $\boldsymbol{\varrho}_{d+1}$ is a $\dfrac{p\lambda}{16e(\ln m_{d+1}+1)}$-random refinement of $\boldsymbol{\varrho}_d$.

Note that $\boldsymbol{\varrho}_{d+1}$ is a p-random restriction.

For all integers $k \geq 1$ and $s_1,\ldots,s_k \geq 1$ and bitstrings $a_h \in \{0,1\}^{s_h}$ and $c_h \in \{0,1\}^{s_1+\cdots+s_{h-1}}$ ($h \in [k]$), we have:

- for all depth-1 subformulas $F_1,\ldots,F_k$ of F, Lemma 28 implies

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1}\left[\, \exists \alpha_1,\ldots,\alpha_k \bigwedge_{h\in[k]} \left(\mathcal{T}_{\boldsymbol{\varrho}_1}^{(a_h)}(F_h\restriction_{(\alpha_1\circ\cdots\circ\alpha_{h-1})\leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_1)}c_h}) = \alpha_h\right) \,\right] \leq \left(\frac{1}{2e}\right)^{s_1+\cdots+s_k},$$

- for all $i \in \{2,\ldots,d\}$ and depth-i subformulas $F_1,\ldots,F_k$ of F, Proposition 27 implies

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_i}\left[\, \exists \alpha_1,\ldots,\alpha_k \bigwedge_{h\in[k]} \left(\mathcal{T}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_i}^{(a_h)}(F_h\restriction_{(\alpha_1\circ\cdots\circ\alpha_{h-1})\leftarrow_{\mathrm{Stars}(\boldsymbol{\varrho}_i)}c_h}) = \alpha_h\right) \,\right] \leq \left(\frac{1}{2e}\right)^{s_1+\cdots+s_k},$$

- finally, Proposition 25 (or Proposition 27) implies

$$\mathop{\mathbb{P}}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d+1}}\left[\, \mathcal{T}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d+1}}^{(a_1)}(F) \text{ exists } \,\right] \leq \left(\frac{p\lambda}{4}\right)^{s_1}.$$

Therefore, for all $t \geq 1$, we have

$$\begin{aligned}\mathbb{P}\left[\, \mathsf{DT}_{\mathsf{depth}}(F\restriction \boldsymbol{R}_p) \geq t \,\right] &\leq \sum_{u=t}^{\infty} \mathbb{P}\left[\, \mathcal{T}^{*}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d+1}}(F) \text{ has depth } u \,\right] \\ &\leq \sum_{u=t}^{\infty} \sum_{a\in\{0,1\}^u} \mathbb{P}\left[\, \mathcal{T}^{(a)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_{d+1}}(F) \text{ exists} \,\right] \\ &\leq \sum_{u=t}^{\infty} 2^u \left(\frac{p\lambda}{4}\right)^u \leq (p\lambda)^t.\end{aligned}$$

This shows that F is λ-critical, and since $\lambda = O(\frac{1}{d}\log s)^d$, the theorem is proved. ◀

9 Satisfiability Algorithms

The following theorem gives a randomized #SAT algorithm for regular AC^0 formulas. For AC^0 circuits of size $\Omega(n^2)$ (after converting to regular formulas), this matches the runtime of the #SAT algorithm of Impagliazzo, Matthews and Paturi [7].

▶ **Theorem 29.** *There is a randomized, zero-error algorithm which, given a regular* AC^0 *formula* F *of depth* $d+1$ *and size* s *on* n *variables, outputs a decision tree for* F *of size* $O(sn\cdot 2^{(1-\varepsilon)n})$ *where* $\varepsilon = 1/O(\frac{1}{d}\log s)^d$. *This algorithm also solves the* #SAT *problem, that is, it counts the number of satisfying assignments for* F.

Proof. First we require a lemma that $|\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F\restriction_\gamma)| \leq |\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)|$ for any depth-d formulas F, restrictions $\varrho_1 \subseteq \cdots \subseteq \varrho_d$ and ϱ_d-consistent restriction γ. This is straightforward from Definition 19 of $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(\cdot)$, but requires a careful argument (by induction on a stronger statement) to make precise. We omit the details.

Accepting this claim, we can extract from Definition 19 an algorithm which, given any depth-d formula F and restrictions $\varrho_1 \subseteq \cdots \subseteq \varrho_d$, computes the set $\mathcal{T}_{\varrho_1,\ldots,\varrho_d}(F)$ in time $O(n)\cdot\sum_{i=1}^{d}\sum_{F_i}|\mathcal{T}_{\varrho_1,\ldots,\varrho_i}(F_i)|$ where F_i ranges over depth-i subformulas of F.

