

Pairwise Independent Random Walks Can Be Slightly Unbounded

Shyam Narayanan

Massachusetts Institute of Technology, Cambridge, Massachusetts, USA
shyam.s.narayanan@gmail.com

Abstract

A family of problems that have been studied in the context of various streaming algorithms are generalizations of the fact that the expected maximum distance of a 4-wise independent random walk on a line over n steps is $O(\sqrt{n})$. For small values of k , there exist k -wise independent random walks that can be stored in much less space than storing n random bits, so these properties are often useful for lowering space bounds. In this paper, we show that for all of these examples, 4-wise independence is required by demonstrating a pairwise independent random walk with steps uniform in ± 1 and expected maximum distance $\Omega(\sqrt{n} \lg n)$ from the origin. We also show that this bound is tight for the first and second moment, i.e. the expected maximum square distance of a 2-wise independent random walk is always $O(n \lg^2 n)$. Also, for any even $k \geq 4$, we show that the k th moment of the maximum distance of any k -wise independent random walk is $O(n^{k/2})$. The previous two results generalize to random walks tracking insertion-only streams, and provide higher moment bounds than currently known. We also prove a generalization of Kolmogorov's maximal inequality by showing an asymptotically equivalent statement that requires only 4-wise independent random variables with bounded second moments, which also generalizes a result of Błasiok.

2012 ACM Subject Classification Theory of computation → Random walks and Markov chains; Mathematics of computing → Probabilistic algorithms

Keywords and phrases k -wise Independence, Random Walks, Moments, Chaining

Digital Object Identifier 10.4230/LIPIcs.APPROX-RANDOM.2019.63

Category RANDOM

Related Version <https://arxiv.org/abs/1807.04910>

Funding This research was funded by the PRISE program at Harvard University and by Harvard's Herchel Smith Fellowship.

Acknowledgements I would like to thank Prof. Jelani Nelson for advising this work, as well as for problem suggestions, forwarding me many papers from the literature, and providing helpful feedback on my writeup.

1 Introduction

Random walks are well-studied stochastic processes with numerous applications in physics [14], math [17], computer science [2], economics [13], and biology [4]. A commonly studied random walk on \mathbb{Z} is a process that starts at 0 and at each step independently moves either $+1$ or -1 with equal probability. In this paper, we do not study this random walk but instead study k -wise independent random walks, meaning that steps are not totally independent but that any k steps are completely independent. In many low-space randomized algorithms, information is tracked with processes similar to random walks, but simulating a totally random walk of n steps is known to require $O(n)$ bits while there exist k -wise independent families which can be simulated with $O(k \lg n)$ bits [10]. As a result, understanding properties of k -wise independent random walks have applications to streaming algorithms, such as heavy-hitters [8, 9], distinct elements [5], and ℓ_p tracking [6].



© Shyam Narayanan;

licensed under Creative Commons License CC-BY

Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019).

Editors: Dimitris Achlioptas and László A. Végh; Article No. 63; pp. 63:1–63:19



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

For any k -wise independent random walk, where $k \geq 2$, it is well-known that after n steps, the expected squared distance from the origin is exactly n , since $\mathbb{E}_{h \in \mathcal{H}}(h(1) + \dots + h(n))^2 = n$ for any 2-wise independent hash family \mathcal{H} . One can see this by expanding and applying linearity of expectation. This property provides good bounds for the distribution of the final position of a 2-wise independent random walk. However, we study the problem of bounding the position throughout the random walk, by providing comparable moment bounds for $\sup_{1 \leq i \leq n} |h(1) + \dots + h(i)|$ rather than just for $|h(1) + \dots + h(n)|$ and determining an example of a 2-wise independent random walk where the expected bounds do not hold, even though very strong bounds for even 4-wise independent random walks can be established.

Two more general questions that have been studied in the context of certain streaming algorithms are random walks corresponding to insertion-only streams, and random walks with step sizes corresponding to random variables. These are useful generalizations as the first proves useful in certain algorithms with insertion stream inputs, and the second allows for a setup similar to Kolmogorov's inequality [16], which we will generalize to 4-wise independent random variables. To understand these two generalizations, consider a k -wise independent family of random variables X_1, \dots, X_n and an insertion stream $p_1, \dots, p_m \in [n]$, where now seeing p_j means that our random walk moves by X_{p_j} on the j th step. The insertion stream can be thought of as keeping track of a vector z in \mathbb{R}^n where seeing p_j increments the p_j th component of z by 1, and \vec{X} can be thought of as a vector in \mathbb{R}^n with i th component X_i . Then, one goal is to bound for appropriate values of k'

$$\mathbb{E}_{h \in \mathcal{H}} \left[\sup_{1 \leq t \leq m} \left| \langle \vec{X}, z^{(t)} \rangle \right|^{k'} \right],$$

where $z^{(t)}$ is the vector z after seeing only the first t elements of the insertion stream. Notice that bounding the k' th moment of the furthest distance from the origin in a k -wise independent random walk is the special case of $m = n$, $p_j = j$ for all $1 \leq j \leq n$, and the X_i 's are uniform random signs.

1.1 Main Results

Intuitively, even in a pairwise independent random walk, since the positions at various times have strong correlations with each other, the expectation of the furthest we ever get from the origin should not be much more than the expectation of than our distance from the origin after n steps. But surprisingly, we show in Section 2 that there is a pairwise independent family \mathcal{H} such that

$$\mathbb{E}_{h \in \mathcal{H}} \left[\sup_{1 \leq t \leq n} |h_1 + \dots + h_t| \right] = \Omega(\sqrt{n} \lg n), \quad (1)$$

meaning there is a uniform pairwise independent ± 1 -valued random walk which is not continuously bounded in expectation by $O(\sqrt{n})$. Furthermore, this bound of $\sqrt{n} \lg n$ is tight up to the first and second moments, because in Section 3 we prove that for any pairwise independent family \mathcal{H} from $[n]$ to $\{-1, 1\}$ with $\mathbb{E}[h_i] = 0$ for all i ,

$$\mathbb{E}_{h \in \mathcal{H}} \left[\sup_{1 \leq t \leq n} (h_1 + \dots + h_t)^2 \right] = O(n \lg^2 n). \quad (2)$$

In Section 4, we uniformly bound random walks corresponding to insertion-only streams and random walks with step sizes not necessarily uniform ± 1 variables. We first generalize Kolmogorov's inequality [16] by proving that for any 4-wise independent random variables X_1, \dots, X_n with mean 0 and finite variance,

$$\mathbb{P} \left(\sup_{1 \leq i \leq n} |X_1 + \dots + X_i| \geq \lambda \right) \leq \frac{\sum \mathbb{E}[X_i^2]}{\lambda^2} \quad (3)$$

for all $\lambda > 0$. In Appendix A, we generalize Equation (2) by proving for any family X_1, \dots, X_n of pairwise independent variables such that $\mathbb{E}[X_i] = 0, \mathbb{E}[X_i^2] \leq 1$, and for any insertion stream $p_1, \dots, p_m \in [n]$,

$$\mathbb{E} \left[\sup_{1 \leq t \leq m} \left| \langle \vec{X}, z^{(t)} \rangle \right|^2 \right] = O(\|z\|_2^2 \lg^2 m) \quad (4)$$

where $z = z^{(m)}$ is the final position of the vector. Finally, we show that for any even $k \geq 4$, any k -wise independent family X_1, \dots, X_n such that $\mathbb{E}[X_i] = 0, \mathbb{E}[X_i^k] \leq 1$, and any insertion stream $p_1, \dots, p_m \in [n]$,

$$\mathbb{E} \left[\sup_{1 \leq t \leq m} \left| \langle \vec{X}, z^{(t)} \rangle \right|^k \right] = O(\|z\|_2^k). \quad (5)$$

Equations (3), (4), and (5) are interesting together as they provide various bounds on the supremum of generalized random walks under differing moment bounds and degrees of independence.

Finally, we note that to prove Equation (1), we create a complicated pairwise independent hash function, which suggests that standard pairwise independent hash functions do not have this property. Indeed, for many such families, such as some types of codes constructed from Hadamard matrices or random linear functions from $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$, we have $\mathbb{E} \sup_i |h_1 + \dots + h_i| = o(n \lg n)$. (See the appendices in the arXiv version of this paper.) However, we note that for some standard pairwise independent hash functions, it is difficult to provide either an upper or lower bound for $\mathbb{E} \sup_i |h_1 + \dots + h_i|$. Therefore, even if some simpler pairwise independent hash function satisfies Equation 1, our hash family has the advantage that the analysis is simpler, even if the construction of the family is not.

1.2 Motivation and Relation to Previous Work

The primary motivation of this paper comes from certain theorems that provide strong bounds for certain variants of 4-wise independent random walks, which raised the question of whether any of these bounds can be extended to 2-wise independence. For example, Theorem 1 in [8] proves for any family \mathcal{H} of $h \in \{-1, 1\}^n$ with 4-wise independent coordinates, $\mathbb{E}_{h \in \mathcal{H}} (\sup_t \langle h, z^{(t)} \rangle) = O(\|z\|_2)$. This result generalizes a result from [9] which proves the same but only if h is uniformly chosen from $\{-1, 1\}^n$. [8] provides an algorithm that successfully finds all ℓ_2 ε -heavy hitters in an insertion-only stream in $O(\varepsilon^{-2} \log \varepsilon^{-1})$ space, in which the above result was crucial for analysis of a subroutine which attempts to find bit-by-bit the index of a single “super-heavy” heavy hitter if one exists. Theorem 1 in [8] also proved valuable for an algorithm for continuous monitoring of ℓ_p norms in insertion-only data streams [6]. Lemma 18 in [5] shows that even without bounded fourth moments, given 4-wise independent random variables X_1, \dots, X_n , each with mean 0 and finite variance,

$$\mathbb{P} \left(\max_{1 \leq i \leq n} |X_1 + \dots + X_i| \geq \lambda \right) = O \left(\frac{n \cdot \max_i \mathbb{E}[X_i^2]}{\lambda^2} \right).$$

This theorem was crucial in analyzing an algorithm tracking distinct elements that provides a $(1 + \varepsilon)$ -approximation with failure probability δ in $O(\varepsilon^{-2} \lg \delta^{-1} + \lg n)$ bits of space. Notice that our Equation (3) is stronger than the above equation and is asymptotically equivalent to Kolmogorov’s inequality, though under much weaker assumptions.

