Report from Dagstuhl Seminar 19092

Beyond-Planar Graphs: Combinatorics, Models and Algorithms

Edited by

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Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 19092 "Beyond-Planar Graphs: Combinatorics, Models and Algorithms" which brought together 36 researchers in the areas of graph theory, combinatorics, computational geometry, and graph drawing. This seminar continued the work initiated in Dagstuhl Seminar 16452 "Beyond-Planar Graphs: Algorithmics and Combinatorics" and focused on the exploration of structural properties and the development of algorithms for so-called beyond-planar graphs, i.e., non-planar graphs that admit a drawing with topological constraints such as specific types of crossings, or with some forbidden crossing patterns. The seminar began with four talks about the results of scientific collaborations originating from the previous Dagstuhl seminar. Next we discussed open research problems about beyond planar graphs, such as their combinatorial structures (e.g., book thickness, queue number), their topology (e.g., simultaneous embeddability, gap planarity, quasi-quasiplanarity), their geometric representations (e.g., representations on few segments or arcs), and applications (e.g., manipulation of graph drawings by untangling operations). Six working groups were formed that investigated several of the open research questions. In addition, talks on related subjects and recent conference contributions were presented in the morning opening sessions. Abstracts of all talks and a report from each working group are included in this report.

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1 Executive Summary

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Most big data sets are relational, containing a set of objects and relations between the objects. This is commonly modeled by graphs, with the objects as the vertices and the relations as the edges. A great deal is known about the structure and properties of special types of graphs, in particular *planar graphs* which are fundamental for both Graph Theory, Graph Algorithms and Automatic Layout. Structural properties of planar graphs can often be expressed, for example, in terms of excluded minors, low density, and small separators. These properties lead to efficient algorithms; consequently a number of fundamental algorithms for planar graphs have been discovered. As many of the characteristic properties of planar graphs have been generalized (e.g., graph minor theory, topological obstructions, χ-boundedness), these algorithms also extend in various directions to broad families of graphs.

Typical real world graphs, such as social networks and biological networks, are *nonplanar*. In particular, the class of scale-free networks, which can be used to model web-graphs, social networks and many kinds of biological networks, are sparse nonplanar graphs, with globally sparse and locally dense structure. To analyze and visualize such real world networks, we need to formulate and solve fundamental mathematical and algorithmic research questions on *sparse nonplanar* graphs. Sparsity, in most cases, is explained by properties that generalize those of planar graphs: in terms of topological obstructions or forbidden intersection patterns among the edges. These are called *beyond-planar graphs*. Important beyond-planar graph classes include the following:

- *k-planar graphs*: graphs that can be drawn with at most k crossings per edge;
- *k-quasi-planar graphs*: graphs which can be drawn without k mutually crossing edges;
- *k-gap-planar graphs*: graphs that admit a drawing in which each crossing is assigned to one of the two involved edges and each edge is assigned at most k of its crossings;
- *RAC (Right Angle Crossing) graphs*: graphs that have straight-line drawings in which any two crossing edges meet in a right angle;
- *bar k-visibility graphs*: graphs whose vertices are represented as horizontal segments (bars) and edges are represented as vertical lines connecting bars, intersecting at most k bars;
- *fan-crossing-free graphs*: graphs which can be drawn without fan-crossings; and
- *fan-planar graphs*: graphs which can be drawn such that every edge is crossed only by pairwise adjacent edges (fans).

Compared to the first edition of the seminar, we planned to focus more on aspects of computational geometry. Therefore, we included one new organizer as well as some more participants from this field.

Thirty-six participants met on Sunday afternoon for a first informal get-together and reunion since the last workshop which took place more than two years ago. From that event, the four working groups nearly all have completed and published subsequent work. We decided to build on the achievements of the previous meeting and scheduled short talks recalling the previous seminar's results. On Monday afternoon, we held an engaging open problems session and formed new working groups. Notably, this time, more problems related

to computational geometry as well as questions from combinatorics have been proposed. Open problems included questions about the combinatorial structures (e.g, book thickness, queue number), the topology (e.g., simultaneous embeddability, gap planarity, quasi-quasiplanarity), the geometric representations (e.g., representations on few segments or arcs), and applications (e.g., manipulation of graph drawings by untangling operations) of beyond-planar graphs.

In the opening session of every morning, we have drawn inspiration from additional talks, fresh conference contributions on related topics (see abstracts). An impressive session on the last day was devoted to progress reports that included plans for publications and follow-up projects among researchers that would have been highly unlikely without this seminar. From our personal impression and the feedback of the participants, the seminar has initiated collaboration and lead to new ideas and directions.

We thank all the people from Schloss Dagstuhl for providing a positive environment and hope to repeat this seminar, possibly with some new focus, for a third time.

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3 Overview of Talks

3.1 On the relationship between *k*-planar and *k*-quasiplanar graphs

Patrizio Angelini (Universität Tübingen, DE)

Joint work of Patrizio Angelini, Michael A. Bekos, Franz J. Brandenburg, Giordano Da Lozzo, Giuseppe Di Battista, Walter Didimo, Giuseppe Liotta, Fabrizio Montecchiani, Ignaz Rutter, Michael Hoffmann, Csaba Tóth
Main reference Patrizio Angelini, Michael A. Bekos, Franz J. Brandenburg, Giordano Da Lozzo, Giuseppe Di Battista, Walter Didimo, Giuseppe Liotta, Fabrizio Montecchiani, Ignaz Rutter: "On the Relationship Between k-Planar and k-Quasi-Planar Graphs", in Proc. of the Graph-Theoretic Concepts in Computer Science – 43rd International Workshop, WG 2017, Eindhoven, The Netherlands, June 21-23, 2017, Revised Selected Papers, Lecture Notes in Computer Science, Vol. 10520, pp. 59–74, Springer, 2017.
URL https://doi.org/10.1007/978-3-319-68705-6_5
Main reference Michael Hoffmann, Csaba D. Tóth: "Two-Planar Graphs Are Quasiplanar", in Proc. of the 42nd International Symposium on Mathematical Foundations of Computer Science, MFCS 2017, August 21-25, 2017 – Aalborg, Denmark, LIPIcs, Vol. 83, pp. 47:1–47:14, Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, 2017.
URL http://dx.doi.org/10.4230/LIPIcs.MFCS.2017.47

In the area of beyond planarity, the two most studied families of graph classes are those of k-planar and k-quasiplanar graphs. A graph is k-planar if it admits a drawing in the plane so that no edge is crossed by more than k edges, while it is k-quasiplanar if it admits a drawing that contains no set of pairwise crossing edges.

We are interested in inclusion relationships between the classes belonging to these two families. Clearly, every k-planar graph is $(k+1)$-planar, and every k-quasiplanar graph is $(k+1)$-quasiplanar, and hence the two families define proper hierarchies. On the other hand, the relationship between these two hierarchies is not well established yet. The only result, which follows from the definitions, is that every k-planar graph is $(k+2)$-quasiplanar.

In this work we prove that every k-planar graph is also $(k+1)$-quasiplanar. This result is obtained by a rerouting technique that solves all sets of $k+1$ pairwise crossing edges without introducing new ones. The question whether every k-planar graph is also k-quasiplanar, for $k > 2$, remains open.

3.2 Z$_2$-genus of graphs and minimum rank of partial symmetric matrices

Radoslav Fulek (IST Austria – Klosterneuburg, AT)

Joint work of Radoslav Fulek, Jan Kynčl
Main reference Radoslav Fulek, Jan Kynčl: "Z$_2$-Genus of Graphs and Minimum Rank of Partial Symmetric Matrices", in Proc. of the 35th International Symposium on Computational Geometry, SoCG 2019, LIPIcs, Vol. 129, pp. 39:1–39:16, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2019.
URL http://dx.doi.org/10.4230/LIPIcs.SoCG.2019.39

The *genus* $\mathrm{g}(G)$ of a graph G is the minimum g such that G has an embedding on the orientable surface M_g of genus g. A drawing of a graph on a surface is *independently even* if every pair of nonadjacent edges in the drawing crosses an even number of times. The $\mathbb{Z}_2$*-genus* of a graph G, denoted by $\mathrm{g}_0(G)$, is the minimum g such that G has an independently even drawing on M_g. By a result of Battle, Harary, Kodama and Youngs from 1962, the graph genus is additive over 2-connected blocks. In 2013, Schaefer and Štefankovič proved that the

$\mathbb{Z}_2$-genus of a graph is additive over 2-connected blocks as well, and asked whether this result can be extended to so-called 2-amalgamations, as an analogue of results by Decker, Glover, Huneke, and Stahl for the genus. We give the following partial answer. If $G = G_1 \cup G_2$, G_1 and G_2 intersect in two vertices u and v, and $G - u - v$ has k connected components (among which we count the edge uv if present), then $|\mathrm{g}_0(G) - (\mathrm{g}_0(G_1) + \mathrm{g}_0(G_2))| \leq k + 1$. For complete bipartite graphs $K_{m,n}$, with $n \geq m \geq 3$, we prove that $\frac{\mathrm{g}_0(K_{m,n})}{\mathrm{g}(K_{m,n})} = 1 - O(\frac{1}{n})$. Similar results are proved also for the Euler $\mathbb{Z}_2$-genus. We express the $\mathbb{Z}_2$-genus of a graph using the minimum rank of partial symmetric matrices over $\mathbb{Z}_2$; a problem that might be of independent interest.

3.3 Planar Graphs of Bounded Degree have Bounded Queue Number

Henry Förster (Universität Tübingen, DE), Michael Bekos (Universität Tübingen, DE), Martin Gronemann (Universität Köln, DE), Tamara Mchedlidze (KIT – Karlsruher Institut für Technologie, DE), Fabrizio Montecchiani (University of Perugia, IT), Chrysanthi Raftopoulou (National Technical University of Athens, GR), and Torsten Ueckerdt (KIT – Karlsruher Institut für Technologie, DE)

Main reference Michael Bekos, Henry Förster, Martin Gronemann, Tamara Mchedlidze, Fabrizio Montecchiani, Chrysanthi Raftopoulou, and Torsten Ueckerdt: "Planar Graphs of Bounded Degree Have Bounded Queue Number", in Proc. of the 51st Annual ACM SIGACT Symposium on the Theory of Computing (STOC'19), June 23–26, 2019, Phoenix, AZ, USA. ACM, New York, NY, USA, 9 pages.
URL https://doi.org/10.1145/3313276.3316324

A *queue layout* of a graph consists of a *linear order* of its vertices and a partition of its edges into *queues*, so that no two independent edges of the same queue are nested. The *queue number* of a graph is the minimum number of queues required by any of its queue layouts.

