Single-Elimination Brackets Fail to Approximate Copeland Winner

Reyna Hulett
Department of Computer Science, Stanford University, CA, USA
rmhulett@stanford.edu

Abstract

Single-elimination (SE) brackets appear commonly in both sports tournaments and the voting theory literature. In certain tournament models, they have been shown to select the unambiguously-strongest competitor with optimum probability. By contrast, we reevaluate SE brackets through the lens of approximation, where the goal is to select a winner who would beat the most other competitors in a round robin (i.e., maximize the Copeland score), and find them lacking. Our primary result establishes the approximation ratio of a randomly-seeded SE bracket is $2^{-\Theta(\sqrt{\log n})}$; this is underwhelming considering a $\frac{1}{2}$ ratio is achieved by choosing a winner uniformly at random. We also establish that a generalized version of the SE bracket performs nearly as poorly, with an approximation ratio of $2^{-\Omega(\sqrt[4]{\log n})}$, addressing a decade-old open question in the voting tree literature.

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1 Introduction

The *round robin* and the *single-elimination bracket* are two common formats for sporting competitions. In a round robin, every competitor plays against every other competitor once. The outcome of a round robin can be represented as a *tournament graph*, a directed complete graph where an edge from A to B means A defeats B. A single-elimination bracket can be represented by a balanced binary tree with the leaves labeled by a permutation of the competitors. Each internal node is then labeled with the winner of a game between the two children of that node, with the root node indicating the overall winner. (For simplicity, assume no ties, deterministic game outcomes, and $n = 2^m$ competitors for some integer $m \geq 2$.)

A round robin effectively gives us complete information; we learn the outcome of all $\binom{n}{2}$ possible games. However, it is not immediately clear how to translate this into a single winner unless one competitor beats every other competitor (known as a *Condorcet winner*). There are various possible solution concepts – such as the Slater set, the uncovered set, and the top cycle – but we will focus on the (far more popular) Copeland solution. Each competitor's *Copeland score* equals its out-degree in the tournament graph, i.e., the number of other competitors it defeats. This gives us a natural, quantitative measure of competitor strength; thus the Copeland winner(s), or Copeland set, is the competitor(s) with the maximum Copeland score.

An SE bracket leaves no such ambiguity in determining a unique winner. It also requires fewer games, and each game has higher stakes, which may explain the popularity of this format! But what are we trading off in exchange for these desirable qualities? Can we still expect a strong competitor to win? We will address this question by considering how well SE brackets approximate the maximum Copeland score, for both worst-case and random seeding.

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1.1 Related Work

A slightly different version of this question is long resolved. Namely, that line of work assumes the game outcomes are probabilistic *but* there exists an unambiguously strongest competitor (who beats every other competitor with probability $> 1/2$). Competition formats can then be evaluated based on their probability of selecting this strongest competitor, relative to the number of games or rounds played. This evaluation criteria has variously been referred to as "predictive power" or "effectiveness" [12, 1, 14]. Under certain models in this setting, a balanced SE bracket has the highest predictive power of any competition format with at most $n-1$ games [12]. However, the predictive power of *any* knockout format (where a competitor is eliminated after a single loss) will in general be sub-constant in the number of competitors. This is one motivation for evaluating formats based on the expected "strength" of the winner, rather than just the (vanishing) probability of selecting a single strongest competitor.

The question of *seeding* an SE bracket has also received significant attention in the probabilistic setting, both in terms of *designing* a fair seeding [21] and *manipulating* the seeding to help a particular competitor [14, 20] – in general, it is NP-hard to find a seeding which maximizes a given competitor's win probability. Perhaps surprisingly, it is even NP-hard to determine whether there exists a winning seeding for a given competitor with *deterministic* game outcomes [2], although many special cases have been identified where a polynomial time algorithm exists [19, 16, 18, 17, 2, 10, 9]. We ignore questions of seeding in the present work, considering only worst-case and random seeding. However, it is worth noting that a winning seeding *does* exist for any Copeland winner [19].

Finding or approximating the Copeland winner(s) of a tournament graph with deterministic game outcomes has been studied more generally, although not specifically for SE brackets. Finding the whole Copeland set requires all $\binom{n}{2}$ games to be played in the worst-case [4, 7]. If we only wish to select a single Copeland winner, that still requires at least $\binom{n}{2} - 2$ games (for odd n) [7]. By contrast, finding the Condorcet winner (or determining none exists) requires only $2n - \lfloor \log n \rfloor - 2$ games to be played [3, 13]. The number of games required to *approximate* the Copeland winner is not well studied, but it is known that finding a competitor with Copeland score *exactly* $k \leq (n-1)/2$ requires $\Theta(nk)$ games to be played in the worst-case [3].

The most direct predecessor of the present work concerns approximating the Copeland winner using a broad category of tournament formats known as *voting trees*, which include SE brackets. Whereas an SE bracket corresponds to a balanced binary tree with n leaves, a voting tree can be any binary tree with any number of leaves (the same competitor can label multiple leaves). For $n \leq 7$ competitors, there exists a voting tree which can select a Copeland winner, but not so for 8 or more competitors [15]. However, voting trees can *approximate* the maximum Copeland score, with an approximation ratio of $2/3$ [8], although this result is non-constructive. The best-known upper bound on the approximation ratio achievable by voting trees is $3/4$ [5, 6]. The situation changes slightly if, instead of a single voting tree, we are allowed to specify a distribution over voting trees. For instance, we could consider a randomly-seeded SE bracket. In this case, we want the *expected* Copeland score of the winner to approximate the maximum Copeland score. Naturally, the $2/3$ lower bound still applies, but the best-known upper bound for randomized voting trees is $5/6$ [5]. In additional, for randomized voting trees there is a constructive lower bound with certain nice properties which obtains an approximation ratio approaching $1/2$ [5]. The relevant paper concludes by conjecturing that SE brackets, or certain sizes of balanced voting trees more generally, may be able to obtain good approximation ratios, although they note "[t]he analysis of this type of randomization is closely related to the theory of dynamical systems, and we expect it to be rather involved" [5]. We answer this conjecture in the negative.

It is worth noting that randomly-seeded single-elimination brackets have previously been assumed to select "strong" winners. The probability of a competitor winning such a bracket has variously been referred to as "a natural notion of player strength" [10], and proposed as a way to select a winner from a tournament graph [11]. This makes it especially surprising that SE brackets fail to approximate the maximum Copeland score.

1.2 Contributions

We divide our contributions into three primary categories.

In Section 3, we analyze SE brackets with worst-case seeding. Although it will be straightforward to see that they achieve an approximation ratio of only $\frac{\log n}{n-2}$, we provide context for this result by calculating the query complexity (number of games that must be played) to approximate the maximum Copeland score with a given approximation ratio. We argue that, for worst-case seeding, SE brackets can actually be considered optimal among formats with at most $n-1$ games satisfying a basic fairness criterion, in close analogy to the work of [12]. Additionally, our results suggest a "single-elimination into round robin" format as an optimal generalization of SE brackets to more than n games.

Our main result is described in Section 4. Namely, we establish that the approximation ratio of a randomly-seeded SE bracket is $2^{-\Theta(\sqrt{\log n})}$. (For comparision, this is not as bad as the worst non-zero approximation ratio $\Theta(\frac{1}{n})$ but is worse than, say, $\Theta(\frac{1}{\log n})$.) A central lemma used in this proof is based on a result of [5], although we have to redo their analysis more precisely for our purposes. Essentially, we consider a class of tournament graphs where the competitors fall into three categories: "weak", "mediocre", and "strong". We construct a randomized distribution over such tournament graphs, which allows us to use the lemma to show that "weak" competitors usually win the corresponding SE brackets. Additionally, we show that most of the probability mass concentrates on tournament graphs where the "weak" competitors really do have low Copeland score, and thus there exists a specific tournament graph with low approximation ratio. We obtain the corresponding lower bound on the approximation ratio by showing that sufficiently low-scoring competitors are few in number and likely to be eliminated early in an SE bracket.

Finally, in Section 5, we reuse the aforementioned lemma to upper-bound the approximation ratio of *all* randomly-seeded, balanced voting trees of sufficient size as $2^{-\Omega(\sqrt[4]{\log n})}$. This refutes the hope expressed in [5] that carefully choosing the size of a balanced voting tree could result in a good approximation ratio.

2 Preliminaries

Let $S = \{1, \ldots, n = 2^m\}$ be a set of competitors. A *tournament graph*, or just *tournament*, on S is a directed complete graph where S is the vertex set. For any two competitors $c_1, c_2 \in S$, $c_1 \neq c_2$, the corresponding edge is directed from the winner to the loser in a hypothetical game between the two. If c_1 is the winner, we say c_1 beats c_2, or $c_1 \to c_2$. Note that this implies there are no ties, and the game outcomes are deterministic, i.e., if $c_1 \to c_2$, then c_1 beats c_2 always. We use $\mathcal{T}(n)$ to denote the set of all tournaments on n competitors.

