Reducing the Domination Number of Graphs via Edge Contractions

Esther Galby
Department of Informatics, University of Fribourg, Fribourg, Switzerland
esther.galby@unifr.ch

Paloma T. Lima
Department of Informatics, University of Bergen, Bergen, Norway
paloma.lima@uib.no

Bernard Ries
Department of Informatics, University of Fribourg, Fribourg, Switzerland
bernard.ries@unifr.ch

Abstract

In this paper, we study the following problem: given a connected graph G, can we reduce the domination number of G by at least one using k edge contractions, for some fixed integer $k \geq 0$? We show that for $k \leq 2$, the problem is coNP-hard. We further prove that for $k = 1$, the problem is W[1]-hard parameterized by the size of a minimum dominating set plus the mim-width of the input graph, and that it remains NP-hard when restricted to P_9-free graphs, bipartite graphs and $\{C_3, \ldots, C_\ell\}$-free graphs for any $\ell \geq 3$. Finally, we show that for any $k \geq 1$, the problem is polynomial-time solvable for P_5-free graphs and that it can be solved in FPT-time and XP-time when parameterized by tree-width and mim-width, respectively.

2012 ACM Subject Classification Mathematics of computing $\to$ Graph theory

Keywords and phrases domination number, blocker problem, graph classes

Digital Object Identifier 10.4230/LIPIcs.MFCS.2019.41

1 Introduction

In a graph modification problem, we are usually interested in modifying a given graph G, via a small number of operations, into some other graph G' that has a certain desired property. This property often describes a certain graph class to which G' must belong. Such graph modification problems allow to capture a variety of classical graph-theoretic problems. Indeed, if for instance only k vertex deletions are allowed and G' must be a stable set or a clique, we obtain the Stable Set or Clique problem, respectively.

Now, instead of specifying a graph class to which G' should belong, we may ask for a specific graph parameter π to decrease. In other words, given a graph G, a set $\mathcal{O}$ of one or more graph operations and an integer $k \geq 1$, the question is whether G can be transformed into a graph G' by using at most k operations from $\mathcal{O}$ such that $\pi(G') \leq \pi(G) - d$ for some *threshold* $d \geq 0$. Such problems are called *blocker problems* as the set of vertices or edges involved can be viewed as "blocking" the parameter π. Notice that identifying such sets may provide important information about the structure of the graph G.

Blocker problems have been well studied in the literature (see for instance [1, 2, 3, 4, 6, 9, 14, 15, 16, 17, 19]) and their relations to other well-known graph problems have been presented (see for instance [9, 15]). So far, the literature mainly focused on the following graph parameters: the chromatic number, the independence number, the clique number, the matching number and the vertex cover number. Furthermore, the set $\mathcal{O}$ consisted of a single graph operation, namely either vertex deletion, edge contraction, edge deletion or edge

44th International Symposium on Mathematical Foundations of Computer Science (MFCS 2019).
Editors: Peter Rossmanith, Pinar Heggernes, and Joost-Pieter Katoen; Article No. 41; pp. 41:1–41:13
LIPICS Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

addition. Since these blocker problems are usually NP-hard in general graphs, a particular attention has been paid to their computational complexity when restricted to special graph classes.

In this paper, we focus on another parameter, namely the domination number γ, and we restrict $\mathcal{O}$ to a single graph operation, the edge contraction. More specifically, let $G = (V, E)$ be a graph. The *contraction* of an edge $uv \in E$ removes vertices u and v from G and replaces them by a new vertex that is made adjacent to precisely those vertices that were adjacent to u or v in G (without introducing self-loops nor multiple edges). We say that a graph G can be k-*contracted* into a graph G', if G can be transformed into G' by a sequence of at most k edge contractions, for an integer $k \geq 1$ (note that contracting an edge cannot increase the domination number). We will be interested in the following problem, where $k \geq 1$ is a fixed integer.

k-Edge Contraction(γ)
Instance: A connected graph $G = (V, E)$
Question: Can G be k-edge contracted into a graph G' such that $\gamma(G') \leq \gamma(G) - 1$?

In other words, we are interested in a blocker problem with parameter γ, graph operations set $\mathcal{O} = \{$edge contraction$\}$ and threshold $d = 1$. Notice that if $\gamma(G) = 1$, that is, G contains a dominating vertex, then G is always a No-instance for k-Edge Contraction(γ). Reducing the domination number using edge contractions was first considered in [13]; given a graph $G = (V, E)$, the authors denote by $ct_\gamma(G)$ the minimum number of edge contractions required to transform G into a graph G' such that $\gamma(G') \leq \gamma(G) - 1$ and prove that for a connected graph G such that $\gamma(G) \geq 2$, we have $ct_\gamma(G) \leq 3$. It follows that a connected graph G with $\gamma(G) \geq 2$ is always a Yes-instance of k-Edge Contraction(γ), if $k \geq 3$. The authors [13] further give necessary and sufficient conditions for $ct_\gamma(G)$ to be equal to 1, respectively 2.

▶ **Theorem 1** ([13]). *For a connected graph G, the following holds.*
(i) $ct_\gamma(G) = 1$ *if and only if there exists a minimum dominating set in G that is not a stable set.*
(ii) $ct_\gamma(G) = 2$ *if and only if every minimum dominating set in G is a stable set and there exists a dominating set D in G of size $\gamma(G) + 1$ such that $G[D]$ contains at least two edges.*

To the best of our knowledge, a systematic study of the computational complexity of k-Edge Contraction(γ) has not yet been attempted in the literature. We here initiate such a study as it has been done for other parameters and several graph operations. Our paper is organised as follows[1]. In Section 2, we present definitions and notations that are used throughout the paper. In Section 3, we prove the (co)NP-hardness of k-Edge Contraction(γ) for $k = 1, 2$. We further show that 1-Edge Contraction(γ) is W[1]-hard parameterized by the size of a minimum dominating set plus the mim-width of the input graph, and that it remains NP-hard when restricted to P_9-free graphs, bipartite graphs and $\{C_3, \ldots, C_l\}$-free graphs for any $l \geq 3$. Finally, we present in Section 4 some positive results; in particular, we show that for any $k \geq 1$, k-Edge Contraction(γ) is polynomial-time solvable for P_5-free graphs and that it can be solved in FPT-time and XP-time when parameterized by tree-width and mim-width, respectively.