A long-standing conjecture by Heath, Leighton and Rosenberg states that the queue number of planar graphs is bounded. This conjecture has been partially settled in the positive for several subfamilies of planar graphs (most of which have bounded treewidth). In this talk, we present a new important step towards settling this conjecture. We prove that planar graphs of bounded degree (which may have unbounded treewidth) have bounded queue number.

A notable implication of this result is that every planar graph of bounded degree admits a three-dimensional straight-line grid drawing in linear volume. Further implications are that every planar graph of bounded degree has bounded track number, and that every k-planar graph (i.e., every graph that can be drawn in the plane with at most k crossings per edge) of bounded degree has bounded queue number.

3.4 Orthogonal and Smooth Orthogonal Layouts of 1-Planar Graphs with Low Edge Complexity

Chrysanthi Raftopoulou (National Technical University of Athens, GR)

Joint work of Chrysanthi Raftopoulou, Evmorfia Argyriou, Sabine Cornelsen, Henry Förster, Michael Kaufmann, Martin Nöllenburg, Yoshio Okamoto, Alexander Wolff
Main reference Evmorfia N. Argyriou, Sabine Cornelsen, Henry Förster, Michael Kaufmann, Martin Nöllenburg, Yoshio Okamoto, Chrysanthi N. Raftopoulou, Alexander Wolff: "Orthogonal and Smooth Orthogonal Layouts of 1-Planar Graphs with Low Edge Complexity", in Proc. of the Graph Drawing and Network Visualization – 26th International Symposium, GD 2018, Barcelona, Spain, September 26-28, 2018, Proceedings, Lecture Notes in Computer Science, Vol. 11282, pp. 509–523, Springer, 2018.
URL https://doi.org/10.1007/978-3-030-04414-5_36

While orthogonal drawings have a long history, smooth orthogonal drawings have been introduced only recently. So far, only planar drawings or drawings with an arbitrary number of crossings per edge have been studied. Recently, a lot of research effort in graph drawing has been directed towards the study of beyond-planar graphs such as 1-planar graphs, which admit a drawing where each edge is crossed at most once. In this talk, we consider graphs with a fixed embedding. For 1-planar graphs, we present algorithms that yield orthogonal drawings with optimal edge complexity and smooth orthogonal drawings with small edge complexity. For the subclass of outer-1-planar graphs, which can be drawn such that all vertices lie on the outer face, we achieve optimal edge complexity for both, orthogonal and smooth orthogonal drawings.

3.5 Inserting an Edge into a Geometric Embedding

Ignaz Rutter (Universität Passau, DE)

Joint work of Marcel Radermacher, Ignaz Rutter
Main reference Marcel Radermacher, Ignaz Rutter: "Inserting an Edge into a Geometric Embedding", in Proc. of the Graph Drawing and Network Visualization – 26th International Symposium, GD 2018, Barcelona, Spain, September 26-28, 2018, Proceedings, Lecture Notes in Computer Science, Vol. 11282, pp. 402–415, Springer, 2018.
URL https://doi.org/10.1007/978-3-030-04414-5_29

The algorithm to insert an edge e in linear time into a planar graph G with a minimal number of crossings on e [1], is a helpful tool for designing heuristics that minimize edge crossings in drawings of general graphs. Unfortunately, some graphs do not have a geometric embedding Γ such that $\Gamma + e$ has the same number of crossings as the embedding $G + e$. This motivates the study of the computational complexity of the following problem: Given a combinatorially embedded graph G, compute a geometric embedding Γ that has the same combinatorial embedding as G and that minimizes the crossings of $\Gamma + e$. We give polynomial-time algorithms for special cases and prove that the general problem is fixed-parameter tractable in the number of crossings. Moreover, we show how to approximate the number of crossings by a factor $(\Delta - 2)$, where Δ is the maximum vertex degree of G.

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3.6 A crossing lemma for multigraphs

Géza Tóth (Alfréd Rényi Institute of Mathematics – Budapest, HU) and János Pach (EPFL – Lausanne, CH)

Joint work of Géza Tóth, János Pach
Main reference János Pach, Géza Tóth: "A Crossing Lemma for Multigraphs", in Proc. of the 34th International Symposium on Computational Geometry, SoCG 2018, June 11-14, 2018, Budapest, Hungary, LIPIcs, Vol. 99, pp. 65:1–65:13, Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, 2018.
URL http://dx.doi.org/10.4230/LIPIcs.SoCG.2018.65

Let G be a drawing of a graph with n vertices and $e > 4n$ edges, in which no two adjacent edges cross and any pair of independent edges cross at most once. According to the celebrated Crossing Lemma of Ajtai, Chvátal, Newborn, Szemerédi and Leighton, the number of crossings in G is at least $c\frac{e^3}{n^2}$, for a suitable constant $c > 0$. In a seminal paper, Székely generalized this result to multigraphs, establishing the lower bound $c\frac{e^3}{mn^2}$, where m denotes the maximum multiplicity of an edge in G. We get rid of the dependence on m by showing that, as in the original Crossing Lemma, the number of crossings is at least $c'\frac{e^3}{n^2}$ for some $c' > 0$, provided that the "lens" enclosed by every pair of parallel edges in G contains at least one vertex. This settles a conjecture of Bekos, Kaufmann, and Raftopoulou.

This work started at the Dagstuhl Seminar "Beyond-Planar Graphs: Algorithmics and Combinatorics", November 6-11, 2016, in a working group, together with Stefan Felsner, Michael Kaufmann, Vincenzo Roselli, Torsten Ueckerdt, and Pavel Valtr. We are very grateful to them for their valuable comments, suggestions, and for many interesting discussions.

3.7 The Number of Crossings in Multigraphs with No Empty Lens

Torsten Ueckerdt (KIT – Karlsruher Institut für Technologie, DE)

Joint work of Michael Kaufmann, János Pach, Géza Tóth, Torsten Ueckerdt
Main reference Michael Kaufmann, János Pach, Géza Tóth, Torsten Ueckerdt: "The Number of Crossings in Multigraphs with No Empty Lens", in Proc. of the Graph Drawing and Network Visualization – 26th International Symposium, GD 2018, Barcelona, Spain, September 26-28, 2018, Proceedings, Lecture Notes in Computer Science, Vol. 11282, pp. 242–254, Springer, 2018.
URL http://dx.doi.org/10.1007/978-3-030-04414-5_17

Let G be a multigraph with n vertices and $e > 4n$ edges, drawn in the plane such that any two parallel edges form a simple closed curve with at least one vertex in its interior and at least one vertex in its exterior. Pach and Tóth [1] extended the Crossing Lemma of Ajtai *et al.* [2] and Leighton [3] by showing that if no two adjacent edges cross and every pair of nonadjacent edges cross at most once, then the number of edge crossings in G is at least $\alpha e^3/n^2$, for a suitable constant $\alpha > 0$. The situation turns out to be quite different if nonparallel edges are allowed to cross any number of times. It is proved that in this case the number of crossings in G is at least $\alpha e^{2.5}/n^{1.5}$. The order of magnitude of this bound cannot be improved.

This project initiated at the Dagstuhl seminar 16452 "Beyond-Planar Graphs: Algorithmics and Combinatorics," November 2016. We would like to thank all participants, especially Stefan Felsner, Vincenzo Roselli, and Pavel Valtr, for fruitful discussions.

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3.8 Every collinear set in a planar graph is free

Vida Dujmović (University of Ottawa, CA), Fabrizio Frati (Roma Tre University, IT), Günter Rote (FU Berlin, DE)

Joint work of Vida Dujmović, Fabrizio Frati, Daniel Gonçalves, Pat Morin, Günter Rote
Main reference Vida Dujmović, Fabrizio Frati, Daniel Gonçalves, Pat Morin, Günter Rote: "Every Collinear Set in a Planar Graph Is Free", in Proc. of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019, pp. 1521–1538, SIAM, 2019.
URL http://dx.doi.org/10.1137/1.9781611975482.92

We show that if a planar graph G has a plane straight-line drawing in which a subset S of its vertices are collinear, then for any set of points, X, in the plane with $|X| = |S|$, there is a plane straight-line drawing of G in which the vertices in S are mapped to the points in X. This solves an open problem posed by Ravsky and Verbitsky in 2008. In their terminology, we show that every collinear set is free.

This result has applications in graph drawing, including untangling, column planarity, universal point subsets, and partial simultaneous drawings.

Preprint of the full paper: http://arxiv.org/abs/1811.03432

4 Working groups

4.1 Traversing Edges

Eyal Ackerman (University of Haifa, IL), Stefan Felsner (TU Berlin, DE), Radoslav Fulek (IST Austria – Klosterneuburg, AT), Balázs Keszegh (Alfréd Rényi Institute of Mathematics – Budapest, HU), János Pach (EPFL – Lausanne, CH), Günter Rote (FU Berlin, DE), Csaba D. Tóth (California State University – Northridge, US), Géza Tóth (Alfréd Rényi Institute of Mathematics – Budapest, HU), and Torsten Ueckerdt (KIT – Karlsruher Institut für Technologie, DE)

A *geometric graph* is a graph drawn in the plane such that its vertices are distinct points in a general position (no three on a line) and its edges are straight-line segments. Two edges in a geometric graph are either adjacent, crossing or disjoint. Disjoint edges may be further classified as *avoiding* (or *parallel*) and nonavoiding, where two disjoint edges are called avoiding if their endpoints are in convex position. Define two edges to be *traversing* if they are crossing or they are disjoint and nonavoiding. In other words, two edges are traversing if at least one of them contains in its interior the intersection point of the two lines that contain the two edges.