If we are now given a regular AC^0 formula F of depth $d+1$ and size s on n variables, we can compute a decision tree for F as follows. Consider any sets $D_1 \subseteq \cdots \subseteq D_{d+1} \subseteq [n]$ and let $D := D_{d+1}$. We get a decision tree for F by querying all variables in D, receiving answers $\varrho : D \to \{0,1\}$, and then proceeding as the decision tree $\mathcal{T}_{\varrho_1,\ldots,\varrho_{d+1}}(F)$ where ϱ_i is the restriction of ϱ to domain D_i. The time required to construct this decision tree is $O(n)\cdot\sum_{i=1}^{d+1}\sum_{F_i}\sum_{\varrho:D\to\{0,1\}}|\mathcal{T}_{\varrho_1,\ldots,\varrho_i}(F_i)|$. If $\boldsymbol{\varrho}$ is a uniform random restriction with domain D and $|D| \leq (1-\varepsilon)n$ (for a choice of $\delta > 0$ to be determined), then this bound is $O(n\cdot 2^{(1-\delta)n})\cdot\sum_{i=1}^{d+1}\sum_{F_i}\mathbb{E}\,|\mathcal{T}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_i}(F_i)|$.

Let us now randomly generate sets $\boldsymbol{D}_1 \subseteq \cdots \subseteq \boldsymbol{D}_{d+1} \subseteq [n]$ as follows:

- $\boldsymbol{D}_1$ is a $1-\frac{1}{4e}$-random subset of $[n]$ (i.e., $\boldsymbol{D}_1$ includes each variable index in $[n]$ independently with probability $1-\frac{1}{4e}$),
- for each $i \in \{2,\ldots,d+1\}$, $\boldsymbol{D}_i$ is the union of $\boldsymbol{D}_{i-1}$ and a $1-\frac{1}{8e^2(\ln m_i+1)}$-random subset of $[n]\setminus\boldsymbol{D}_{i-1}$ where m_i is the top fan-in of depth-i subformulas of F (equivalently: $[n]\setminus\boldsymbol{D}_i$ is a $\frac{1}{8e^2(\ln m_i+1)}$-random subset of $[n]\setminus\boldsymbol{D}_{i-1}$).

Note that $\boldsymbol{D}$ $(:= \boldsymbol{D}_{d+1})$ is a $(1-\frac{1}{\lambda})$-random subset of $[n]$ where $\lambda = O(\frac{1}{d}\log s)^d$. The proof of Theorem 3 shows that, for every $i \in [d+1]$ and depth-i subformula F_i,

$$\mathbb{E}\,|\mathcal{T}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_i}(F_i)| = \sum_{t=0}^{\infty}\sum_{a\in\{0,1\}^t}\mathbb{P}\left[\, \mathcal{T}^{(a)}_{\boldsymbol{\varrho}_1,\ldots,\boldsymbol{\varrho}_i}(F_i) \text{ exists} \,\right] \leq 1 + \sum_{t=1}^{\infty} 2^t(1/2e)^t = \frac{1}{1-(1/e)}.$$

We may assume that $\lambda \leq n/2$, since otherwise the theorem is trivial. We then have $\mathbb{P}[\ |\boldsymbol{D}| \leq (1-\frac{1}{2\lambda})n\ ]$ with probability $\Omega(1)$ (and in fact $1-o(1)$ when $\lambda \ll n$). It follows that, with at least this probability, our algorithm constructs a decision tree for F in time $O(sn \cdot 2^{(1-\frac{1}{2\lambda})n})$. ◀

As a corollary, we improve the parameters of an algorithm of Santhanam and Williams [14] for the satisfiability problem for q-QB-CNF and q-QB-DNF, the class of quantified CNF and DNF formulas with q quantifier blocks (i.e., q quantifier alternations).

▶ **Corollary 30.** *Satisfiability of* q-QB-CNF *(resp.* q-QB-DNF*) with* n *variables and* $\mathrm{poly}(n)$ *clauses (resp. disjuncts) can be solved probabilistically with zero error in time* $\mathrm{poly}(n) \cdot 2^{n-\Omega(qn^{1/(q+1)})+O(q)}$.

The proof of Corollary 30 is adapted straightforwardly from [14], using Theorem 29 in place of the AC^0-circuit satisfiability algorithm of Impagliazzo, Matthews and Paturi [7]. Corollary 30 extends from $o(\frac{\log n}{\log\log n})$ to $o(\log n)$ the range of q for which the algorithm of [14] beats exhaustive search (i.e., $\mathrm{poly}(n) \cdot 2^n$ time). We remark that a second algorithm in [14] running time $\mathrm{poly}(n) \cdot 2^{n-\Omega(q)}$, which beats exhaustive search when $q = \omega(\log n)$. The range of q where q-QB-CNF and q-QB-DNF is not known to beat exhaustive search by a factor of at least n^k is therefore reduced to between $c_1 \log n$ and $c_2 \log n$ for constants $c_1(k) < c_2(k)$.

