Towards Improving Christofides Algorithm for Half-Integer TSP

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Abstract

We study the traveling salesman problem (TSP) in the case when the objective function of the subtour linear programming relaxation is minimized by a *half-cycle point*: $x_e \in \{0, \frac{1}{2}, 1\}$ where the half-edges form a 2-factor and the 1-edges form a perfect matching. Such points are sufficient to resolve half-integer TSP in general and they have been conjectured to demonstrate the largest integrality gap for the subtour relaxation.

For half-cycle points, the best-known approximation guarantee is $\frac{3}{2}$ due to Christofides' famous algorithm. Proving an integrality gap of α for the subtour relaxation is equivalent to showing that αx can be written as a convex combination of tours, where x is any feasible solution for this relaxation. To beat Christofides' bound, our goal is to show that $(\frac{3}{2} - \epsilon)x$ can be written as a convex combination of tours for some positive constant ϵ. Let $y_e = \frac{3}{2} - \epsilon$ when $x_e = 1$ and $y_e = \frac{3}{4}$ when $x_e = \frac{1}{2}$. As a first step towards this goal, our main result is to show that y can be written as a convex combination of tours. In other words, we show that we can *save on 1-edges*, which has several applications. Among them, it gives an alternative algorithm for the recently studied *uniform cover* problem. Our main new technique is a procedure to glue tours over proper 3-edge cuts that are tight with respect to x, thus reducing the problem to a base case in which such cuts do not occur.

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1 Introduction

In the **traveling salesman problem (TSP)** we are given a complete graph $G = (V, E)$ together with a vector $c \in \mathbb{R}^E_{\geq 0}$ of edge costs satisfying the triangle inequality: $c_{uv} + c_{vw} \geq c_{uw}$ for $u, v, w \in V$. The goal is to find a minimum cost Hamiltonian cycle of G. The following formulation is a classic linear programming relaxation for TSP [9].

$$\min\{\sum_{e \in E} c_e x_e : \sum_{u \in V \setminus \{v\}} x_{vu} = 2 \text{ for } v \in V, \sum_{v \in S, u \notin S} x_{vu} \geq 2 \text{ for } \emptyset \subset S \subset V, x \in \mathbb{R}^E_{\geq 0}\}.$$

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Let $\textsc{Subtour}(G)$ denote the feasible region of this relaxation. We will refer to $\sum_{e\in E} c_e x_e$ as the *objective function*. A *tour* of G is a connected, spanning, Eulerian multi-subgraph of G. It is well known that due to the triangle inequality on the edge costs, a tour of G can be turned into a Hamiltonian cycle of G of no greater cost. For any $x \in \textsc{Subtour}(G)$, the vector $\frac{3}{2}x$ can be decomposed into a convex combination of tours of G. This follows from a polyhedral analysis of Christofides' famous $\frac{3}{2}$-approximation algorithm [7, 21, 20]. For a point $x \in \textsc{Subtour}(G)$, define $G_x = (V, E_x = \{e \in E : x_e > 0\})$ to be the support graph of x. Let $\text{TSP}(G_x)$ be the convex hull of characteristic vectors of tours of G_x. The following conjecture is well-known and widely studied and implies a $\frac{4}{3}$-approximation algorithm for TSP.

▶ **Conjecture 1** (The Four-Thirds Conjecture). *If $x \in \textsc{Subtour}(G)$, then $\frac{4}{3}x \in \text{TSP}(G_x)$.*

However, more than four decades after the publication of Christofides' algorithm, there is still no $(\frac{3}{2} - \epsilon)$-approximation algorithm known for TSP. For special cases, there has been some progress in the past few years. For example, in the *unweighted* case where the edge costs correspond to the shortest path metric of an unweighted graph, a series of papers improved the $\frac{3}{2}$ factor to $\frac{7}{5}$ [16, 15, 19].

One interesting special case of *weighted* TSP is when the solution $x \in \textsc{Subtour}(G)$ that minimizes the objective function is half-integer. In the unweighted case, if a half-integer point $x \in \textsc{Subtour}(G)$ minimizes the objective function, then there is a $\frac{4}{3}$-approximation algorithm for TSP [15].

▶ **Problem 2** (Half-integer TSP). *For $x \in \textsc{Subtour}(G) \cap \{0, \frac{1}{2}, 1\}^E$, henceforth a half-integer point, show $\alpha x \in \text{TSP}(G_x)$ for constant $\alpha \in [1, \frac{3}{2})$.*

Consider a half-integer point $x \in \textsc{Subtour}(G) \cap \{0, \frac{1}{2}, 1\}^E$ and let $H_x = \{e \in E : x_e = \frac{1}{2}\}$ and $W_x = \{e \in E : x_e = 1\}$. Carr and Vempala showed that in Problem 2, we can assume without loss of generality a stronger condition for $x \in \textsc{Subtour}(G)$: a *half-integer Carr-Vempala point* is a half-integer point such that the support graph G_x is a cubic graph and for every vertex $u \in V$, there is exactly one edge e incident on u with $x_e = 1$ and two edges f, g incident on u with $x_f = x_g = \frac{1}{2}$. Moreover, H_x forms a Hamilton cycle of G_x, and W_x forms a perfect matching of G_x. If for any half-integer Carr-Vempala point x we have $\alpha x \in \text{TSP}(G_x)$, then for any half-integer point y we have $\alpha y \in \text{TSP}(G_y)$ [6, 4].

We consider a generalization of a half-integer Carr-Vempala point called a *half-cycle point*, which is a half-integer point $x \in \textsc{Subtour}(G)$ such that the graph G_x is a cubic graph and for every vertex $u \in V$, there is exactly one edge e incident on u with $x_e = 1$ and two edges f, g incident on u with $x_f = x_g = \frac{1}{2}$. This implies that H_x, the half-edges in G_x, forms a 2-factor of G (in which the minimum cycle length is three). Formally, we define a half-cycle point as follows.