A natural follow-up question to the above theorems is whether 4-wise independence is necessary, or whether lesser levels of independence such as 2-wise or 3-wise are required. Equation (1) shows that 2-wise independence does not suffice for any of the above results,

because the random walk on a line case is strictly weaker than all of the above results, though the case of 3-wise independence is still unknown. As a result, we know that the tracking sketches in [8, 6, 5] cannot be extended to 2-wise independent sketches.

However, the results given still have interesting extensions, such as to higher moments. Equation (5) shows a stronger result than the one established in [8], since it not only bounds the first moment of $\sup_t \langle h, z^{(t)} \rangle$ for a 4-wise independent family of uniform ± 1 variables but also bounds the 4th moment equally (as they have mean 0 and k th moment 1). The main methods used for proving most of our upper bounds are based on chaining methods, specifically Dudley chaining, with slight modifications, although the bounds in Section 3 are proved differently from standard chaining methods but are still motivated by similar ideas. Dudley chaining was introduced in [11], and Dudley chaining and other chaining techniques, along with applications, are summarized in [18].

k -wise independence for hash functions was first introduced in [10]. Bounding the amount of independence required for analysis of algorithms has been studied in various contexts, often since k -wise independent hash families can be stored in low space but may provide equally adequate bounds as totally independent families. As further examples, the well-known AMS sketch [1] is a streaming algorithm to estimate the ℓ_2 norm of a vector z to a factor of $1 \pm \varepsilon$ with high probability by multiplying the vector by a sketch matrix $\Pi \in \mathbb{R}^{n \times (1/\varepsilon^2)}$ of 4-wise independent random signs and using $\|\Pi z\|_2$ as an estimate for $\|z\|_2$. It is known from [20, 22] that the accuracy of the AMS sketch can be much worse if 3-wise independent random signs are used instead of 4-wise independent random signs. If z is given as an insertion stream, it is known that the AMS sketch with 8-wise independent random signs can provide weak tracking [8], meaning that $\mathbb{E} \sup_t \left| \|\Pi z^{(t)}\|_2^2 - \|z^{(t)}\|_2^2 \right| \leq \varepsilon \|z\|_2^2$. This implies that the approximation of the ℓ_2 norm with the 8-wise independent AMS sketch is quite accurate at all times t . While one cannot perform weak tracking with 3-wise independence of the AMS sketch, it is unknown for 4-wise independence through 7-wise independence whether the AMS sketch provides weak tracking. Finally, linear probing, a well-known implementation of hash tables, was shown to take $O(1)$ expected update time with any 5-wise independent hash function [19] but was shown to take $\Theta(\lg n)$ expected update time for certain 4-wise independent hash functions and $\Theta(\sqrt{n})$ expected update time for certain 2-wise independent hash functions [20].

Bounding the maximum distance traveled of a random walk has also been studied in probability theory independent of computer science applications, both when the steps are totally independent or k -wise independent. For example, Kolmogorov's inequality [16] provides bounds for $\sup_t (X_1 + \dots + X_t)$ for independent random variables X_1, \dots, X_t even if only the second moments of X_1, \dots, X_t are finite. [3] constructed an infinite sequence $\{X_1, X_2, \dots\}$ of pairwise independent random variables taking on the values ± 1 such that $\sup_t (X_1 + \dots + X_t)$ is bounded almost surely, though the paper also proved that this phenomenon can never occur for 4-wise independent variables taking on the values ± 1 . Finally, the supremum of a random walk with i.i.d. bounded random variable steps was studied in [12], which provided comparisons with the supremum of a Brownian motion random walk regardless of the random variable chosen for step size.

1.3 Notation

We define $[n] := \{1, \dots, n\}$, and treat $p_1, \dots, p_m \in [n]$ as an insertion-only stream that keeps track of a vector z that starts at the origin and increments its p_j th component by 1 after we see p_j .

A k -wise independent family from $[n]$ to $\{-1, 1\}$ is a family \mathcal{H} of functions $h : [n] \rightarrow \{-1, 1\}$ such that for any k distinct indices, their values are independent *Rademachers*, where Rademachers are random variables uniformly selected from $\{-1, 1\}$. A k -wise independent random walk is a random walk where one's position after t steps is $h(1) + \dots + h(t)$, with h chosen from \mathcal{H} . We may also denote a k -wise independent random walk as a random walk where the i th step is a random variable X_i , assuming X_1, \dots, X_n are random variables such that any k distinct X_i 's are totally independent.

In this paper, we think of a hash function $h : [n] \rightarrow \{-1, 1\}$ as a vector in \mathbb{R}^n , where $h_i = h(i)$, for the purpose of denoting inner products. Similarly, treat \vec{X} as the vector (X_1, \dots, X_n) .

Finally, in Section 2, we assume that n is a power of 4, in Section 3, we assume n is a power of 2 and is at least 4, and in Section 4, we assume m is a sufficiently large power of 2. We note that these assumptions can be removed by replacing n with the largest power of 4 less than n or the smallest power of 2 greater than n or m , respectively.

1.4 Overview of Proof Ideas

Here, we briefly outline some of the main ideas behind the proofs of Equations (1) through (5).

The main goal in Section 2 is to establish Equation (1), i.e. construct a pairwise independent \mathcal{H} such that $\mathbb{E}[h_i h_j] = 0$ for all $i \neq j$. In other words, we wish for the covariance matrix $\mathcal{M} = \mathbb{E}[h^T h]$ to be the identity matrix I_n . We also want $\sup_{1 \leq i \leq n} |h_1 + \dots + h_i|$ to be $\Omega(\sqrt{n} \lg n)$ in expectation. The construction has two major steps.

1. Create a hash function such that $\mathbb{E} \sup_{1 \leq i \leq n} |h_1 + \dots + h_i| = \Omega(\sqrt{n} \lg n)$ but rather than have $\mathbb{E}[h_i h_j] = 0$ for all $i \neq j$, have $\sum_{i \neq j} |\mathbb{E}[h_i h_j]| = O(n)$, i.e. the cross terms in total aren't very large in absolute value (this hash function will be \mathcal{H}_2 in our proof). To do this, we first created \mathcal{H}_1 , which certain properties, most notably that $\mathbb{E}[h_1 + \dots + h_n] = 0$ but $\mathbb{E}[h_1 + \dots + h_{n/2}] = \Theta(\sqrt{n} \lg n)$, and rotated the hash family by a uniform index. The rotation allows many of the cross terms to average out, reducing the sum of their absolute values.
2. Remove the cross terms. To do this, we make \mathcal{H} a hash family where with some constant probability, we choose from \mathcal{H}_2 and with some probability, we choose some set of indices and pick a hash function such that $\mathbb{E}[h_i h_j]$ will be the opposite sign of $\mathbb{E}_{h \in \mathcal{H}_2}[h_i h_j]$ for certain indices i, j , so that overall, $\mathbb{E}[h_i h_j]$ will be 0. Certain symmetry properties and most importantly the fact that $\sum_{i \neq j} |\mathbb{E}_{h \in \mathcal{H}_2}[h_i h_j]| = O(n)$ will allow for us to choose from \mathcal{H}_2 with constant probability, which means even for our final hash function, $\mathbb{E} \sup_{1 \leq i \leq n} |h_1 + \dots + h_i| = \Omega(\sqrt{n} \lg n)$.

The goal of Section 3 is to establish Equation (2), i.e. to show that if $\mathcal{M} = \mathbb{E}[h^T h] = I_n$, which is true for any pairwise independent hash function, then $\sup_{1 \leq i \leq n} |h_1 + \dots + h_i|^2 = O(n \lg^2 n)$. To do this, we apply probabilistic method ideas. We notice that for any matrix A , $\mathbb{E}[h^T A h] = \text{Tr}(A)$, and thus, if we can find a matrix such that the trace of the matrix is small, but $h^T A h$ is reasonably large in comparison to $\sup_{1 \leq i \leq n} |h_1 + \dots + h_i|^2$, then $\mathbb{E}[h^T A h]$ is small but is large in comparison to $\mathbb{E}[\sup_{1 \leq i \leq n} |h_1 + \dots + h_i|^2]$. If we assume that n is a power of 2, then the matrix that corresponds to the quadratic form

$$h^T A h = \sum_{r=0}^{\lg n - 1} \sum_{i=0}^{(n/2^r)-1} (h_{i \cdot 2^r + 1} + \dots + h_{(i+1) \cdot 2^r})^2,$$

i.e. $h^T A h = h_1^2 + \dots + h_n^2 + (h_1 + h_2)^2 + \dots + (h_{n-1} + h_n)^2 + \dots + (h_1 + \dots + h_n)^2$ can be shown to satisfy $\text{Tr}(A) = n \lg n$ and for any vector x , $x^T A x \geq \frac{1}{\lg n} \cdot (x_1 + \dots + x_i)^2$ for all $1 \leq i \leq n$, not just in expectation. These conditions will happen to be sufficient for our goals.

This method, in combination with Equation (1), will also allow us to prove an interesting matrix inequality, proven at the end of Section 3. The method above actually generalizes to looking at k th moments of k -wise independent hash functions, as well as random walks corresponding to tracking insertion-only streams, and will allow us to prove Equations (4) and (5). However, these generalizations will also need the construction of ε -nets, which are explained in Appendix A, or in [18].

We finally explain the ideas behind Equation (3), the generalization of Kolmogorov's inequality and Lemma 18 of [5]. We use ideas of chaining, such as in [18], and an idea of [5] that allows us to bound the minimum of $X_{i+1} + \dots + X_j$ and $X_{j+1} + \dots + X_k$ where $i < j < k$, given 4-wise independent functions X_1, \dots, X_n with only bounded second moments. We combine these with another idea, that we can consider distances between i and j for $1 \leq i < j \leq n$ as $\mathbb{E}[X_{i+1}^2 + \dots + X_j^2]$ and that for any $i < j < k$, either $\mathbb{E}[X_{i+1}^2 + \dots + X_j^2]$ is very small and we can bound $X_{i+1} + \dots + X_j$, $\mathbb{E}[X_{j+1}^2 + \dots + X_k^2]$ is very small and we can bound $X_{j+1} + \dots + X_k$, or we can bound $\min(|X_{i+1} + \dots + X_j|, |X_{j+1} + \dots + X_k|)$ with the idea of [5]. These ideas allow for our chaining method to be quite effective, even if the X_i 's do not have bounded 4th moments or if the X_i 's wildly differ in variance.