10 Open Questions

It is an open question whether the assumption of regularity is unnecessary in Theorem 3. We conjecture that our criticality bound for regular formulas holds for all formula.

▶ **Conjecture 1.** *All* AC^0 *formulas of depth* $d+1$ *and size* s *have criticality at most* $O(\frac{1}{d}\log s)^d$.

For (regular) formulas of n variables, can this bound be improved to $O(\frac{1}{d}\log(\frac{s}{n})+\log(d))^d$? (Results in [7] for AC^0 circuits involve the quantity $O(\log(\frac{s}{n})+d\log(d))^d$.)

Since $\mathsf{deg}(f) \leq \mathsf{DT}_{\mathsf{depth}}(f)$ for all boolean functions f, it follows that λ-criticality implies λ-*degree-criticality*, that is, the bound $\mathbb{P}[\ \mathsf{deg}(f\restriction_{\boldsymbol{R}_p}) \geq t\ ] \leq (p\lambda)^t$. What about a reserve implication?

▶ Question 1. Does degree-criticality λ imply criticality $O(\lambda)$?

Tal [16] showed that DeMorgan formulas of size L have degree-criticality $O(\sqrt{L})$. As a special case of Question 1, one can ask:

▶ Question 2. Do DeMorgan formulas of size L have criticality $O(\sqrt{L})$?

Finally, we ask a question that would potentially yield a much simpler and more aesthetic proof of Theorem 3. We will say that a boolean function f is *hereditarily* λ*-critical* if every subfunction of f is λ-critical (i.e., $f\restriction_\varrho$ is λ-critical for every restriction ϱ).

▶ Question 3. Suppose f is the disjunction of boolean functions $f_1 \vee \cdots \vee f_m$ where each f_i is hereditarily λ-critical. Is f necessarily $O(\lambda \ln(m+1))$ critical?

A positive answer to Question 3 implies Theorem 3. If Question 3 could be answered affirmatively, we may then consider the following generalization:

▶ Question 4. Suppose f is the disjunction of boolean functions $f_1 \vee \cdots \vee f_m$ where each f_i is hereditarily λ_i-critical. Let $\lambda \geq \max\{\lambda_1, \ldots, \lambda_m\}$ such that $\sum_{i=1}^m e^{-(\lambda/\lambda_i)} \leq 1$. Is f necessarily $O(\lambda)$ critical?

A positive answer to Question 4 would be very interesting as it implies Conjecture 1.

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A Appendix: Average Sensitivity of Size-m DNF Formulas

As a warm-up for our switching lemma for size-m DNF formulas (Section 4), we present a simple proof that every size-m DNF formula F with expected value $\lambda \in [0,1]$ has average sensitivity at most $\min\{2\log(m+1), 2\lambda\log(m/\lambda)\}$. Up to an $1+o(1)$ factor, these bounds can be derived from known results on the average sensitivity of width-w DNFs (see Remark 33). However, our proof involves different argument based on the entropy of the "first witness function" associated with F. This argument was the starting point for our alternative proof of the switching lemma and provides a simple illustration of the underlying principle.

Recall the definitions of *sensitivity* and *average sensitivity*. For a function f with domain $\{0,1\}^n$ and a point $x \in \{0,1\}^n$, let

$$\mathsf{S}(f,x) := |\{i \in [n] : f(x) \neq f(x \oplus i)\}| \quad \text{and} \quad \mathsf{AS}(f) := \mathop{\mathbb{E}}_{x\in\{0,1\}^n}[\, \mathsf{S}(f,x)\,].$$

The *expected value* of f is $\mathbb{E}_{x\in\{0,1\}^n}[\, f(x)\,]$.

▶ **Theorem 31.** *Every m-clause DNF with expected value λ has average sensitivity at most* $\min\{2\log(m+1), 2\lambda\log(m/\lambda)\}$.

Proof. Let $F = C_1 \vee \cdots \vee C_m$ be an m-clause DNF. Let $\widetilde{F} : \{0,1\}^n \to [m+1]$ be the "first witness function" mapping $x \in \{0,1\}^n$ to the index of the first satisfied clause if any, and otherwise to $m+1$. Let

$$\mathsf{S}_<(\widetilde{F},x) := |\{i \in [n] : \widetilde{F}(x) < \widetilde{F}(x \oplus i)\}| \quad \text{and} \quad \mathsf{AS}_<(\widetilde{F}) := \mathop{\mathbb{E}}_{x\in\{0,1\}^n}[\, \mathsf{S}_<(\widetilde{F},x)\,].$$

Observe that $\mathsf{AS}(F) \le \mathsf{AS}(\widetilde{F}) = 2{\cdot}\mathsf{AS}_<(\widetilde{F})$.