▶ **Definition 3.** *A vector $x \in \textsc{Subtour}(G)$ is called a half-cycle point if the support graph G_x of x is cubic and 2-edge-connected and $x_e \in \{1, \frac{1}{2}\}$ for all $e \in E_x$.*

Half-cycle points have been studied in restricted cases when all cycles in the 2-factor are triangles [2, 3] or squares [4, 11]. Schalekamp, Williamson and van Zuylen conjectured that the largest gap between $\textsc{Subtour}(G)$ and $\text{TSP}(G_x)$ occurs for half-cycle points in which the 2-factor consists of odd-cycles [17].[1] We can restate Problem 1 as follows.

[1] Their precise conjecture is that instances of TSP that have an optimal solution $x \in \textsc{Subtour}(G)$ that is also an optimal *fractional 2-matching* exhibit the largest integrality gap for $\textsc{Subtour}(G)$. The extreme points of the fractional 2-matching polytope are half-cycle points in which all cycles in the 2-factor are odd [1].

▶ **Problem 4** (Half-integer TSP). *Let $x \in \mathbb{R}^{E}_{\geq 0}$ be a half-cycle point. Show $\alpha x \in \text{TSP}(G_x)$ for constant $\alpha \in [1, \frac{3}{2})$.*

We can also state Problem 4 in different way.

▶ **Problem 5** (Half-integer TSP). *Let $x \in \mathbb{R}^{E}_{\geq 0}$ be a half-cycle point. Define vector $y \in \mathbb{R}^{E_x}$ as follows: $y_e = \frac{3}{2} - \epsilon$ for $e \in W_x$ and $y_e = \frac{3}{4} - \delta$ for $e \in H_x$. Show there exists constants $\epsilon, \delta > 0$ such that $y \in \text{TSP}(G_x)$.*

The aforementioned polyhedral analysis of Christofides' algorithm implies the following theorem.

▶ **Theorem 6** ([7, 21, 20]). *Let $x \in \mathbb{R}^{E}_{\geq 0}$ be a half-cycle point. Define vector $y \in \mathbb{R}^{E_x}$ as follows: $y_e = \frac{3}{2}$ for $e \in W_x$ and $y_e = \frac{3}{4}$ for $e \in H_x$. Then $y \in \text{TSP}(G_x)$.*

Our main result is the following.

▶ **Theorem 7.** *Let $x \in \mathbb{R}^{E}_{\geq 0}$ be a half-cycle point. Define vector $y \in \mathbb{R}^{E_x}$ as follows: $y_e = \frac{3}{2} - \frac{1}{20}$ for $e \in W_x$ and $y_e = \frac{3}{4}$ for $e \in H_x$. Then $y \in \text{TSP}(G_x)$.*

While Theorem 7 is not strong enough to resolve Problem 5 (and therefore Problem 4), it does have several applications. For example, given an edge cost function c for which a half-cycle point $x \in \textsc{Subtour}(G)$ minimizes the objective function, if the total edge costs of the 1-edges is a constant fraction of the total cost of the half-edges, then by Theorem 7, we obtain an approximation factor better than $\frac{3}{2}$.

Another application is related to the problem of *uniform covers* posed by Sebő [18]. Let x be a *cubic point* if $x \in \textsc{Subtour}(G) \cap \{0, \frac{2}{3}\}$. Observe that G_x is cubic and 3-edge-connected.

▶ **Problem 8** (Uniform cover problem). *Let x be a cubic point. Show that $\alpha x \in \text{TSP}(G_x)$ for constant $\alpha \in [1, \frac{3}{2})$.*

Recently, Haddadan, Newman and Ravi gave a positive answer to Problem 8 and showed $\alpha \leq \frac{27}{19} \approx 1.421$ [13]. Previously, Boyd and Sebő had shown that $\alpha \leq \frac{9}{7} \approx 1.286$ if G_x is additionally Hamiltonian [4]. In fact, Theorem 7 gives an alternative way to answer Problem 8.

▶ **Lemma 9.** *Let x be a half-cycle point. Define vector $y \in \mathbb{R}^{E_x}$ as follows: $y_e = \frac{3}{2} - \epsilon$ for $e \in W_x$ and $y_e = \frac{3}{4} - \delta$ for $e \in H_x$ for constants $\epsilon, \delta \geq 0$. Suppose $y \in \text{TSP}(G_x)$. Then for any cubic point z, we have $\alpha z \in \text{TSP}(G_z)$ for $\alpha = \frac{3}{2} - \frac{\epsilon}{2} - \delta$.*

In other words, suppose that we can *save either on the 1-edges or on the half-edges*. Then we can solve the uniform cover problem. Moreover, Theorem 7 can be used to slightly improve the currently best-known factors for Problem 8. The proofs of Lemma 9 and Theorem 10 can be found in the full version [12].

▶ **Theorem 10.** *Let x be a cubic point. Then $\alpha x \in \text{TSP}(G_x)$ for $\alpha = 1.416$. If G_x is Hamiltonian, then $1.279x \in \text{TSP}(G_x)$.*

On a high level, our proof of Theorem 7 is based on Christofides' algorithm: We show that a half-cycle point x can be written as a convex combination of spanning subgraphs with certain properties and then we show that vector $y \in \mathbb{R}^{E_x}$, where $y_e = \frac{9}{20}$ for $e \in W_x$ and $y_e = \frac{1}{4}$ for $e \in H_x$, can be used for parity correction. Our main new tool is a procedure to glue tours over *critical cuts*. For $S \subset V$, let $\delta(S) \subset E_x$ denote the subset of edges crossing the cut $(S, V \setminus S)$.