Given a tournament graph T, the *Copeland score* of a competitor c is the out-degree of the corresponding vertex, i.e., $d_+(c) = |\{s \in S \mid c \to s\}|$. The *Copeland winner(s)* of a tournament is the competitor(s) with the highest Copeland score. If some competitor has Copeland score $n-1$, she is called the *Condorcet winner*.

We will be considering various *competition formats*, which are (deterministic or randomized) algorithms that query edges of the tournament graph by running games between pairs of competitors, and return a single winner. For a competition format F, the *query complexity* is the worst-case number of games played under F. For a given set of n competitors S, we also define the *approximation ratio* for the maximum Copeland score as

$$\min_{T \in \mathcal{T}(n)} \frac{\mathbb{E}[d_+(F(T))]}{\max_{s \in S} d_+(s)}$$

where $F(T)$ denotes the winner of tournament T under format F (possibly randomized).

A *single-elimination bracket* is a competition format represented by a balanced binary tree with $n = 2^m$ leaves labeled by a permutation or *seeding* of the competitors. For each level of the tree, moving up from the leaves – that is, for each round of the bracket – for each internal node, it labels the node with the winner of a game between the node's two children. The root node's label indicates the winner. We will analyze both worst-case seeding (i.e., the deterministic competition format with an arbitrary labeling of the leaves) and random seeding (i.e., the randomized competition format where a random permutation of competitors is chosen to label the leaves).

3 Deterministic Approximation of Copeland Winner

We first consider how well SE brackets with worst-case seeding approximate the Copeland winner, and for comparison, we establish the deterministic query complexity required for any competition format with a given approximation ratio. This reveals that SE brackets require significantly more games than the optimal query complexity. However, in Section 3.1, we reanalyze the query complexity of approximation subject to a basic fairness constraint, the "Condorcet property." Under this constraint, SE brackets actually achieve optimal query complexity, and can be generalized into an asymptotically optimal "single-elimination into round robin" format for more than $n - 1$ games.

We begin by calculating the approximation ratio of SE brackets.

▶ **Theorem 1.** *The single-elimination bracket on $n = 2^m$ competitors achieves a deterministic approximation ratio of exactly $\frac{\log n}{n-2}$ for the maximum Copeland score.*

Proof. Observe that (1) the SE winner must have a Copeland score of at least $\log n$, since she must beat one competitor per level for $\log n$ levels, and (2) if a Condorcet winner exists, he must be the SE winner, or conversely, a non-SE-winner has Copeland score at most $n - 2$. Taken together, these observations imply that the approximation ratio of the SE bracket is *at least* $\frac{\log n}{n-2}$.

Now, we construct a tournament graph showing that the approximation ratio is *at most* $\frac{\log n}{n-2}$. Consider any tournament graph G on $n - 1$ competitors with a Condorcet winner, c. Let the n^{th} competitor be s. Fix any seeding for the bracket. In order to win the bracket, s must defeat $\log n$ competitors, whose identities are fully determined by the graph G and the seeding. Thus we may assume s defeats exactly this set of $\log n$ competitors, loses to every other competitor, and wins the bracket. By construction, s has Copeland score $\log n$ and the maximum Copeland score is (at least) $n - 2$, so the approximation ratio is (at most) $\frac{\log n}{n-2}$, as desired. ◀

In order to evaluate how good or bad this approximation ratio is, we will consider how well an arbitrary competition format can approximate the maximum Copeland score, trading off against the total number of games played. This can be thought of as the query complexity of approximation. I.e., if we are allowed arbitrary and adaptive “queries”, how many games must be played (outcomes queried) to find a competitor with at least $0 < r \leq 1$ times the maximum Copeland score?

▶ **Theorem 2.** *The deterministic query complexity to find a competitor with at least* $0 < r \leq 1$ *times the maximum Copeland score in a tournament on* n *vertices is* $\Theta(\max(1, rn)^2)$.

Proof. We will use the fact that, for any tournament on n vertices, the maximum Copeland score M lies in $[\frac{n-1}{2}, n-1]$, i.e., between the average Copeland score and the maximum possible.

The upper bound is simple: to obtain an approximation ratio r, pick an arbitrary $n' = \min(2\lceil r(n-1)\rceil, n)$ competitors, and query all games within this sub-tournament. If $n' = n$, you find the true Copeland winner, so the approximation ratio is $1 \geq r$. Otherwise, $n' = 2\lceil r(n-1)\rceil$, so the average Copeland score within this sub-tournament is $\frac{2\lceil r(n-1)\rceil - 1}{2} = \lceil r(n-1)\rceil - \frac{1}{2}$. The maximum Copeland score within this sub-tournament is at least the average score rounded up, so at least $\lceil r(n-1)\rceil \geq r(n-1) \geq rM$, as desired. The total number of games played is at most

$$\binom{2\lceil r(n-1)\rceil}{2} \leq 2\lceil rn\rceil^2 = O(\max(1, rn)^2).$$

For the lower bound, we will define a simple adversarial strategy for answering the queries: when two competitors are queried, give the win to whichever has *fewer* wins so far, breaking ties arbitrarily. The competitor c returned by the optimal algorithm must have been shown to beat at least $k = \lceil \frac{r(n-1)}{2} \rceil$ distinct other competitors, since any un-queried game could be a loss for c. Now, because of the adversary's strategy, when the algorithm discovers the i^{th} win for c, it must be beating a competitor which already had $i-1$ wins queried. Thus, counting the number of wins for c and for the k competitors she defeats, the algorithm must have queried at least

$$k + \sum_{i=1}^{k}(i-1) = \frac{k(k+1)}{2} \geq \left\lceil \frac{r(n-1)}{2} \right\rceil^2 \div 2 = \Omega(\max(1, rn)^2)$$

games. ◀

As a sanity check, we know that finding a competitor with maximum Copeland score requires at least $\binom{n}{2} - 2$ games to be played (for odd n) [4], and indeed plugging in $r = 1$ we get a query complexity of $\Theta(n^2)$.

Numerically, SE brackets do not look very good at this point! The sub-tournament strategy described above can obtain the same approximation ratio $r = \frac{\log n}{n-2}$ in only $\Theta(\log^2 n) \ll n - 1$ games. Conversely, if we allow ourselves $n - 1$ games, we should be able to obtain an approximation ratio of $\Theta(\frac{1}{\sqrt{n}}) \gg \frac{\log n}{n-2}$.

Of course, this sub-tournament strategy is a deeply unsatisfying format for any kind of competition or election. 64 teams qualify for March Madness; should we suggest the NCAA just pick 12 at random and then play a round robin?

3.1 “Fair” Deterministic Approximation

There are multiple reasons why we might prefer an SE bracket over this strange sub-tournament round robin, but the most glaring is fairness. An SE bracket may be more or less “fair” depending on how the competitors are seeded, but at least it doesn't eliminate a majority of the competitors from the get-go.

In this section, we will investigate the complexity of approximation restricted to *Condorcet* competition formats. That is, if one competitor beats every other competitor in the underlying tournament graph (i.e., if there is a Condorcet winner) then he must be chosen as the winner of the competition. This is only a slightly stronger requirement than insisting that no competitor be eliminated a priori, since it specifies an intuitively obvious win condition. Additionally, this "Condorcet property" is well-studied in voting theory, has previously been described for tournaments under the name "unbiasedness" [12], and is closely related to the concept of "admissibility" from the voting tree literature [5]. Clearly, the SE bracket is a Condorcet competition format, while the sub-tournament round robin discussed above is not.

As a warm-up, observe that any Condorcet format must query at least $n-1$ game outcomes. This holds because every competitor except the winner must have been observed to lose at least one game; otherwise, a non-winner could violate the Condorcet property. Thus if we want to obtain an approximation ratio of $\frac{\log n}{n-2}$ for the maximum Copeland score, the SE bracket is optimal among Condorcet competition formats. In fact, the following result implies that SE brackets achieve the optimal approximation ratio among all Condorcet formats with exactly $n-1$ games.

▶ **Theorem 3.** *The deterministic query complexity, restricted to Condorcet competition formats, to find a competitor with at least* $0 < r \leq 1$ *times the maximum Copeland score in a tournament on* $n = 2^m \geq 4$ *vertices is* $n-1$ *if* $r \leq \frac{\log n}{n-2}$ *or* $n-1+\Theta(\max(1, r(n-2)-\log n)^2)$ *otherwise.*

Proof. As noted above, $n-1$ games are required for any Condorcet format, so when $r \leq \frac{\log n}{n-2}$, the desired approximation ratio is achieved in the optimal $n-1$ games by an SE bracket. For the remainder of the proof, we consider $r > \frac{\log n}{n-2}$.

The upper bound is similar to the sub-tournament round robin approach from Theorem 2, except that we use a partial single-elimination bracket to select the competitors for the sub-tournament. Define $\delta = \max(1, r(n-2) - \log n)$. If $\delta \geq \frac{n}{8}$, then we simply run a round robin on all n competitors and return the Copeland winner. This is clearly a Condorcet format, achieves approximation ratio $1 \geq r$, and requires $\binom{n}{2} \leq \frac{n^2}{2} = n - 1 + O(\delta^2)$ games.