2 Lower Bounds for Pairwise Independence

In this section, we construct a 2-wise independent family \mathcal{H} such that the furthest distance traveled by the random walk is $\Omega(\sqrt{n} \lg n)$ in expected value. In other words, we prove the following:

► **Theorem 1.** *There exists a 2-wise independent hash family \mathcal{H} from $[n] \rightarrow \{-1, 1\}$ such that*

$$\mathbb{E}_{h \in \mathcal{H}} \left[\sup_{1 \leq t \leq n} \left| \sum_{1 \leq j \leq t} h_j \right| \right] = \Omega(\sqrt{n} \lg n).$$

To actually construct this counterexample, we proceed by a series of families and tweak each family accordingly to get to the next one, until we get the desired \mathcal{H} .

We start by creating \mathcal{H}_1 . First, split $[n]$ into blocks of size \sqrt{n} so that $\{(c-1)\sqrt{n} + 1, \dots, c\sqrt{n}\}$ form the c th block for each $1 \leq c \leq \sqrt{n}$. Also, define $\ell = \frac{\sqrt{n}}{2}$. Now, to pick a function h from \mathcal{H}_1 , choose the value of h_i for each $1 \leq i \leq n$ independently, but if i is in the c th block for some $1 \leq c \leq \ell$, make $\mathbb{P}[h_i = 1] = \frac{1}{2} + \frac{1}{2(\ell+1-c)}$ and if i is in the c th block for some $\ell+1 \leq c \leq \sqrt{n}$, make $\mathbb{P}[h_i = 1] = \frac{1}{2} - \frac{1}{2(c-\ell)}$. This way, $\mathbb{E}[h_i] = \frac{1}{\ell+1-c}$ if i is in the c th block for $c \leq \ell$ and $\mathbb{E}[h_i] = -\frac{1}{c-\ell}$ if i is in the c th block for $c > \ell$.

From now on, assume that h_i is periodic modulo n , i.e. $h_{i+n} = h_i$ for all integers i . We first prove the following about \mathcal{H}_1 :

► **Lemma 2.** *Suppose that $1 \leq i < j \leq n$. Suppose that i is in block c_1 and j is in block c_2 , where c_1 and c_2 are not necessarily distinct. Define $r = \min(c_2 - c_1, \sqrt{n} - (c_2 - c_1))$. Then,*

$$\sum_{d=0}^{\sqrt{n}-1} \mathbb{E}_{h \in \mathcal{H}_1} (h_{i+d\sqrt{n}} h_{j+d\sqrt{n}}) = O\left(\frac{\lg(r+2)}{(r+1)^2}\right).$$

Proof. For $1 \leq c \leq \sqrt{n}$, define f_c to equal $\frac{1}{\ell+1-c}$ if $1 \leq c \leq \ell$ and to equal $-\frac{1}{c-\ell}$ if $\ell+1 \leq c \leq \sqrt{n}$. In other words, $f_c = \mathbb{E}[h_i]$ if i is in the c th block. Furthermore, assume that f is periodic modulo \sqrt{n} , i.e. $f_c = f_{c+\sqrt{n}}$ for all integers c . Then,

$$\begin{aligned} \sum_{d=0}^{\sqrt{n}-1} \mathbb{E}_{h \in \mathcal{H}_1}(h_{i+d\sqrt{n}} h_{j+d\sqrt{n}}) &= \sum_{d=0}^{\sqrt{n}-1} \mathbb{E}_{h \in \mathcal{H}_1}(h_{i+d\sqrt{n}}) \mathbb{E}_{h \in \mathcal{H}_1}(h_{j+d\sqrt{n}}) \\ &= \sum_{d=0}^{\sqrt{n}-1} f_{c_1+d} f_{c_2+d} = \sum_{d=1}^{\sqrt{n}} f_d f_{r+d}. \end{aligned}$$

Now, since $r \leq \ell$, if we assume $r \geq 1$, this sum can be explicitly written as

$$\begin{aligned} & 2 \cdot \sum_{d=1}^{\ell-r} \frac{1}{d(d+r)} - \sum_{d=1}^r \frac{1}{d(r+1-d)} - \sum_{d=1}^r \frac{1}{(n+1-d)(n+1-(r+1-d))} \\ & \leq 2 \sum_{d=1}^{\infty} \frac{1}{d(d+r)} - \sum_{d=1}^r \frac{1}{d(r+1-d)} \\ & = \frac{2}{r} \sum_{d=1}^{\infty} \left(\frac{1}{d} - \frac{1}{d+r} \right) - \frac{1}{r+1} \sum_{d=1}^r \left(\frac{1}{d} + \frac{1}{r+1-d} \right) \\ & = \frac{2}{r} \left(\sum_{d=1}^r \frac{1}{d} \right) - \frac{2}{r+1} \left(\sum_{d=1}^r \frac{1}{d} \right) \\ & = \frac{2}{r(r+1)} \left(\sum_{d=1}^r \frac{1}{d} \right) \leq \frac{C_1 \lg(r+2)}{(r+1)^2} \end{aligned}$$

for some constant C_1 . If we assume $r = 0$, then this sum can be explicitly written as

$$2 \cdot \sum_{d=1}^{\ell} \frac{1}{d^2} \leq C_2 = \frac{(C_2) \cdot \lg(0+2)}{(0+1)^2}$$

for some constant C_2 . Therefore, setting $C_3 = \max(C_1, C_2)$ as our constant, we are done. \blacktriangleleft

To construct \mathcal{H}_2 , first choose $h \in \mathcal{H}_1$ at random, and then choose an index d between 0 and $\sqrt{n} - 1$ uniformly at random. Our chosen function h' will then be the function that satisfies $h'_i = h_{i+d\sqrt{n}}$ for all i . We show the following about \mathcal{H}_2 :

► Lemma 3. *The following three statements are true:*

- For all $i, j \in \mathbb{Z}$, $\mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) = \mathbb{E}_{h \in \mathcal{H}_2}(h_{i+\sqrt{n}} h_{j+\sqrt{n}})$.
- Suppose that $1 \leq i, i', j, j' \leq n$, where i, i' are in blocks c_1 , j, j' are in blocks c_2 , and $i \neq j, i' \neq j'$. Then, $\mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) = \mathbb{E}_{h \in \mathcal{H}_2}(h_{i'} h_{j'})$.
- $\sum_{i \neq j} |\mathbb{E}_{h \in \mathcal{H}_2} h_i h_j| = O(n)$.

Proof. Part a) is quite straightforward, since

$$\begin{aligned} \mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) &= \frac{1}{\sqrt{n}} \sum_{d=0}^{\sqrt{n}-1} \mathbb{E}_{h \in \mathcal{H}_1}(h_{i+d\sqrt{n}} h_{j+d\sqrt{n}}) \\ &= \frac{1}{\sqrt{n}} \sum_{d=0}^{\sqrt{n}-1} \mathbb{E}_{h \in \mathcal{H}_1}(h_{i+(d+1)\sqrt{n}} h_{j+(d+1)\sqrt{n}}) = \mathbb{E}_{h \in \mathcal{H}_2}(h_{i+\sqrt{n}} h_{j+\sqrt{n}}) \end{aligned}$$

by periodicity of h modulo n .

63:8 Pairwise Independent Random Walks Can Be Slightly Unbounded

For part b), for all $d \in \mathbb{Z}$, note that $i + d\sqrt{n}$ and $i' + d\sqrt{n}$ are in the same blocks, $j + d\sqrt{n}$ and $j' + d\sqrt{n}$ are in the same blocks, $i + d\sqrt{n} \neq j + d\sqrt{n}$ and thus $h_{i+d\sqrt{n}}, h_{j+d\sqrt{n}}$ are independent, and $i' + d\sqrt{n} \neq j' + d\sqrt{n}$ and thus $h_{i'+d\sqrt{n}}, h_{j'+d\sqrt{n}}$ are independent. Therefore, $\mathbb{E}_{h \in \mathcal{H}_1}(h_{i+d\sqrt{n}}h_{j+d\sqrt{n}}) = \mathbb{E}_{h \in \mathcal{H}_1}(h_{i'+d\sqrt{n}}h_{j'+d\sqrt{n}})$ for all d . Because of the way we constructed \mathcal{H}_2 , part b) is immediate from these observations.

We use Lemma 2 to prove part c). First note that for all $i \neq j$,

$$\mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) = \frac{1}{\sqrt{n}} \sum_{d=0}^{\sqrt{n}-1} \mathbb{E}_{h \in \mathcal{H}_1}(h_{i+d\sqrt{n}} h_{j+d\sqrt{n}}) \leq \frac{C_3 \lg(r+2)}{\sqrt{n} \cdot (r+1)^2},$$

where i is in block c_1 , j is in block c_2 , and $r = \min(|c_1 - c_2|, \sqrt{n} - |c_1 - c_2|)$. Now, there are exactly $n(\sqrt{n} - 1)$ pairs (i, j) where $1 \leq i, j \leq n$, $i \neq j$, and $r = 0$. This is because we can choose from \sqrt{n} blocks for the value of $c_1 = c_2$, and then choose from $\sqrt{n}(\sqrt{n} - 1)$ possible pairs (i, j) in each block. For a fixed $0 < r < \ell$, there are exactly $2n^{3/2}$ pairs (i, j) , since there are $2\sqrt{n}$ choices for blocks c_1 and c_2 and \sqrt{n} choices for each of i and j after that, for $r = \ell$, there are exactly $n^{3/2}$ such pairs, since there are $2\sqrt{n}$ choices for blocks c_1 and c_2 and \sqrt{n} choices for each of i and j after that, and finally we cannot have $r > \ell$. Therefore,

$$\sum_{i \neq j} \max(0, \mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j)) \leq 2n^{3/2} \cdot \sum_{r=0}^{\ell} \frac{C_3 \lg(r+2)}{\sqrt{n}(r+1)^2} \leq C_4 n$$

for some constant C_4 , since $\sum \frac{\lg(r+2)}{(r+1)^2}$ is a convergent series.