▶ **Definition 11.** *Let x be a half-cycle point. A proper cut*[2] *$S \subset V$ in G_x is called* critical *if $|\delta(S)| = 3$ and $\delta(S)$ contains exactly one edge e with $x_e = 1$. Moreover, for each pair of edges in $\delta(S)$, their endpoints in S (and in $V \setminus S$) are distinct.*

Observe that a critical cut in G_x is a proper 3-edge cut that is *tight*: the x-values of the three edges crossing the cut sum to 2. Thus, critical cuts are difficult to handle using an approach based on Christofides' algorithm. In particular, using $(\frac{1}{2} - \epsilon)x$ would be insufficient for parity correction of a critical cut if it is crossed by an odd number of edges in the spanning subgraph.

Applying our gluing procedure, we can reduce TSP on half-cycle points to a problem (i.e., base case) where there are only two types of tight 3-edge cuts. The first type of cut is a *vertex cut*, which we show are easier to handle. In particular, the parity of vertex cuts can be addressed with a key tool used by Boyd and Sebő [4] called *rainbow v-trees* (see Theorem 17). We refer to the second type of cut as a *degenerate tight cut*, which is a cut $S \subset V$ such that $|\delta(S)| = 3$, $|S| > 3$ and $|V \setminus S| > 3$ and the two half-edges in $\delta(S)$ share an endpoint in either S or $V \setminus S$. (Observe that for every degenerate tight cut in G_x, there is a 2-edge cut in G_x.) These cuts are also easier to handle. Using this in combination with a decomposition of the 1-edges into few *induced matchings* (see Definition 18), which have some additional required properties, we can prove Theorem 7 for the base case. We discuss gluing procedures in more detail in Section 1.1.

Let us look back at Problem 2. Let x be a *quartic point* if $x \in \text{SUBTOUR}(G) \cap \{0, \frac{1}{2}\}$. Observe that G_x is 4-regular and 4-edge-connected. Yet another equivalent version of Problem 2 is as follows.

▶ **Problem 12** (Half-integer TSP). *Let x be a quartic point. Show $\alpha x \in \text{TSP}(G_x)$ for $\alpha \in [1, \frac{3}{2})$.*

If we assume that the only 4-edge cuts of G_x are its vertex cuts and the number of vertices is even, we can answer this problem.

▶ **Theorem 13.** *Let x be a quartic point. If G_x has an even number of vertices, and G_x does not have any proper 4-edge cuts, then $(\frac{3}{2} - \frac{1}{42})x \in \text{TSP}(G_x)$.*

In the case of a quartic point, Theorem 13 could serve as the base case for if we were able to glue over proper 4-edge cuts of G_x. However, the main difference here is that the gluing arguments we presented for half-cycle points can not easily be extended to this case due to the increased complexity of the distribution of patterns. The proof of Theorem 13 can be found in the full version [12].

1.1 Gluing tours over cuts

The approach of gluing solutions over (often) 3-edge cuts and thereby reducing to an instance without such cuts has been used previously for TSP (e.g., [8]) and extensively in the case of two related problems: the **2-edge-connected multigraph problem (2EC)** and the **2-edge-connected subgraph problem (2ECSS)**. In 2EC, we want to find a minimum cost 2-edge-connected spanning multi-subgraph (henceforth, multigraph for brevity), and in 2ECSS, we want to find a minimum cost 2-edge-connected spanning subgraph (i.e., we are not allowed to double edges). Let $2\text{EC}(G_x)$ and $2\text{ECSS}(G_x)$ denote that convex hulls of characteristic vectors of 2-edge-connected multigraphs and subgraphs, respectively, of G_x. Observe that $\text{TSP}(G_x) \subseteq 2\text{EC}(G_x)$ and $2\text{ECSS}(G_x) \subseteq 2\text{EC}(G_x)$.

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[2] A cut $S \subset V$ is *proper* if $|S| \geq 2$ and $|V \setminus S| \geq 2$.

For example, consider the problem of showing $\frac{6}{5}x \in 2\textsc{ecss}(G_x)$ for a cubic point x [3]. Here, we can assume that G_x is essentially 4-edge-connected due to the following commonly used observation. Let $S \subset V$ be a subset of vertices such that $|\delta(S)| = 3$ in G_x. We construct graphs, G_S and $G_{\bar{S}}$ by contracting the sets $\bar{S}$ and S, respectively, in G_x to a *pseudovertex*. Suppose that the graphs G_S and $G_{\bar{S}}$ contain no proper 3-edge cuts and suppose we can write αx restricted to the edge set of each graph as a convex combination of 2-edge-connected subgraphs of the respective graph. Let us consider the patterns around the pseudovertices; each vertex can be adjacent to two or three edges and therefore, there are only four possible patterns around a vertex. Moreover, since each pattern appears the same percentage of time (in the respective convex combinations) for each pseudovertex, tours with corresponding patterns can be *glued* over the 3-edge cut. (For a more formal presentation of this argument, see Lemma 3.3 in [11] or Case 2 in Section 3.1.2 in [14].) Thus, for 2ECSS, this gluing procedure is quite straightforward. Gluing has also been used for 2EC, but here it is necessary to make certain extra assumptions to control the number of patterns around a vertex, due to the fact that the distribution of possible patterns is more complex. Carr and Ravi proved that the vector $\frac{4}{3}x \in 2\textsc{ec}(G_x)$ for a half-integer point x [5]. To control the number of patterns so that they can use gluing, they require some strong assumptions on the multigraphs in their convex combinations: for example, no edge e with $x_e = \frac{1}{2}$ is doubled and some arbitrarily chosen edge is never used.

In contrast, it appears that no such gluing procedure has been used in approximation algorithms for TSP. Indeed, gluing proofs for 2ECSS and 2EC [5, 3, 14] can not be easily extended to TSP for several reasons: (1) As just discussed, they are used for gluing subgraphs (no doubled edges), while for multigraphs, there are often too many different patterns around a vertex. (For TSP, we must allow doubled edges.) (2) They do not necessarily preserve parity of the vertex degrees. Finally, (3) many of the results for 2ECSS and 2EC based on gluing do not result in polynomial-time algorithms.