Otherwise, we will play the first $\log n - \lceil \log \delta \rceil - 3$ rounds of a single-elimination bracket, then run a round robin among the remaining $2^{\lceil \log \delta \rceil + 3}$ competitors and return a competitor with highest number of wins. Observe that this is a Condorcet format, and it returns a competitor with Copeland score at least $\log n - \lceil \log \delta \rceil - 3 + 2^{\lceil \log \delta \rceil + 2} \geq \log n + 2^{\lceil \log \delta \rceil} \geq \log n + \delta \geq r(n-2)$. Therefore, we obtain an r-approximation of the maximum Copeland score, because either (1) there is a Condorcet winner, she wins the tournament, and the ratio is $1 \geq r$, or (2) there is no Condorcet winner, so the maximum Copeland score is $M \leq n-2$. Finally, the number of games played is less than

$$n - 1 + \binom{8\delta}{2} = n - 1 + O(\delta^2).$$

For the lower bound, we will reuse the adversarial strategy from Theorem 2: whenever a query is made, the winner of the game will be whichever competitor has *fewer* wins so far, with ties broken arbitrarily. Suppose the competitor c chosen as the winner of the competition has had k wins queried. Because of the adversary's strategy, these k competitors must have already had $0, 1, 2, \dots, k-1$ wins (out-edges), respectively, at the time they were beaten. In fact, the same logic extends to these k competitors and their wins, etc., forming a cascade of 2^k vertices. However, these 2^k vertices need not necessarily be distinct competitors (except the top $k+1$, which must). Moreover, every vertex except the winner must have at least one in-edge (otherwise it could be a Condorcet winner), so the number of vertex-reuses in this cascade is at most the query complexity less $n-1$, since every reuse increases the in-degree of some vertex by 1.

We would like to lower-bound the query complexity in terms of n; to do this, we will initially frame it as lower-bounding n in terms of the query complexity, for fixed k. Let $i \in \mathbb{R}_{\geq 0}$ be such that the total number of games played will be $n - 1 + \lceil i(i+1)/2 \rceil$. Since k is fixed, we need a cascade of 2^k vertices; however $\lceil i(i+1)/2 \rceil$ can be "reuses." Note that if a vertex is reused, it *and* its children appear only once in the resulting cascade. Any valid reuse can be captured by "erasing" a sub-tree, with the interpretation that the dangling edge that led to that sub-tree now points somewhere else. However, none of the top $k+1$ vertices can be erased in this way, since they must be distinct. (Other vertices can be erased and have their edges pointed to one of these vertices, however.)

Thus, to lower-bound n, we can equivalently upper-bound the number of vertices erased with $\lceil i(i+1)/2 \rceil$ reuses, since there need to be at least enough distinct vertices to constitute all the non-erased vertices in the cascade. How can we maximize the number of vertices erased? The top two layers (top $k+1$ vertices) cannot be erased, so the optimal strategy is to erase vertices from the layer directly below, in order of decreasing size of their sub-trees, since it always removes more vertices to erase the root of a sub-tree than any of its children. In particular, we would first erase the 3^{rd} level sub-tree of size 2^{k-2}, then the two sub-trees of size 2^{k-3}, then the three sub-trees of size 2^{k-4}, etc. For ease of accounting, let us assume we remove all the 3^{rd}-level sub-trees of size at least $2^{k-\lceil i \rceil -1}$. Observe that this comes to $1 + 2 + \cdots + \lceil i \rceil \geq \lceil i(i+1)/2 \rceil$ reuses. We need to erase at least $2^k - n$ vertices from the cascade, so

$$
\begin{aligned}
2^k - n &\leq 2^{k-2} + 2 \times 2^{k-3} + \cdots + \lceil i \rceil \times 2^{k-\lceil i \rceil - 1} \\
&= 2^{k-\lceil i \rceil -1} \left(2^{\lceil i \rceil - 1} + 2 \times 2^{\lceil i \rceil -2} + \cdots + \lceil i \rceil \times 2^0 \right) \\
&= 2^{k-\lceil i \rceil -1} \left(2^{\lceil i \rceil + 1} - \lceil i \rceil - 2 \right) \\
&= 2^k - (\lceil i \rceil + 2) 2^{k - \lceil i \rceil - 1} \\
\log n &\geq \log(\lceil i \rceil + 2) + k - \lceil i \rceil - 1 \\
\lceil i \rceil &\geq k - \log n + \log(\lceil i \rceil + 2) - 1 \geq k - \log n.
\end{aligned}
$$

Note that this means any Condorcet competition format using $n - 1 + \lceil i(i+1)/2 \rceil$ games returns a winner with at most $k \leq \log n + \lceil i \rceil$ wins queried. What does this mean for our approximation ratio? By the pigeonhole principle, there is some non-winner with no more than $1 + \left\lfloor \frac{\lceil i(i+1)/2 \rceil}{n-1} \right\rfloor$ losses queried. Thus the maximum Copeland score could be as high as $M \geq n - 2 - \left\lfloor \frac{\lceil i(i+1)/2 \rceil}{n-1} \right\rfloor$. In particular, observe that if only $n-1$ games are played, then $i = 0$ and $r = \frac{k}{M} \leq \frac{\log n}{n-2}$. This confirms that for $r > \frac{\log n}{n-2}$, $n - 1 + \Omega(1)$ games are required, implying SE brackets achieve the optimal approximation ratio for Condorcet formats with $n-1$ games.

We have essentially calculated a bound on the approximation ratio in terms of i, but we want to turn this into an asymptotic bound on query complexity for a given approximation ratio. Assuming $0 < i \leq n-1$ (since we can only have $\binom{n}{2}$ games in total), we have

$$
\begin{aligned}
r &\leq \frac{\log n + \lceil i \rceil}{n - 2 - \left\lfloor \frac{\lceil i(i+1)/2 \rceil}{n-1} \right\rfloor} \\
r(n-2) - \log n &\leq r \left\lfloor \frac{\lceil i(i+1)/2 \rceil}{n-1} \right\rfloor + \lceil i \rceil \\
(r(n-2) - \log n)^2 &\leq \left(r \left\lfloor \frac{\lceil i(i+1)/2 \rceil}{n-1} \right\rfloor + \lceil i \rceil \right)^2 \\
&\leq \lceil i(i+1)/2 \rceil^2 \left(\frac{1}{n-1} + \frac{\lceil i \rceil}{\lceil i(i+1)/2 \rceil} \right)^2 \\
&\leq \lceil i(i+1)/2 \rceil^2 \left(\frac{1}{n-1} + \min\left(1, \frac{2}{i}\right) \right)^2 \\
&\leq \lceil i(i+1)/2 \rceil^2 \left(2 \min\left(1, \frac{2}{i}\right) \right)^2 \\
&\leq 4 \lceil i(i+1)/2 \rceil \left(\frac{i(i+1)}{2} + 1 \right) \min\left(1, \frac{4}{i^2}\right) \leq 16 \lceil i(i+1)/2 \rceil .
\end{aligned}
$$

Thus the query complexity is at least

$$
n - 1 + \lceil i(i+1)/2 \rceil \geq n - 1 + \frac{1}{16}(r(n-2) - \log n)^2 = n - 1 + \Omega(\max(1, r(n-2) - \log n)^2)
$$

as desired. ◀

Interestingly, this implies that a "single-elimination into round robin" format achieves asymptotically optimal query complexity, with very simple structure. The initial single-elimination rounds could still benefit from seeding (to make stronger competitors more likely to survive the early rounds), while the round-robin phase ensures the eventual winner is reasonably strong, regardless of any manipulation in the seeding. Both single-elimination and round robin are common formats for sporting competitions, but they are rarely if ever employed together in this order.

In the following section, we move on to analyze SE brackets with random seeding (rather than worst-case). Note, however, that coming up with a good *randomized* approximation of the Copeland winner is much easier than the deterministic case considered above. In fact, we can achieve an approximation ratio of $r = \frac{1}{2}$ with a query complexity of *zero* – the average Copeland score is $\frac{n-1}{2}$, so simply returning a random competitor achieves this objective! This makes it especially surprising that randomly-seeded SE brackets cannot even achieve a constant approximation of the maximum Copeland score.

4 SE Brackets Fail to Approximate Copeland Winner

In this section, we prove our main result: the approximation ratio of SE brackets for the maximum Copeland score is $2^{-\Theta(\sqrt{\log n})}$.

To obtain our upper bound on the approximation ratio, we consider tournament graphs consisting of 3 groups ("components") of competitors: a small set of "strong" competitors, a small set of "weak" competitors, and a majority of "mediocre" competitors. We assume every strong competitor beats every mediocre competitor, who beats every weak competitor; however, the weak beat the strong (a related concept has been analyzed under the term "choking" [10]). Note that the weak competitors will have low Copeland scores, while the strong have high scores. The idea is that, even though weak competitors have low scores, they can beat the strongest competitors in the tournament and thus come out on top. In fact, the key observation follows one made in Theorem 5 of [5]: as the depth of a balanced bracket grows, the likely winner oscillates between these three components.