To finish, note that $|x| = 2 \cdot \max(0, x) - x$, so

$$\sum_{i \neq j} |\mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j)| \leq 2 \cdot C_4 n - \sum_{i \neq j} \mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) \leq (2C_4 + 1)n,$$

since

$$\sum_{i \neq j} \mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) = \sum_{i, j} \mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) - \sum_i \mathbb{E}_{h \in \mathcal{H}_2} h_i^2 = \mathbb{E}_{h \in \mathcal{H}_2}(h_1 + \dots + h_n)^2 - n \geq -n.$$

Thus, setting $C_5 = 2C_4 + 1$ gets us our desired result. \blacktriangleleft

Next, we tweak \mathcal{H}_2 to create a new family \mathcal{H}_3 . First, notice that we can define $g_{c_1 c_2}$ for $1 \leq c_1, c_2 \leq \sqrt{n}$ to equal $\mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j)$ for some i in the c_1 th block and j in the c_2 th block such that $i \neq j$. This is well defined by Lemma 3 b), and as $1 \leq c_1, c_2 \leq \sqrt{n}$, there always exist $i \neq j$ with i in the c_1 th block and j in the c_2 th block, as long as $n \geq 4$. Now, to create \mathcal{H}_3 , define $g = 1 + \sum_{c_1 < c_2} |g_{c_1 c_2}|$. Then, with probability $\frac{1}{g}$, we choose a hash function from \mathcal{H}_2 . With probability $\frac{|g_{c_1 c_2}|}{g}$ for each $1 \leq c_1 < c_2 \leq \sqrt{n}$, we choose $h_i = 1$ for all i in the c_1 th bucket, if $g_{c_1 c_2} \geq 0$, we make $h_i = -1$ for all i in the c_2 th bucket and if $g_{c_1 c_2} < 0$, we make $h_i = 1$ for all i in the c_2 th bucket, and if i is not in either the c_1 th or the c_2 th bucket, we let h_i be an independent Rademacher. We prove the following about \mathcal{H}_3 :

► Lemma 4. *If i and j are in different buckets, then $\mathbb{E}_{h \in \mathcal{H}_3}(h_i h_j) = 0$. If i, j are in the same bucket but $i \neq j$, then there is some constant $0 \leq C_6 \leq C_5$ such that $\mathbb{E}_{h \in \mathcal{H}_3}(h_i h_j) = \frac{C_6}{\sqrt{n}}$.*

Proof. Assume WLOG that $i < j$. If i, j are in different buckets, then we compute $\mathbb{E}_{h \in \mathcal{H}_3}(h_i h_j)$ as follows. With probability $\frac{1}{g}$, we are choosing h from \mathcal{H}_2 , and if i is in the c_1 th bucket and j is in the c_2 th bucket, then $\mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) = g_{c_1 c_2}$. With probability $\frac{|g_{c_1 c_2}|}{g}$ we have $h_i h_j = 1$ with probability 1 if $g_{c_1 c_2} < 0$ and $h_i h_j = -1$ with probability 1 if

$g_{c_1 c_2} \geq 0$. In all other scenarios, either h_i or h_j is a Rademacher completely independent of all other elements, which means that $\mathbb{E}[h_i h_j] = 0$. Therefore, the overall expected value of $h_i h_j$ equals $g_{c_1 c_2} \cdot \frac{1}{g} + \frac{|g_{c_1 c_2}|}{g} \cdot \pm 1$ where the ± 1 is positive if and only if $g_{c_1 c_2} \leq 0$, so the expected value is 0.

If i, j are in the same bucket, then we can compute $\mathbb{E}_{h \in \mathcal{H}_3}(h_i h_j)$ as follows. With probability $\frac{1}{g}$, we are choosing h from \mathcal{H}_2 , and if i, j are in the c th bucket, then $\mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) = g_{cc}$. For all $c' \neq c$, there is a $\frac{|g_{cc'}|}{g}$ probability of everything in the c th block having the same sign and everything in the c' th block having the same sign. For the other cases, i, j are independent Rademachers. Therefore,

$$\mathbb{E}_{h \in \mathcal{H}_3}(h_i h_j) = \frac{g_{cc}}{g} + \sum_{c' \neq c} \frac{|g_{cc'}|}{g} = \frac{1}{g} \left(g_{cc} + \sum_{c' \neq c} |g_{cc'}| \right).$$

However, note that $g_{cc} \geq 0$ since $\mathbb{E}_{h \in \mathcal{H}_2}(h_i h_j) = \frac{1}{\sqrt{n}} \sum_d \mathbb{E}_{h \in \mathcal{H}_1}(h_{i+d\sqrt{n}} h_{j+d\sqrt{n}})$ and for all d , we have $\mathbb{E}_{h \in \mathcal{H}_1}(h_{i+d\sqrt{n}} h_{j+d\sqrt{n}}) \geq 0$ since $i + d\sqrt{n}, j + d\sqrt{n}$ are in the same block for all d . Furthermore, for all indices c_1, c_2 , $g_{c_1 c_2} = g_{(c_1+1)(c_2+1)}$, where indices are taken modulo \sqrt{n} , by Lemma 3 a). Combining these gives

$$\mathbb{E}_{h \in \mathcal{H}_3}(h_i h_j) = \frac{1}{\sqrt{n}} \cdot \left(\frac{1}{g} \cdot \left(\sum_{c_1, c_2} |g_{c_1 c_2}| \right) \right).$$

However, we know that $g \geq 1$ and $\sum_{c_1, c_2} |g_{c_1 c_2}| \leq C_5$ by the arguments of Lemma 3 c), so the lemma follows. \blacktriangleleft

Now, we are almost done. To create \mathcal{H} , with probability $p = \frac{1}{1+C_6(\sqrt{n}-1)/\sqrt{n}} \geq \frac{1}{1+C_6}$, choose h from \mathcal{H}_3 , and assuming we chose from \mathcal{H}_3 , with probability $\frac{1}{2}$ negate h_1, \dots, h_n . With probability $1 - p$, for each block of \sqrt{n} elements, choose uniformly at random a subset of size ℓ from the block, and make the corresponding elements 1 and the remaining elements -1 . It is easy to see that now, $\mathbb{E}_{h \in \mathcal{H}}(h_i) = 0$ because of the possibility of negating. Moreover, $\mathbb{E}_{h \in \mathcal{H}}(h_i h_j) = 0$ for all $i \neq j$. To see why, if i and j are in different blocks then $\mathbb{E}_{h \in \mathcal{H}_3}(h_i h_j) = 0$ and if we do not choose h from \mathcal{H}_3 , then h_i and h_j are independent. If i, j are in the same block, then if we condition on choosing from \mathcal{H}_3 , $\mathbb{E}(h_i h_j) = \frac{C_6}{\sqrt{n}}$. If we condition on not choosing from \mathcal{H}_3 , the probability of i, j being the same sign is $\frac{(\sqrt{n}/2)-1}{\sqrt{n}-1} = \frac{\ell-1}{2\ell-1}$, meaning $\mathbb{E}(h_i h_j) = -\frac{1}{\sqrt{n}-1}$. Therefore, $\mathbb{E}_{h \in \mathcal{H}}(h_i h_j) = p \cdot \frac{C_6}{\sqrt{n}} - (1-p) \cdot \frac{1}{\sqrt{n}-1} = 0$.

To finish, it suffices to show that

$$\mathbb{E}_{h \in \mathcal{H}} \left[\sup_{1 \leq t \leq n} |h_1 + \dots + h_t| \right] = \Omega(\sqrt{n} \lg n).$$

To check this, note that with probability at least $\frac{1}{1+C_6}$ we are picking something from \mathcal{H}_3 , so we need to just verify that

$$\mathbb{E}_{h \in \mathcal{H}_3} \left[\sup_{1 \leq t \leq n} |h_1 + \dots + h_t| \right] = \Omega(\sqrt{n} \lg n).$$

But for \mathcal{H}_3 , we are choosing something from \mathcal{H}_2 with probability $\frac{1}{g}$ but $g \leq 1 + C_5$ by the arguments of Lemma 3 c), so it suffices to verify that

$$\mathbb{E}_{h \in \mathcal{H}_2} \left[\sup_{1 \leq t \leq n} |h_1 + \dots + h_t| \right] = \Omega(\sqrt{n} \lg n).$$

63:10 Pairwise Independent Random Walks Can Be Slightly Unbounded

But for \mathcal{H}_2 , if we condition on the shifting index d , we know that

$$\mathbb{E}[h_{1+d\sqrt{n}} + h_{2+d\sqrt{n}} + \cdots + h_{(d+\ell)\sqrt{n}}] \geq \sqrt{n} \left(1 + \cdots + \frac{1}{\ell}\right) \geq C_7 \sqrt{n} \lg n$$

for some C_7 , and likewise

$$\mathbb{E}[h_{1+(d+\ell)\sqrt{n}} + h_{2+(d+\ell)\sqrt{n}} + \cdots + h_{(d+2\ell)\sqrt{n}}] \leq \sqrt{n} \left(-1 - \cdots - \frac{1}{\ell}\right) \leq -C_7 \sqrt{n} \lg n,$$

which means that regardless of whether $d \leq \ell$ or $d > \ell$,

$$\mathbb{E}_{h \in \mathcal{H}_2} \left[\max(|h_1 + \cdots + h_{d\sqrt{n}}|, |h_1 + \cdots + h_{(d+\ell)\sqrt{n}}|) \right] \geq \frac{C_7}{2} \sqrt{n} \lg n$$

by the triangle inequality. But for any $h \in \mathcal{H}_2$,

$$\max(|h_1 + \cdots + h_{d\sqrt{n}}|, |h_1 + \cdots + h_{(d+\ell)\sqrt{n}}|) \leq \sup_{1 \leq t \leq n} (h_1 + \cdots + h_t),$$

so the result follows by taking the expected value of both sides, which proves our upper bound is tight in the case of a random walk. Thus, we have proven Theorem 1.