The main technical contribution of this paper is to show that for a carefully chosen set of tours, we can design a gluing procedure over critical cuts. In particular, we can fix a critical cut $S \subset V$ in G_x and find a convex combination of tours for G_S. Then we can find a set of tours for $G_{\bar{S}}$ such that the distribution of patterns around the pseudovertex corresponding to S matches that of the pseudovertex corresponding to $\bar{S}$ in G_S. This is done by separately matching the pattern for the spanning subgraphs and for the parity correction. In fact, while each vertex may have a different set of patterns around it, we show that the patterns around each vertex can be encapsulated by a single parameter: the fraction of times in the convex combination of spanning subgraphs that a vertex is a leaf. There can be some flexibility in this degree distribution for some arbitrarily chosen vertex, and this is what we exploit to sufficiently control the patterns around a pseudovertex to enable gluing.

1.2 Definitions, tools and notation

▶ **Definition 14.** *Let $G = (V, E)$ be a graph. For a vertex $v \in V$, a v-tree of G is a subset F of E such that $|F \cap \delta(v)| = 2$ and $F \setminus \delta(v)$ induces a spanning tree of $V \setminus \{v\}$.*

Denote by v-TREE(G) the convex hull of incidence vectors of v-trees of G. The v-TREE(G) is characterized by the following linear inequalities.

$$v\text{-TREE}(G) = \{x \in [0,1] : x(\delta(v)) = 2,$$
$$x(E[U]) \leq |U| - 1 \text{ for all } \emptyset \subset U \subseteq V \setminus \{v\},\ x(E) = |V|\}.$$

▶ **Definition 15.** *Let $G = (V, E)$ and v be a vertex of G. Let $\mathcal{P}$ a collection of disjoint subsets of E. A $\mathcal{P}$-rainbow v-tree of G, is a v-tree of G such that $|T \cap P| = 1$ for $P \in \mathcal{P}$.*

▶ **Definition 16.** *Let $G = (V, E)$ and let x be a vector $(0, 1]^E$. Let $\mathcal{S}$ denote a set of subgraphs of G (i.e., each $S \subseteq E$ for each $S \in \mathcal{S}$). If there is a probability distribution $\lambda = \{\lambda_S\}_{S \in \mathcal{S}}$ such that $x = \sum_{S \in \mathcal{S}} \lambda_S \chi^S$, then we say $\{\lambda, \mathcal{S}\}$ is a* convex combination *for x. If such a probability distribution exists, then we say that x* can be decomposed into (or written as) a convex combination of subgraphs in *$\mathcal{S}$.*

▶ **Theorem 17** (Boyd, Sebő [4]). *Let $x \in \textsc{Subtour}(G)$ and $\mathcal{P}$ be a collection of disjoint subsets of E such that $x(P) = 1$ for $P \in \mathcal{P}$. Then, x can be decomposed into a convex combination of $\mathcal{P}$-rainbow v-trees of G_x for any $v \in V$.*

▶ **Definition 18.** *Given a graph $G = (V, E)$, a set of edges $M \subseteq E$ forms an* induced matching *in G if the subgraph of G induced on the endpoints of M forms a matching (i.e., if edges e and f belong to an induced matching M, then there is no 3-edge path in G containing both e and f).*

Consider a half-cycle point x. For a vertex u in G_x we denote by e_u the unique 1-edge incident on u and by $\gamma(u)$ the two vertices that are the other endpoints of the half-edges incident on v. In other words, suppose $\delta(u) = \{e_u, f, g\}$ and suppose that w_1 and w_2 are the other endpoints of f and g, respectively. Then $\gamma(u) = \{w_1, w_2\}$.

2 Saving on 1-edges for half-cycle points

Let x be a half-cycle point. In this section, we present an algorithm to write x as a convex combination of tours of G_x. Following Christofides' algorithm, we first construct a convex combination of spanning subgraphs in Section 2.1. Next, we address parity correction in Section 2.2. We combine these two steps in Section 2.3 for the base case, in which G_x contains no critical cuts. In Section 2.4, we show how to iteratively glue tours for base cases together to construct tours for general G_x.

2.1 Convex combinations of spanning subgraphs

▶ **Definition 19.** *Let x be a half-cycle point and let v be a vertex of G_x. Suppose $M \subset W_x$ is a subset of 1-edges of G_x. Let $0 \leq \Lambda \leq \frac{1}{2}$. Let $\mathcal{T}$ be a set of spanning connected subgraphs of G_x and let $\lambda = \{\lambda_T\}_{T \in \mathcal{T}}$ be a probability distribution such that $\{\lambda, \mathcal{T}\}$ is a convex combination for x. Then we say $P(v, M, \Lambda)$ holds for the convex combination $\{\lambda, \mathcal{T}\}$ if it has the following properties.*

1. $\sum_{T \in \mathcal{T}: |\delta_T(v)|=1} \lambda_T = \sum_{T \in \mathcal{T}: |\delta_T(v)|=3} \lambda_T = \Lambda$ *and* $\sum_{T \in \mathcal{T}: |\delta_T(v)|=2} \lambda_T = 1 - 2\Lambda$.
2. *For each edge $st \in M$, $|\delta_T(s)| = |\delta_T(t)| = 2$ for $T \in \mathcal{T}$.*
3. *$T \setminus \delta_T(v)$ induces a spanning subgraph on $V \setminus \{v\}$.*

▶ **Lemma 20.** *Suppose $M \subset W_x$ forms an induced matching in G_x and edge $e_v \in M$. Then there is a set of spanning connected subgraphs $\mathcal{T}$ of G_x and a probability distribution $\lambda = \{\lambda_T\}_{T \in \mathcal{T}}$ such that $\{\lambda, \mathcal{T}\}$ is a convex combination for x for which $P(v, M, 0)$ holds.*