[1] Proofs marked by ♠ are omitted due to space constraints.

2 Preliminaries

Throughout the paper, we only consider finite, undirected, connected graphs that have no self-loops or multiple edges. We refer the reader to [8] for any terminology and notation not defined here and to [7] for basic definitions and terminology regarding parameterized complexity.

Let $G = (V, E)$ be a graph and let $u \in V$. We denote by $N_G(u)$, or simply $N(u)$ if it is clear from the context, the set of vertices that are adjacent to u i.e., the *neighbors* of u, and let $N[u] = N(u) \cup \{u\}$. Two vertices $u, v \in V$ are said to be *true twins* (resp. *false twins*), if $N[u] = N[v]$ (resp. if $N(u) = N(v)$).

For a family $\{H_1, \ldots, H_p\}$ of graphs, G is said to be $\{H_1, \ldots, H_p\}$*-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$ we may write H_1-free instead of $\{H_1\}$-free. For a subset $V' \subseteq V$, we let $G[V']$ denote the subgraph of G *induced* by V', which has vertex set V' and edge set $\{uv \in E \mid u, v \in V'\}$.

We denote by $d_G(u, v)$, or simply $d(u, v)$ if it is clear from the context, the length of a shortest path from u to v in G. Similarly, for any subset $V' \subseteq V$, we denote by $d_G(u, V')$, or simply $d(u, V')$ if it is clear from the context, the minimum length of a shortest path from u to some vertex in V' i.e., $d(u, V') = \min_{v \in V'} d(u, v)$.

For a vertex $v \in V$, we write $G - v = G[V \setminus \{v\}]$ and for a subset $V' \subseteq V$ we write $G - V' = G[V \setminus V']$. For an edge $e \in E$, we denote by $G \backslash e$ the graph obtained from G by contracting the edge e. The k*-subdivision* of an edge uv consists in replacing it by a path u-v_1-...-v_k-v, where $v_1, \ldots, v_k$ are new vertices.

For $n \geq 1$, the path and cycle on n vertices are denoted by P_n and C_n respectively. A graph is *bipartite* if every cycle contains an even number of vertices.

A subset $S \subseteq V$ is called a *stable set* of G if any two vertices in S are nonadjacent; we may also say that S is *stable*. A subset $D \subseteq V$ is called a *dominating set*, if every vertex in $V \setminus D$ is adjacent to at least one vertex in D; the *domination number* $\gamma(G)$ is the number of vertices in a minimum dominating set. For any $v \in D$ and $u \in N[v]$, v is said to *dominate* u (in particular, v dominates itself); furthermore, u is a *private neighbor of* v *with respect to* D if u has no neighbor in $D \backslash \{v\}$. We say that D *contains an edge* (or more) if the graph $G[D]$ contains an edge (or more). The Dominating Set problem is to test whether a given graph G has a dominating set of size at most ℓ, for some given integer $\ell \geq 0$.

3 Hardness results

In this section, we present hardness results for the k-Edge Contraction(γ) problem. Recall that for $k \geq 3$, the problem is trivial; we show that for $k = 1, 2$, it becomes (co)NP-hard. To this end, we introduce the following problem.

Contraction Number(γ,k)	
Instance:	A connected graph $G = (V, E)$.
Question:	Is $ct_\gamma(G) = k$?

▶ **Theorem 2.** Contraction Number(γ,3) *is* NP*-hard.*

Proof. We reduce from 1-in-3 Positive 3-Sat, where each variable occurs only positively, each clause contains exactly three positive literals, and we want a truth assignment such that each clause contains exactly one true variable. This problem is known to be NP-complete [11]. Given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance G_Φ of Contraction Number(γ,3) as follows. For any variable $x \in X$, we introduce a copy of C_3, which we denote by G_x, with two distinguished *truth vertices* T_x and F_x (see Fig. 1); in the following, the third vertex of G_x is denoted by u_x. For any clause $c \in C$ containing variables x_1, x_2 and x_3, we introduce the gadget G_c depicted in Fig. 1 (where it is connected to the corresponding variable gadgets). The vertex set of the clique K_c corresponds to the set of subsets of size 1 of $\{x_1, x_2, x_3\}$ (hence the notation); for any $i \in \{1,2,3\}$, the vertex x_i (resp. x'_i) is connected to every vertex $v_S \in K_c$ such that $x_i \notin S$ (resp. $x_i \in S$). Finally, for $i = 1,2,3$, we add an edge between t_i (resp. x'_i) and the truth vertex T_{x_i} (resp. F_{x_i}). Our goal now is to show that Φ is satisfiable if and only if $ct_\gamma(G_\Phi) = 3$. In the remainder of the proof, given a clause $c \in C$, we denote by x_1, x_2 and x_3 the variables occuring in c and thus assume that t_i (resp. x'_i) is adjacent to T_{x_i} (resp. F_{x_i}) for $i \in \{1,2,3\}$. Let us first start with some easy observations.

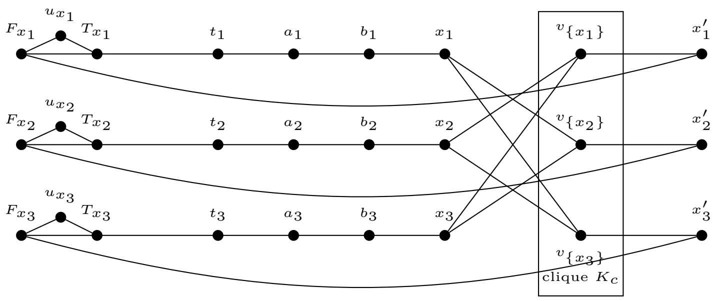

Figure 1 The gadget G_c together with G_{x_i}, $i = 1,2,3$, for a clause $c \in C$ containing variables x_1, x_2 and x_3 (the rectangle indicates that the corresponding set of vertices induces a clique).

▶ **Observation 1.** *Let D be a dominating set of G_Φ. Then for any $x \in X$, $|D \cap V(G_x)| \geq 1$ and for any $c \in C$, $|D \cap V(G_c)| \geq 4$. In particular, $|D| \geq |X| + 4|C|$.*

Clearly, for any $x \in X$, $|D \cap V(G_x)| \geq 1$ since u_x must be dominated. Also, in order to dominate vertices a_1, a_2, a_3 and $v_{\{x_1\}}$ in some gadget G_c, we need at least 4 distinct vertices, since their neighborhoods are pairwise disjoint and so, $|D \cap V(G_c)| \geq 4$, for any $c \in C$.