3 Moment Bounds for Pairwise Independence

We show that the bound established in Section 2 and the induced bound on the second moment are tight for the 2-wise independent random walk case by proving Equation (2) in Section 1.1:

► **Theorem 5.** *For all 2-wise families \mathcal{H} from $[n]$ to $\{-1, 1\}$,*

$$\mathbb{E}_{h \in \mathcal{H}} \left(\sup_{1 \leq i \leq n} (h_1 + \cdots + h_i)^2 \right) = O(n \lg^2 n).$$

We provide a generalization of this theorem in Section 4, with a slightly different method. To prove this, we first establish the following lemma:

► **Lemma 6.** *Suppose that there exists a positive definite matrix $A \in \mathbb{R}^{n \times n}$ such that $\text{Tr}(A) = d_1$ for some $d_1 > 0$ and there exists some function f such that for all vectors $x \in \mathbb{R}^n$ and integers $1 \leq i \leq n$, if $x_1 + \cdots + x_i = 1$, then $x^T A x \geq \frac{1}{d_2}$ for some $d_2 > 0$. Then, for all 2-wise families \mathcal{H} ,*

$$\mathbb{E}_{h \in \mathcal{H}} \left(\sup_{1 \leq i \leq n} (h_1 + \cdots + h_i)^2 \right) \leq d_1 d_2.$$

Proof. Note that $\mathbb{E}_{h \in \mathcal{H}} h_i^2 = 1$ for all i and $\mathbb{E}_{h \in \mathcal{H}} (h_i h_j) = 0$ for all $i \neq j$. Therefore,

$$\begin{aligned} \mathbb{E}_{h \in \mathcal{H}} (h^T A h) &= \sum_{1 \leq i, j \leq n} \mathbb{E}_{h \in \mathcal{H}} (h_i h_j A_{ij}) = \sum_{1 \leq i, j \leq n} A_{ij} (\mathbb{E}_{h \in \mathcal{H}} (h_i h_j)) \\ &= \sum_{1 \leq i \leq n} A_{ii} = \text{Tr}(A) = d_1. \end{aligned}$$

However, for any $1 \leq i \leq n$, for any $h \in \mathcal{H}$, if $h_1 + \cdots + h_i \neq 0$, then

$$h^T A h \geq (h_1 + \cdots + h_i)^2 \cdot \frac{1}{d_2},$$

since the vector $\frac{1}{h_1 + \dots + h_i} \cdot h$ has its first i components sum to 1, so we can let this vector equal x to get $x^T A x \geq \frac{1}{f(n)}$. If $h_1 + \dots + h_i = 0$, then the above inequality is still true as A is positive definite.

Therefore,

$$h^T A h \geq \frac{1}{d_2} \cdot \sup_{1 \leq i \leq n} (h_1 + \dots + h_i)^2,$$

which means that

$$d_1 = \mathbb{E}_{h \in \mathcal{H}}(h^T A h) \geq \frac{1}{d_2} \cdot \mathbb{E}_{h \in \mathcal{H}} \left(\sup_{1 \leq i \leq n} (h_1 + \dots + h_i)^2 \right),$$

so we are done. ◀

► **Lemma 7.** *There exists a positive definite matrix $A \in \mathbb{R}^{n \times n}$ such that $\text{Tr}(A) = n \lg n$ and for all $x \in \mathbb{R}^n$ and $1 \leq i \leq n$, if $x_1 + \dots + x_i = 1$, then $x^T A x \geq \frac{1}{\lg n}$. This clearly implies Theorem 5.*

Proof. Consider the matrix A such that for all $1 \leq i, j \leq n$, $A_{ij} = \lg n - k$ if k is the smallest nonnegative integer such that $\lfloor \frac{i-1}{2^k} \rfloor = \lfloor \frac{j-1}{2^k} \rfloor$. Alternatively, we can think of A as the sum of all matrices B^{ij} , where B^{ij} is a matrix such that $B_{kl}^{ij} = 1$ if $i \leq k, l \leq j$ and 0 otherwise. However, we sum this not over all $1 \leq i, j \leq n$ but for $i = 2^r \cdot (s-1) + 1, j = 2^r \cdot s$ for $0 \leq r \leq \lg n - 1$ and $1 \leq s \leq 2^{\lg n - r}$. As an illustrative example, for $n = 8$, A equals

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is easy to see that $\text{Tr}(A) = n \lg n$, since $A_{ii} = \lg n$ for all i . For any $1 \leq i < n$, define $i_0 = 0$ and for any $1 \leq r \leq \lg n$, define $i_r = 2^{\lg n - r} \cdot \lfloor \frac{i}{2^{\lg n - r}} \rfloor$. Then, for any $1 \leq i < n$, one can see that $i_{\lg n} = i$ and for any $1 \leq i \leq n$, $A = B^{1i_1} + B^{(i_1+1)i_2} + \dots + B^{(i_{\lg n-1}+1)i_{\lg n}} + C$, where C is some positive semidefinite matrix and we assume B^{ij} is the 0 matrix if $i = j + 1$, because B^{1i_1} and $B^{(i_{r-1}+1)i_r}$ for all $1 \leq r \leq \lg n$ are verifiable as matrices in the summation of A . Therefore, if $x_1 + \dots + x_i = 1$,

$$\begin{aligned} x^T A x &\geq \sum_{i=1}^r x^T B^{(i_{r-1}+1)i_r} x \\ &= (x_1 + \dots + x_{i_1})^2 + (x_{i_1+1} + \dots + x_{i_2})^2 + \dots + (x_{i_{\lg n-1}+1} + \dots + x_{i_{\lg n}})^2 \\ &\geq \frac{1}{\lg n}, \end{aligned}$$

since $(x_1 + \dots + x_{i_1}) + (x_{i_1+1} + \dots + x_{i_2}) + \dots + (x_{i_{\lg n-1}+1} + \dots + x_{i_{\lg n}}) = 1$ and by Cauchy-Schwarz.

63:12 Pairwise Independent Random Walks Can Be Slightly Unbounded

Finally, if $i = n$, then $A = B^{1(n/2)} + B^{(n/2+1)n} + C$, where C is some positive semidefinite matrix. Therefore, if $x_1 + \dots + x_n = 1$,

$$\begin{aligned} x^T A x &\geq x^T B^{1(n/2)} x + x^T B^{(n/2+1)n} x \\ &= (x_1 + \dots + x_{n/2})^2 + (x_{n/2+1} + \dots + x_n)^2 \geq \frac{1}{2} \geq \frac{1}{\lg n}. \end{aligned} \quad \blacktriangleleft$$

As a final note, for any positive definite matrix A and vector v , the minimum value of $w^T A w$ over all w such that $w^T v = 1$ is known to equal $(v^T A^{-1} v)^{-1}$. This can be checked with Lagrange Multipliers, since the Lagrangian $f(w, \lambda)$ of $f(w) = w^T A w$ subject to $w^T v = 1$ equals $w^T A w - \lambda(w^T v - 1)$, which is a convex function in w and has its derivatives vanish on the hyperplane $w^T v = 1$ when $\lambda = 2(v^T A^{-1} v)^{-1}$, $w = \frac{\lambda}{2}(A^{-1} v)$ (See for example [7], Chapter 5, for more details of Lagrange Multipliers). By Lemma 6 and Theorem 1, we have the following corollary:

► **Corollary 8.** *For all positive definite A , if we define v^i as the vector with first i components 1 and last $n - i$ components 0,*

$$\text{Tr}(A) \cdot \max_{1 \leq i \leq n} (v^i A^{-1} v^i) = \Omega(n \lg^2 n)$$

and this bound is tight for the matrix of Lemma 7.

Proof. If the first part were not true, then there would be matrices A_n such that $\text{Tr}(A) = d_1$, $w^T A w = \frac{1}{d_2}$ where $w^T v^i = 0$ for some i , and $d_1 d_2 = o(n \lg^2 n)$. However, this would mean by Lemma 6 that for all pairwise independent \mathcal{H} ,

$$\mathbb{E}_{h \in \mathcal{H}} \left(\sup_{1 \leq i \leq n} (h_1 + \dots + h_i)^2 \right) \leq d_1 d_2 = o(n \lg^2 n),$$

contradicting Theorem 1. The second part is immediate by the analysis of Lemma 7. ◀

4 Generalized Upper Bounds

In this section, our goal is to prove Equation (3) of Section 1.1.

4.1 Proof of Equation 3

In this subsection, we prove a generalization of Kolmogorov's inequality [16] by proving an identical result even if we only know that our random variables X_1, \dots, X_n are 4-wise independent.

► **Theorem 9.** *Suppose that X_1, \dots, X_n are 4-wise independent random variables satisfying $\mathbb{E}[X_i] = 0$ and $\text{Var}(X_i) < \infty$ for all i . Then, for all $\lambda > 0$,*

$$\mathbb{P} \left(\sup_{1 \leq i \leq n} (X_1 + \dots + X_i) \geq \lambda \right) \leq \frac{\sum \mathbb{E}[X_i^2]}{\lambda^2}.$$

Proof. Assume WLOG that $\lambda \geq 1$, $\sum \mathbb{E}[X_i^2] = 1$, and $\mathbb{E}[X_i^2] > 0$ for all i , i.e. none of the variables are almost surely 0. Also, define $S_i = X_1 + \dots + X_i$ and $T_i = \mathbb{E}[X_1^2 + \dots + X_i^2]$ for $0 \leq i \leq n$. Note that $T_0 = 0$ and $T_n = 1$.

We proceed by constructing a series of nested intervals $[a_{r,s}, b_{r,s}]$ and our analysis will be similar to that of Lemma 18 in [5]. We construct $a_{r,s}$ and $b_{r,s}$ for $0 \leq r \leq d = \Theta(\max_i \lg(\mathbb{E}[X_i^2]^{-1}))$ and $1 \leq s \leq 2^r$, as integers between 0 and n , inclusive. First define $a_{0,1} = 0$ and $b_{0,1} = n$. Next, we inductively define $a_{r,s}, b_{r,s}$. Define $a_{r+1,2s-1} := a_{r,s}$ and $b_{r+1,2s} := b_{r,s}$. Then, if there exists any index $a_{r,s} \leq t \leq b_{r,s}$ such that

$$0.45 \cdot |T_{b_{r,s}} - T_{a_{r,s}}| \leq |T_t - T_{a_{r,s}}| \leq 0.55 \cdot |T_{b_{r,s}} - T_{a_{r,s}}|,$$

let $a_{r+1,2s} = b_{r+1,2s-1} = t$ (if there are multiple such indices t , choose any one). Else, define $b_{r+1,2s-1}$ to be the largest index $t \geq a_{r,s}$ such that

$$|T_t - T_{a_{r,s}}| \leq 0.45 \cdot |T_{b_{r,s}} - T_{a_{r,s}}|$$

and similarly define $a_{r+1,2s}$ to be the smallest index $t \leq b_{r,s}$ such that

$$|T_t - T_{a_{r,s}}| \geq 0.55 \cdot |T_{b_{r,s}} - T_{a_{r,s}}|.$$

Note that in this case, $a_{r,2s} = b_{r,2s-1} + 1$.