Proof. For each $st \in M$, pair the half-edges incident on s and pair those incident on t to obtain disjoint subsets of edges $\mathcal{P}$ and decompose x into a convex combination of $\mathcal{P}$-rainbow v-trees $\mathcal{T}$ (i.e., $x = \sum_{T \in \mathcal{T}} \lambda_T \chi^T$) via Theorem 17. This is the desired convex combination since for all $T \in \mathcal{T}$, we have $|\delta_T(v)| = 2$ and $|\delta_T(u)| = 2$ for all endpoints u of edges in M. Thus, the first and second conditions are satisfied. The third condition holds by definition of v-trees. ◀

▶ **Lemma 21.** *Let $\gamma(v) = \{w_1, w_2\}$ and let Λ be any constant such that $0 \leq \Lambda \leq \frac{1}{2}$. If $M \subset W_x$ forms an induced matching in G_x, $e_v \notin M$ and $|M \cap \{e_{w_1}, e_{w_2}\}| \leq 1$. Then there is a set of spanning connected subgraphs $\mathcal{T}$ of G_x and a probability distribution $\lambda = \{\lambda_T\}_{T \in \mathcal{T}}$ such that $\{\lambda, \mathcal{T}\}$ is a convex combination for x for which $P(v, M, \Lambda)$ holds.*

Proof. As in the proof of Lemma 20, for each $st \in M$, pair the half-edges incident on s and pair those incident on t to obtain a collection of disjoint subsets of edges $\mathcal{P}$. Apply Theorem 17 to obtain $\{\lambda, \mathcal{T}\}$ which is a convex combination for x, where $\mathcal{T}$ is a set of $\mathcal{P}$-rainbow v-trees (i.e., $x = \sum_{T \in \mathcal{T}} \lambda_T \chi^T$). Notice that this convex combination clearly satisfies the second requirement in Definition 19.

Now let $\delta(v) = \{e_v, f, g\}$, where w_1 and w_2 are the other endpoints of f and g, respectively. Without loss of generality, assume $e_{w_1} \notin M$. Since $x = \sum_{T \in \mathcal{T}} \lambda_T \chi^T$, we have $e_v \in T$ for $T \in \mathcal{T}$, since $x_{e_v} = 1$. In addition, we have $|\delta_T(v)| = 2$ for all $T \in \mathcal{T}$ by the definition of v-trees. Hence, $\sum_{T \in \mathcal{T}: f \in T, g \notin T} \lambda_T = \sum_{T \in \mathcal{T}: f \notin T, g \in T} \lambda_T = x_f = \frac{1}{2}$. Without loss of generality, assume $f \in T$ and $g \notin T$ for $T \in \mathcal{T}_f$, and $f \notin T$ and $g \in T$ for $T \in \mathcal{T}_g$, where $\mathcal{T}_f \cup \mathcal{T}_g = \mathcal{T}$ and $\mathcal{T}_f \cap \mathcal{T}_g = \emptyset$.

We can also assume that there are subsets $\mathcal{T}_f^1 \subseteq \mathcal{T}_f$ and $\mathcal{T}_g^1 \subseteq \mathcal{T}_g$ such that $\sum_{T \in \mathcal{T}_f^1} \lambda_T = \Lambda$ and $\sum_{T \in \mathcal{T}_g^1} \lambda_T = \Lambda$, since $\Lambda \leq \frac{1}{2}$. For $T \in \mathcal{T}_f^1$, replace T with $T - f$. Similarly, for $T \in \mathcal{T}_g^1$, replace T with $T + f$. For all $T \in \mathcal{T} \setminus (\mathcal{T}_f^1 \cup \mathcal{T}_g^1)$, keep T as is. Observe that $T \setminus \delta_T(v)$ still induces a spanning subgraph on $V \setminus \{v\}$ since we did not remove any edge in $T \setminus \delta(v)$ from the v-tree T. We want to show that the new convex combination $\{\lambda, \mathcal{T}\}$ is the desired convex combination for x. Notice that

$$\begin{aligned}\sum_{T \in \mathcal{T}} \lambda_T \chi_f^T &= \sum_{T \in \mathcal{T}_f^1} \lambda_T \chi_f^T + \sum_{T \in \mathcal{T}_f \setminus \mathcal{T}_f^1} \lambda_T \chi_f^T + \sum_{T \in \mathcal{T}_g^1} \lambda_T \chi_f^T + \sum_{T \in \mathcal{T}_g \setminus \mathcal{T}_g^1} \lambda_T \chi_f^T \\ &= 0 + (\frac{1}{2} - \Lambda) + \Lambda + 0 = x_f.\end{aligned}$$

So $x = \sum_{T \in \mathcal{T}} \lambda_T \chi^T$. Also, $T \in \mathcal{T}$ is a connected subgraph of G_x since each $T \in \mathcal{T}_f^1$ is obtained by removing an edge incident on v, which does not disconnect it. Finally, for each vertex s with $e_s \in M$, we have $|\delta_T(s)| = 2$ for all $T \in \mathcal{T}$. To observe this, notice that the initial convex combination satisfies this property for vertex s (since the convex combination is obtained via Theorem 17). In the transformation of the convex combination we only change edges incident on w_1 and w_2, so if $s \neq w_1, w_2$ the property clearly still holds after the transformation. If $s = w_1$ or w_2, we only remove or add an edge incident on s if $e_s \neq M$. ◀

2.2 Tools for parity correction

Let $G = (V, E)$ be an arbitrary graph and $O \subseteq V$ where $|O|$ is even. An *O-join* of G is a subgraph J of G in which the set of odd-degree vertices of J are exactly O. The convex hull of characteristic vectors of O-joins of G, denoted by O-JOIN(G) can be described as follows.