▶ **Observation 2.** *Let D be a dominating set of G_Φ. For any clause gadget G_c and $i \in \{1,2,3\}$, $D \cap \{a_i, b_i, x_i\} \neq \emptyset$.*

This immediately follows from the fact that every vertex b_i needs to be dominated and its neighbors are a_i and x_i for $i \in \{1,2,3\}$.

▶ **Observation 3.** *Let D be a dominating set of G_Φ. For any clause gadget G_c, if $|D \cap V(G_c)| = 4$, then $D \cap \{t_i, x'_i\} = \emptyset$ and $|D \cap \{a_i, b_i, x_i\}| = 1$, for any $i \in \{1,2,3\}$.*

If $t_i \in D$ for some $i \in \{1,2,3\}$, then it follows from Observation 2 that $|D \cap \{a_j, b_j, x_j\}| = 1$ for any $j \in \{1,2,3\}$. This implies that at least two vertices among x_1, x_2 and x_3 belong to D for otherwise there would exist $j \in \{1,2,3\}$ such that $v_{\{x_j\}}$ is not dominated. In particular, there must exist $j \neq i$ such that $x_j \in D$; but then, a_j is not dominated. Similarly, if $x'_i \in D$ for some $i \in \{1,2,3\}$, it follows from Observation 2 that $|D \cap \{a_j, b_j, x_j\}| = 1$ for any $j \in \{1,2,3\}$. But then, in order to dominate the vertices of K_c, either $x_i \in D$ in which case a_i is not dominated; or $\{x_j, j \neq i\} \subset D$ and a_j with $j \neq i$, is not dominated.

Now suppose that $|D \cap \{a_i, b_i, x_i\}| \geq 2$ for some $i \in \{1,2,3\}$. Then by Observation 2, we conclude that $|D \cap \{a_k, b_k, x_k\}| = 1$ for $k \neq i$ and $|D \cap \{a_i, b_i, x_i\}| = 2$. This implies that

$D \cap V(K_c) = \emptyset$ for otherwise we would have $|D \cap V(G_c)| \geq 5$. But then, since $x'_i \notin D$, D must contain at least two vertices among x_1, x_2 and x_3 in order to dominate the vertices of K_c; in particular, there exists $j \neq i$ such that $x_j \in D$ and so, a_j is not dominated.

▶ **Observation 4.** *Let D be a minimum dominating set of G_Φ and suppose that $ct_\gamma(G_\Phi) = 3$. Then for any vertices $u, v \in D$, we have $d(u, v) \geq 3$.*

Indeed, if u, v are adjacent, we conclude by Theorem 1(i) that $ct_\gamma(G_\Phi) = 1$; and if u, v are at distance 2 then $D \cup \{w\}$, where w is the vertex on a shortest path from u to v, contains two edges and we conclude by Theorem 1(ii) that $ct_\gamma(G_\Phi) = 2$.

▶ **Observation 5.** *Let D be a minimum dominating set of G_Φ and suppose that $ct_\gamma(G_\Phi) = 3$. Then for any clause gadget G_c and $i \in \{1, 2, 3\}$, $a_i \in D$ if and only if $T_{x_i} \notin D$.*

This readily follows from Observation 4. Further note that we may assume that for any $i \in \{1, 2, 3\}$, $a_i \in D$ if and only if $F_{x_i} \in D$; $T_{x_i} \notin D$ is equivalent to $\{F_{x_i}, u_{x_i}\} \cap D \neq \emptyset$ and if $T_{x_i} \notin D$, we may always replace D by $(D \backslash \{u_{x_i}\}) \cup \{F_{x_i}\}$.

▶ **Observation 6.** *Let D be a minimum dominating set of G_Φ and suppose that $ct_\gamma(G_\Phi) = 3$. Then for any clause gadget G_c, $|D \cap \{a_1, a_2, a_3\}| \leq 2$.*

If it weren't the case then, by Observation 4, no x_i or b_i ($i = 1, 2, 3$) would belong to D. But since x_1, x_2 and x_3 must be dominated, it follows that $D \cap V(K_c) \neq \emptyset$ and by Observation 5, we conclude that D contains two vertices at distance two (namely, $v_{\{x_i\}} \in D \cap V(K_c)$ and F_{x_i} for some $i \in \{1, 2, 3\}$), which contradicts Observation 4.

▶ **Observation 7.** *Let D be a minimum dominating set of G_Φ and suppose that $ct_\gamma(G_\Phi) = 3$. Then for any clause gadget G_c, $|D \cap \{b_1, b_2, b_3\}| \leq 1$.*

Indeed, if we assume, without loss of generality, that $b_1, b_2 \in D$, then by Observation 4, $D \cap V(K_c) = \emptyset$. It then follows from Observation 4 that $x'_3 \in D$ for otherwise $V_{\{x_3\}}$ would not be dominated. But then $D \cap V(G_{x_3}) = \emptyset$ by Observation 4, which contradicts Observation 1.

▶ **Claim 1.** *$\gamma(G_\Phi) = |X| + 4|C|$ if and only if $ct_\gamma(G_\Phi) = 3$.*

Assume that $\gamma(G_\Phi) = |X| + 4|C|$ and consider a minimum dominating set D of G_Φ. We first show that D is a stable set which would imply that $ct_\gamma(G_\Phi) > 1$ (see Theorem 1(i)). First note that Observation 1 implies that $|D \cap V(G_x)| = 1$ and $|D \cap V(G_c)| = 4$, for any variable $x \in X$ and any clause $c \in C$. It then follows from Observation 3 that no truth vertex is dominated by some vertex t_i or x'_i in some clause gadget G_c with $i \in \{1, 2, 3\}$; in particular, this implies that there can exist no edge in D having one endvertex in some gadget G_x ($x \in X$) and the other in some gadget G_c ($c \in C$). Hence, it is enough to show that for any $c \in C$, $D \cap V(G_c)$ is a stable set.