It is clear that intervals are all nested in each other and for every r , all integers between 0 and n are in an interval $[a_{r,s}, b_{r,s}]$ for some s (possibly at an endpoint). Also, we always have $a_{r,0} \leq b_{r,0} \leq a_{r,1} \leq \dots \leq b_{r,2^r}$, and any interval $[a_{r,s}, b_{r,s}]$ satisfies $T_{b_{r,s}} - T_{a_{r,s}} \leq 0.55^r$. The previous point implies that since $d = \Theta(\max_i (\lg \mathbb{E}[X_i^2]^{-1}))$, every integer equals $a_{d,s} = b_{d,s}$ for some s .

We now call an interval $[a_{r,s}, b_{r,s}]$ *bad* if either s is odd and $b_{r,s} \neq a_{r,s+1}$ or s is even and $a_{r,s} \neq b_{r,s-1}$. Define the *rank* $q_{r,s}$ of a bad interval as the number of distinct $r' \leq r$ such that $[a_{r,s}, b_{r,s}] \subseteq [a_{r',s'}, b_{r',s'}]$ for some bad interval $[a_{r',s'}, b_{r',s'}]$, which may equal $[a_{r,s}, b_{r,s}]$. Define the *relative rank* of a bad interval with respect to some interval $[a, b]$ as the number of distinct $r' \leq r$ such that $[a_{r,s}, b_{r,s}] \subseteq [a_{r',s'}, b_{r',s'}] \subsetneq [a, b]$ for some bad interval $[a_{r',s'}, b_{r',s'}]$. Note that $[a_{r,2s-1}, b_{r,2s-1}]$ and $[a_{r,2s}, b_{r,2s}]$ are either both bad or both *good*, i.e. not bad. We now show the following:

► **Lemma 10.** *Given distinct bad intervals $[a_{r_i,s_i}, b_{r_i,s_i}]$ for $1 \leq i \leq \ell$ all contained in some interval $[a_{r,s}, b_{r,s}]$, where each interval has relative rank exactly q with respect to $[a_{r,s}, b_{r,s}]$,*

$$\sum_{i=1}^{\ell} (T_{b_{r_i,s_i}} - T_{a_{r_i,s_i}}) \leq 0.9^q \cdot (T_{a_{r,s}} - T_{b_{r,s}}).$$

As an immediate consequence, given distinct bad intervals $[a_{r_i,s_i}, b_{r_i,s_i}]$ with absolute rank q ,

$$\sum_{i=1}^{\ell} (T_{b_{r_i,s_i}} - T_{a_{r_i,s_i}}) \leq 0.9^q.$$

Proof. First, note that the bad intervals cannot overlap, except at endpoints, as the only way for such intervals to overlap is for one to be contained in another, which would mean they have different ranks. Now, we prove this by induction on $b_{r,s} - a_{r,s}$. If $b_{r,s} - a_{r,s} = 1$, then for any value of q , this is quite straightforward, since there cannot exist bad intervals of nonzero length with positive relative rank. Now, given $b_{r,s} - a_{r,s} > 1$, then $a_{r,s} = a_{r+1,2s-1} \leq b_{r+1,2s-1} \leq a_{r+1,2s} \leq b_{r+1,2s} = b_{r,s}$, and at least one of the two outer inequalities must be strict. If $b_{r+1,2s-1} = a_{r+1,2s}$, then neither $[a_{r+1,2s-1}, b_{r+1,2s-1}]$ nor $[a_{r+1,2s}, b_{r+1,2s}]$ are bad intervals. We can separately look at intervals which are subintervals of $[a_{r+1,2s-1}, b_{r+1,2s-1}]$ or $[a_{r+1,2s}, b_{r+1,2s}]$ to see which ones have rank q . By induction on $b_{r,s} - a_{r,s}$, the total length of the subintervals of relative rank q is at most

$$0.9^q \cdot (T_{b_{r+1,2s-1}} - T_{a_{r+1,2s-1}}) + 0.9^q \cdot (T_{b_{r+1,2s}} - T_{a_{r+1,2s}}) = 0.9^q \cdot (T_{b_{r,s}} - T_{a_{r,s}}).$$

63:14 Pairwise Independent Random Walks Can Be Slightly Unbounded

If $b_{r+1,2s-1} \neq a_{r+1,2s}$, then if $q = 1$, we can only choose the subintervals $[a_{r+1,2s-1}, b_{r+1,2s-1}]$ and $[a_{r+1,2s}, b_{r+1,2s}]$, and clearly

$$(T_{b_{r+1,2s-1}} - T_{a_{r+1,2s-1}}) + (T_{b_{r+1,2s}} - T_{a_{r+1,2s}}) \leq (0.45 + 0.45) \cdot (T_{b_{r,s}} - T_{a_{r,s}}) = 0.9 \cdot (T_{b_{r,s}} - T_{a_{r,s}}).$$

If $q > 1$, we can separately look at intervals which are subintervals of $[a_{r+1,2s-1}, b_{r+1,2s-1}]$ and $[a_{r+1,2s}, b_{r+1,2s}]$ to see which ones have relative rank $q - 1$, where we have to subtract one from the rank since $[a_{r+1,2s-1}, b_{r+1,2s-1}]$ and $[a_{r+1,2s}, b_{r+1,2s}]$ are both bad. Then, the total length of the subintervals of relative rank q is at most

$$\begin{aligned} & 0.9^{q-1} \cdot (T_{b_{r+1,2s-1}} - T_{a_{r+1,2s-1}}) + 0.9^{q-1} \cdot (T_{b_{r+1,2s}} - T_{a_{r+1,2s}}) \\ & \leq 0.9^{q-1} \cdot (0.45 + 0.45) \cdot (T_{b_{r,s}} - T_{a_{r,s}}) = 0.9^q (T_{b_{r,s}} - T_{a_{r,s}}). \quad \blacktriangleleft \end{aligned}$$

Next, for any λ , we bound the probability that there exists either a bad interval $[a_{r,s}, b_{r,s}]$ with rank q such that $|S_{b_{r,s}} - S_{a_{r,s}}| \geq 0.99^q \cdot \lambda$ or good intervals $[a_{r,2s-1}, b_{r,2s-1}]$, $[a_{r,2s}, b_{r,2s}]$ such that $\min(|S_{b_{r,2s-1}} - S_{a_{r,2s-1}}|, |S_{b_{r,2s}} - S_{a_{r,2s}}|) \geq 0.99^r \cdot \lambda$. Note that by the Chebyshev inequality,

$$\mathbb{P}(|S_{b_{r,s}} - S_{a_{r,s}}| \geq 0.99^q \cdot \lambda) \leq \frac{T_{b_{r,s}} - T_{a_{r,s}}}{0.99^{2q} \cdot \lambda^2},$$

since $\mathbb{E}[(S_{b_{r,s}} - S_{a_{r,s}})^2] = T_{b_{r,s}} - T_{a_{r,s}}$ by pairwise independence. Therefore, the probability of us having this for any bad interval is at most

$$\sum_{q=1}^{\infty} \sum_{\substack{\text{bad interval} \\ \text{rank } q}} \frac{T_{b_{r,s}} - T_{a_{r,s}}}{0.99^{2q} \cdot \lambda^2} \leq \sum_{q=1}^{\infty} \frac{0.9^q}{0.99^{2q} \cdot \lambda^2} = O(\lambda^{-2}).$$

Next, note that for any good intervals $[a_{r,2s-1}, b_{r,2s-1}]$ and $[a_{r,2s}, b_{r,2s}]$, we have that

$$\begin{aligned} & \mathbb{P}(\min(|S_{b_{r,2s-1}} - S_{a_{r,2s-1}}|, |S_{b_{r,2s}} - S_{a_{r,2s}}|) \geq \lambda \cdot 0.99^r) \\ & \leq \mathbb{P}((S_{b_{r,2s-1}} - S_{a_{r,2s-1}})^2 (S_{b_{r,2s}} - S_{a_{r,2s}})^2 \geq \lambda^4 \cdot 0.99^{4r}) \\ & \leq \frac{\mathbb{E}[(S_{b_{r,2s-1}} - S_{a_{r,2s-1}})^2 (S_{b_{r,2s}} - S_{a_{r,2s}})^2]}{\lambda^4 \cdot 0.99^{4r}} \\ & \leq \frac{(T_{b_{r,2s-1}} - T_{a_{r,2s-1}})(T_{b_{r,2s}} - T_{a_{r,2s}})}{\lambda^4 \cdot 0.99^{4r}} \leq \frac{0.55^{2r}}{\lambda^4 \cdot 0.99^{4r}} \end{aligned}$$

using 4-wise independence of X_1, \dots, X_n . Since there are at most 2^r such pairs of good intervals for any r , the probability of $|S_{b_{r,2s-1}} - S_{a_{r,2s-1}}|, |S_{b_{r,2s}} - S_{a_{r,2s}}|$ both being greater than $\lambda \cdot 0.99^r$ for any pair of good intervals, is at most

$$\sum_{r=1}^{\infty} 2^r \cdot \frac{0.55^{2r}}{\lambda^4 \cdot 0.99^{4r}} = O(\lambda^{-4}).$$

Finally, the probability of $|S_n - S_0| = |S_{b_{0,1}} - S_{a_{0,1}}| > \lambda$ is at most $\frac{\mathbb{E}[S_n^2]}{\lambda^2} = O(\lambda^{-2})$.