$$\begin{aligned}O\text{-JOIN}(G) = \{&x \in [0,1]^E : \\ &x(\delta(S) \setminus U) - x(U) \geq 1 - |U| \text{ for } S \subseteq V,\ U \subseteq \delta(S),\ |S \cap O| + |U| \text{ odd}\}.\end{aligned}$$

▶ **Lemma 22.** *Let x be a half-cycle point and assume that $G_x = (V, E_x)$ has no critical cuts. Let $M \subset W_x$ be a subset of 1-edges of G_x such that each 3-edge cut in G_x contains at most one edge from M. Let $O \subseteq V$ be a subset of vertices such that $|O|$ is even and for all $e = st \in M$, neither s nor t is in O. Also for any set $S \subseteq V$ such that $|\delta(S)| = 2$, both $|S \cap O|$ and $|\delta(S) \cap M|$ are even. Define vector z as follows: $z_e = \frac{1}{2}$ if $e \in W_x$ and $e \notin M$, and $z_e = \frac{1}{4}$ otherwise. Then vector $z \in O$-JOIN(G_x).*

The proof of Lemma 22 can be found in the full version [12].

▶ **Observation 1.** *Let $G = (V, E)$ be a cubic graph, and let $O \subseteq V$ be a subset of vertices such that $|O|$ is even. Let $z \in O\text{-JOIN}(G)$, and $z(\delta(u)) \leq 1$ for all $u \in V$. Then there exists a set of O-joins of G, namely $\mathcal{J}$, and a probability distribution $\psi = \{\psi_J\}_{J\in\mathcal{J}}$ such that $\{\psi, \mathcal{J}\}$ is a convex combination for z. Moreover, for each vertex $v \in V$, the following properties hold.*

1. *If $u \in O$, then we have $|J \cap \delta(u)| = 1$ for each $J \in \mathcal{J}$. (Notice that in this case we must have $z(\delta(u)) = 1$.)*
2. *If $u \notin O$ and $\delta(u) = \{e, f, g\}$, then we have the following (four) cases. (Notice that sum of the right hand sides is exactly 1.)*

$$\sum_{J\in\mathcal{J}:J\cap\delta(u)=\emptyset} \psi_J = 1 - \frac{z(\delta(u))}{2},$$

$$\sum_{J\in\mathcal{J}:J\cap\delta(u)=\{h,h'\}} \psi_J = \frac{z(\delta(u))}{2} - z_{h''} \quad \textit{for any distinct } h, h', h'' \in \delta(u).$$

The proof of this observation follows from the fact that if $z \in O\text{-JOIN}(G)$, then it can be efficiently decomposed into a convex combination of O-joins of G [10].

2.3 Convex combinations of tours: Base case

Let x be a half-cycle point such that $G_x = (V, E_x)$ has no critical cuts. Let v be a fixed vertex in V and let $\gamma(v) = \{w_1, w_2\}$. Let $\{M_1, \dots, M_h\}$ be a partition of W_x into induced matchings such that $|M_i \cap \{e_v, e_{w_1}, e_{w_2}\}| \leq 1$ for all $i \in [h]$, $e_v \in M_1$, each 3-edge cut of G_x contains at most one edge from each M_i, and each 2-edge cut of G_x contains an even number of edges from each M_i. Let $\alpha = \frac{1}{h}$ and Λ be some constant where $0 \leq \Lambda \leq \frac{1-\alpha}{2}$.

For $i = 1$, let $\mathcal{T}_1$ be a set of spanning subgraphs of G_x and let $\{\theta, \mathcal{T}_1\}$ be a convex combination for x for which $P(v, M_1, 0)$ holds (by Lemma 20). For $i \in \{2, \dots, \ell\}$, let $\mathcal{T}_i$ be a set of spanning subgraphs of G_x and let $\{\theta, \mathcal{T}_i\}$ be a convex combination for x for which $P(v, M_i, \frac{\Lambda}{1-\alpha})$ holds (by Lemma 21). Notice that $\frac{\Lambda}{1-\alpha} \leq \frac{1}{2}$ since $\Lambda \leq \frac{1-\alpha}{2}$. Let $\mathfrak{T} = \cup_{i\in[h]}\mathcal{T}_i$.

We can write x as a convex combination of the spanning subgraphs in $\mathfrak{T}$, by weighting each set $\mathcal{T}_i$ by α. In particular, we have $x = \alpha \sum_{i=1}^{h} \sum_{T\in\mathcal{T}_i} \theta_T \chi^T$. For each $T \in \mathfrak{T}$, let $\sigma_T = \alpha \cdot \theta_T$. Then $\{\sigma, \mathfrak{T}\}$ is a convex combination for x. From Definition 19 and Lemmas 20 and 21, we observe the following.

▷ Claim 23. For each $T \in \mathfrak{T}$, $T \setminus \delta(v)$ induces a connected, spanning subgraph on $V \setminus \{v\}$.

For each $i \in [h]$, define $z^i_e = \frac{1}{2}$ if $e \in W_x \setminus M_i$ and $z^i_e = \frac{1}{4}$ otherwise. For each $T \in \mathcal{T}_i$, let $O_T \subseteq V$ be the set of odd-degree vertices of T. By construction, we have $V(M_i) \cap O_T = \emptyset$. By Lemma 22, we have $z^i \in O_T\text{-JOIN}(G)$, so there exists a set of O-joins $\mathcal{J}_T$ and a probability distribution $\psi = \{\psi_J\}_{J\in\mathcal{J}_T}$ such that $\{\psi, \mathcal{J}_T\}$ is a convex combination for z^i. This implies that $x + z^i$ can be written as a convex combination of tours of G_x. We denote this set of tours by $\mathcal{F}_i$ and we let $\mathfrak{F} = \cup_{i\in[h]}\mathcal{F}_i$. We claim that $\sum_{i=1}^{h} \alpha(x + z^i)$ can be written as a convex combination of tours of G_x in $\mathfrak{F}$ using the probability distribution $\phi = \{\phi_F\}_{F\in\mathfrak{F}}$, constructed as follows: For a tour F that is the union of $T \in \mathfrak{T}$ and $J \in \mathcal{J}_T$, set $\phi_F = \sigma_T \cdot \psi_J$.