The construction of this upper bound is similar to the proof of Theorem 5 in [5], with a few key differences. First, they consider a tournament with only a single "strong" and a single "weak" competitor, and label each leaf of the bracket *independently and uniformly at random.* Since an SE bracket must be labeled with a random *permutation* of the competitors, we instead have to construct a distribution over tournament graphs with varying numbers of strong and weak competitors in order to simulate each leaf being labeled independently. Because of this, even after showing that a "weak" competitor is likely to win the bracket, we have to prove this still holds when we restrict our distribution to tournaments where the "weak" competitors really have low Copeland score, and thus there exists some specific tournament graph where the SE bracket has a poor approximation ratio.

Second, [5] shows that the winner of a bracket oscillates between these three components, but does not establish the rate of oscillation. Because we need to show a weak competitor is likely to win after *precisely* $\log n$ rounds, we have to repeat their analysis with significant additional bookkeeping.

The result of this bookkeeping is the following lemma, analogous to Lemma G.1 from [5]. Roughly, it says: Suppose after some number of rounds of an SE bracket, practically all the remaining competitors come from the "strong" component. Nevertheless, after a specific number of additional rounds, practically all the remaining competitors will come from the *"weak"* component. Furthermore, the "weak" competitors continue to dominate for many rounds before the oscillation repeats. The proof consists of analyzing a simple recursive formula for the likelihood of a "strong", "mediocre", or "weak" player winning an SE bracket after k rounds, in order to give painstaking bounds on the magnitude and rate of oscillation of these probabilities.

The rather lengthy and unenlightening proof has been relegated to the appendix.

▶ **Lemma 4.** *Let S be a set of competitors partitioned into three components C_1, C_2, C_3 such that every member of component C_i beats every member of component $C_{(i \bmod 3)+1}$. Fix probabilities $p_i^{(0)}$ summing to 1, and let $p_i^{(k)}$ denote the probability that a member of component C_i wins a balanced bracket of height k where each leaf is labeled independently according to $p_i^{(0)}$. If for some $K \in \mathbb{N}$ and $0 < \epsilon \leq 2^{-10}$, $\epsilon^2 \leq p_3^{(K)} \leq \epsilon$ and $\epsilon \leq p_1^{(K)} \leq 2\epsilon$, then there exists $K + \log(\frac{1}{\epsilon}) \leq K' \leq K + 3\log(\frac{1}{\epsilon})$ and $\epsilon^{2\log(\frac{1}{\epsilon})} \leq \delta \leq \epsilon^{\log(\frac{1}{\epsilon})/4}$ such that $\delta^2 \leq p_2^{(K')} \leq \delta$ and $\delta \leq p_3^{(K')} \leq 2\delta$. Furthermore, if $\epsilon \leq 2^{-75}$ then for any $K'' \in [K', K' + 2^5\log(\frac{1}{\epsilon})]$, $p_2^{(K'')}, p_3^{(K'')} \leq \epsilon^{\log(\frac{1}{\epsilon})/2^5}$.*

We are now ready to prove our upper bound on the approximation ratio for SE brackets.

▶ **Theorem 5.** *The approximation ratio of a randomly-seeded single-elimination bracket on $n = 2^m$ competitors for the maximum Copeland score is $O(2^{-\sqrt{\log(n)/7}})$.*

Proof. For any $n = 2^m$ with $\log n \geq 2^{12}$, pick $0 < \delta \leq 2^{-18}$ such that $7\log^2\frac{1}{\delta} \leq \log n \leq 8\log^2\frac{1}{\delta}$. Define a distribution over tournaments $D(n,\delta)$, with $p_s^{(0)} = p_w^{(0)} = \delta, p_m^{(0)} = 1 - 2\delta$. $D(n,\delta)$ is supported over tournaments of size n with three (possibly empty) components s, m, w where s beats m, m beats w, and w beats s. Internally each component is a regular tournament, meaning the difference between the maximum and minimum Copeland scores is 1 or 0. For any fixed size of the components, summing to n, the weight of the corresponding tournament in $D(n,\delta)$ is equal to the probability that those fixed sizes are achieved by assigning each of n competitors independently to a component according to $p_s^{(0)}, p_m^{(0)}, p_w^{(0)}$.

We will now analyze the winner of a bracket where each leaf is labeled independently with a component according to $p_s^{(0)}, p_m^{(0)}, p_w^{(0)}$. Observe that this is equivalent to choosing a random tournament according to $D(n,\delta)$ and then labeling n leaves with a random permutation of the competitors. In particular, for any given tournament graph in $D(n,\delta)$, every permutation of the competitors appears with equal probability.

Letting $C_1 = s, C_2 = m$, and $C_3 = w$ we apply Lemma 4. Since $0 < \delta \le 2^{-10}$ and $\delta^2 \le p_w^{(0)} \le \delta \le p_s^{(0)} \le 2\delta$, we obtain that $(\delta')^2 \le p_m^{(K)} \le \delta' \le p_w^{(K)} \le 2\delta'$ for some $\delta^{2\log(\frac{1}{\delta})} \le \delta' \le \delta^{\log(\frac{1}{\delta})/4}$ and $\log(\frac{1}{\delta}) \le K \le 3\log(\frac{1}{\delta})$. We apply Lemma 4 once more, now with $C_1 = w, C_2 = s$, and $C_3 = m$ and starting from K, to find that a weak competitor wins with overwhelming probability after K' rounds, with

$$0 \le K' \le K + 3\log(\frac{1}{\delta'}) \le 3\log(\frac{1}{\delta}) + 6\log^2(\frac{1}{\delta}).$$

We will use the final part of Lemma 4 to increase this to a bracket of depth $\log n$. Observe that no more than $8\log^2(\frac{1}{\delta})$, but more than 0, additional rounds are required. Furthermore, note that

$$\delta' \le \delta^{\log(\frac{1}{\delta})/4} \le 2^{-18^2/4} < 2^{-75}$$

as required for this part of the lemma. Finally, the number of additional rounds required is at most

$$8\log^2(\frac{1}{\delta}) \le 2^5\log(\frac{1}{\delta'})$$

and thus by Lemma 4, after $\log n$ rounds we have

$$p_s^{(\log n)}, p_m^{(\log n)} \le \delta'^{\log(\frac{1}{\delta'})/2^5} \le 2^{-\log^2(2^{\log^2(\delta)/4})/2^5} = 2^{-(\log^2(\delta)/4)^2)/2^5} = 2^{-\log^4(\delta)/2^9}$$

so also $p_3^{(\log n)} \ge 1 - 2 \times 2^{-\log^4(\delta)/2^9}$.

We have established the winning probability of a "weak" competitor, over distribution $D(n,\delta)$. However, some tournaments with non-zero weight in the distribution have "weak" competitors with high Copeland score (those in which either the weak or strong component is large). Next, we bound the probability that this happens in order to establish a distribution $D'(n,\delta)$, where a "weak" competitor still wins almost always *and* the weak competitors all have low Copeland score.

First, we separate out the high-scoring weak and low-scoring weak cases from $D(n,\delta)$, where $\Pr[w]$ represents the probability of a weak competitor winning:

$$\begin{aligned}\Pr[w] = \Pr[w : |w|, |s| < 10\delta n]\,\Pr[|w|, |s| < 10\delta n] + \\ \Pr[w : |w| \text{ or } |s| \ge 10\delta n]\,\Pr[|w| \text{ or } |s| \ge 10\delta n]\end{aligned}$$

Rearranging,

$$\begin{aligned}&\Pr[w \text{ wins} : |w|, |s| < 10\delta n] \\ &\quad \ge \Pr[w \text{ wins} : |w|, |s| < 10\delta n]\,\Pr[|w|, |s| < 10\delta n] \\ &\quad = \Pr[w \text{ wins}] - \Pr[w \text{ wins} : |w| \text{ or } |s| \ge 10\delta n]\,\Pr[|w| \text{ or } |s| \ge 10\delta n] \\ &\quad \ge 1 - 2 \times 2^{-\log^4(\delta)/2^9} - \Pr[|w| \text{ or } |s| \ge 10\delta n] \\ &\quad \ge 1 - 2 \times 2^{-\log^4(\delta)/2^9} - \Pr[|s| \ge 10\delta n] - \Pr[|w| \ge 10\delta n] \\ &\quad \ge 1 - 2 \times 2^{-\log^4(\delta)/2^9} - 2e^{-\frac{9\delta n}{3}} := p\end{aligned}$$

using the Chernoff bound $\Pr[|C| \ge 10\delta n] \le e^{-\frac{9\delta n}{3}}$ (since the expectation of $|w|, |s|$ is δn).

Let $D'(n,\delta)$ equal $D(n,\delta)$ restricted to tournaments where $|w|,|s| < 10\delta n$. A weak competitor wins the SE bracket on a tournament graph drawn from $D'(n,\delta)$ with probability at least p.