These imply the following result:

► **Lemma 11.** *The probability of there existing a bad interval $[a_{r,s}, b_{r,s}]$ with rank q such that $|S_{b_{r,s}} - S_{a_{r,s}}| \geq 0.99^q \cdot \lambda$, or good intervals $[a_{r,2s-1}, b_{r,2s-1}]$ and $[a_{r,2s}, b_{r,2s}]$ such that $|S_{b_{r,2s-1}} - S_{a_{r,2s-1}}|, |S_{b_{r,2s}} - S_{a_{r,2s}}|$ are both greater than $\lambda \cdot 0.99^r$, or of $|S_n - S_0| \geq \lambda$ is $O(\lambda^{-2})$.*

Next, we prove the following:

► **Lemma 12.** *For any $0 \leq i \leq n$, there exists a sequence $0 \leq i_0, i_1, i_2, \dots, i_d \leq n$ with $i_0 = 0, i_d = i$, and a sequence of nested intervals $[a_{0,s_0}, b_{0,s_0}] \supset \dots \supset [a_{d,s_d}, b_{d,s_d}]$ such that for any $1 \leq j \leq d-1$, i_j is an endpoint of the interval $[a_{j,s_j}, b_{j,s_j}]$ and of the interval $[a_{j-1,s_{j-1}}, b_{j-1,s_{j-1}}]$. Furthermore, for any $1 \leq j \leq d$, either $i_{j-1} = i_j$, or $[a_{j,s_j}, b_{j,s_j}]$ is a bad interval, or i_j equals $a_{j,2s} = b_{j,2s-1}$ and i_{j-1} is either $a_{j,2s-1}$ or $b_{j,2s}$ such that $|S_{i_j} - S_{i_{j-1}}| = \min(|S_{b_{j,2s}} - S_{a_{j,2s}}|, |S_{a_{j,2s-1}} - S_{a_{j,2s-1}}|)$. The intervals and values i_0, \dots, i_d may depend on the actual values of X_1, \dots, X_n .*

Proof. We know that $i = i_d$ equals $a_{d,s_d} = b_{d,s_d}$ for some s_d , and thus must also equal either $a_{d-1,s_{d-1}}$ or $b_{d-1,s_{d-1}}$ for some s_{d-1} . If we are given i_{j+1} for some $1 \leq j < d$, if i_{j+1} equals $a_{j,2s-1}$ or $b_{j,2s}$ for some s , then let $i_j = i_{j+1}$ which equals $a_{j-1,s}$ or $b_{j-1,s}$, respectively. If i_j equals $a_{j,2s}$ or $b_{j,2s-1}$ for some s , then if $a_{j,2s} = b_{j,2s-1}$, we can choose i_{j-1} accordingly as either $a_{j,2s-1} = a_{j-1,s}$ or $b_{j,2s} = b_{j-1,s}$ based on whether $|S_{b_{j,2s-1}} - S_{a_{j,2s-1}}|$ or $|S_{b_{j,2s}} - S_{a_{j,2s}}|$ is smaller. If $a_{j,2s} \neq b_{j,2s-1}$, then if $i_j = a_{j,2s}$ we choose $i_{j-1} = a_{j-1,s}$ and if $i_j = b_{j,2s-1}$ then we choose $i_{j-1} = b_{j-1,s}$. ◀

As a result, we have that if the conditions of Lemma 11 do not hold, which happens with probability $1 - O(\lambda^{-2})$, then for any i , then every $|S_i|$ satisfies

$$|S_i| \leq \sum_{j=1}^d |S_{i_j} - S_{i_{j-1}}| \leq \lambda + \sum_{q=1}^{\infty} 0.99^q \cdot \lambda + \sum_{r=1}^{\infty} 0.99^r \cdot \lambda = O(\lambda),$$

where I am using the fact that the intervals $[a_{j,s_j}, b_{j,s_j}]$ are nested in each other, so no two bad intervals can have the same rank.

In summary, we have with probability at least $1 - O(\lambda^{-2})$, the supremum of $|S_i| = |X_1 + \dots + X_i|$ over all i doesn't exceed $O(\lambda)$, so we have proven Theorem 9. ◀

5 Open Problems

We note a few further directions that could be taken after this work. The first main open problem is whether 3-wise independent random walks on n steps have supremum distance bounded by $O(\sqrt{n})$ in expectation. As all 3-wise independent random walks are also 2-wise independent, we know that the supremum distance is bounded by $O(\sqrt{n} \log n)$ in expectation and the supremum square distance is bounded by $O(n \log^2 n)$ in expectation. However, we do not know if better bounds are true for 3-wise independent random walks. Likewise, for odd $k \geq 5$, we could ask if the k th moment of the supremum distance is bounded by $O(n^{k/2})$ in expectation, as we only know that the $(k-1)$ th moment of the supremum distance is bounded by $O(n^{(k-1)/2})$ in expectation by Equation (5), since $k-1$ is even and at least 4.

Finally, we could ask how the constants increase as k grows. By our proof of Equation (5) and the constants in Khintchine's inequality [15], we know that the k th moment of the supremum distance of a random walk is $O(k)^{3k/2} \cdot n^{k/2}$. Therefore, we could ask if the constant $O(k)^{3k/2}$ could be improved if k grows with respect to n (such as if $k = \Theta(\log n)$).

References

- 1 Noga Alon, Yossi Matias, and Mario Szegedy. The Space Complexity of Approximating the Frequency Moments. *J. Comput. Syst. Sci.*, 58(1):137–147, 1999. doi:10.1006/jcss.1997.1545.
- 2 Christophe Andrieu, Nando de Freitas, Arnaud Doucet, and Michael I. Jordan. An Introduction to MCMC for Machine Learning. *Machine Learning*, 50(1):5–43, January 2003. doi:10.1023/A:1020281327116.

63:16 Pairwise Independent Random Walks Can Be Slightly Unbounded

- 3 Itai Benjamini, Gady Kozma, and Dan Romik. Random walks with k -wise independent increments. *Electron. Commun. Probab.*, 11:100–107, 2006. doi:10.1214/ECP.v11-1201.
- 4 Howard Berg. *Random Walks in Biology*. Princeton University Press, 1993.
- 5 Jarosław Błasiok. Optimal streaming and tracking distinct elements with high probability. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 2432–2448, 2018. doi:10.1137/1.9781611975031.156.
- 6 Jarosław Błasiok, Jian Ding, and Jelani Nelson. Continuous Monitoring of ℓ_p Norms in Data Streams. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2017, August 16-18, 2017, Berkeley, CA, USA*, pages 32:1–32:13, 2017. doi:10.4230/LIPIcs.APPROX-RANDOM.2017.32.
- 7 Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.
- 8 Vladimir Braverman, Stephen R. Chestnut, Nikita Ivkin, Jelani Nelson, Zhengyu Wang, and David P. Woodruff. BPTree: An ℓ_2 heavy hitters algorithm using constant memory. In *Proceedings of the 36th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS 2017, Chicago, IL, USA, May 14-19, 2017*, pages 361–376, 2017. doi:10.1145/3034786.3034798.
- 9 Vladimir Braverman, Stephen R. Chestnut, Nikita Ivkin, and David P. Woodruff. Beating CountSketch for heavy hitters in insertion streams. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 740–753, 2016. doi:10.1145/2897518.2897558.
- 10 Larry Carter and Mark N. Wegman. Universal Classes of Hash Functions. *J. Comput. Syst. Sci.*, 18(2):143–154, 1979. doi:10.1016/0022-0000(79)90044-8.
- 11 R. M. Dudley. The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *Journal of Functional Analysis*, 1(3):290–330, 1967. doi:10.1016/0022-1236(67)90017-1.
- 12 M. P. Etienne and P. Vallois. Approximation of the Distribution of the Supremum of a Centered Random Walk. Application to the Local Score. *Methodology And Computing In Applied Probability*, 6(3):255–275, September 2004. doi:10.1023/B:MCAP.0000026559.87023.ec.
- 13 Eugene F. Fama. Random Walks in Stock Market Prices. *Financial Analysts Journal*, 21(5):55–59, 1965. URL: <http://www.jstor.org/stable/4469865>.
- 14 Pierre-Gilles Gennes. *Scaling Concepts in Polymer Physics*. Cornell University Press, 1 edition, November 1979.
- 15 Uffe Haagerup. The best constants in the Khintchine inequality. *Studia Mathematica*, 70(3):231–283, 1981. URL: <http://eudml.org/doc/218383>.
- 16 A. Kolmogoroff. Über die summen durch den zufall bestimmter unabhängiger größen. *Mathematische Annalen*, 99:309–319, 1928. URL: <http://eudml.org/doc/159258>.
- 17 László Lovász. *Random Walks on Graphs: A Survey*, 1993.
- 18 Jelani Nelson. Chaining introduction with some computer science applications. *Bulletin of the EATCS*, 120, 2016. URL: <http://eatcs.org/beatcs/index.php/beatcs/article/view/450>.
- 19 Anna Pagh, Rasmus Pagh, and Milan Ružić. Linear Probing with Constant Independence. *SIAM J. Comput.*, 39(3):1107–1120, 2009. doi:10.1137/070702278.
- 20 Mihai Pătraşcu and Mikkel Thorup. On the k -Independence Required by Linear Probing and Minwise Independence. *ACM Trans. Algorithms*, 12(1):8:1–8:27, 2016. doi:10.1145/2716317.
- 21 Yao-Feng Ren and Han-Ying Liang. On the best constant in Marcinkiewicz–Zygmund inequality. *Statistics & Probability Letters*, 53(3):227–233, 2001. doi:10.1016/S0167-7152(01)00015-3.
- 22 Mathias Knudsen (via Jelani Nelson). Personal communication.

A Generalized Upper Bounds: Proof of Equations 4 and 5

Before we prove Equations (4) and (5), we construct $2^{-r/2}$ -nets for $0 \leq r \leq 2 \lg m + 1$ in a very similar way as in Theorem 1 in [8]. We define an ε -net to be a finite set of points $a_{r,0}, a_{r,1}, \dots, a_{r,d_r}$ such that for every $z^{(t)}$, $\|z^{(t)} - a_{r,s}\|_2 \leq \varepsilon \|z\|_2$ for some $0 \leq s \leq d_r$. The constructions are defined identically for both equations. Define $a_{0,0} := z^{(0)}$ as the only element of the $2^{-0/2} = 1$ -net. For $r \geq 1$, define $a_{r,0} = z^{(0)}$, and given $a_{r,s} = z^{(t_1)}$ then define $a_{r,s+1}$ as the smallest $t > t_1$ such that

$$\|z^{(t)} - z^{(t_1)}\|_2 > 2^{-r/2} \cdot \|z\|_2^2,$$

unless such a t does not exist, in which case let $s = d_r$ and do not define $a_{r,s'}$ for any $s' > s$.