Now consider a clause gadget G_c. It follows from Observation 3 that if there exists $i \in \{1, 2, 3\}$ such that $a_i \notin D$ then $b_i \in D$ since a_i must be dominated (also note that by Observation 3, if $a_i \in D$ then $b_i \notin D$). Hence, for any $i \in \{1, 2, 3\}$, exactly one of a_i and b_i belongs to D. But then, by Observation 3 and since $|D \cap V(G_c)| = 4$, we immediately conclude that $D \cap V(G_c)$ is a stable set and so, D is a stable set.

Now, suppose to the contrary that $ct_\gamma(G_\Phi) = 2$ i.e., there exists a dominating set D' of G_Φ of size $\gamma(G_\Phi) + 1$ containing two edges e and e' (see Theorem 1(ii)). First assume that there exists $x \in X$ such that $|D' \cap V(G_x)| = 2$. Then, for any $x' \neq x$, $|D' \cap V(G_{x'})| = 1$; and for any $c \in C$, $|D' \cap V(G_c)| = 4$ which by Observation 3 implies that $\{t_i, x'_i\} \cap D' = \emptyset$ for any $i \in \{1, 2, 3\}$. Since as shown previously, $D' \cap V(G_c)$ is then a stable set, it follows that D' contains at most one edge, a contradiction.

Thus, there must exist some $c \in C$ such that $|D' \cap V(G_c)| = 5$. We then claim that $\{a_1, a_2, a_3\} \not\subset D'$. Indeed, since $x_1, x_2, x_3, v_{\{x_1\}}, v_{\{x_2\}}$ and $v_{\{x_3\}}$ must be dominated, $D' \cap V(K_c) \neq \emptyset$ (otherwise, at least three additional vertices of G_c would be required to dominate x_1, x_2 and x_3), say $v_{\{x_1\}} \in D'$ without loss of generality. But then, $|N[x_1] \cap D'| = 1$ as x_1 must be dominated and $|D' \cap V(G_c)| = 5$ and so, D' contains at most one edge. Therefore, there must exist $i \in \{1, 2, 3\}$ such that $a_i \notin D'$, say $a_1 \notin D'$ without loss of generality. Then, since a_1 must be dominated, either $t_1 \in D'$ or $b_1 \in D'$.

Assume first that t_1 belongs to D' (note that $\{b_1, x_1\} \cap D' \neq \emptyset$ by Observation 2). We then claim that either e or e' has an endvertex in $\{a_j, b_j, x_j\}$ for some $j \neq 1$. Indeed, if it weren't the case, then t_1 would be an endvertex of neither e nor e' for otherwise $T_{x_1} \in D'$ which implies that $D' \cap \{v_{\{x_1\}}, x'_1\} \neq \emptyset$ as $|D' \cap V(G_{x_1})| = 1$ and x'_1 should be dominated. But then, D' contains at most one edge as $5 = |D' \cap V(G_c)| \geq |\{t_1\}| + |D' \cap \{b_1, x_1\}| + |D' \cap \{v_{\{x_1\}}, x'_1\}| + |D' \cap \{a_j, b_j, x_j, j \neq 1\}| \geq 1 + 1 + 1 + 2$ and neither e nor e' has an endvertex in $\{a_j, b_j, x_j\}$ for some $j \neq 1$ by assumption, a contradiction. Since e and e' have at most one common endvertex, it then follows that $|D' \cap V(G_c)| \geq |\{t_1\}| + |D' \cap \{a_j, b_j, x_j, j \neq 1\}| + 3 \geq 1 + 2 + 3$, a contradiction. Thus, either e or e' has an endvertex in $\{a_j, b_j, x_j\}$ for some $j \neq 1$, say $j = 2$ without loss of generality. Suppose that x_2 is an endvertex of e. Then the other endvertex of e should be b_2 for otherwise it belongs to K_c and thus, a_2 would not be dominated. But then, we conclude by Observation 2 and the fact that $|D' \cap V(G_c)| = 5$, that D' contains only one edge. Thus, $e = a_2b_2$ or $e = a_2t_2$ and since $v_{\{x_1\}}$ must be dominated, necessarily $x_3 \in D'$; but then, a_3 is not dominated. Therefore, it must be that b_1 belongs to D'; and we conclude similarly that if a_2 (resp. a_3) is not in D' then b_2 (resp. b_3) belongs to D'.

Now, since $t_1, a_1 \notin D'$, it follows that $T_{x_1} \in D'$ for otherwise t_1 would not be dominated. But $|D' \cap V(G_x)| = 1$ and so, $F_{x_1} \notin D'$; thus, $D' \cap \{x'_1, v_{\{x_1\}}\} \neq \emptyset$ as x'_1 must be dominated and we may assume, without loss of generality, that in fact, $v_{\{x_1\}} \in D'$. Then, if $D' \cap \{v_{\{x_2\}}, v_{\{x_3\}}\} = \emptyset$, necessarily $F_{x_2}, F_{x_3} \in D'$; indeed, since $|D' \cap V(G_c)| = 5$, at least one among x'_2 and x'_3 does not belong to D', say x'_2 without loss of generality. But if $x'_3 \in D'$, then exactly one of a_j and b_j, for $j \neq 1$ belongs to D' (recall that if $a_j \notin D'$ then $b_j \in D'$) and therefore, D' contains at most one edge. Thus, $F_{x_2}, F_{x_3} \in D'$ which implies that $D' \cap \{t_j, a_j\} \neq \emptyset$ for $j \neq 1$ as t_j must be dominated. But by Observation 2 and the fact that $|D' \cap V(G_c)| = 5$, we have that $|D' \cap \{t_2, t_3\}| \leq 1$ and so, D' contains at most one edge. Thus, $D' \cap \{v_{\{x_2\}}, v_{\{x_3\}}\} \neq \emptyset$ and since by Observation 2 $|D' \cap V(K_c)| \leq 2$, we conclude that in fact $|D' \cap V(K_c)| = 2$. But then, exactly one among a_j and b_j belongs to D' for $j \neq 1$ and so, D' contains only one edge. Consequently, no such dominating set D' exists and thus, $ct_\gamma(G_\Phi) = 3$.