Finally, we observe that the Copeland score of any member of the weak component of any tournament with non-zero support on $D'(n,\delta)$ is less than $\frac{3}{2}10\delta n$ (consisting of less than $10\delta n$ edges to the strong component and less than $10\delta n/2$ edges within the weak component). Thus the expected Copeland score of the winner of a randomly-seeded SE bracket over a tournament drawn from $D'(n,\delta)$ is less than

$$\begin{aligned}
&p\frac{30\delta n}{2} + (1-p)(n-1)\\
&\quad\leq 15\delta n + 2n \times 2^{-\log^4(\delta)/2^9} + 2ne^{-3\delta n};
\end{aligned}$$

recall $\log n \leq 8\log^2(\frac{1}{\delta})$, so $\delta \leq 2^{-\sqrt{\log(n)/8}}$:

$$\begin{aligned}
&\leq 15 \times 2^{\log n - \sqrt{\log(n)/8}} + 2n \times 2^{-\log^4(2^{-\sqrt{\log(n)/8}})/2^9} + 2ne^{-3\times 2^{\log n - \sqrt{\log(n)/8}}}\\
&\leq 15 \times 2^{\log n - \sqrt{\log(n)/8}} + 2n \times 2^{-(\sqrt{\log(n)/8})^4/2^9} + 2ne^{-3\times 2^{\log(n)/2}}\\
&\leq 15 \times 2^{\log n - \sqrt{\log(n)/8}} + 2 \times 2^{\log n-(\sqrt{\log(n)/8})^4/2^9} + 2ne^{-3\sqrt{n}}\\
&\leq 15 \times 2^{\log n - \sqrt{\log(n)/8}} + 2 \times 2^{\log n - \sqrt{\log(n)/8}} + 2
\end{aligned}$$

This implies that some *individual* tournament with non-zero support on $D'(n,\delta)$ achieves expected Copeland score at most $O(2^{\log n - \sqrt{\log(n)/8}})$. Thus the approximation ratio is $O(2^{-\sqrt{\log(n)/8}})$, completing the proof. ◀

In fact, the upper bound shown above is "nearly" tight, as the following theorem establishes.

▶ **Theorem 6.** *The approximation ratio of a randomly-seeded single-elimination bracket on $n = 2^m$ competitors for the maximum Copeland score is $\Omega(2^{-\sqrt{2\log n}})$.*

Proof. For any $k < n$, at most $2k$ competitors in the tournament can have Copeland score less than k – otherwise, the average score amongst these competitors alone would be at least $\frac{(2k+1)-1}{2} = k$, a contradiction. If k is sufficiently small, it becomes quite likely that these few competitors will be eliminated early in a randomly-seeded SE bracket.

We will capture this idea by union-bounding over the probability that an individual competitor c with Copeland score $d_+(c) < k$ survives $\lfloor \log(\frac{n}{k}) \rfloor + 1$ rounds. Each round, c must face one of the $< k$ competitors he can beat (not including those he has already beaten) – even assuming every other competitor he can beat advances. Therefore,

$$\begin{aligned}
&\Pr\left(c \text{ survives } \left\lfloor \log\left(\frac{n}{k}\right)\right\rfloor + 1 \text{ rounds} : d_+(c) < k\right)\\
&\quad\leq \prod_{i=0}^{\lfloor \log(\frac{n}{k})\rfloor} \frac{k-i-1}{\frac{n}{2^i}-1}\\
&\quad\leq \prod_{i=0}^{\lfloor \log(\frac{n}{k})\rfloor} 2^i\frac{k}{n}\\
&\quad\leq 2^{\frac{\log(\frac{n}{k})(\log(\frac{n}{k})+1)}{2}}\left(\frac{k}{n}\right)^{\log(\frac{n}{k})+1}\\
&\quad= \left(\frac{k}{n}\right)^{\frac{-1-\log(\frac{n}{k})}{2}}\left(\frac{k}{n}\right)^{\log(\frac{n}{k})+1} = \left(\frac{k}{n}\right)^{\frac{\log(\frac{n}{k})+1}{2}}
\end{aligned}$$

Even if we assume that every one of these low-scoring competitors wins with the probability calculated above, and contributes nothing to the expected Copeland score of the SE winner, we still know that the remaining probability belongs to competitors with Copeland score at least k. Thus,

$$\mathbb{E}[d_+(\text{winner})] \geq k\left(1 - 2k\left(\frac{k}{n}\right)^{\frac{\log(\frac{n}{k})+1}{2}}\right).$$

Let us plug in $k = n2^{-\sqrt{2\log n}} = 2^{\log n - \sqrt{2\log n}}$. Then,

$$\begin{aligned}\mathbb{E}[d_+(\text{winner})] &\geq 2^{\log n - \sqrt{2\log n}}\left(1 - 2 \times 2^{\log n - \sqrt{2\log n}}\left(2^{-\sqrt{2\log n}}\right)^{\frac{\sqrt{2\log n}+1}{2}}\right)\\ &\geq 2^{\log n - \sqrt{2\log n}}\left(1 - 2 \times 2^{\log n - \sqrt{2\log n}} \times 2^{-\log n - \sqrt{\frac{\log n}{2}}}\right)\\ &\geq 2^{\log n - \sqrt{2\log n}}\left(1 - 2^{-3\sqrt{\frac{\log n}{2}}+1}\right)\\ &\geq n\left(2^{-\sqrt{2\log n}} - 2^{-5\sqrt{\frac{\log n}{2}}+1}\right) \geq n2^{-\sqrt{2\log n}-1}\end{aligned}$$

which establishes that the approximation ratio is $\Omega(2^{-\sqrt{2\log n}})$, as desired. ◀

Taken together, these bounds establish that the approximation ratio of randomly-seeded SE brackets for the maximum Copeland score is $2^{-\Theta(\sqrt{\log n})}$.

5 Balanced Voting Trees Fail to Approximate Copeland Winner

In this section, we derive an upper bound on the approximation ratio for the maximum Copeland score of a generalized version of SE brackets from the voting tree literature. Recall that a voting tree is any binary tree with leaves labeled by the n competitors. The *randomized perfect voting tree* of depth k (k-RPT) is a class of voting trees introduced by [5], consisting of a balanced binary tree of depth k, with each leaf labeled uniformly at random from the set of n competitors. The k-RPT is similar to an SE bracket, except (1) the number of leaves may be larger (or smaller) than the number of competitors, and (2) the random seeding process does not require every competitor to appear on the leaves. However, as noted by [5], as k grows, the probability of any competitor not appearing on the leaves vanishes.

[5] established that, for infinitely many k, the k-RPT has an approximation ratio of $O(1/n)$ for the maximum Copeland score – essentially the worst possible! However, they left open the question of whether, for some carefully chosen $k = f(n)$, the k-RPT might achieve a good approximation ratio. We certainly shouldn't expect the approximation ratio to be as low as $O(1/n)$ for *every* k, since technically the 1-RPT corresponds to randomly choosing a winner and so has approximation ratio $\frac{1}{2}$, while the $(\log n)$-RPT is closely related to the SE bracket which has a ratio of $2^{-\Theta(\sqrt{\log n})}$. However, we can at least show that the approximation ratio of a k-RPT for any $k \geq \log n$ is sub-constant.

▶ **Theorem 7.** *The approximation ratio of the k-RPT with $k \geq \log n$ is $O(2^{-\sqrt[4]{\log n}/4})$*

Proof. We use the same tournament structure as in the previous section, with strong, mediocre, and weak components s, m, w. Because the labeling of leaves in a k-RPT is uniformly random, there is no need to define a distribution over such tournaments; the components have fixed proportions $p_s^{(0)}, p_m^{(0)}, p_w^{(0)}$ to be specified later.

We again make repeated use of Lemma 4. In this setting, however, we do not have arbitrary control over ϵ, the probability of a competitor being weak; it must initially be some integer multiple of $1/n$. Thus we will first establish, for any sufficiently small ϵ, an infinite set of ranges for which the probability of a weak competitor winning must be high. We will then argue that for sufficiently large n, we can vary $\epsilon = \ell/n$ enough that these ranges collectively cover every $k \geq \log n$.

Let $\epsilon_0 \leq 2^{-2^6}$ equal $p_w^{(0)}$, i.e., it will represent the fraction of competitors that are weak; note that this satisfies the requirement $\epsilon_0 \leq 2^{-10}$ for Lemma 4. Each time we apply Lemma 4, we will obtain a new ϵ_i, so for instance, $\epsilon_1 \in [2^{-2\log^2(\epsilon_0)}, 2^{-\log^2(\epsilon_0)/4}]$. We claim that

$$\log \epsilon_i \in \left[-2^{2^i-1}\left(\log\frac{1}{\epsilon_0}\right)^{2^i}, -\left(\frac{1}{4}\right)^{2^i-1}\left(\log\frac{1}{\epsilon_0}\right)^{2^i}\right]$$

We can verify this by induction – it clearly holds for $i = 0$. Assume it holds for $i-1$. By Lemma 4,

$$\begin{aligned}
\log \epsilon_i &\in \left[-2\log^2(\frac{1}{\epsilon_{i-1}}), -\frac{1}{4}\log^2(\frac{1}{\epsilon_{i-1}})\right] \\
&\in \left[-2\left(-2^{2^{i-1}-1}\left(\log\frac{1}{\epsilon_0}\right)^{2^{i-1}}\right)^2, -\frac{1}{4}\left(-\left(\frac{1}{4}\right)^{2^{i-1}-1}\left(\log\frac{1}{\epsilon_0}\right)^{2^{i-1}}\right)^2\right] \\
&\in \left[-2\times 2^{2^i-2}\left(\log\frac{1}{\epsilon_0}\right)^{2^i}, -\frac{1}{4}\left(\frac{1}{4}\right)^{2^i-2}\left(\log\frac{1}{\epsilon_0}\right)^{2^i}\right],
\end{aligned}$$

as desired.

