

# A Modal Logic for Subject-Oriented Spatial Reasoning

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## Abstract

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We present a modal logic for representing and reasoning about space seen from the subject’s perspective. The language of our logic comprises modal operators for the relations “in front”, “behind”, “to the left”, and “to the right” of the subject, which introduce the intrinsic frame of reference; and operators for “behind an object”, “between the subject and an object”, “to the left of an object”, and “to the right of an object”, employing the relative frame of reference. The language allows us to express nominals, hybrid operators, and a restricted form of distance operators which, as we demonstrate by example, makes the logic interesting for potential applications. We prove that the satisfiability problem in the logic is decidable and in particular PSPACE-complete.

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## 1 Introduction

Spatial reasoning is one of the most interesting abilities that humans possess and whose modeling is still a great challenge for AI. Research in this area is interesting from a theoretical point of view, for it may help us understand how people reason and use language, and from a practical perspective, as it may result in methods with a broad range of applications, e.g., in robotics [10] or geographical information systems [9].

There is a great variety of logical approaches to spatial reasoning which exploit the machinery of modal logics [32, 19, 21, 17, 5, 20, 22], relational algebras [24, 16, 28, 23], and first-order theories [8, 28, 27], among others. Various aspects of space are modeled by these systems, such as topology, directions, distance, orientation, size, shape, etc.; some use quantitative and other qualitative methods which, by the way, are often inspired by human-like reasoning. What is common for many of these approaches (especially for the ones based on modal logics) is that a *frame of reference* imposed on objects in order to describe and reason about their location is *absolute*. However, various studies show that users of Indo-European languages (e.g., English and Dutch) or Japanese mostly adopt the *intrinsic* frame of reference (also called *object-centered*, e.g., “something is to the left of me”) or the *relative* one (called *egocentric*, e.g., “something is to the left of a given object with respect to my location”), when performing both linguistic and non-linguistic tasks [14, 18, 15].

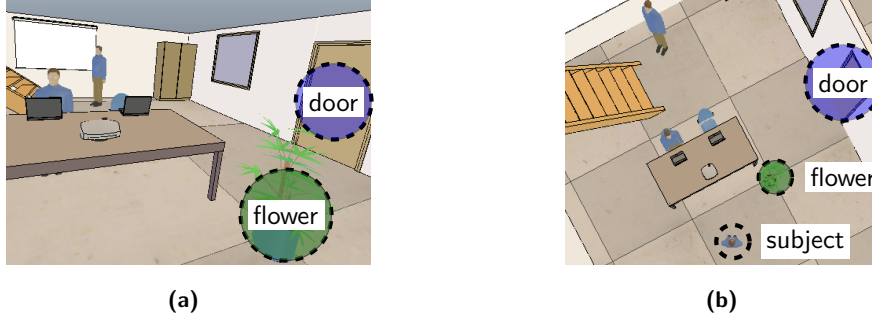
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## 4:2 Subject-Oriented Spatial Reasoning

To illustrate how space can be perceived from different points of view consider the spatial arrangement depicted in Figure 1. From the subject’s point of view “the flower is in front of me” (intrinsic frame of reference) and “the flower is between the door and myself” (relative frame of reference). On the other hand, from the aerial perspective “the flower is to the south of the door” (absolute frame of reference).



■ **Figure 1** A spatial configuration viewed from the subject’s (a) and aerial (b) perspectives.

A logical system modeling such a subject-oriented way of representing and reasoning about space could have a number of practical applications, e.g., in designing architectural objects which are required to be perceived by humans in a specific way, assisting blind persons, or giving directions to someone who is lost. In this paper we address this topic by constructing a modal logic for subject-oriented spatial reasoning (denoted by SOSL) which allows us to express spatial relations in both intrinsic and relative frames of reference, and which, to the best of our knowledge, is the first modal logic of this type.