Conversely, assume that $ct_\gamma(G_\Phi) = 3$ and consider a minimum dominating set D of G_Φ. It readily follows from Observations 1 and 4 that for any variable $x \in X$, $|D \cap V(G_x)| = 1$. Now consider a clause gadget G_c. Then, by Observation 4, we obtain that $t_i \notin D$ (resp. $x'_i \notin D$) for $i \in \{1, 2, 3\}$, as otherwise it would be within distance at most 2 from the vertex in D belonging to the gadget G_{x_i}.

Now since for any $i \in \{1, 2, 3\}$, $t_i \notin D$, if $a_i \notin D$ then $b_i \in D$ as a_i must be dominated (also note that by Observation 4, if $a_i \in D$ then $b_i \notin D$). Thus, by Observations 6 and 7, we conclude that for any clause gadget G_c, $|D \cap \{a_1, a_2, a_3\}| = 2$ and $|D \cap \{b_1, b_2, b_3\}| = 1$, say $a_1, a_2, b_3 \in D$ without loss of generality. But then, $v_{\{x_3\}}$ must belong to D; indeed, since $b_3 \in D$, it follows that $T_{x_3} \in D$ for otherwise t_3 is not dominated. Observation 4 then implies that $x'_3 \notin D$ and thus, it can only be dominated by $v_{\{x_3\}}$. But then, it follows from Observation 5 that every vertex in G_c is dominated and we conclude that $|D \cap V(G_c)| = 4$ by minimality of D. Consequently, $|D| = |X| + 4|C|$ which concludes the proof of Claim 1.

▶ **Claim 2.** $\gamma(G_\Phi) = |X| + 4|C|$ *if and only if* Φ *is satisfiable.*

Assume first that $\gamma(G_\Phi) = |X| + 4|C|$ and consider a minimum dominating set D of G_Φ. We construct a truth assignment from D satisfying Φ as follows. For any $x \in X$, if $T_x \in D$, set x to true; otherwise, set x to false. We claim that each clause $c \in C$ has exactly one true variable. Indeed, it follows from Observation 1 that $|D \cap V(G_c)| = 4$ for any $c \in C$, and from Claim 1 that $ct_\gamma(G_\Phi) = 3$. But then, by Observation 3, for any $i \in \{1,2,3\}$, $a_i \notin D$ if and only if $b_i \in D$ (a_i would otherwise not be dominated). It then follows from Observations 6 and 7 that $|D \cap \{a_1, a_2, a_3\}| = 2$ and $|D \cap \{b_1, b_2, b_3\}| = 1$ for any $c \in C$; but by Observation 5 we conclude that $b_i \in D$ if and only if $T_{x_i} \in D$, which proves our claim.

Conversely, assume that Φ is satisfiable and consider a truth assignment satisfying Φ. We construct a dominating set D of G_Φ as follows. If variable x is set to true, we add T_x to D; otherwise, we add F_x to D. For any clause $c \in C$ and $i \in \{1,2,3\}$, if $T_{x_i} \in D$, then add b_i to D; otherwise, add a_i to D. Since every clause has exactly one true variable, it follows that $|D \cap \{b_1, b_2, b_3\}| = 1$ and $|D \cap \{a_1, a_2, a_3\}| = 2$; finally add $v_{\{x_i\}}$ to D where $b_i \in D$. Now clearly $|D \cap V(G_c)| = 4$ and every vertex in G_c is dominated. Thus, $|D| = |X| + 4|C|$ and so by Observation 1, $\gamma(G_\Phi) = |X| + 4|C|$, which concludes this proof.

Now combining Claims 1 and 2, we have that Φ is satisfiable if and only if $ct_\gamma(G_\Phi) = 3$ which completes the proof of Theorem 2. ◀

By observing that for any graph G, G is a Yes-instance for Contraction Number(γ,3) if and only if G is a No-instance for 2-Edge Contraction(γ), we deduce the following corollary from Theorem 2.

▶ **Corollary 3.** 2-Edge Contraction(γ) *is* coNP*-hard.*

It is thus coNP-hard to decide whether $ct_\gamma(G) \leq 2$ for a graph G; and in fact, it is NP-hard to decide whether equality holds, as stated in the following.

▶ **Theorem 4 (♠).** Contraction Number(γ,2) *is* NP*-hard.*

We finally consider the case $k = 1$.

▶ **Theorem 5.** 1-Edge Contraction(γ) *is* NP*-hard even when restricted to* P_t*-free graphs, with* $t \geq 9$.

Proof. We reduce from Dominating Set: given an instance (G, ℓ) of this problem, we construct an equivalent instance G' of 1-Edge Contraction(γ) as follows. We denote by $\{v_1, \ldots, v_n\}$ the vertex set of G. The graph G' consists of $\ell + 1$ copies of G, denoted by $G_0, \ldots, G_\ell$, connected in such a way that for any $1 \leq i \leq \ell$ and $1 \leq k \leq n$, the copies $v_k^i \in V(G_i)$ and $v_k^0 \in V(G_0)$ of a vertex v_k of G are true twins in the subgraph of G' induced by $V(G_0) \cup V(G_i)$; and for any $1 \leq i, j \leq \ell$ and $1 \leq k \leq n$, the copies $v_k^i \in V(G_i)$ and $v_k^j \in V(G_j)$ of a vertex v_k of G are false twins in the subgraph of G' induced by $\bigcup_{1 \leq p \leq \ell} V(G_p)$. Next, we add $\ell + 1$ pairwise nonadjacent vertices $x_1, \ldots, x_{\ell+1}$, which are made adjacent to every vertex in G_0; x_i is further made adjacent to every vertex in G_i, for all $1 \leq i \leq \ell$. Finally, we add a vertex y adjacent to only $x_{\ell+1}$ (see Fig. 2). Note that the fact that for all $1 \leq k \leq n$ and $1 \leq i, j \leq \ell$, v_k^i and v_k^j (resp. v_k^i and v_k^0) are false (resp. true) twins within the graph induced by $\bigcup_{1 \leq p \leq \ell} V(G_p)$ (resp. $V(G_0) \cup V(G_i)$) is not made explicit on Fig. 2 for the sake of readability. In the following, we denote by $X = \{x_1, \ldots, x_{\ell+1}\}$ and $V = \bigcup_{0 \leq p \leq \ell} V(G_p)$. We now claim the following.

▶ **Claim 3.** $\gamma(G') = \min\{\gamma(G) + 1, \ell + 1\}$.