Let t_i be the step (bracket depth) at which the i^{th} application of Lemma 4 begins, $t_0 = 0$. Note that $p_w^{(t_i)}$ is high when $i \bmod 3 = 2$. We claim that

$$t_{i+1} \in \left[\log(\frac{1}{\epsilon_i}), 4\log(\frac{1}{\epsilon_i})\right].$$

The lower bound is immediate from Lemma 4 because the i^{th} oscillation takes at least $\log(\frac{1}{\epsilon_i})$ steps. The upper bound we again prove inductively; for t_1 it likewise holds directly from Lemma 4. Assuming it holds for t_i, and since $\log \epsilon_i \in [-2\log^2(\epsilon_{i-1}), -\frac{1}{4}\log^2(\epsilon_{i-1})]$,

$$\begin{aligned}
t_{i+1} &\leq t_i + 3\log(\frac{1}{\epsilon_i}) \\
&\leq 4\log(\frac{1}{\epsilon_{i-1}}) + 3\log(\frac{1}{\epsilon_i}) \\
&\leq 4\sqrt{4\log(\frac{1}{\epsilon_i})} + 3\log(\frac{1}{\epsilon_i}) \\
&\leq 4\log(\frac{1}{\epsilon_i})
\end{aligned}$$

where the last line holds because $\log(\frac{1}{\epsilon_i}) \geq \log(\frac{1}{\epsilon_0}) \geq 2^6$ by assumption.

Next, recall that by Lemma 4, the two smaller probabilities at t_{i+1} sum to at most $3\epsilon_{i+1}$. If we allow this sum to increase slightly, say to ϵ_i, we can go additional steps beyond t_{i+1}. Specifically, knowing the probability at most doubles each time step, for any $t \leq \frac{1}{8}\log^2(\frac{1}{\epsilon_i})$ we have

$$\begin{aligned}\log\left(2^t \times 3\epsilon_{i+1}\right) &\le t - 1 + \log(\epsilon_{i+1}) \\ &\le \frac{1}{8}\log^2(\frac{1}{\epsilon_i}) - 1 - \frac{1}{4}\log^2(\frac{1}{\epsilon_i}) \\ &\le -\log(\frac{1}{\epsilon_i})\end{aligned}$$

where the last line holds because $\log(\frac{1}{\epsilon_i}) \ge \log(\frac{1}{\epsilon_0}) \ge 2^6$. Removing the logarithm, the above implies that for any $t \le \frac{1}{8}\log^2(\frac{1}{\epsilon_i})$, the two smaller probabilities at time t_{i+1} still sum to at most ϵ_i at time $t_{i+1} + \lfloor t \rfloor$. Also, for $t = \frac{1}{8}\log^2(\frac{1}{\epsilon_i})$, observe that

$$\begin{aligned}t_{i+1} &\le 4\log(\frac{1}{\epsilon_i}) \le 2\left(2\log\frac{1}{\epsilon_0}\right)^{2^i} \\ t_{i+1} + \lfloor t \rfloor &\ge \log(\frac{1}{\epsilon_i}) + \frac{1}{8}\log^2(\frac{1}{\epsilon_i}) - 1 \\ &\ge \frac{1}{8}\log^2(\frac{1}{\epsilon_i}) \ge 2\left(\frac{\log(1/\epsilon_0)}{4}\right)^{2^{i+1}}.\end{aligned}$$

Thus for any $\epsilon_0 \le 2^{-2^6}$ and $i \in \mathbb{N}$, we have established an interval on which the largest probability is at least $1 - \epsilon_i$. In particular, whenever $i + 1 = 2 \pmod 3$, this gives an interval on which a weak competitor wins with overwhelming probability.

Next, we want to show that, for any sufficiently large n, these intervals can be made to cover every depth $k \ge \log n$, even with the limitation that ϵ_0 must equal ℓ/n for some $\ell \in \mathbb{N}$. In fact, we claim that letting ℓ take on values $1, 2, \ldots, \left\lceil 2^{\log n - \sqrt[4]{\log n}/4} \right\rceil := L$ covers every $k \ge \log n$ for any n sufficiently large that $\left\lceil 2^{\log n - \sqrt[4]{\log n}/4} \right\rceil / n \le 2^{-2^6}$ – this is necessary to ensure that $\epsilon_0 \in [1/n, L/n]$ will be at most 2^{-2^6} as assumed above.

First, let us verify that the lowest k contained in one of these intervals is sufficiently small. k will be smallest when i is small $(i + 1 = 2)$ and when ϵ_0 is large $(\epsilon_0 = L/n)$. Thus the first interval will start at

$$2\left(2\log\frac{n}{L}\right)^2 \le 2\left(\log(2^{\sqrt[4]{\log n}/4})\right)^2 = \sqrt{\log n}/8 \ll \log n,$$

so indeed our overlapping intervals start below $k = \log n$.

Next, we need to establish that for a fixed i, the adjacent intervals with $\epsilon_0 = \ell/n, \epsilon_0 = (\ell - 1)/n$ overlap. That is, the lower bound of the later interval needs to be below the upper bound of the earlier. I.e., we require

$$\begin{aligned}2\left(2\log(\frac{n}{\ell - 1})\right)^2 &\le 2\left(\log(\frac{n}{\ell})/4\right)^4 \\ 2^5\log(\frac{n}{\ell - 1}) &\le \log^2(\frac{n}{\ell})\end{aligned}$$

and indeed,

$$2^5\log(\frac{n}{\ell - 1}) \le 2^5\left(1 + \log\frac{n}{\ell}\right) \le \log^2(\frac{n}{\ell})$$

where the final inequality holds because $\log(\frac{n}{\ell}) \ge 2^6$.

Finally, we will show that the latest interval for i, $\epsilon_0 = 1/n$, overlaps with the earliest interval for $i+3$, $\epsilon_0 = L/n$, and thus that there is no gap between intervals where a weak competitor can win with high probability. We require

$$
\begin{aligned}
2\left(2\log\frac{n}{L}\right)^{2^{i+3}} &\leq 2\left(\frac{\log n}{4}\right)^{2^{i+1}} \\
\left(2\log\frac{n}{L}\right)^{4} &\leq \frac{\log n}{4}
\end{aligned}
$$

and indeed,

$$
\left(2\log\frac{n}{L}\right)^{4} \leq \left(2\sqrt[4]{\log n}/4\right)^{4} = \frac{\log n}{16} < \frac{\log n}{4}
$$

as desired.

Having shown that a weak competitor can win with overwhelming probability for sufficiently large n and any $k \geq \log n$, we can upper-bound the expected Copeland score of the winner as

$$
\mathbb{E}[d_+(\text{winner})] \leq (1-\epsilon_i)(\ell+\ell/2)+\epsilon_i(n-1) \leq L+L/2+\frac{L}{n}(n-1) \leq 3L = 3\left\lceil 2^{\log n - \sqrt[4]{\log n}/4}\right\rceil .
$$

Equivalently, the approximation ratio of any k-RPT is $O(2^{-\sqrt[4]{\log n}/4})$. ◀

Interestingly, although this bound holds for all $k \geq \log n$, significantly tighter bounds can be obtained for certain k by the same method. In particular, the bound is made much looser by increasing the size of the weak component up to L/n, which is only necessary when trying to cover every possible k. When the weak component has size $1/n$, we recover the $O(1/n)$ approximation ratio of [5]. Without matching lower bounds, it is unclear to what extent this oscillating approximation ratio is real versus an artifact of the proof method. Regardless, this upper bound settles the question of whether any k-RPT can obtain a decent approximation ratio.

6 Conclusion

In this work, we establish that randomly-seeded single-elimination brackets are surprisingly bad at approximating the maximum Copeland score, as is a generalized version of SE brackets from the voting theory literature, the k-RPT. However, we show that SE brackets have optimal approximation ratio for *worst-case*/deterministic seeding among Condorcet competition formats.

Despite their sub-constant approximation ratio, single-elimination brackets are widely used; perhaps quirks of their occurrence in practice could improve the approximation ratio? For instance, one could consider the impact of seeding based on some measure of competitor ability, or investigate whether SE brackets perform better on tournament graphs generated from some random model. Alternatively, one could investigate other existing competition formats (e.g., double-elimination, Swiss-system) to see if they better approximate the Copeland winner.