It is a natural question to ask for the density of a geometric graph with no k pairwise conflicting edges, where 'conflicting' refers to one of the above-mentioned relations between two edges.[1] The case of no k pairwise adjacent edges is not interesting as it implies that the maximum degree is $k-1$. Considering geometric graphs with no k pairwise disjoint edges, it was first proved by Pach and Törőcsik [11] that they have linearly many edges. The best bound is due to G. Tóth [13]:

▶ **Theorem 1** ([13]). *An n-vertex geometric graph with no k pairwise disjoint edges has $O(k^2 n)$ edges.*

Valtr [15] proved a linear bound considering pairwise avoiding edges.

▶ **Theorem 2** ([15]). *An n-vertex geometric graph with no k pairwise avoiding edges has $O_k(n)$ edges.*

The case of pairwise crossing edges is a special case of a famous and rather old conjecture [6, 8] concerning the density of *k-quasi-planar* graphs.[2]

▶ **Conjecture 3.** *An n-vertex k-quasi-planar graph has $O_k(n)$ edges.*

This conjecture is known to hold for $k \leq 4$ [1, 4, 5] but for $k > 4$ it is open even for geometric graphs. The best bound is due to Valtr:

▶ **Theorem 4** ([14]). *An n-vertex geometric graph with no k pairwise crossing edges has $O_k(n \log n)$ edges.*

The main goal of our workgroup was to prove the following relaxed variant of Conjecture 3:

▶ **Conjecture 5** ([3]). *An n-vertex geometric graph with no k pairwise traversing edges has $O_k(n)$ edges.*

An n-vertex geometric graph with no pair of traversing edges is outerplanar and therefore has at most $2n-3$ edges (for $n > 1$). For $k \leq 4$ Conjecture 5 holds since Conjecture 3 holds. For $k > 4$ a possible approach to prove Conjecture 5 would have been to provide a positive answer to the following question.

▶ **Problem 6.** *Is it true that every set of m segments in the plane without k pairwise traversing segments contains a subset of $\Omega_k(m)$ segments no two of which are traversing?*

Indeed, if this question had an affirmative answer, then it would imply Conjecture 5 as follows. Given an n-vertex geometric graph with m edges, no k of which are pairwise traversing, one can slightly shorten each edge and obtain a set of m segments, no k of which are pairwise traversing. Suppose that this set contains $c_k m$ segments such that no two of them are traversing, for some $c_k > 0$. Then the corresponding edges of the graph are pairwise nontraversing and hence $c_k m \leq 2n-3$ and Conjecture 5 follows. Unfortunately, by modifying a construction by Pawlik et al. [12] and Walczak [16] we provide a negative answer to Problem 6.

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[1] We consider k to be a fixed integer and use the notation $O_k(\cdot)$ to indicate that the constant hiding in the big O notation depends only on k.

[2] Recall that a graph is *k-quasi-planar* if it admits a drawing in which no k edges are pairwise crossing.

▶ **Theorem 7.** *There exist sets of m segments, no three of which are pairwise traversing, such that the maximum size of a pairwise nontraversing subset is $o(m)$.*

The maximum size of a subset with no two traversing segments in this construction is $O(m/\log\log m)$. It is an interesting problem to determine the maximum size of such a subset in any set of m segments no k of which are pairwise traversing. The best lower bound we were able to find was $\Omega(\sqrt{m})$

In the special case of a (bipartite) geometric graph G in which all edges cross a single line ℓ, we were able to prove Conjecture 5. In fact in this case, a linear upper bound is known even when no k edges pairwise cross [14]. However, for traversing edges we have devised a simpler proof: Denote by $\overline{e}$ the complement of an edge e on the line that supports e and observe that e_1 and e_2 are traversing if and only if $\overline{e_1}$ and $\overline{e_2}$ are disjoint. Therefore, as ℓ goes to infinity we obtain a graph with no k pairwise disjoint edges and the linear bound on its density follows from Theorem 1. This result, along with a standard divide-and-conquer argument, shows that an n-vertex geometric graph with no k pairwise traversing edges has $O_k(n\log n)$ edges, without relying on the same known bound for k-quasi-planar graphs.

Alas, we were unable to make any further progress on Conjecture 5. Still, to get a better understanding of the notion of traversing edges we reverted to simpler questions involving such edges. Recall that an embedded graph is k-plane if each of its edges is crossed at most k times. The maximum densities of n-vertex k-plane graphs for $k = 1, 2, 3, 4$ are known to be $4n-8$ [10], $5n-10$ [10], $5.5n-11$ [9], and $6n-O(1)$ [2], respectively. We considered analogue graphs with respect to traversing edges, that is, the density of *k-traversing* geometric graphs – graphs in which each edge is involved in at most k traversings. Since, by definition, these graphs are k-plane we are interested in exact bounds on their densities.

▶ **Theorem 8.** *Let G be an n-vertex 1-traversing geometric graph. Then $|E(G)| \leq \lfloor 2.5n\rfloor - 4$, if $n \geq 2$. This bound is tight.*

Note that there might be asymmetry when two edges e_1 and e_2 are traversing according to which of them contains the intersection point of the two supporting lines. Suppose that e_1 contains that point. Then we say that e_1 is *traversed* by e_2 and that e_2 is *traversing* e_1. Note that if e_1 and e_2 are crossing, then each of them is traversing and traversed by the other. Theorem 8 is in fact implied by each of following two variants.

▶ **Theorem 9.** *Let G be an n-vertex geometric graph in which each edge is traversing at most one edge. Then $|E(G)| \leq \lfloor 2.5n\rfloor - 4$, if $n \geq 2$. This bound is tight.*

▶ **Theorem 10.** *Let G be an n-vertex geometric graph in which each edge is traversed by at most one edge. Then $|E(G)| \leq \lfloor 2.5n\rfloor - 4$, if $n \geq 2$. This bound is tight.*

The upper bound $\lfloor 2.5n\rfloor - 4$ matches the maximum size of an n-vertex outer 1-plane graph [7] (an *outer* k-plane graph is a geometric k-plane graph in which the vertices are in convex position). Note that for a convex geometric graph the notions of crossing and traversing edges coincide. We only found one example of a nonconvex 1-traversing geometric graph with the maximum possible density, namely a nonconvex drawing of K_4. Call a k-traversing geometric graph *optimal* if there is no other k-traversing geometric graph with the same number of vertices and a greater number of edges.

▶ **Problem 11.** *Is it true that for every integer k there is an integer n_k such that every optimal k-traversing graph with more than n_k vertices is an outer k-plane graph?*

A possible way to provide a negative answer to this question would be to show that in some cases we get different maximum densities for the different notions of traversing. Perhaps an easier problem would be to show that the class of graphs that can be drawn such that every edge is traversing at most k other edges and the class of graphs that can be drawn such that every edge is traversed by at most k other edges are not the same.

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4.2 Variants of the Segment Number of a Graph

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When drawing a graph, a way to keep the visual complexity low is to use few geometric objects for drawing the edges. This idea is captured by the *segment number* of a graph, that is, the smallest number of line segments that together constitute a straight-line drawing of the given graph. The *arc number* of a graph is defined analogously with respect to circular-arc drawings. For a graph G, we denote its segment number by $\mathrm{seg}(G)$ and its arc number by $\mathrm{arc}(G)$. So far, both numbers have only been studied for planar graphs. Two obvious lower bounds for $\mathrm{seg}(G)$ are known [1]: the slope number of G and $\eta(G)/2$, where $\eta(G)$ is the number of odd-degree vertices of G. Dujmović et al. [1], who introduced slope and segment number, showed among others that trees can be drawn such that the optimum segment number and the optimum slope number are achieved simultaneously. In other words, any tree T admits a drawing with $\eta(T)/2$ segments and $\Delta(T)/2$ slopes, where $\Delta(T)$ is the maximum degree of T. Unfortunately, these drawings need exponential area. Therefore, Schulz [9] suggested to study the arc number of planar graphs. Among others, he showed that any n-vertex tree can be drawn on a polynomial-size grid ($O(n^{1.81}) \times n$) using at most $3n/4$ arcs.

Upper bounds for the segment number and the arc number (in terms of the number of vertices, n, ignoring constant additive terms) are known for series-parallel graphs ($3n/2$ vs. n), planar 3-trees ($2n$ vs. $11n/6$), and triconnected planar graphs ($5n/2$ vs. $2n$) [1, 9]. The upper bound on the segment number for triconnected planar graphs has been improved for the special cases of triangulations and 4-connected triangulations (from $5n/2$ to $7n/3$ and $9n/4$, respectively) by Durocher and Mondal [2]. Hültenschmidt et al. [4] provided bounds for segment and arc number under the additional constraint that vertices must lie on a polynomial-size grid. They also showed that n-vertex triangulations can be drawn with at most $5n/3$ arcs, which is better than the lower bound of $2n$ for the segment number on this class of graphs. For 4-connected triangulations, they need at most $3n/2$ arcs. Kindermann et al. [6] recently strengthened some of these results by showing that many classes of planar graphs admit non-trivial bounds on the segment number even when restricting vertices to a grid of size $O(n) \times O(n^2)$. For drawing n-vertex trees with at most $3n/4$ segments, they reduced the grid size to $n \times n$. Durocher et al. [3] showed that the segment number is NP-hard to compute, even in the special case of arrangement graphs. It is still open, however, whether the segment number is fixed-parameter tractable.

In this report, we consider several variants of the planar segment number seg that has been studied extensively. In particular, we study the *3D segment number* seg_3, which is the most obvious generalization of the planar segment number. It is the smallest number of straight-line segments needed for a crossing-free straight-line drawing of a given graph in 3D. We also study the *crossing segment number* $\mathrm{seg}_\times$ in 2D, where edges are allowed to cross, but they are not allowed to overlap or to contain vertices in their interiors. Finally, for planar graphs, we study the *bend segment number* $\mathrm{seg}_\angle$ in 2D, which is the smallest number of straight-line segments needed for a crossing-free polyline drawing of a given graph in 2D. For a given polyline drawing δ of a graph in 2D or 3D, let $\mathrm{seg}(\delta)$ be the number of straight-line segments of which the drawing δ consists.