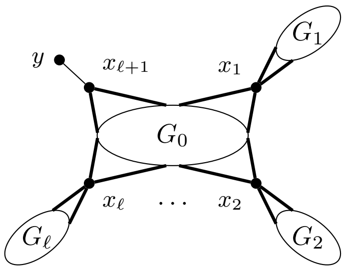

Figure 2 The graph G' (thick lines indicate that the vertex x_i is adjacent to every vertex in G_0 and G_i, for $i = 1, \ldots, \ell + 1$).

It is clear that $\{x_1, \ldots, x_{\ell+1}\}$ is a dominating set of G'; thus, $\gamma(G') \leq \ell + 1$. If $\gamma(G) \leq \ell$ and $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimum dominating set of G, it is easily seen that $\{v^0_{i_1}, \ldots, v^0_{i_k}, x_{\ell+1}\}$ is a dominating set of G'. Thus, $\gamma(G') \leq \gamma(G) + 1$ and so, $\gamma(G') \leq \min\{\gamma(G) + 1, \ell + 1\}$. Now, suppose to the contrary that $\gamma(G') < \min\{\gamma(G) + 1, \ell + 1\}$ and consider a minimum dominating D' set of G'. We first make the following simple observation.

▶ **Observation 8.** *For any dominating set D of G', $D \cap \{y, x_{\ell+1}\} \neq \emptyset$.*

Now, since $\gamma(G') < \ell + 1$, there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$ (otherwise, $\{x_1, \ldots, x_\ell\} \subset D'$ and combined with Observation 8, D' would be of size at least $\ell + 1$). But then, $D'' = D' \cap V$ must dominate every vertex in G_i, and so $|D''| \geq \gamma(G)$. Since $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_{\ell+1}\} \neq \emptyset$), we then have $\gamma(G) \leq |D'| - 1$, a contradiction. Thus, $\gamma(G') = \min\{\gamma(G) + 1, \ell + 1\}$.

We now show that (G, ℓ) is a YES-instance for DOMINATING SET if and only if G' is a YES-instance for 1-EDGE CONTRACTION(γ).

First assume that $\gamma(G) \leq \ell$. Then, $\gamma(G') = \gamma(G) + 1$ by the previous claim, and if $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimum dominating set of G, then $\{v^0_{i_1}, \ldots, v^0_{i_k}, x_{\ell+1}\}$ is a minimum dominating set of G' which is not stable. Hence, by Theorem 1(i), G' is a YES-instance for 1-EDGE CONTRACTION(γ).

Conversely, assume that G' is a YES-instance for 1-EDGE CONTRACTION(γ) i.e., there exists a minimum dominating set D' of G' which is not stable (see Theorem 1(i)). Then, Observation 8 implies that there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$; indeed, if it weren't the case, then by Claim 3 we would have $\gamma(G') = \ell + 1$ and thus, D' would consist of $x_1, \ldots, x_\ell$ and either y or $x_{\ell+1}$. In both cases, D' would be stable, a contradiction. It follows that $D'' = D' \cap V$ must dominate every vertex in G_i and thus, $|D''| \geq \gamma(G)$. But $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_{\ell+1}\} \neq \emptyset$) and so by Claim 3, $\gamma(G) \leq |D'| - 1 \leq (\ell + 1) - 1$ that is, (G, ℓ) is a YES-instance for DOMINATING SET.

▷ Claim 6 (♠). If G is a $2K_2$-free graph, then G' is a P_9-free graph.

Since DOMINATING SET is NP-complete on $2K_2$-free graphs [5], the above claim concludes the proof of Theorem 5. ◀

Given the NP-hardness of 1-EDGE CONTRACTION(γ) and its close relation to DOMINATING SET, it is natural to consider the complexity of the problem when parameterized by the size of a minimum dominating set of the input graph. In the following, we show that 1-EDGE CONTRACTION(γ) is W[1]-hard when parameterized by $\gamma + mimw$, where $mimw$ denotes the maximum induced matching-width parameter (in short, mim-width). For a formal definition and basic properties of this width measure we refer the reader to [18]. We first state two simple facts regarding the mim-width of a graph.

▶ **Observation 9.** *Let G be a graph and $u, v \in V(G)$ be two vertices that are true (resp. false) twins in G. Then $mimw(G - v) = mimw(G)$.*

▶ **Observation 10.** *Let G be a graph and $v \in V(G)$. Then $mimw(G) \leq mimw(G - v) + 1$.*

▶ **Theorem 7.** 1-Edge Contraction(γ) *is* W[1]*-hard parameterized by $\gamma + mimw$.*

Proof. We give a parameterized reduction from Dominating Set parameterized by solution size plus mim-width, which is a problem that was recently shown to be W[1]-hard by Fomin et al. [10]. Given an instance (G, ℓ) of Dominating Set, the construction of the equivalent instance G' for 1-Edge Contraction(γ) is the same as the one introduced in the proof of Theorem 5; and it is there shown that G is a Yes-instance for Dominating Set if and only if G' is a Yes-instance for 1-Edge Contraction(γ). Now, note that G' can be obtained from G by the addition of true twins (the set $V(G_1)$), the addition of false twins (the sets $V(G_2), \ldots, V(G_\ell)$), and the addition of $\ell + 2$ vertices $(x_1, \ldots, x_{\ell+1}, y)$. By Observation 9, the addition of true (resp. false) twins does not increase the mim-width of a graph and, by Observation 10, the addition of a vertex can only increase the mim-width of G by one; thus, $mimw(G') \leq mimw(G) + \ell + 2$ and since $\gamma(G') \leq \ell + 1$ by Claim 3, we conclude that $mimw(G') + \gamma(G') \leq mimw(G) + 2\ell + 3$. ◀

In order to obtain complexity results for further graph classes, let us now consider subdivisions of edges.

▶ **Lemma 8 (♠).** *Let G be a graph and let G' be the graph obtained by 3-subdividing every edge of G. Then G is a* Yes*-instance for* 1-Edge Contraction(γ) *if and only if G' is a* Yes*-instance for* 1-Edge Contraction(γ).

By 3-subdividing every edge of a graph G sufficiently many times, we deduce the following two corollaries from Lemma 8.