Let $\mu = (\mu_1, \ldots, \mu_{m+1})$ be the probability distribution induced by $\widetilde{F}$ under the uniform distribution on $\{0,1\}^n$, that is, $\mu_\ell := \mathbb{P}_{x\in\{0,1\}^n}[\, \widetilde{F}(x) = \ell\,]$. For each $\ell \in [m]$, we have

$$\begin{aligned}
2^{\mathbb{E}_{y\in\widetilde{F}^{-1}(\ell)}[\,\mathsf{S}_<(\widetilde{F},y)\,]} &\le \mathop{\mathbb{E}}_{y\in\widetilde{F}^{-1}(\ell)}[\, 2^{\mathsf{S}_<(\widetilde{F},y)}\,] && \text{by Jensen's inequality}\\
&\le 2^{|C_\ell|} && \text{since } \mathsf{S}_<(\widetilde{F},y) \le |C_\ell| \text{ for all } y \in \widetilde{F}^{-1}(\ell)\\
&\le \frac{1}{\mu_\ell} && \text{since } \mu_\ell \le \mathop{\mathbb{P}}_{x\in\{0,1\}^n}[\, C_\ell(x) = 1\,] = 2^{-|C_\ell|}.
\end{aligned}$$

Therefore, $\mathbb{E}_{y\in\widetilde{F}^{-1}(\ell)}[\, \mathsf{S}_<(\widetilde{F},y)\,] \le \log(1/\mu_\ell)$.

Using the fact that μ has entropy at most $\log(m+1)$, we have

$$\begin{aligned}
\mathsf{AS}_<(\widetilde{F}) &= \mathop{\mathbb{E}}_{x\in\{0,1\}^n}[\, \mathsf{S}_<(\widetilde{F},x)\,]\\
&= \sum_{\ell\in[m]} \mu_\ell \mathop{\mathbb{E}}_{y\in\widetilde{F}^{-1}(\ell)}[\, \mathsf{S}_<(\widetilde{F},y)\,]\\
&\le \sum_{\ell\in[m]} \mu_\ell \log(1/\mu_\ell) \;\le\; \sum_{\ell\in[m+1]} \mu_\ell \log(1/\mu_\ell) \;=\; \mathbb{H}(\mu) \;\le\; \log(m+1).
\end{aligned}$$

We conclude that $\mathsf{AS}(F) \le 2\log(m+1)$.

If F has expected value λ, then letting $\mu'_\ell := \mu_\ell/\lambda$ (and noting that $\lambda = \sum_{\ell\in[m]} \mu_\ell$), we have

$$\sum_{\ell\in[m]} \mu_\ell \log(1/\mu_\ell) = \lambda \sum_{\ell\in[m]} \mu'_\ell\Big(\log(1/\mu'_\ell) - \log(\lambda)\Big) = \lambda\Big(\mathbb{H}(\mu') - \log(\lambda)\Big) \le \lambda\log(m/\lambda).$$

This gives the bound $\mathsf{AS}(F) \le 2\lambda\log(m/\lambda)$. ◀

For $k, t \in \mathbb{N}$, observe that the function $\mathrm{PARITY}(x_1, \ldots, x_k) \wedge \mathrm{AND}(x_{k+1}, \ldots, x_{k+t})$ is equivalent to a DNF with $m := 2^{k-1}$ clauses and has expected value $\lambda := (1/2)^{t+1}$ and average sensitivity $(k+t)2^{-t}$ $(= 2\lambda \log(m/\lambda))$. This shows that Theorem 31 is tight whenever $\lambda \in [0, \frac{1}{2}]$ is an inverse power of two.

▶ Remark 32. Theorem 31 has a (weak) converse: Keller and Lifshitz [8] showed that every boolean function with expected value λ and average sensitivity at most $2\lambda \log(m/\lambda)$ is $\varepsilon\lambda$-approximated by a DNF of size $2^{m^{O(1/\varepsilon)}}$.

▶ Remark 33. The average sensitivity of a width-w DNF with expected value λ is known to be at most the minimum of w (Amano [1]), $2\lambda w$ (Boppana [4]) and $2(1-\lambda)w/\log(\frac{1}{1-\lambda})$ (Traxler [18]). Each of these bounds is tight for a certain values of λ. Extending all three bounds, Scheder and Tan [15] proved an upper bound of $\beta(\lambda)w$ for a certain piecewise linear function $\beta : [0,1] \to [0,1]$; this bound is asymptotically tight for all values of λ. By approximating any m-clause by a DNF of width $\lceil \log m \rceil$, they also observe that $(1 + o(1))\beta(\lambda) \log(m+1)$ is an upper bound on the average sensitivity of m-clause DNFs.