Table 1 Overview over our results for cubic graphs. The lower and upper bounds depend on the vertex connectivity γ of the given n-vertex graph G. Note that seg and $\mathrm{seg}_\angle$ are defined only for planar graphs.

γ	$\mathrm{seg}(G)$	$\mathrm{seg}_3(G)$	$\mathrm{seg}_\angle(G)$	$\mathrm{seg}_\times(G)$
1	$\geq 5n/6$ (Prop. 4)	$\geq 5n/6$ (Prop. 4)	$\geq 5n/6$ (Prop. 4)	$\geq 5n/6$ (Prop. 4)
2		$\leq 5n/4 + 1/2$ (Thm. 5)	$\leq n+1$ (Prop. 6)	
	$\geq 3n/4$ (Prop. 8)	$\geq 5n/6$ (Prop. 7)	$\geq 3n/4$ (Prop. 8)	$\geq 3n/4$ (Prop. 8)
3	$n/2 + 3$ [5, 8]	$\leq n$ (except $K_{3,3}$; Thm. 9)		
		$\geq 9n/14$ (Prop. 10)	$\mathrm{seg}_\angle(G) = \mathrm{seg}(G)$	

Table 1 gives an overview over our results for connected ($\gamma = 1$), biconnected ($\gamma = 2$), and triconnected ($\gamma = 3$) cubic graphs. We sketch some of the proofs in Section 4.2.2. First, however, we establish some relationships between the variants of the segment number; see Section 4.2.1.

4.2.1 Relationships Between Variants of the Segment Number

▶ **Proposition 1.** *For any graph G it holds that $\mathrm{seg}_\times(G) \leq \mathrm{seg}_3(G)$.*

Proof. Let δ be a (crossing-free) straight-line drawing of G in 3D with $\mathrm{seg}(\delta) = \mathrm{seg}_3(G)$. For each triple u, v, w of three distinct vertices of G in δ let $P(u, v, w)$ be a plane spanned by the vectors $u - v$ and $w - v$ and let $\mathcal{P}$ be the set of all such planes. Choose a point A in $\mathbb{R}^3 \setminus \bigcup \mathcal{P}$ that does not lie in the xy-plane. Let δ' be the drawing that results from projecting δ parallel to the vector OA onto the xy-plane. Due to the choice of our projection, δ' may contain crossings, but no edge contains a vertex it is not incident to and no two edges overlap. Hence, $\mathrm{seg}_\times(G) \leq \mathrm{seg}_3(G)$. ◀

▶ **Proposition 2.** *There is an infinite family of planar graphs $(T_i)_{i\geq 4}$ such that T_i has i vertices and the ratios $\mathrm{seg}(T_i)/\mathrm{seg}_3(T_i)$, $\mathrm{seg}(T_i)/\mathrm{seg}_\angle(T_i)$, and $\mathrm{seg}(T_i)/\mathrm{seg}_\times(T_i)$ all tend to 2 with increasing i.*

Proof sketch. We construct the graph T_i starting from a triangulation with maximum degree 6 and $t_i = i$ vertices (and, hence, $3i - 6$ edges and $2i - 4$ faces). For example, take two triangular grids and glue their boundaries. We assume that i is even. To each vertex v of the triangulation, we attach an *i-fan*, that is, a path of length i each of whose vertices is connected to v. Now the idea of the proof is that, for every i-fan that must be drawn inside one of the interior faces, we need roughly i segments if we cannot bend edges, use crossings, or exploit 3D. Otherwise, we need only about $i/2$ segments. ◀

4.2.2 Cubic Graphs

Now we turn to cubic graphs. Consider a straight-line drawing δ of a cubic graph (in 2D or 3D). Note that there are two types of vertices; those where exactly one segment ends and those where three segments end. We call these vertices *flat vertices* and *tripods*, respectively. Let $f(\delta)$ be the number of flat vertices, and let $t(\delta)$ be the number of tripods in δ.

▶ **Lemma 3.** *For any straight-line drawing δ of a cubic graph with n vertices, $\mathrm{seg}(\delta) = 3n/2 - f(\delta) = n/2 + t(\delta)$.*

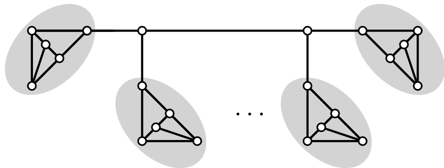

Figure 1 The graph G_k (here $k = 4$) is a caterpillar with $k - 2$ inner vertices of degree 3 where each leaf has been replaced by a copy of the 5-vertex graph H (shaded gray).

Proof. The number of "segment ends" is $3t(\delta) + f(\delta) = 3n - 2f(\delta) = n + 2t(\delta)$. The claim follows since every segment has two ends. ◀

▶ **Proposition 4.** *There is an infinite family $(G_k)_{k\geq 1}$ of connected cubic graphs such that G_k has $n_k = 6k - 2$ vertices and $\mathrm{seg}(G_k) = \mathrm{seg}_3(G_k) = \mathrm{seg}_\angle(G_k) = \mathrm{seg}_\times(G_k) = 5k - 1 = 5n_k/6 + 2/3$.*

Proof sketch. Consider the graph G_k depicted in Fig. 1 (for $k = 4$). In each gray-shaded subgraph, at least two vertices are tripods. Hence, for any drawing δ of G, $t(\delta) \geq 2k$. Now Lemma 3 yields that $\mathrm{seg}(\delta) \geq 5k - 1$. For the drawing in Fig. 1, the bound is tight. ◀

Every biconnected cubic graph G admits an st-ordering, that is, an ordering $\langle v_1, \ldots, v_n\rangle$ of the vertex set $\{v_1, \ldots, v_n\}$ of G such that for every $j \in \{2, n-1\}$ vertex v_j has at least one predecessor (that is, a neighbor v_i with $i < j$) and at least one successor (that is, a neighbor v_k with $k > j$). Using an st-ordering of the given graph, we can construct a straight-line drawing of the graph in 3D and bound the number of segments in the drawing as follows.

▶ **Theorem 5.** *For any biconnected cubic graph G with n vertices, $\mathrm{seg}_3(G) \leq 5n/4 + 1/2$.*

▶ **Proposition 6.** *For any biconnected planar cubic graph G with n vertices, it holds that $\mathrm{seg}_\angle(G) \leq n + 1$. A corresponding drawing can be found in linear time.*

Proof. We draw G using the algorithm of Liu et al. [7] that draws any planar biconnected cubic graph except the tetrahedron orthogonally with at most one bend per edge and at most $n/2 + 1$ bends in total. It remains to count the number of segments in this drawing. In any vertex exactly one segment ends; in any bend exactly two segments end. In total, this yields at most $n + 2 \cdot (n/2 + 1) = 2n + 2$ segment ends and at most $n + 1$ segments.

Concerning the special case of the tetrahedron (K_4), note that it can be drawn with five segments when bending one of its six edges. ◀

▶ **Proposition 7.** *There is an infinite family of cubic graphs $(H_k)_{k\geq 3}$ such that H_k has $n_k = 6k$ vertices, $\mathrm{seg}_3(H_k) = 5k = 5n_k/6$, and $\mathrm{seg}_\times(H_k) = 4k = 2n_k/3$.*

Proof sketch. Consider the graph H_k depicted in Fig. 2 (for $k = 4$). It is a k-cycle where each vertex is replaced by a copy of a 6-vertex graph K ($K_{3,3}$ minus an edge). The graph H_k has $n_k = 6k$ vertices and is not planar. In any 2D drawing with crossings at least one vertex in each copy of K is a tripod; in 3D at least two vertices in each copy are tripods. Now Lemma 3 yields that $\mathrm{seg}_\times(H_k) \geq 4k$ and $\mathrm{seg}_3(H_k) \geq 5k$.

Figure 2 shows that $\mathrm{seg}_\times(H_k) \leq 4k$ and, by lifting in each copy of K the white vertex that is not on the convex hull out of the drawing plane, that $\mathrm{seg}_3(H_k) \leq 5k$. ◀

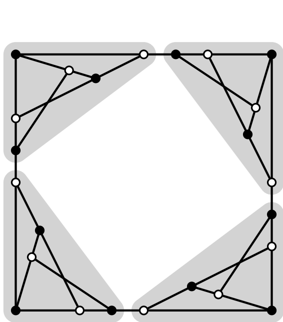

Figure 2 The cubic graph H_k (here $k = 4$) is a k-cycle whose vertices are replaced by the subgraphs in the gray shaded regions ($K_{3,3}$ minus an edge). The graph H_k has $n_k = 6k$ vertices, $\text{seg}_3(H_k) = 5n_k/6$, and $\text{seg}_\times(H_k) = 2n_k/3$.

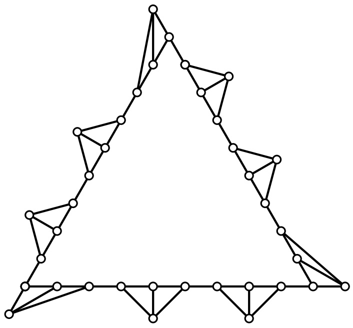

Figure 3 The planar Hamiltonian cubic graph I_k (here $k = 9$) is a k-cycle whose vertices are replaced by copies of K_4 minus an edge. The graph I_k has $n_k = 4k$ vertices and $\text{seg}(I_k) = \text{seg}_3(I_k) = \text{seg}_\angle(I_k) = \text{seg}_\times(I_k) = 3n_k/4$.

▶ **Proposition 8.** *There is an infinite family of planar cubic Hamiltonian graphs $(I_k)_{k\geq 3}$ such that I_k has $n_k = 4k$ vertices and $\text{seg}(I_k) = \text{seg}_3(I_k) = \text{seg}_\angle(I_k) = \text{seg}_\times(I_k) = 3k = 3n_k/4$.*

Proof sketch. Consider the graph I_k depicted in Fig. 3 (for $k = 9$). The proof is similar to that of the crossing case in Proposition 7. ◀

▶ **Theorem 9.** *Every triconnected cubic n-vertex graph admits a straight-line drawing in 3D with at most n segments – except $K_{3,3}$, which needs seven segments.*

Proof sketch. Partition the given graph into a perfect matching and a collection of pairwise disjoint cycles. Treat each cycle separately and draw it on a copy of the moment curve. ◀

▶ **Proposition 10.** *There is an infinite family of triconnected cubic graphs $(F_k)_{k\geq 4}$ such that F_k has $n_k = 14k$ vertices and $\text{seg}_3(F_k) = 9k = 9n_k/14$.*

Proof sketch. Let K' be the graph that results from removing one edge from $K_{3,3}$ and subdividing another edge. Now take any triconnected cubic graph with $2k$ vertices and replace each of its vertices by a copy of the 7-vertex graph K'. The resulting graph F_k has $n_k = 14k$ vertices and is not planar.