▶ **Corollary 9.** 1-Edge Contraction(γ) *is* NP*-hard when restricted to bipartite graphs.*

▶ **Corollary 10.** *For any $\ell \geq 3$,* 1-Edge Contraction(γ) *is* NP*-hard when restricted to $\{C_3, \ldots, C_\ell\}$-free graphs.*

We finally observe that, even if an edge is given, deciding whether contracting this particular edge decreases the domination number is unlikely to be solvable in polynomial time as shown in the following result.

▶ **Theorem 11.** *There exists no polynomial-time algorithm deciding whether contracting a given edge decreases the domination number, unless* P = NP.

Proof. We denote by Edge Contraction(γ) the problem that takes as an input a graph $G = (V, E)$ and an edge $e \in E$, and asks whether $\gamma(G \backslash e) \leq \gamma(G) - 1$. We show that if Edge Contraction(γ) can be solved in polynomial time, then Dominating Set can also be solved in polynomial time. Since Dominating Set is a well-known NP-complete problem, the result follows.

Let (G, ℓ) be an instance for Dominating Set and let e be an edge of G. We run the polynomial time algorithm for Edge Contraction(γ) to determine if $\gamma(G \setminus e) = \gamma(G) - 1$; we then have two possible scenarios.

Case 1. (G, e) is a YES-instance for EDGE CONTRACTION(γ). Since $\gamma(G \setminus e) = \gamma(G) - 1$, we know that G has a dominating set of size ℓ if and only if $G \setminus e$ has a dominating set of size $\ell - 1$. Hence, we obtain that $(G \setminus e, \ell - 1)$ is an equivalent instance for DOMINATING SET.

Case 2. (G, e) is a NO-instance for EDGE CONTRACTION(γ). Since $\gamma(G \setminus e) = \gamma(G)$, we know that G has a dominating set of size ℓ if and only if $G \setminus e$ has a dominating set of size ℓ. In this case, we obtain that $(G \setminus e, \ell)$ is an equivalent instance for DOMINATING SET.

In both cases, the ensuing equivalent instance has one less vertex. Thus, by applying the polynomial-time algorithm for EDGE CONTRACTION(γ) at most n times, we obtain a trivial instance for DOMINATING SET and can therefore correctly determine its answer. ◀

4 Algorithms

We now deal with cases in which k-EDGE CONTRACTION(γ) is tractable, for $k = 1, 2$. A first simple approach to the problem, from which we obtain Proposition 12, is based on brute force.

▶ **Proposition 12.** *For $k = 1, 2$, k-*EDGE CONTRACTION(γ) *can be solved in polynomial time for a graph class $\mathcal{C}$, if either*

(a) *$\mathcal{C}$ is closed under edge contractions and* DOMINATING SET *can be solved in polynomial time on $\mathcal{C}$; or*

(b) *for every $G \in \mathcal{C}$, $\gamma(G) \leq q$, where q is some fixed constant; or*

(c) *$\mathcal{C}$ is the class of $(H + K_1)$-free graphs, where $|V_H| = q$ is a fixed constant and k-*EDGE CONTRACTION(γ) *is polynomial-time solvable on H-free graphs.*

Proof. In order to prove item (a), it suffices to note that if we can compute $\gamma(G)$ and $\gamma(G \setminus e)$, for any edge e of G, in polynomial time, then we can determine whether a graph G is a YES-instance for 1-EDGE CONTRACTION(γ) in polynomial time (we may proceed in a similar fashion for 2-EDGE CONTRACTION(γ)).

For item (b), we proceed as follows. Given a graph G of $\mathcal{C}$, we first check whether G has a dominating vertex. If it is the case, then G is a NO-instance for k-EDGE CONTRACTION(γ) for both $k = 1, 2$. Otherwise, we may consider any subset $S \subset V(G)$ with $|S| \leq q$ and check whether it is a dominating set of G. Since there are at most $\mathcal{O}(n^q)$ possible such subsets, we can determine the domination number of G and check whether the conditions given in Theorem 1 (i) or (ii) are satisfied in polynomial time.

Finally, so as to prove item (c), we provide the following algorithm that works similarly for $k = 1$ and $k = 2$. Let H and q be as stated and let G be an instance of k-EDGE CONTRACTION(γ) on $(H + K_1)$-free graphs. We first test whether G is H-free (note that this can be done in time $\mathcal{O}(n^q)$). If this is the case, we use the polynomial-time algorithm for k-EDGE CONTRACTION(γ) on H-free graphs. Otherwise, G has an induced subgraph isomorphic to H; but since G is a $(H+K_1)$-free graph, $V(H)$ must then be a dominating set of G and so, $\gamma(G) \leq q$. We then conclude by Proposition 12(b) that k-EDGE CONTRACTION(γ) is also polynomial-time solvable in this case. ◀

Proposition 12(b) provides an algorithm for 1-EDGE CONTRACTION(γ) parameterized by the size of a minimum dominating set of the input graph running in XP-time. Note that this result is optimal as 1-EDGE CONTRACTION(γ) is W[1]-hard with such parameterization from Theorem 7.

We further show that even though simple, this brute force method provides polynomial-time algorithms for a number of relevant classes of graphs, such as graphs of bounded tree-width and graphs of bounded mim-width. We first state the following result and observation.

▶ **Theorem 13.** [18] *Given a graph G and a decomposition of width t,* Dominating Set *can be solved in time $\mathcal{O}^*(3^t)$ when parameterized by tree-width, and in time $\mathcal{O}^*(n^{3t})$ when parameterized by mim-width.*

▶ **Observation 11.** $mimw(G \setminus e) \leq mimw(G) + 1$.

Indeed, note that the graph $G \setminus e$ can be obtained from G by the removal of the vertices u and v where $e = uv$, and the addition of a new vertex whose neighborhood is $N_G(u) \cup N_G(v)$. The result then follows from Observation 10 and the fact that vertex deletion does not increase the mim-width of a graph.