▷ Claim 24. Let x be a half-cycle point such that $G_x = (V, E_x)$ contains no critical cuts. Define vector $y \in \mathbb{R}^E$ as $y_e = \frac{3}{2} - \frac{\alpha}{4}$ for $e \in W_x$ and $y_e = \frac{3}{4}$ for $e \in H_x$. Then $\{\phi, \mathfrak{F}\}$ is a convex combination for y.

Proof. We need to show that $y = \sum_{i=1}^{h} \alpha(x+z^i)$. First, let e be a 1-edge of G_x and M_j be the induced matching that contains e. Then, $x_e = 1$, $z_e^i = \frac{1}{2}$ for $i \in [h] \setminus \{j\}$ and $z_e^j = \frac{1}{4}$. Hence,

$$\sum_{i=1}^{h} \alpha(x_e + z_e^i) = \sum_{\ell=1}^{h} \alpha \cdot \frac{3}{2} - \alpha \cdot \frac{1}{4} = \frac{3}{2} - \frac{\alpha}{4}.$$

For a half-edge e of G_x, we have $x_e = \frac{1}{2}$ and $z_e^i = \frac{1}{4}$ for $i \in [h]$, so $\sum_{i=1}^{h} \alpha(x_e + z_e^i) = \frac{3}{4}$. ◁

Now we prove some additional useful properties of the convex combination $\{\phi, \mathcal{F}\}$. For a vertex u such that $\delta(u) = \{e_u, f, g\}$ (i.e., where e_u is a 1-edge and f and g are half-edges), let $\mathbb{P}_u$ denote the following set of patterns of edges such that u has even degree and the 1-edge e_u is included at least once.

$$\mathbb{P}_u = \{\{2e_u\}, \{e_u, f\}, \{e_u, g\}, \{2e_u, 2f\}, \{2e_u, 2g\}\}, \{2e_u, f, g\}, \{e_u, 2f, g\}, \{e_u, f, 2g\}\}.$$

Let $\mathbb{P} = \cup_{u \in V} \mathbb{P}_u$. For $0 \le \alpha, \rho \le 1$, define the function $\zeta_{\alpha,\rho} : \mathbb{P} \to [0, 1]$ as follows.

$$\zeta_{\alpha,\rho}(p_u) = \begin{cases} \frac{2-\alpha}{8} & \text{for } p_u = \{2e_u, f, g\}; \\ \frac{\rho}{2} & \text{for } p_u = \{2e_u\}; \\ \frac{\alpha+4\rho}{16} & \text{for } p_u \in \{\{e_u, 2f, g\}, \{e_u, f, 2g\}\}; \\ \frac{4+\alpha-4\rho}{16} & \text{for } p_u \in \{\{e_u, f\}, \{e_s, g\}\}; \\ \frac{2-\alpha-4\rho}{16} & \text{for } p_u \in \{\{2e_u, 2f\}, \{2e_u, 2g\}\}. \end{cases}$$

▷ Claim 25. The convex combination $\{\phi, \mathcal{F}\}$, has the following properties.

(i) For each vertex $u \in V$ there is a some constant η_u where $0 \le \eta_u \le \frac{1-\alpha}{2}$ and

$$\sum_{F \in \mathcal{F} : F \cap \delta(u) = p_u} \phi_F = \zeta_{\alpha,\eta_u}(p_u) \text{ for } p_u \in \mathbb{P}_u.$$

(ii) $\eta_v = \Lambda$.

The proof of Claim 25 can be found in the full version [12].

▶ **Lemma 26.** *Let x be a half-cycle point, and assume $G_x = (V, E_x)$ does not have any critical cuts. Let r be a vertex in V and let $\gamma(r) = \{w_1, w_2\}$. The set of 1-edges in G_x, W_x, can be partitioned into five induced matchings $\{M_1, \dots, M_5\}$ such that for $i \in \{1, \dots, 5\}$, the following properties hold.*

(i) $M_i \cap \{e_r, e_{w_1}, e_{w_2}\}| \le 1$,

(ii) *For $S \subseteq V$ such that $|\delta(S)| = 3$, $|\delta(S) \cap M_i| \le 1$.*

(iii) *For $S \subseteq V$ such that $|\delta(S)| = 2$, $|\delta(S) \cap M_i|$ is even.*

The proof of Lemma 26, which can be found in the full version, uses induction (i.e., gluing solutions for base cases) and the proof of the base case is an application of Brooks' theorem on a slight modification of the line graph of G_x [12].

Let $\gamma_v = \{w_1, w_2\}$. By Lemma 26, there are $\{M_1, \dots, M_5\}$ that partition W_x into induced matchings such that $|M_i \cap \{e_v, e_{w_1}, e_{w_2}\}| \le 1$ for all $i \in [5]$, and each induced matching intersects a 3-edge-cut at most once and a 2-edge cut an even number of times. The following Lemma follows from Claim 25 by setting $\alpha = \frac{1}{5}$.

▶ **Lemma 27.** *Let x be a half-cycle point such that $G_x = (V, E_x)$ contains no critical cuts. Fix any vertex in $v \in V$ and Λ with $0 \le \Lambda \le \frac{2}{5}$. Define $y \in \mathbb{R}^E$ as $y_e = \frac{3}{2} - \frac{1}{20}$ for $e \in W_x$ and $y_e = \frac{3}{4}$ if $e \in H_x$. Then there is a set of tours of G_x denoted by $\mathcal{F}$ and a probability distribution $\phi = \{\phi_F\}_{F \in \mathcal{F}}$ such that $\{\phi, \mathcal{F}\}$ is a convex combination for y. Moreover, this convex combination has the following properties.*

(i) *For each vertex $u \in V$, there is a some constant η_u where $0 \leq \eta_u \leq \frac{2}{5}$ and*

$$\sum_{F \in \mathcal{F}: F \cap \delta(u) = p_u} \phi_F = \zeta_{\frac{1}{5}, \eta_u}(p_u) \text{ for } p_u \in \mathcal{P}_u.$$

(ii) $\eta_v = \Lambda$.