The proof that $\text{seg}_3(F_k) = 9k$ is similar to that of the 3D case in Proposition 7. ◀

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4.3 Simultaneous Graph Embedding Beyond Planarity

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Abstract. Simultaneous Graph Embedding asks the question whether a set of graphs $\mathcal{G}$ with shared vertex set V can be embedded in the plane such that each graph in $\mathcal{G}$ is drawn planar. We study this problem in the beyond planarity framework by allowing the graphs in $\mathcal{G}$ to have crossings between their edges as long as they respect certain crossing configurations. We call this setting `Beyond-Simultaneous`. In addition, we also study a setting called `Beyond-Union`, where we require the union of all graphs in $\mathcal{G}$ to fulfill restrictions on the crossing configurations.

We show that in setting `Beyond-Simultaneous` two planar graphs and a tree can always be realized such that each of the graphs is drawn quasiplanar, we also prove that the same holds for a 1-planar graph and a planar graph. Further, we show that in setting `Beyond-Union`, a path and a matching cannot always be embedded such that their union is k-planar for a fixed k whereas five cycles cannot always be drawn such that their union is quasiplanar.

4.3.1 Introduction

Simultaneous Graph Embedding is a family of problems where you are given a set of graphs $\mathcal{G} = \{G_1, \ldots, G_k\}$ with shared vertex set V and you are required to produce drawings $\{\Gamma_1, \ldots, \Gamma_k\}$ of them in such a way that each vertex has the same position in every Γ_i and each Γ_i satisfies certain readability properties. Usually, the readability property that is pursued while searching for a simultaneous embedding is planarity and a large body of research has been dedicated to the complexity of deciding whether a set of graphs admits such simultaneous embeddings or to determine if such embeddings always exist given the number and the types of the input graphs; for a survey refer to [6].

Simultaneous Graph Embedding has been studied both from a geometric point of view (Geometric Simultaneous Embedding – `GSE`) [5, 10] and from a topological point of view (Simultaneous Embedding with Fixed Edges – `SEFE`) [7, 8]. In particular, in `GSE`, the edges are required to be straight-line segments while in `SEFE` they can be drawn as topological curves, but the edges shared between two graphs G_i and G_j have to be drawn in the same way in Γ_i and Γ_j. In the following, we focus on the topological setting unless otherwise specified.

We study two variants of the simultaneous embedding problem in the beyond planarity framework by allowing the graphs in $\mathcal{G}$ to be drawn non-planar. In the first problem, we only restrict the crossings in each of the graphs $G \in \mathcal{G}$.

► **Problem 1** (`Beyond-Simultaneous`). *Is it possible to simultaneously embed a set of graphs $\mathcal{G}$ with shared vertex set V in the plane such that each graph $G \in \mathcal{G}$ is drawn k-(quasi)planar?*

Recall that in a k-planar drawing, each edge is crossed at most k times whereas in a k-quasiplanar drawing, there is no k-tuple of pairwise intersecting edges. Also recall, that 3-quasiplanar is often referred to as quasiplanar. In the second problem, we additionally restrict the crossings in the union of all graphs in $\mathcal{G}$.

► **Problem 2** (`Beyond-Union`). *Is it possible to simultaneously embed a set of graphs $\mathcal{G}$ with shared vertex set V in the plane such that the union graph $G_\cup = \bigcup_{G\in\mathcal{G}} G$ is drawn k-(quasi)planar?*

Note that in setting `Beyond-Union` we could also ask each $G \in \mathcal{G}$ to satisfy stronger restrictions on the crossing configurations.

In the remainder of this report, we first present preliminary results in Section 4.3.2 which will be used in our proofs. Then, we investigate the more restricted `Beyond-Union` setting in Section 4.3.3 and show very restrictive negative results. Afterwards, we show positive results in the `Beyond-Simultaneous` setting in Section 4.3.4. We conclude the report by listing open problems in Section 4.3.5.

4.3.2 Preliminaries

We make use of a result on the partially embedded planarity problem (`PEP`) which is defined as follows.

► **Problem 3** (`PEP`). *Let G be a planar graph, H a subgraph of G and $\mathcal{H}$ an embedding of H. Can G be embedded in the plane such that H is drawn with embedding $\mathcal{H}$?*

Problem `PEP` has been introduced and studied in [4] where a linear-time algorithm is presented. In particular, this algorithm is based on a characterization that we will exploit in the following.

► **Lemma 4** ([4]). *Let $(G, H, \mathcal{H})$ be an instance of `PEP` and let $\mathcal{G}$ be a planar embedding of G. $\mathcal{G}$ is a solution for $(G, H, \mathcal{H})$ if and only if the following conditions hold:*

1. *for every vertex $v \in V$, the edges incident to v in H appear in the same cyclic order in the rotation schemes of v in $\mathcal{H}$ and in $\mathcal{G}$; and*
2. *for every cycle C of H, and for every vertex v of $H \setminus C$, we have that v lies in the interior of C in $\mathcal{G}$ if and only if it lies in the interior of C in $\mathcal{H}$.*

Another important tool that we will exploit is the following theorem due to Pach and Wenger [12].

▶ **Theorem 5** ([12]). *Every planar graph on n vertices admits a planar embedding which maps each vertex to an arbitrarily prespecified distinct location and each edge to a polygonal curve with $O(n)$ bends. Further, there exists a path, whose vertices are mapped to a point set in convex position, such that in any embedding of this graph that respects the mapping of vertices to points there exists one edge with a linear number of bends.*

4.3.3 Setting `Beyond-Union`

Here, we first attempt to maintain k-planarity for a fixed k. Unfortunately, this already fails for a path and a matching.

▶ **Theorem 6.** *There exists a family of paths $\mathcal{P}$ and a family of matchings $\mathcal{M}$ such that $P \in \mathcal{P}$ and $M \in \mathcal{M}$ on n shared vertices cannot be simultaneously embedded such that their union is k-planar for any $k \in o(n/\log^2 n)$.*

Proof. To prove the theorem, we exploit a family of 3-regular graphs which is known to be not k-planar for any $k \in o(n/\log^2 n)$ [3]. Consider the hypercube graph $\mathcal{H}_d$ of dimension d. Let v be a vertex of $\mathcal{H}_d$ and let $u_1, \ldots, u_d$ be its neighbors. We replace v by a cycle $(v_1, \ldots, v_d)$ such that v_i is connected to u_i for $1 \leq i \leq d$. By repeating this procedure for all vertices of $\mathcal{H}_d$, we obtain the cube connected cycle graph CCC_d of dimension d which is a cubic graph on $n = d \cdot 2^d$ vertices.

It is known that the crossing number $cr(CCC_d) = \Omega(4^d)$ [13]. Hence, the average number of crossings per edge is $\Omega(2^d/d) = \Omega(n/\log^2 n)$. Further, it is known, that CCC_d is a Hamiltonian graph [11]. Hence, CCC_d is composed of a cycle (the Hamiltonian cycle) and a matching. To obtain the the statement of the theorem, it is possible to show that removing one edge does not alter the arguments. ◀

In addition, when further restricting each of the subgraphs to be drawn planar, there exist even two paths that cannot be drawn with a sublinear number of crossings per edge. We state this fact in the following Theorem, which can be proved with the same reasoning used to prove Lemma 10 and Theorem 8 of a recent manuscript [9]. We repeat the argument for completeness.

▶ **Theorem 7.** *There exist two families of paths $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $P_1 \in \mathcal{P}_1$ and $P_2 \in \mathcal{P}_2$ on n shared vertices cannot be simultaneously embedded such that their union is k-planar for any $k \in o(n)$ if P_1 and P_2 are embedded planar.*

Proof. Assume for contradiction that every two paths P_1 and P_2 on n shared vertices admit a simultaneous embedding such that both are drawn planar and that their union is $o(n)$-planar. Since we have a simultaneous embedding we can construct a drawing on a point set so that P_1 is drawn monotone and straight-line and each edge of P_2 has as many bends as it has intersections with P_1. In such a drawing, P_1 describes a convex point set for P_2. Hence, every path P_2 admits a planar drawing on every point set such that each of its edges is only bent $o(n)$ times. This is a contradiction to Theorem 5. ◀

In the next step, we shift our attention to quasiplanar embeddings of unions of graphs. Since the union of two planar graphs has thickness two, two planar graphs can always be simultaneously embedded such that their union is quasiplanar [12]. We show however, that even for a few cycles quasiplanarity cannot be maintained:

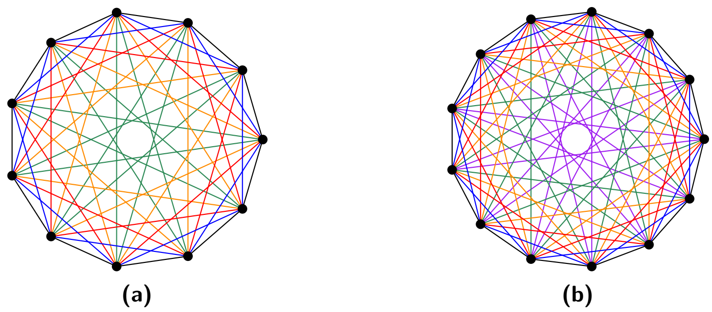

Figure 4 (a) K_{11} is the union of five cycles. (b) K_{13} is the union of six cycles.