▶ **Proposition 14.** *Given a decomposition of width t, k-*Edge Contraction*(γ) can be solved in time $\mathcal{O}^*(3^t)$ in graphs of tree-width at most t and in time $\mathcal{O}^*(n^{3t})$ in graphs of mim-width at most t, for $k = 1, 2$.*

Proof. We use the above-mentioned brute force approach and Theorem 13. That is, for $k = 1$, the algorithm first computes $\gamma(G)$ and then computes $\gamma(G \setminus e)$ for every $e \in E(G)$. For $k = 2$, the algorithm proceeds similarly for every pair of edges. We next show that the width parameters increase by a constant when contracting at most two edges. It is a well-known fact that $tw(G \setminus e) \leq tw(G)$ and so, $tw(G \setminus \{e, f\}) \leq tw(G)$. By Observation 11, $mimw(G \setminus e) \leq mimw(G) + 1$ which implies that $mimw(G \setminus \{e, f\}) \leq mimw(G) + 2$. Also note that, given a tree (resp. mim) decomposition of width t for G, we can construct in polynomial time decompositions of width t (resp. at most $t+2$) for $G \setminus e$ and $G \setminus \{e, f\}$. This implies that $\gamma(G \setminus e)$ and $\gamma(G \setminus \{e, f\})$ can also be computed in time $\mathcal{O}^*(3^t)$ if G is a graph of tree-width at most t, and in time $\mathcal{O}^*(n^{3t})$ if G is a graph of mim-width at most t. ◀

Proposition 14 provides an algorithm for 1-Edge Contraction(γ) parameterized by mim-width running in XP-time; this result is optimal as 1-Edge Contraction(γ) is W[1]-hard parameterized by mim-width from Theorem 7.

Since Dominating Set is polynomial-time solvable in P_4-free graphs (see [12]), it follows from Proposition 12(a) that k-Edge Contraction(γ) can also be solved efficiently in this graph class. However, Dominating Set is NP-complete for P_5-free graphs (see [5]) and thus, it is natural to examine the complexity of k-Edge Contraction(γ) for this graph class. As we next show, k-Edge Contraction(γ) is in fact polynomial-time solvable on P_5-free graphs, for $k = 1, 2$.

▶ **Lemma 15.** *If G is a P_5-free graph with $\gamma(G) \geq 3$, then $ct_\gamma(G) = 1$.*

Proof. Let $G = (V, E)$ be a P_5-free graph and D be a minimum dominating set of G. Suppose that D is a stable set and consider $u, v \in D$ such that $d(u, v) = \max_{x,y \in D} d(x, y)$. Since G is P_5-free, $d(u, v) \leq 3$ and, since D is stable, $d(u, v) \geq 2$. We distinguish two cases depending on this distance.

Case 1. $d(u,v) = 3$. Let x (resp. y) be the neighbor of u (resp. v) on a shortest path from u to v. Then, $N(u) \cup N(v) \subseteq N(x) \cup N(y)$; indeed, if a is a neighbor of u, then a is nonadjacent to v (recall that $d(u,v) = 3$) and thus, a is adjacent to either x or y for otherwise a, u, x, y and v would induce a P_5 in G. The same holds for any neighbor of v. Consequently, $(D \backslash \{u,v\}) \cup \{x,y\}$ is a minimum dominating set of G which is not stable; the result then follows from Theorem 1(i).

Case 2. $d(u,v) = 2$. Since D is stable and $d(u,v) = \max_{x,y \in D} d(x,y) = 2$, it follows that every $w \in D \backslash \{u,v\}$ is at distance two from both u and v. Let x (resp. y) be the vertex on a shortest path from u (resp. v) to some vertex $w \in D \backslash \{u,v\}$.

Suppose first that $x = y$. If every private neighbor of w with respect to D is adjacent to x then $(D \backslash \{w\}) \cup \{x\}$ is a minimum dominating set of G which is not stable; the result then follows from Theorem 1(i). We conclude similarly if every private neighbor of u or v with respect to D is adjacent to x. Thus, we may assume that w (resp. u; v) has a private neighbor t (resp. r; s) with respect to D which is nonadjacent to x. Since G is P_5-free, it then follows that r, s and t are pairwise adjacent. But then, t, r, u, x and v induce a P_5, a contradiction.

Finally, suppose that $x \neq y$ (we may also assume that $uy, vx \notin E$ as we otherwise fall back in the previous case). Then, $xy \in E$ for u, x, w, y and v would otherwise induce a P_5. Now, if a is a private neighbor of u with respect to D then a is adjacent to either x or y (a, u, x, y and v otherwise induce a P_5); we conclude similarly that any private neighbor of v with respect to D is adjacent to either x or y. If b is adjacent to both u and v but not w, then it is adjacent to x (and y) as v, b, u, x and w (u, b, v, y and w) would otherwise induce a P_5. But then, $(D \backslash \{u,v\}) \cup \{x,y\}$ is a minimum dominating set of G which is not stable; thus, by Theorem 1(i), $ct_\gamma(G) = 1$ which concludes the proof. ◀

▶ **Theorem 16.** *k-Edge Contraction(γ) is polynomial-time solvable on P_5-free graphs, for $k = 1, 2$.*

Proof. If G has a dominating vertex, then G is clearly a No-instance for both $k = 1, 2$. Now, for every $uv \in E(G)$, we check whether $\{u,v\}$ is a dominating set. If it is the case, then by Theorem 1(i), G is a Yes-instance for k-Edge Contraction(γ) for $k = 1, 2$. If no edge of G is dominating, we consider all the pairs of nonadjacent vertices of G. If there exists such a pair dominating G and $k = 1$ then by Theorem 1(i), we have a No-instance for 1-Edge Contraction(γ) since this implies that every minimum dominating set of G is stable. For the case $k = 2$, if G has two nonadjacent vertices dominating G, we then consider all triples of vertices of G to check whether there exists one which is dominating and contains at least two edges (see Theorem 1(ii)). Finally, both for $k = 1$ and $k = 2$, if G has no dominating set of size at most two, then by Lemma 15, G is a Yes-instance for k-Edge Contraction(γ). ◀

5 Conclusion

In this paper, we studied the k-Edge Contraction(γ) problem and provided the first complexity results. In particular, we showed that 1-Edge Contraction(γ) is NP-hard for P_t-free graphs, $t \geq 9$, but polynomial-time solvable for P_5-free graphs; it would be interesting to determine the complexity status for P_ℓ-free graphs, for $\ell \in \{6,7,8\}$. Similarly, the complexity of 2-Edge Contraction(γ) for P_t-free graphs, with $t \geq 6$, remains an interesting open problem.

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