We define the set $A_r = \{a_{r,s} : 0 \leq s \leq d_r\}$. The following is directly true from our construction:

► **Proposition 13.** *For any $0 \leq t \leq m$ and fixed r , if $t_1 \leq t$ is the largest t_1 such that $z^{(t_1)} = a_{r,s}$ for some s , then $\|z^{(t)} - z^{(t_1)}\|_2 \leq 2^{-r/2} \cdot \|z\|_2$. Consequently, $A_r = \{a_{r,0}, \dots, a_{r,d_r}\}$ is a $2^{-r/2}$ -net.*

The above proposition implies the following:

► **Proposition 14.** *For all $1 \leq t \leq m$, $z^{(t)} = a_{2 \lg m + 1, s}$ for some s .*

Proof. Let t_1 be the largest integer at most t such that $z^{(t_1)} = a_{2 \lg m + 1, s}$ for some s . Then, $\|z^{(t)} - a_{2 \lg m + 1, s}\|_2^2 \leq 2^{-(2 \lg m + 1)} \cdot \|z\|_2^2 < 1$, which is clearly impossible unless $z^{(t)} = a_{2 \lg m + 1, s}$. ◀

Next, to prove Equations (4) and (5), we will need the Marcinkiewicz–Zygmund inequality (see for example [21]), which is a generalization of Khintchine’s inequality (see for example [15]):

► **Theorem 15.** *For any even $k \geq 2$, there exists a constant B_k only depending on k such that for any fixed vector v and totally independent random variables $\bar{Y} = (Y_1, \dots, Y_n)$,*

$$\mathbb{E} \left[\left(\sum_{i=1}^n Y_i \right)^k \right] \leq B_k \mathbb{E} \left[\left(\sum_{i=1}^n Y_i^2 \right)^{k/2} \right].$$

This implies the following result:

► **Proposition 16.** *For any $k \geq 2$ and vector v , there exists a B_k only dependent on k such that*

$$\mathbb{E} \left[\langle v, \bar{X} \rangle^k \right] = \mathbb{E} \left[\left(\sum_{i=1}^n v_i X_i \right)^k \right] \leq B_k \|v\|_2^k.$$

Proof. Since the expected value of $(\sum v_i X_i)^k$ is only dependent on k -wise independence, we can assume that the X_i ’s are totally independent but have the same marginal distribution. This implies

$$\mathbb{E} \left[\left(\sum_{i=1}^n v_i X_i \right)^k \right] \leq B_k \mathbb{E} \left[\left(\sum_{i=1}^n v_i^2 X_i^2 \right)^{k/2} \right]$$

by Theorem 15. However, we know that $\mathbb{E}[X_i^{2d}] \leq 1$ for all i and all $1 \leq d \leq k/2$, since $\mathbb{E}[X_i^k] \leq 1$ and $\mathbb{E}[X_i^{2d}]^{k/d} \leq \mathbb{E}[X_i^k]$ by Jensen’s inequality, so simply expanding and using independence and linearity of expectation gets us the desired result. ◀

63:18 Pairwise Independent Random Walks Can Be Slightly Unbounded

We now prove equations (4) and (5).

Proof of Equation (4). For $r \geq 1$ and s , suppose $a_{r,s} = z^{(t)}$ and $t_1 \leq t$ is the largest index such that $z^{(t_1)} \in A_{r-1}$. Then, define $f(s, t)$ to be the index s' such that $z^{(t_1)} = a_{r-1,s'}$. Consider the quadratic form

$$\sum_{r=1}^{2 \lg m + 1} \sum_{s=0}^{d_r} \langle (a_{r,s} - a_{r-1,f(r,s)}), \vec{X} \rangle^2.$$

By Proposition 13, $\|a_{r,s} - a_{r-1,f(r,s)}\|_2 \leq 2^{-(r-1)/2} \cdot \|z\|_2$. Thus, by Proposition 16, we get the expected value of the quadratic form equals

$$\begin{aligned} \sum_{r=1}^{2 \lg m + 1} \sum_{s=0}^{d_r} \mathbb{E}[\langle (a_{r,2s+1} - a_{r,2s}), \vec{X} \rangle^2] &\leq B_2 \sum_{r=1}^{2 \lg m + 1} \sum_{s=0}^{d_r} \|a_{r,2s+1} - a_{r,2s}\|_2^2 \\ &\leq B_2 \sum_{r=1}^{2 \lg m + 1} \left(2^r \cdot 2^{-(r-1)} \|z\|_2^2 \right) \leq 2B_2(2 \lg m + 1)(\|z\|_2^2). \end{aligned}$$

Here, I am using the fact that an ε -net has size at most ε^{-2} , which is easy to see since $z^{(0)}, \dots, z^{(m)}$ is tracking an insertion stream (it is proven, for example, in Theorem 1 of [8]), and thus $d_r \leq 2^r$.

Now, for any $0 \leq i \leq n$, consider $z^{(i)}$ and let $z^{(i)} = a_{2 \lg m + 1, s}$. Then, define $s_r = s$ if $r = 2 \lg m + 1$ and $s_{r-1} = f(r, s_r)$ for $1 \leq r \leq 2 \lg m + 1$. Note that $s_0 = 0$ and for any $r \geq 1$, if $a_{r,s_r} \in A_{r-1}$, then $a_{r,s_r} = a_{r-1,s_{r-1}}$. Thus, each $\langle (a_{r,s_r} - a_{r-1,s_{r-1}}), \vec{X} \rangle^2$ for $1 \leq r \leq 2 \lg m + 1$ is either 0 (because $a_{r,s_r} - a_{r-1,s_{r-1}} = 0$) or is a summand in our quadratic form. Therefore,

$$\sum_{r=1}^{2 \lg m + 1} \sum_{s=0}^{d_r} \langle (a_{r,s} - a_{r,f(r,s)}), \vec{X} \rangle^2 \geq \sum_{r=1}^{2 \lg m + 1} \langle (a_{r,s_r} - a_{r-1,s_{r-1}}), \vec{X} \rangle^2 \geq \frac{1}{2 \lg m + 1} \cdot \langle z^{(i)}, \vec{X} \rangle^2,$$

with the last inequality true since $a_{2 \lg m + 1, s_{2 \lg m + 1}} = z^{(i)}$, $a_{0,s_0} = z^{(0)}$, and by the Cauchy-Schwarz inequality. As this is true for all i , taking the supremum over i and then expected values gives us

$$\begin{aligned} 2B_2(2 \lg m + 1)(\|z\|_2^2) &\geq \mathbb{E} \left[\sum_{r=1}^{2 \lg m + 1} \sum_{s=0}^{d_r} \langle (a_{r,2s+1} - a_{r,2s}), \vec{X} \rangle^2 \right] \\ &\geq \frac{1}{2 \lg m + 1} \cdot \sup_i \mathbb{E} \left[\langle z^{(i)}, \vec{X} \rangle^2 \right], \end{aligned}$$

and therefore,

$$\mathbb{E} \left[\langle z^{(i)}, \vec{X} \rangle^2 \right] = O(\|z\|_2^2 \cdot \lg^2 m). \quad \blacktriangleleft$$

Proof of Equation (5). Consider the form

$$\sum_{r=1}^{2 \lg m + 1} 2^{r/2} \sum_{s=0}^{d_r} \langle (a_{r,s} - a_{r-1,f(r,s)}), \vec{X} \rangle^k,$$

with $f(r, s)$ defined as in the proof of Equation (4). We again note that by Proposition 13, $\|a_{r,s} - a_{r-1,f(r,s)}\|_2 \leq 2^{-(r-1)/2} \cdot \|z\|_2$. Thus, by Proposition 16, we get the expected value of the form equals

$$\sum_{r=1}^{2 \lg m + 1} 2^{r/2} \sum_{s=0}^{d_r} \mathbb{E}[\langle (a_{r,s} - a_{r-1,f(r,s)}), \vec{X} \rangle^k] \leq B_k \sum_{r=1}^{2 \lg m + 1} 2^{r/2} \sum_{s=0}^{d_r} \|a_{r,s} - a_{r-1,f(r,s)}\|_2^k$$

$$\leq B_k \sum_{r=0}^{2 \lg m+1} 2^{r/2} \cdot 2^r \cdot 2^{-(r-1)k/2} \|z\|_2^k \leq B_k \cdot 2^{k/2} \sum_{r=0}^{\infty} 2^{r(3-k)/2} \|z\|_2^k = O(2^{k/2} B_k) \|z\|_2^k,$$

since $k \geq 4$. Again, I am using the fact that $d_r \leq 2^r$ as an ε -net has size at most ε^{-2} .

Now, for any $0 \leq i \leq n$, suppose s satisfies $z^{(i)} = a_{2 \lg m+1, s}$. Define $s_r = s$ if $r = 2 \lg m + 1$ and $s_{r-1} = f(r, s_r)$ for $1 \leq r \leq 2 \lg m + 1$. Then, similarly to in the proof of Equation (4),

$$\begin{aligned} \sum_{r=1}^{2 \lg m+1} 2^{r/2} \sum_{s=0}^{d_r} \langle (a_{r,s} - a_{r-1, f(r,s)}), \vec{X} \rangle^k &\geq \sum_{r=1}^{2 \lg m+1} 2^{r/2} \langle (a_{r,s_r} - a_{r-1, s_{r-1}}), \vec{X} \rangle^k \\ &\geq \Omega(k^{-1})^k \cdot \langle z^{(i)}, \vec{X} \rangle^k. \end{aligned}$$

The last inequality requires justification, specifically that if $x_1 + \dots + x_{2 \lg m+1} = 1$, $\sum 2^{r/2} x_r^k = \Omega(k^{-1})^k$. This is sufficient since we can let $x_r = \langle (a_{r,s_r} - a_{r-1, s_{r-1}}), \vec{X} \rangle$. To prove this, define $x'_1, x'_2, \dots, x'_{2 \lg m+1}$ such that $x'_1 + \dots + x'_{2 \lg m+1} = 1$ and $x'_1 > \dots > x'_{2 \lg m+1}$ are in a geometric series with common ratio $2^{-1/(2k)} = 1 - \Theta(1/k)$. Then, note that for any $x_1, \dots, x_{2 \lg m+1}$ such that $x_1 + \dots + x_{2 \lg m+1} = 1$, $x_i \geq x'_i$ for some i . But note that $(x'_r)^k 2^{r/2}$ are equal for all r because of our geometric series, and equals $(x'_1)^k = \Omega(k^{-1})^k$ since $x'_1 = \Omega(k^{-1})^k$ is clearly true. Thus, $\sum 2^{r/2} x_r^k \geq 2^{i/2} (x'_i)^k = \Omega(k^{-1})^k$, so we are done.

As this is true for all i , we can take the supremum over i and then take expected values to get

$$2^{k/2} B_k \cdot \|z\|_2^k \geq \mathbb{E} \left[\sum_{r=1}^{2 \lg m+1} 2^{r/2} \sum_{s=0}^{d_r} \langle (a_{r,s} - a_{r-1, f(r,s)}), \vec{X} \rangle^k \right] \geq \Omega(k^{-1})^k \cdot \sup_i \mathbb{E} \left[\langle z^{(i)}, \vec{X} \rangle^k \right]$$

and therefore, for a fixed k ,

$$\mathbb{E} \sup_i \left[\langle z^{(i)}, \vec{X} \rangle^k \right] = O(\|z\|_2^k). \quad \blacktriangleleft$$