(iii) *$F \setminus \delta_F(v)$ induces a connected multigraph on $V \setminus \{v\}$ for each $F \in \mathcal{F}$.*

2.4 Convex combinations of tours: Gluing over critical cuts

▶ **Theorem 7.** *Let $x \in \mathbb{R}^E_{\geq 0}$ be a half-cycle point. Define vector $y \in \mathbb{R}^{E_x}$ as follows: $y_e = \frac{3}{2} - \frac{1}{20}$ for $e \in W_x$ and $y_e = \frac{3}{4}$ for $e \in H_x$. Then $y \in \mathrm{TSP}(G_x)$.*

For a graph $G = (V, E)$ and nonempty subset of vertices $S \subset V$, contract the component induced on $\bar{S} = V \setminus S$ into a vertex and call this vertex $v_{\bar{S}}$. We define the graph G_S to be the graph induced on vertex set $S \cup v_{\bar{S}}$. The graph $G_{\bar{S}}$ is analogously defined on the vertex set $\bar{S} \cup v_S$.

▶ **Lemma 28.** *Consider a graph $G = (V, E)$ and nonempty $S \subset V$ such that $\delta(S)$ is a minimum cardinality cut in $G = (V, E)$. Let F_S be a tour in G_S and let $F_{\bar{S}}$ be a tour in $G_{\bar{S}}$ such that $\chi_e^{F_S} = \chi_e^{F_{\bar{S}}}$ for $e \in \delta(S)$. Moreover, assume that $F_S \setminus \delta(v_{\bar{S}})$ induces a connected multigraph on S. Then the multiset of edges F defined as $\chi_e^F = \chi_e^{F_S}$ for $e \in E(G_S)$ and $\chi_e^F = \chi_e^{F_{\bar{S}}}$ for $e \in E(G_{\bar{S}})$ is a tour of G.*

Proof. It is clear that F induces an Eulerian multigraph on G, but we need to ensure that F is connected. For example, the tour induced on $F_{\bar{S}} \setminus \delta(v_S)$ might not be connected. However, since the subgraph of F_S induced on the vertex set S is connected, the tour F is connected: each vertex in $\bar{S}$ is connected to some vertex in S. ◀

▶ **Lemma 29.** *Let x be a half-cycle point such that $G_x = (V, E_x)$. Define $y \in \mathbb{R}^E$ as $y_e = \frac{3}{2} - \frac{1}{20}$ for $e \in W_x$ and $y_e = \frac{3}{4}$ if $e \in H_x$. Then there is a set of tours of G_x denoted by $\mathcal{F}$ and a probability distribution $\phi = \{\phi_F\}_{F \in \mathcal{F}}$ such that $\{\phi, \mathcal{F}\}$ is a convex combination for y. Moreover, this convex combination has the following property.*

For each vertex $u \in V$, there is a some constant η_u where $0 \leq \eta_u \leq \frac{2}{5}$ and

$$\sum_{F \in \mathcal{F}: F \cap \delta(u) = p_u} \phi_F = \zeta_{\frac{1}{5}, \eta_u}(p_u) \text{ for } p_u \in \mathcal{P}_u.$$

Proof. If G_x does not contain a critical cut, we apply Lemma 27. Otherwise, set $G := G_x$ and conduct the following procedure: Find a cut $S_1 \subset V(G)$ such that $G_1 = G_{S_1}$ contains no critical cuts. Then set $G := G_{\bar{S}}$ and find a cut $S_2 \in V(G)$ such that $G_2 = G_{S_2}$ contains no critical cuts, etc.

At the end of this procedure, we have a series of graphs $\{G_1, \ldots, G_k\}$ such that for each $j \in [k]$, G_j is the support graph of a half-cycle point and contains no critical cuts. Therefore, each G_j is a base case and we can find a convex combination of tours applying the procedure described in Section 2.3.

We glue the tours together in reverse order according to their index beginning with G_k and G_{k-1}. The graph G_{k-1} corresponds to G_S for some vertex set S of G, where G is the graph at the beginning of iteration $k-1$ of the above procedure. Note that $G_{\bar{S}}$ equals G_k and it has no critical cuts. Therefore, after invoking Lemma 27 to find a convex combination of tours for $G_{\bar{S}}$, we invoke Lemma 27 on G_S with $v = v_{\bar{S}}$ and $\Lambda = \eta_{v_S}$ based on the convex combination of tours returned for $G_{\bar{S}}$. Now in the tours returned, the patterns on vertex $v_{\bar{S}}$ match those of v_S in the convex combination of tours previously found for $G_{\bar{S}}$.

After having glued together the tours from G_{k-1} and G_k in this manner, we glue the resulting tours with those in G_{k-2}, etc., until we have found a convex combination of tours for G_x. ◀

3 Discussion

In this paper, we presented an algorithm to save on 1-edges for a half-cycle point. To fully resolve half-integer TSP, we need to be able to save on half-edges. Towards this goal, we proposed a "base case" when there is no proper minimum cut (Theorem 13). It is not clear how to combine this with a gluing approach similar to the one for half-cycle points described in Section 2. Thus, we close with the following open problem.

▶ **Problem 30.** *Let x be a half-cycle point. vector $y \in \mathbb{R}^{E_x}$ as follows: $y_e = \frac{3}{2}$ for $e \in W_x$ and $y_e = \frac{3}{4} - \delta$ for $e \in H_x$. Show there exists a constant $\delta > 0$ such that $y \in \mathrm{TSP}(G_x)$.*

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