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A Proof of Lemma 4

▶ **Lemma 4.** *Let S be a set of competitors partitioned into three components C_1, C_2, C_3 such that every member of component C_i beats every member of component $C_{(i \bmod 3)+1}$. Fix probabilities $p_i^{(0)}$ summing to 1, and let $p_i^{(k)}$ denote the probability that a member of component C_i wins a balanced bracket of height k where each leaf is labeled independently according to $p_i^{(0)}$. If for some $K \in \mathbb{N}$ and $0 < \epsilon \leq 2^{-10}$, $\epsilon^2 \leq p_3^{(K)} \leq \epsilon$ and $\epsilon \leq p_1^{(K)} \leq 2\epsilon$, then there exists $K + \log(\frac{1}{\epsilon}) \leq K' \leq K + 3\log(\frac{1}{\epsilon})$ and $\epsilon^{2\log(\frac{1}{\epsilon})} \leq \delta \leq \epsilon^{\log(\frac{1}{\epsilon})/4}$ such that $\delta^2 \leq p_2^{(K')} \leq \delta$ and $\delta \leq p_3^{(K')} \leq 2\delta$. Furthermore, if $\epsilon \leq 2^{-75}$ then for any $K'' \in [K', K' + 2^5\log(\frac{1}{\epsilon})]$, $p_2^{(K'')}, p_3^{(K'')} \leq \epsilon^{\log(\frac{1}{\epsilon})/2^5}$.*

Proof. Since we have labeled each leaf independently, the two children of a node are independent, so we can easily calculate $p_i^{(k)}$ recursively. For all $k \geq 0$,

$$p_i^{(k+1)} = \left(p_i^{(k)}\right)^2 + 2p_i^{(k)}p_{(i \bmod 3)+1}^{(k)} = p_i^{(k)}\left(p_i^{(k)} + 2p_{(i \bmod 3)+1}^{(k)}\right).$$

We will proceed by phases. Phase 1 will be the time during which $p_2^{(k)}$ shrinks to 1/2; phase 2 will extend from there to the time when $p_2^{(k)}$ shrinks to less than $p_3^{(k)}$ (this will be K'); and phase 3 will be the additional $2^5\log(\frac{1}{\epsilon})$ steps after K'.

Phase 1. Let $K_1 > K$ be the first step for which $p_1^{(k)} + p_3^{(k)} > 1/2$. Such a step must exist because for $K \leq k < K_1$, $p_1^{(k)} + p_3^{(k)} \leq 1/2$, and thus

$$p_3^{(k+1)} = p_3^{(k)}\left(p_3^{(k)} + 2p_1^{(k)}\right) \leq p_3^{(k)} \leq \epsilon,$$

i.e., $p_3^{(k)}$ is weakly decreasing on this interval. Thus also,

$$p_1^{(k+1)} = p_1^{(k)}\left(p_1^{(k)} + 2p_2^{(k)}\right) = p_1^{(k)}(1 - p_3^{(k)} + p_2^{(k)}) \geq p_1^{(k)}(1.5 - \epsilon) \geq p_1^{(k)}\sqrt{2}$$

for k in this interval, since $p_1^{(k)} + p_3^{(k)} \leq 1/2$ implies $p_2^{(k)} \geq 1/2$, and since $\epsilon \leq 2^{-10} < 1.5 - \sqrt{2}$. Therefore, $p_1^{(k)}$ is increasing by at least a constant factor every step, and so eventually $p_1^{(k)} + p_3^{(k)}$ will exceed 1/2. Note also that $p_i^{(k+1)} \leq 2p_i^{(k)}$ for all i, k, so $p_1^{(k)}$ is increasing by at least a factor of $\sqrt{2}$ and at most a factor of 2 on this interval.

This leads to the following observations about phase 1:

1. $\epsilon 2^{(k-K)/2} \leq p_1^{(k)} \leq \epsilon 2^{k-K+1}$ for any $k \in [K, K_1]$
2. $\epsilon^2\left(\epsilon 2^{(K_1-K+3)/4}\right)^{K_1-K} \leq p_3^{(K_1)} \leq \epsilon\left(\epsilon 2^{(K_1-K+5)/2}\right)^{K_1-K}$
3. $\log(\frac{1}{\epsilon}) - 3 \leq K_1 - K \leq 2\log(\frac{1}{\epsilon})$

Observation 1 follows directly from our initial assumption on $p_1^{(K)}$ and the bounds on the factor by which it increases each step.

Observation 2 can be shown via observation 1 and our initial assumptions as follows, using the fact that $0 \le p_3^{(k)} \le p_1^{(k)}$ on this interval.

$$p_3^{(K_1)} = p_3^{(K)} \prod_{t=K}^{t=K_1-1} \left(p_3^{(t)} + 2p_1^{(t)}\right)$$

$$\begin{aligned}
p_3^{(K)} \prod_{t=K}^{t=K_1-1} 2p_1^{(t)} \le & p_3^{(K_1)} \le p_3^{(K)} \prod_{t=K}^{t=K_1-1} 3p_1^{(t)} \\
p_3^{(K)} \prod_{t=0}^{t=K_1-K-1} 2p_1^{(K)} 2^{t/2} \le & p_3^{(K_1)} \le p_3^{(K)} \prod_{t=0}^{t=K_1-K-1} 3p_1^{(K)} 2^{t} \\
p_3^{(K)} \left(2p_1^{(K)}\right)^{K_1-K} 2^{(K_1-K-1)(K_1-K)/4} \le & p_3^{(K_1)} \le p_3^{(K)} \left(3p_1^{(K)}\right)^{K_1-K} 2^{(K_1-K-1)(K_1-K)/2} \\
\epsilon^2 \left(\epsilon 2^{(K_1-K+3)/4}\right)^{K_1-K} \le & p_3^{(K_1)} \le \epsilon \left(\epsilon 2^{(K_1-K+5)/2}\right)^{K_1-K}
\end{aligned}$$

Finally, observation 3 is obtained from observation 1 based on how long it would take for $p_1^{(k)}$ to get to $1/2$ (or rather in the range $[1/2-\epsilon, 1]$). Specifically, plugging in $k = K_1$ to observation 1 and taking the log of both sides:

$$\begin{aligned}
\log \epsilon + \frac{K_1 - K}{2} \le & \log p_1^{(K_1)} \le \log \epsilon + K_1 - K + 1 \\
\log p_1^{(K_1)} - \log \epsilon - 1 \le & K_1 - K \le 2 \log p_1^{(K_1)} - 2 \log \epsilon \\
\log(\frac{1}{2} - \epsilon) - \log \epsilon - 1 \le & K_1 - K \le 2 \log 1 - 2 \log \epsilon \\
\log(\frac{1}{\epsilon}) - 3 \le & K_1 - K \le 2 \log(\frac{1}{\epsilon})
\end{aligned}$$

Phase 2. Let $K_2 > K_1$ be the first step for which $p_3^{(k)} > p_2^{(k)}$. We claim that $K' = K_2$ is as required in the statement of the lemma.

To show that such a step must exist, note that at step K_1, we have $1/4 < p_2^{(K_1)} < 1/2$, $p_3^{(K_1)} \le \epsilon$, and therefore $p_1^{(K_1)} > 1/2 - \epsilon$. Furthermore, since $p_3^{(k)} \le p_2^{(k)}$ on this interval, for $K_1 \le k < K_2$,

$$p_1^{(k+1)} = p_1^{(k)} \left(p_1^{(k)} + 2p_2^{(k)}\right) = p_1^{(k)} \left(1 - p_3^{(k)} + p_2^{(k)}\right) \ge p_1^{(k)}$$

i.e., $p_1^{(k)}$ is weakly increasing. In particular,

$$p_1^{(K_1+1)} = p_1^{(K_1)} \left(p_1^{(K_1)} + 2p_2^{(K_1)}\right) \ge (1/2 - \epsilon)(1/2 - \epsilon + 1/4) > 0.6 > 1/2.$$

Thus for every step after the first, $p_3^{(k)}$ is increasing by a factor of $p_3^{(k)} + 2p_1^{(k)} > 1.2$, while $p_2^{(k)}$ is multiplied by a factor of $p_2^{(k)} + 2p_3^{(k)} = 1 - p_1^{(k)} + p_3^{(k)} < 1$, and they will eventually cross.