▶ **Theorem 8.** *There exist five cycles $C_1 = (V, E_1)$, $C_2 = (V, E_2)$, $C_3 = (V, E_3)$, $C_4 = (V, E_4)$ and $C_5 = (V, E_5)$ on $|V| = 11$ vertices which cannot be simultaneously embedded such that their union is simple quasiplanar. In addition, there exist six cycles $C'_1 = (V', E'_1), \dots, C'_6 = (V', E'_6)$ on $|V'| = 13$ vertices which cannot be simultaneously embedded such that their union is quasiplanar.*

Proof. Consider K_{11}. It has $\binom{11}{2} = 55$ edges. Since simple quasiplanar graphs have density $6.5n - 20$ [2], K_{11} cannot be quasiplanar. Further K_{11} is the union of the following five cycles:

- $C_1 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11})$
- $C_2 = (v_1, v_3, v_5, v_7, v_9, v_{11}, v_2, v_4, v_6, v_8, v_{10})$
- $C_3 = (v_1, v_4, v_7, v_{10}, v_2, v_5, v_8, v_{11}, v_3, v_6, v_9)$
- $C_4 = (v_1, v_5, v_9, v_2, v_6, v_{10}, v_3, v_7, v_{11}, v_4, v_8)$
- $C_5 = (v_1, v_6, v_{11}, v_5, v_{10}, v_4, v_9, v_3, v_8, v_2, v_7)$

For an illustration, refer to Fig. 4a.

Similar arguments for K_{13} apply for the non-simple case; see Fig. 4b. ◀

4.3.4 Setting `Beyond-Simultaneous`

▶ **Theorem 9.** *Let $G_1 = (V, E_1)$ and $G_3 = (V, E_3)$ be planar graphs and $T_2 = (V, E_2)$ be a tree with shared vertex set V. Then G_1, T_2 and G_3 can be simultaneously embedded in the plane such that G_1 and T_2 are drawn planar and G_3 is drawn quasiplanar.*

Proof. Our strategy is to construct first a simultaneous embedding of G_1 and T_2 and then of the resulting graph with G_3. When constructing a simultaneous embedding of two graphs, we consider the graph induced by their common edges as a subgraph for which we want to satisfy the conditions of Lemma 4. Since this subgraph is always a forest due to the fact that T_2 is a tree, Condition 2 is always satisfied. For Condition 1, we already take into account the conditions imposed by the planar embedding of G_3 to the embedding of T_2 while constructing the simultaneous embedding of G_1 and T_2. Namely, we first embed G_1 in the plane such that G_1 is planar. Then, we add the edges $E_2 \setminus E_1$ without intersecting an edge of $E_2 \cap E_1$. Finally, we draw $G'_3 = (V, E_3 \setminus E_1)$ planar. Hence, edges of G'_3 can only intersect edges of G_3 which are part of G_1 resulting in a quasiplanar drawing of G_3.

We draw the remaining edges of T_2 without intersecting $E_2 \cap E_1$ as follows: We observe, that $(V, E_2 \cap E_1)$ is a planar drawn subforest of T_2. Since T_2 is a tree any of its embeddings is planar. Hence, Condition 1 stated in Lemma 4 is trivially fulfilled. Moreover, for edges

$E_3 \cap (E_2 \setminus E_1)$, we can chose an ordering around each vertex such that it corresponds to a planar embedding $\mathcal{G}'_3$ of planar graph G'_3. The remaining edges of T_2 can be arbitrarily embedded.

When embedding G'_3, we already have embedded edges $E_3 \cap (E_2 \setminus E_1)$. Since we have chosen the embedding of these edges such that they respect the proper planar embedding $\mathcal{G}'_3$ of G'_3, Condition 1 stated in Lemma 4 is again fulfilled. Thus, we can extend the partial embedding of G'_3 to planar embedding $\mathcal{G}'_3$. ◀

▶ **Corollary 10.** *Let $G_1 = (V, E_1)$ be a 1-planar graph and $G_2 = (V, E_2)$ be a planar graph. Then G_1 and G_2 can be simultaneously embedded in the plane such that both G_1 and G_2 are drawn quasiplanar.*

Proof. Since G_1 is 1-planar, it is the union of a planar graph G'_1 and a forest F_1 with shared vertex set [1]. By Theorem 9, there exists a simultaneous embedding of G'_1, F_1, and G_2 such that G'_1 and F_1 are drawn planar and G_2 is drawn quasiplanar. Since the union of two planar drawings with same vertex set is quasiplanar, G_1 is drawn quasiplanar, as well. ◀

4.3.5 Open Problems

- Our results show that asking for k-planarity is too restrictive in setting `Beyond-Union`, while for quasiplanarity we have a counterexample for a set of five cycles. What about the quasiplanarity of the union of a small set of paths (e.g. 3 or 4)?
- In the setting `Beyond-Simultaneous`, we ask what is the smallest set of graph families which cannot be always simultaneously embedded so that each graph is quasiplanar. In particular, can three planar graphs (or two 1-planar graphs, or four paths) always be simultaneously embedded such that each one is drawn quasiplanar?
- How difficult is it to test whether a given set of graphs admits a `Beyond-Union` or `Beyond-Simultaneous` embedding?

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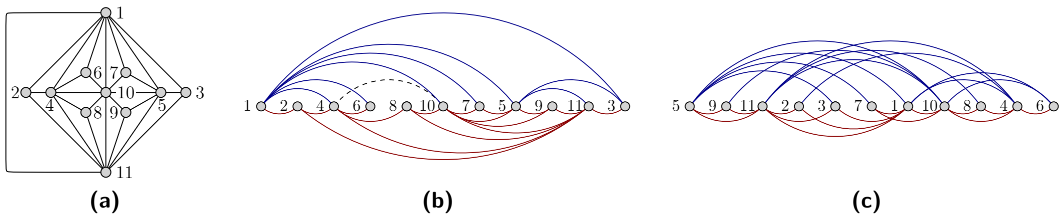

Figure 5 (a) The Goldner-Harary graph; (b) a 3-stack layout; and (c) a 2-queue layout.

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4.4 On Linear Layouts of Planar and *k*-Planar Graphs

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4.4.1 Introduction and Related Work

A *linear layout* of a graph G consists of a linear order of the vertices of G and of a partition of the edges of G that satisfies a certain property and whose size is given. In what follows, we study two well-known types of linear layouts, namely stack and queue layouts. Moreover, we consider linear layouts in which these two types are mixed.

4.4.1.1 Stack Layouts

We first consider stack layouts, also known as *book embeddings*, which form a fundamental problem in graph theory (see, e.g., [7] for an overview). In a stack layout, the edge partition is such that no two edges of the same part, which is called *stack*, cross; see Figure 5b. The *stack number*, or *book thickness*, of a graph is the smallest number of stacks that are required by any stack layout of the graph.

Problems on stack layouts are mainly classified into two categories based on whether the graph to be embedded is planar or not. For non-planar graphs, it is known that there exist graphs on n vertices that have stack number $\Theta(n)$, e.g., the stack number of the complete graph K_n is $\lceil n/2 \rceil$ [6]. Sublinear stack number is achieved by graphs with, e.g., subquadratic number of edges [28], subquadratic genus [27] or sublinear treewidth [13]. Constant stack number is achieved by graphs that are, e.g., in a minor-closed family [8] or in a bounded-treewidth family [21]. Another class of non-planar graphs that was proved to have constant stack number is the class of 1-planar graphs [4].

For planar graphs, a remarkable result is due to Yannakakis, who back in 1986 proved that for any planar graph four stacks suffice [34]. However, more restricted subclasses of planar graphs allow layouts with fewer stacks. Bernhart and Kainen [6] showed that the graphs which can be embedded using a single stack are the outerplanar graphs, while the graphs which can be embedded using two stacks are the subhamiltonian ones.

It is known that not all planar graphs are subhamiltonian and the corresponding decision problem whether a maximal planar graph is Hamiltonian (and therefore admits a 2-stack layout) is $\mathcal{NP}$-complete [32]. However, several subclasses of planar graphs are known to be Hamiltonian or subhamiltonian, see, e.g., [3, 9, 10, 22, 25, 29].

4.4.1.2 Queue Layouts

A *queue layout* is a linear layout such that no two independent edges that are assigned to the same part, which is called a *queue*, are nested [24]; see Figure 5c for an illustration. The *queue number* of a graph G is the minimum number of queues in any queue layout of G.

It is known that there exist non-planar graphs on n vertices with $\Theta(n)$ queue number, for example, the queue number of the complete graph K_n is $\lfloor n/2 \rfloor$ [24]. Moreover, there exist graphs of bounded degree that may require arbitrarily many queues [33]. Among the graphs having sublinear queue number are those with a subquadratic number of edges [23], and those that belong to any minor-closed graph family [17]. Bounded queue number is achieved by all graphs of bounded treewidth [16]. In particular, a graph with treewidth w has queue number $\mathcal{O}(2^w)$ [31]. Improved bounds (linear in the parameter) are known for graphs of bounded pathwidth [16], bounded track number [19], bounded bandwidth [23], or bounded layered pathwidth [2]; for a survey we refer the reader to [17].

A rich body of literature focuses on planar graphs. In fact, it is known that the graphs that admit queue layouts with only one queue are the arched-level planar graphs [24], which are planar graphs with at most $2n - 3$ edges over n vertices (note that testing whether a graph is arched-level planar is $\mathcal{NP}$-complete [23]). Trees are arched-level planar and therefore have queue number one [24]. Outerplanar graphs have queue number at most two [23], Halin graphs and series-parallel graphs have queue number at most three [20, 30], and planar 3-trees have queue number at most five [1]. Back in 1992, Heath, Leighton and Rosenberg [23] conjectured that every planar graph has bounded queue number. Notably, this conjecture has been an open problem for almost three decades. Recently, the conjecture was settled in the positive first for planar graphs with bounded degree [5, 18], and subsequently for general planar graphs [15], thus improving the previous logarithmic and poly-logarithmic upper bounds [2, 11, 14]. On the other hand, the best-known lower bound is due to a family of planar 3-trees that require four queues [1].

4.4.2 Problems and Progress

In what follows we give a high-level description of the problems we studied and of the progress we made for them. In particular, we mainly focused on two research problems: linear layouts of directed planar graphs and nonplanar graphs that can be drawn with few crossings per edge.

4.4.2.1 Upward Planar Graphs

An *upward stack (queue) layout* of a directed graph G is a stack (queue) layout of G such that the linear ordering of the vertices is a linear extension of the partial order induced by the directions on the edges of G; that is, for any edge directed from a vertex u to a vertex v, we have that u precedes v in the linear ordering. We consider upward planar graphs, that is, planar directed graphs that can be drawn without crossings and such that each edge is a y-monotone curve from its source to its target. It is a longstanding open question to determine the asymptotic behavior of the upward stack number of upward planar graphs. Surprisingly, the best known bounds are only the trivial ones, $\mathcal{O}(n)$ and $\Omega(1)$. Contrastingly, it is known that the upward queue number of upward planar graphs is $\Theta(n)$ in the worst case.

During the Dagstuhl seminar, we proved that every n-vertex upward planar graph has a *mixed* layout with $\mathcal{O}(\sqrt{n})$ stacks and $\mathcal{O}(\sqrt{n})$ queues. We proved that this bound is tight if the vertex ordering is fixed in advance. We also proved that $\mathcal{O}(\log n)$ stacks are enough to construct stack layouts of n-vertex upward outerplanar graphs. Constant bounds can be achieved for upward outerplanar st-graphs and upward outerplanar single-source graphs.

4.4.2.2 k-Planar Graphs

A graph is *k-planar*, for a positive integer k, if it can be drawn in the plane such that each edge is crossed at most k times (see [12, 26] for surveys). Recall that every 1-planar graph admits a stack layout with a constant number of stacks [4]. Moreover, for a fixed value of k, every k-planar graph admits a queue layout with a constant number of queues [15].

During the Dagstuhl seminar, we sketched a proof that every graph that admits a drawing in the plane such that the uncrossed edges form a biconnected planar drawing in which each face has length at most ℓ admits a stack layout with a number of stacks that depends polynomially in ℓ and that does not depend on the size of the graph. Observe that any such a graph is also k-planar, where $k \leq \frac{\ell^2}{4}$.

4.4.3 Open Problems

The main objectives for our research are the following open problems.

- What is the asymptotic behavior of the upward stack number of n-vertex upward planar graphs? The question is interesting even for n-vertex upward planar graphs without transitive edges.
- What is the largest integer k such that every directed acyclic graph whose underlying graph has treewidth at most k has upward stack number in $O(1)$? We proved that $k \leq 2$; further, it is known that $k \geq 1$. We conjecture that $k = 2$; this strengthens a conjecture of Heath, Pemmaraju and Trenk on the upward stack number of directed outerplanar graphs. The above question is interesting even for upward planar graphs whose underlying graph has treewidth at most k, where we are not aware of any upper bound on k.

- Establish a worst-case optimal upper bound for the stack number of general k-planar graphs, ideally $\mathcal{O}(k)$.
- Establish upper bounds for the stack number of other families of nonplanar graphs, such as fan-planar graphs, fan-crossing-free graphs and k-quasiplanar graphs (see [12] for definitions and results about these families of graphs).

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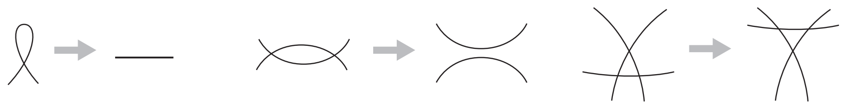

Figure 6 Homotopy moves $1 \to 0$, $2 \to 0$, and $3 \to 3$.

4.5 Monotone Untangling of Graph Drawings

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Abstract. Given a planar graph drawn in the plane with edge crossings, our goal is to untangle it to a crossing-free drawing using a sequence of moves that never increase the number of crossings. We consider two types of moves: continuous *homotopy moves*; and the more general *edge moves* that remove and redraw one edge at a time. We call a move *monotone* if it does not increase the number of crossings. Thus our goal is to untangle the graph drawing using monotone homotopy moves or monotone edge moves.

4.5.1 Our Results

1. With homotopy moves, if the tangled drawing has been created from a planar drawing with homotopy moves that never move an edge across a vertex, then the drawing can be untangled using monotone homotopy moves that never move an edge across a vertex.
2. With monotone edge moves we can untangle any drawing of a cactus graph, and we can untangle any drawing of a *banana cactus* graph if the drawing has a planar rotation system.
3. Not every drawing of a planar graph can be untangled with monotone edge moves.

4.5.2 Background and Concepts

In one well-studied version of untangling a straight-line graph drawing, the goal is to move as few vertices as possible in order to get a planar straight-line drawing, see [9, 1] and references therein. In this version, the vertices are re-positioned all at once. By contrast, we fix the vertices and consider a sequence of incremental changes to the curves representing the edges.

There is considerable work on untangling a curve or a set of curves using incremental homotopy moves. Homotopy moves, which are the "shadows" of the classical Reidemeister moves, are defined as follows:

These moves are monotone, but the reversals of $1 \to 0$ and $2 \to 0$ are not.

Our preliminary understanding of the relevant background work is as follows. An algorithm to simplify any planar closed curve using at most $O(n^2)$ monotone homotopy moves is implicit in Steinitz's proof [10, 11] that every 3-connected planar graph is the 1-skeleton of a convex polyhedron. For more information, see [5, 7, 4, 2]. Chang and Erickson [2] improved the number of moves to a tight bound of $\Theta(n^{3/2})$ but at the expense of losing monotonicity. For monotone moves, no bound better than $O(n^2)$ is known.

One of Steinitz's basic ideas was extended by Hass and Scott [6, 7] to Theorem 3 (stated below), which has been used in many subsequent works and which we will also use.

For generalizations to multiple curves, tangles, and other surfaces (with boundary and/or of higher genus), see [2] and the references therein. Chang's thesis [3] is an excellent resource.

Homotopy moves, and in particular, the result of Hass and Scott, have been applied to graph drawings, for example in the work by Kynčl [8] on simple realizability of complete abstract topological graphs. Proofs of versions of the Hanani-Tutte theorem may also be relevant.

4.5.3 Untangling via Monotone Homotopy Moves

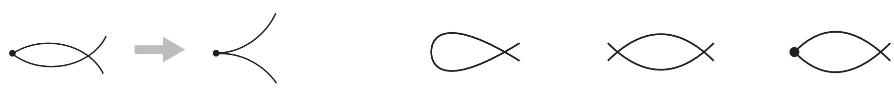
We may interpret a drawing of the graph G as a set of curves on the punctured plane, where each curve starts and ends at a boundary component. With this interpretation, two curves are homotopic if one can be continuously deformed to the other in the punctured plane (i.e., this replaces the condition that we never move an edge across a vertex). In order to deal with multiple curves (edges), we allow the homotopy move shown in Figure 7a.

▶ **Theorem 1.** *Let D and D^* be two drawings of a planar graph G such that D^* is plane, the vertex positions coincide, and for each edge e of G the curve representing e in D is homotopic to the curve representing e in D^*.*

Then D can be transformed into D^* by a sequence of monotone homotopy moves. The number of moves is at most $k + \frac{1}{4}k^2$ where k is the number of crossings in D.

We start with some definitions: A curve is *simple* if it has no self-intersections. A *loop* is a section of a curve such that its endpoints coincide and it has no other self-intersections, i.e., it is simple.

A *lens* consists of two sections of curves (either two disjoint sections of the same curve or sections of distinct curves) such that each is simple and connects two distinct points; moreover, between the two sections there exist no other intersections. See Fig. 7b.



(a) A $1 \to 0$ homotopy move in the presence of a vertex.

(b) An empty loop and two empty lenses.

Figure 7

Note that a loop or lens forms a simple closed curve which has an interior and an exterior. We say that a loop is *empty* if neither its interior nor the loop itself contains a vertex; a lens is *empty* if its interior does not contain any vertex and the lens is incident to at most one vertex placed on an intersection point of the two sections (of two different curves) of the lens. We call a loop or a lens *clean* with respect to a set of curves if it is not involved in any crossing (except for the one crossing of a loop and the two crossings of the lens by definition). Note that a clean lens does not need to be empty; it may contain many vertices in its interior.

The idea of our procedure is to untangle empty lenses until we arrive at a crossing-free drawing. To do so, we first show that a non-simple curve (edge) guarantees the existence of an empty loop or empty lens (Lemma 2). Moreover, two simple and intersecting curves guarantee the existence of an empty lens (Lemma 4). Finally, we show that given an empty lens or empty loop, there exists a sequence of monotone homotopy moves that removes the empty lens or loop and reduces the number of crossings (Lemma 5).

▶ **Lemma 2.** *If a curve in D representing an edge of G (a curve homotopic to a simple curve) is not simple then it contains an empty loop or an empty lens (not incident to any vertex).*

Lemma 2 follows from a theorem of Hass and Scott. We state their theorem in the original language of the paper: an *embedded 1-gon* is an empty loop and an *embedded 2-gon* is an empty lens; two arcs between vertices are *homotopic rel boundary* if they are homotopic (in the punctured plane).

▶ **Theorem 3** (Hass and Scott [6], Theorem 2.1). *Let f be a general position arc on a surface F such that f is homotopic rel boundary to a simple arc g on F, but f is not simple. Then, the arc f has an embedded 1-gon or 2-gon.*

When we overlay D and D^*, every edge is represented by two curves which together form a closed curve that is incident to two vertices. We call the curves from D *curvy* and the curves from D^* *straight*.

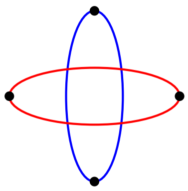
▶ **Lemma 4.** *Let C_1 and C_2 be two simple closed curves that contain no vertex and such that each curve C_i is incident to two distinct vertices that split it into two parts, the* straight *and the* curvy *part. If C_1 and C_2 intersect but their straight parts do not intersect, then (parts of) C_1 and C_2 form an empty lens (that may be incident to one vertex).*

Proof Sketch. Let e be a part of C_1 that intersects C_2. We keep track of the sequence of intersections with the parts of C_2 by a word over the alphabet $\{s, c\}$, where s represents an intersection with the straight part and c an intersection with the curvy part of C_2. First we consider the case that the vertices of C_1 and C_2 are distinct. Thus, in particular, e starts and ends outside of C_2. Note that a subword ss or cc represents a lens (since both curves are simple) which is empty (since it is contained in the interior of C_2 which contains no vertices). Assume for the sake of a contradiction that any two consecutive letters are different in the word. Note that the word has even length, otherwise C_2 contains a vertex of e. Thus the word has the form $(sc)^k$ or $(cs)^k$ for some $k \in \mathbb{N}$. Since e intersects s, it must be the curvy part of C_1. Consequently, $C_1 - e$ is the straight part of C_1, and does not intersect s by assumption.