We make the following observations about phase 2:

1. $\left(\frac{1}{4}\right)^{2^{k-K_1}} \le p_2^{(k)} \le \left(\frac{1}{2}\right)^{2^{k-K_1-1}}$ for any $k \in [K_1 + 1, K_2]$
2. $1.5^{k-K_1-2} p_3^{(K_1)} \le p_3^{(k)} \le 2^{k-K_1} p_3^{(K_1)}$ for any $k \in [K_1 + 2, K_2]$
3. $\log\log(\frac{1}{\epsilon}) \le K_2 - K_1 \le \log(\frac{1}{\epsilon})$

Observation 1 follows from the fact that $\frac{1}{4} \leq p_2^{(K_1)} \leq \frac{1}{2}$ and $p_3^{(K_1)} \leq \epsilon \leq 2^{-10}$. Recall that no probability can more than double in a single step. Thus

$$
\begin{aligned}
p_2^{(K_1+1)} &\leq p_2^{(K_1)}\left(p_2^{(K_1)} + 2p_3^{(K_1)}\right) \leq \frac{1}{2}\left(\frac{1}{2} + 2^{-9}\right) \leq 0.251 \\
p_2^{(K_1+2)} &\leq p_2^{(K_1+1)}\left(p_2^{(K_1+1)} + 2p_3^{(K_1+1)}\right) \leq 0.251(0.251 + 2^{-8}) \leq 0.064 \\
p_2^{(K_1+3)} &\leq p_2^{(K_1+2)}\left(p_2^{(K_1+2)} + 2p_3^{(K_1+2)}\right) \leq 0.064(0.064 + 2^{-7}) \leq 0.0046.
\end{aligned}
$$

Therefore, since $p_2^{(k+1)} = p_2^{(k)}\left(p_2^{(k)} + 2p_3^{(k)}\right)$, we have for any $k \geq K_1 + 3$,

$$
\begin{aligned}
\left(p_2^{(k)}\right)^2 &\leq p_2^{(k+1)} \leq 3\left(p_2^{(k)}\right)^2 \\
\left(\frac{1}{4}\right)^{2^{k-K_1}} &\leq p_2^{(k+1)} \leq (3 \times 0.0046)^{2^{k-K_1-3}} \leq \left(\frac{1}{2}\right)^{2^{k-K_1-1}}.
\end{aligned}
$$

Since $p_2^{(k)}$ is decreasing we have $p_3^{(k)} \leq p_2^{(k)} \leq \frac{1}{8}$ for $K_1 + 2 \leq k \leq K_2$. Thus $p_3^{(k)}$ increases by at most a factor of 2, and at least a factor of $2p_1^{(k)}$ which is at least 1.5 after the first two steps. ($p_3^{(k)}$ may decrease by a factor no smaller than $1 - \epsilon$ in the first step, but the next step more than cancels this out.) This yields observation 2.

Observation 3 we'll just show by plugging in the given values for $K_2 - K_1$ into the bounds for $p_2^{(k)}$ and $p_3^{(k)}$ and showing that they either must have, or must not have crossed by the given time.

First let us verify that after another $\log(\frac{1}{\epsilon})$ steps it must be the case that $p_3^{(k)} > p_2^{(k)}$. Specifically, if we assume that $K_1 + \log(\frac{1}{\epsilon}) < K_2$, we obtain the following contradiction:

$$
\begin{aligned}
p_3^{(K_1+\log(\frac{1}{\epsilon}))} &\geq 1.5^{\log(\frac{1}{\epsilon})-2} p_3^{(K_1)} \geq 1.5^{\log(\frac{1}{\epsilon})-2}\epsilon^2\left(\epsilon 2^{(K_1-K+3)/4}\right)^{K_1-K} \\
&\geq 1.5^{\log(\frac{1}{\epsilon})-2}\epsilon^2\left(\epsilon\left(\frac{1}{\epsilon}\right)^{1/2} 2^{3/4}\right)^{2\log(\frac{1}{\epsilon})} \\
&= \left(\frac{2}{3}\right)^2\left(\frac{3}{2}\right)^{\log(\frac{1}{\epsilon})} \epsilon^{\log(\frac{1}{\epsilon})+2} 2^{3\log(\frac{1}{\epsilon})/2} \\
&> \left(\frac{1}{2}\right)^{\log(\frac{1}{\epsilon})(\log(\frac{1}{\epsilon})+2)} > \left(\frac{1}{2}\right)^{1/2\epsilon} \geq p_2^{(K_1+\log(\frac{1}{\epsilon}))}
\end{aligned}
$$

since $1/2\epsilon > \log^2(\frac{1}{\epsilon}) + 2\log(\frac{1}{\epsilon})$ for $\epsilon \leq 2^{-7}$. This establishes the upper bound on $K_2 - K_1$.

To establish that $K_2 - K_1 \geq \log\log(\frac{1}{\epsilon})$, we need to show that after that many steps, it still holds that $p_3^{(k)} \leq p_2^{(k)}$.

$$
\begin{aligned}
p_3^{(K_1+\log\log(\frac{1}{\epsilon}))} &\leq 2^{\log\log(\frac{1}{\epsilon})} p_3^{(K_1)} \leq 2^{\log\log(\frac{1}{\epsilon})+2}\epsilon\left(\epsilon 2^{(\log(\frac{1}{\epsilon})-3+5)/2}\right)^{\log(\frac{1}{\epsilon})-3} \\
&= \log(\frac{1}{\epsilon})\epsilon\left(2\epsilon^{1/2}\right)^{\log(\frac{1}{\epsilon})-3} \\
&= \frac{1}{8}\log(\frac{1}{\epsilon})\epsilon^{\log(\frac{1}{\epsilon})/2-3/2} \\
&\leq \epsilon^{\log(\frac{1}{\epsilon})/4} \leq \left(\frac{1}{4}\right)^{\log(\frac{1}{\epsilon})} \leq p_2^{(K_1+\log\log(\frac{1}{\epsilon}))}
\end{aligned}
$$

where the last line holds for $\epsilon \leq 2^{-10}$. Thus phase 2 must proceed for more than $\log\log(\frac{1}{\epsilon})$ steps.

Note also that observation 3 (together with observation 3 of phase 1) implies that $K + \log(\frac{1}{\epsilon}) \le K' \le K + 3\log(\frac{1}{\epsilon})$, as required.

Finally, we need to establish that there exists a δ as required in the statement of the lemma. If $p_3^{(K'-1)} \le p_2^{(K')} \le p_3^{(K')}$, let $\delta = p_3^{(K')}$. Then trivially $\delta \le p_3^{(K')} \le 2\delta$ and also

$$\delta^2 \le \delta/2 \le p_3^{(K'-1)} \le p_2^{(K')} \le p_3^{(K')} = \delta$$

since $\delta \le 1/2$. Otherwise, $p_2^{(K')} < p_3^{(K'-1)}$; in this case, let $\delta = p_3^{(K'-1)}$. Again trivially, $\delta \le p_3^{(K')} \le 2\delta$. Additionally,

$$\delta^2 = \left(p_3^{(K'-1)}\right)^2 \le \left(p_2^{(K'-1)}\right)^2 \le p_2^{(K')} \le p_3^{(K'-1)} = \delta.$$

Now to lower-bound δ, using observations 2 and 3, we have

$$\begin{aligned}
2\delta &\ge p_3^{(K')} \ge 1.5^{\log\log(\frac{1}{\epsilon})-2} p_3^{(K_1)} \\
&\ge \left(\frac{2}{3}\right)^2 \log(\frac{1}{\epsilon})^{1/2}\epsilon^2 \left(\epsilon 2^{(2\log(\frac{1}{\epsilon})+1)/4}\right)^{2\log(\frac{1}{\epsilon})} \\
&\ge \left(\frac{2}{3}\right)^2 \log(\frac{1}{\epsilon})^{1/2}\epsilon^{\log(\frac{1}{\epsilon})+3/2} \ge 2\epsilon^{2\log(\frac{1}{\epsilon})} \\
\delta &\ge \epsilon^{2\log(\frac{1}{\epsilon})}
\end{aligned}$$

where the penultimate line holds for $\epsilon \le 2^{-4}$.

As for an upper bound:

$$\begin{aligned}
\delta &\le p_3^{(K')} \le p_3^{(K_1)} 2^{\log(\frac{1}{\epsilon})} \\
&\le \epsilon \left(\epsilon 2^{(\log(\frac{1}{\epsilon})-3+5)/2}\right)^{\log(\frac{1}{\epsilon})-3} 2^{\log(\frac{1}{\epsilon})} \\
&= \left(2\epsilon^{1/2}\right)^{\log(\frac{1}{\epsilon})-3} = 2^{-3}\epsilon^{(\log(\frac{1}{\epsilon})-5)/2} \le \epsilon^{\log(\frac{1}{\epsilon})/4}
\end{aligned}$$

where the last line holds for $\epsilon \le 2^{-10}$. Thus we have shown that $K' = K_2$ is as required in the statement of the lemma.

Phase 3. Finally, we want to establish that $p_3^{(k)}$ and $p_2^{(k)}$ stay relatively small for $2^5\log(\frac{1}{\epsilon})$ steps past K', provided ϵ is small enough. In particular, we know that $p_3^{(K')} > p_2^{(K')}$, and that no probability can more than double at each time step. Thus

$$\begin{aligned}
p_3^{(K'+t)}, p_2^{(K'+t)} &\le 2^t p_3^{(K')} \le 2^t \left(2\epsilon^{1/2}\right)^{\log(\frac{1}{\epsilon})-3} \\
&= 2^{t-3}\epsilon^{(\log(\frac{1}{\epsilon})-5)/2} \\
&\le \epsilon^{\log(\frac{1}{\epsilon})/2^5}
\end{aligned}$$

where the last line holds if

$$\begin{aligned}
2^{t-3}\epsilon^{-5/2} &\le \epsilon^{-(2^4-1)\log(\frac{1}{\epsilon})/2^5} \\
t - 3 - \frac{5}{2}\log(\epsilon) &\le (2^4-1)\log^2(\epsilon)/2^5 \\
t &\le \frac{2^4-1}{2^5}\log^2(\epsilon) + \frac{5}{2}\log(\epsilon) + 3
\end{aligned}$$

In particular, since we want this to hold up to $t = 2^5\log(\frac{1}{\epsilon})$, $\epsilon \le 2^{-75}$ suffices. This completes the proof. ◀