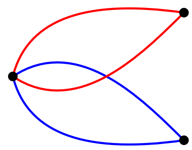
Analogous argument for C_2 implies that all intersections between C_1 and C_2 are in their curvy parts. Consequently, the word representing the intersections of e is s^k for some $k \in \mathbb{N}$, which is a contradiction.

Now, we consider the case that C_1 and C_2 share a vertex v. If both edges of C_1 start and end outside of C_2, the above argument yields an empty lens. Thus, at least one edge e of C_1 starts inside C_2 at v. Clearly, it ends outside of C_2 since its endvertex is not incident to C_2 and C_2 does not contain any vertex. The first intersection point on e from v certifies an empty lens; as before, it is a lens since the curves are simple and it is empty since it is contained in C_2. ◀

Remark: Note that the fact that no two straight parts intersect is necessary for the existence of a lens, see Fig. 8a.



(a) An example without a lens. However, the two straight parts intersect.



(b) A 1-0-move is necessary to change the rotation system.

Figure 8

▶ **Lemma 5.** *If there exists an empty loop or an empty lens (in the union of D and D^*) of one or two curves (possibly with one vertex), then there exists a sequence of monotone homotopy moves to decrease the number of crossings of the set of curves.*

Note that a monotone homotopy move never introduces a new pair of crossing curves; hence even the (multi-)set of crossings is monotonically non-increasing. Thus artificial/imaginary crossings between curves of D and D^* will never make up for real crossings. The proof of Lemma 5 is similar to a result by Hass and Scott [7, Lemma 2.6], but we also handle the case that one crossing of the lens is a vertex.

Proof Sketch. Suppose we are given an empty loop. We may assume that it contains neither a loop nor a lens in its interior; otherwise we consider the minimal such loop or lens, which would be empty and not contain any vertex on the boundary. Thus this loop is clean and the loop can be removed with a 1-0-move.

Suppose we are given an empty lens. Without loss of generality we consider the case with precisely one vertex (formed by different curves); the case with no vertex can be handled by inserting an artificial vertex on one of the two intersection points. We may assume that the lens contains no loop; otherwise we take a minimal loop which is empty and has no vertex on the boundary. We may further assume that it contains no lens, since every contained lens is empty and not incident to the vertex since its sections belong to different curves (or the vertex is artificial). By definition, the lens consists of two parts which meet at a (artificial) vertex v and a further intersection point p. Note that every section of a curve intersecting the interior of the lens connects the two parts of the lens. We *clean* the lens by moving the crossings inside the lens outside, by 3-3-moves. (Here we use the fact that any arrangement of chords in a circle where at least two chord cross contains at least two triangles incident to the circle. One of these triangles is not incident to the vertex on the loop and a crossing can be moved outside the loop by a 3-3-move. Consequently, there is a linear order of the edges of the arrangement in the interior of the lens. They can be moved outside the lens one by one with 3-3-moves over p in this order. Thus, we have a clean and empty lens, which can be removed via a 2-0 or 1-0 move. ◀

With these lemmas at hand, we are ready to prove the theorem.

Proof of Theorem 1. As long as the drawing has crossings, Lemmas 2 and 4 guarantee the existence of an empty loop or lens. Given an empty loop or lens, there exists a sequence of monotone homotopy moves to reduce the number of crossings by Lemma 5. Thus, in the end, we have transformed $D + D^*$ into a crossing-free drawing $\tilde{D} + \tilde{D}^*$. Since $\tilde{D} + \tilde{D}^*$ is crossing-free, $\tilde{D}$ and $\tilde{D}^+$ have the same rotation system and are thus equivalent drawings. Moreover, since the set of crossing is monotonically decreasing, the drawings D'^* and D^+ have the same rotation system. Consequently, we have transformed D into D^* by a sequence of monotone homotopy moves.

Number of moves: Note that the number of moves to clean an empty lens is upper bounded by the number of remaining crossings. Untangling the lens by a 2-0 move reduces the number of crossings by 2. Recall that a minimal loop is already clean and hence, it takes one move to reduce the number of crossings by one. Consequently, the number of homotopy moves is upper bounded by $k + \frac{1}{4}k^2$. ◀

4.5.4 Untangling via Edge Moves

In this section we consider a more general move called an *edge move* that removes one edge and then redraws it. Note that the vertices remain fixed (as for homotopy moves). We examine the power of monotone edge moves to untangle some special graphs. Firstly, we observe that

▶ **Theorem 6.** *Not every drawing of a planar graph can be untangled with monotone edge moves.*

Proof. We show that the statement holds for two interlaced K_4's as depicted in Fig. 9a. Note that every edge is involved in exactly one crossing and redrawing it in a non-equivalent way introduces at least two crossings. ◀

(a) A drawing of two interlaced K_4's that cannot be untangled with monotone edge moves, nor, consequently, with monotone homotopy moves.

(b) The rotation system is that of a planar drawing, but it must change during homotopy moves.

Figure 9

A *cactus* graph is a graph such that every 2-connected component is an edge or a cycle. A *banana* graph is a graph with two vertices joined by an edge that has multiplicity at least 2. A *banana cactus* is a cactus graph in which each edge may be replaced by a banana. We prove:

▶ **Lemma 7.** *The following can be untangled using monotone edge moves:*

- *Any drawing of a cactus graph.*
- *Any drawing of a banana graph.*
- *Any drawing of a banana cactus in which the rotation system of the drawing (the cyclic order of edges incident to each vertex) belongs to a planar drawing.*

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4.6 Gap Planarity

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Recently, Bae et al. [2] defined k-gap-planar graphs, for $k \in \mathbb{N}$, that admit drawings in which each edge is "responsible" for up to k crossings. By Hall's matching theorem, it is equivalent to the following.

▶ **Definition 1.** A graph $G = (V, E)$ is *k-gap-planar* if it has a drawing in the plane so that for every subgraph $G = (V', E')$ there are at most $k|E'|$ crossings between the edges in E'.

Ossona de Mendez, Oum, and Wood [3] introduced a similar definition.

▶ **Definition 2.** A graph $G = (V, E)$ is *k-close-to-planar* if every subgraph $G' = (V', E')$ has a drawing in the plane with at most $k|E'|$ crossings (i.e., $\mathrm{cr}(G') \leq k|E'|$).

It is clear that every k-gap-planar graph is k-close-to-planar. Is the converse true?

Answer: No. We show that graph $K_{6,6}$ is a counterexample. Bachmaier, Rutter, and Stumpf [1] show that $K_{6,6}$ is not 1-gap-planar. We claim that $K_{6,6}$ is 1-close-to-planar.

First, $\mathrm{cr}(K_{6,6}) = 36 = |E(K_{6,6})|$; see Fig. 10 for a crossing-minimal drawing of $K_{6,6}$. In the drawing in Fig. 10 the set of edges with precisely 4 crossings contains both adjacent and independent pairs of edges. By symmetry, we have $\mathrm{cr}(K_{6,6} - e) \leq 32$ for any edge e, and $\mathrm{cr}(K_{6,6} - \{e, f\}) \leq 28$ for any pair of distinct edges e, f. In particular, the crossing number of $K_{6,6} - e$ and $K_{6,6} - \{e, f\}$, resp., is clearly less than the number of edges in these graphs. It follows that any subgraph $G' = (V', E')$ obtained by removing 3 or more edges from $K_{6,6}$ satisfies $\mathrm{cr}(G') \leq 28$. Hence, if $\mathrm{cr}(G') > |E'|$ it follows that $|E'| \leq 28$, and hence $|E \setminus E'| \geq 8$.

An easy counting argument shows that $E \setminus E'$ contains a set A of three edges that are incident to a common vertex or a set B of three edges, two of which are adjacent and the third is independent from the other two. However, Figure 11 shows that both $K_{6,6} - A$ and $K_{6,6} - B$ are 1-gap planar, and hence also 1-close-to-planar. Both drawings are based on a 1-gap-planar drawing of $K_{5,6}$ by Bae et al. [2].

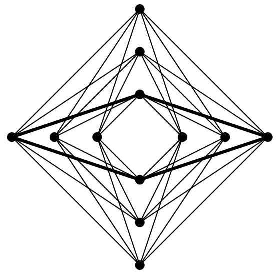

Figure 10 Crossing-minimal drawing of $K_{6,6}$. The bold edges are pairwise noncrossing and they each have four crossings, which shows that removing any two of these edges (and by symmetry any two edges) decreases the crossing number by at least 8.

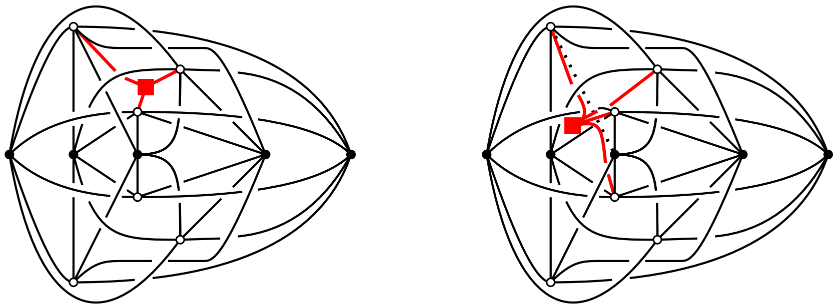

Figure 11 1-gap-planar drawings of $K_{6,6} - A$ and $K_{6,6} - B$, respectively. In the left drawing, A contains three edges incident to the square vertex. In the right drawing, B contains two edges incident to the square vertex and the dashed edge.

Open Problem. The negative answer raises the following question. Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that every k-close-to-planar graph is $f(k)$-gap-planar?

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