To represent the two-dimensional space in a subject-oriented way, we use the polar coordinate system with a subject located at its center and with a spatial universe consisting of cells denoted by the polar coordinates of their central points, i.e., by a pair of a radius-coordinate, and a linear-coordinate (see Figure 2). To capture the intrinsic frame of reference we will introduce modal operators for the relations “in front of the subject” ( $\blacklozenge$ ), “behind the subject” ( $\blacklozenge$ ), “to the left of the subject” ( $\blacklozenge$ ), and “to the right of the subject” ( $\blacklozenge$ ), and to capture the relative frame of reference we will use operators expressing the relations “behind an object” ( $\blacklozenge$ ), “between an object and the subject” ( $\blacklozenge$ ), “to the left of an object” ( $\blacklozenge$ ), and “to the right of an object” ( $\blacklozenge$ ). Moreover, we will employ operators expressing the single-step relations between adjacent cells, namely “away from the subject” ( $\oplus$ ), “towards the subject” ( $\ominus$ ), “counter-clockwise” ( $\ominus$ ), and “clockwise” ( $\oplus$ ). As we will show in the paper, such a language is expressive enough to define the universal modality (A), where  $A\varphi$  states that  $\varphi$  holds in all cells, nominals, i.e., atoms which hold in exactly one cell, satisfaction operators of the form  $@_i$ , where  $i$  is a nominal, stating that  $\varphi$  is satisfied in the cell in which  $i$  holds, and operators expressing distance, e.g.,  $\blacklozenge_{=d}\varphi$  for “ $\varphi$  holds in a cell with a not-smaller radius-coordinate which is in distance  $d$ ” or  $\blacklozenge_{<d}\varphi$  for “ $\varphi$  holds in a cell with a not-smaller radius-coordinate which is closer than  $d$ ” etc. Then, in the intrinsic frame of reference the relation between the flower and the door from Figure 1 is described by the SOSL-formula  $\blacklozenge\text{flower}$ , and in the relative frame of reference by  $@_{\text{door}}\blacklozenge\text{flower}$ .

We will prove that the satisfiability problem of SOSL-formulas is decidable (in contrast to many two-dimensional modal logics [32, 21, 12]) and in particular PSPACE-complete (the same as for one-dimensional Linear Temporal Logic [11, 3]). To show the lower bound we reduce the satisfiability problem of Linear Temporal Logic, and for the upper bound we present a relatively complex (due to the fact that SOSL is a two-dimensional logic with 12 modal

operators) construction of a generalized nondeterministic Büchi automaton which accepts precisely those words which describe models of a given SOSL-formula. As a result, we obtain a two-dimensional and subject-oriented logic which is decidable and enables us to express and reason about complex spatial configurations in the intrinsic and relative frames of reference. In what follows, we present examples demonstrating the expressive power of the language and its potential applications.

**Modeling human cognition.** Modeling human-centered cognition is required in various applications, e.g., in Computer-Aided Architecture Design (CAAD) systems to address problems such as indoor spatial awareness, visibility analysis, or wayfinding [6, 7]. The subject-oriented representation of space makes SOSL an adequate tool for the above-mentioned applications. For the sake of example consider the following SOSL-formulas specifying how household goods should be located with respect to a person sitting on a sofa, where *sofa*, *tv*, and *window* are propositional variables and  $d \in \mathbb{R}$ :

$$\begin{aligned} & \text{sofa} \wedge \blacklozenge(\text{tv} \wedge \blacklozenge\text{window}) \wedge \blacklozenge_{\leq d}\text{lamp}; \\ & A(\text{tv} \vee \text{window} \rightarrow \neg\blacklozenge\text{lamp}) \wedge A(\text{tv} \rightarrow \neg\blacklozenge\text{window}). \end{aligned}$$

The first formula states that the configuration is described from the perspective of a person sitting on a *sofa*. There should be a *tv* standing in front of ( $\blacklozenge$ ) the *sofa* and a *window* located to the right of ( $\blacklozenge$ ) this *tv* with respect to the *sofa*. Furthermore, there should be a *lamp* located in a distance at most  $d$  ( $\blacklozenge_{\leq d}$ ) from the *sofa*. The second formula describes the following universal constraints: no *lamp* can stand between ( $\blacklozenge$ ) any *tv* and the *sofa* or between any *window* and the *sofa* (not to block the view); and no *window* can be located behind ( $\blacklozenge$ ) any *tv* (for a better visibility). To determine whether the specification is spatially possible one can check whether the conjunction of presented SOSL-formulas is satisfiable.

**Preprocessing spatial data.** Spatial data gathered by artificial visual systems (e.g., in robotics) is often represented in the polar coordinate system [31], hence SOSL can be used to reason about such data. Indeed, the following example shows how SOSL-formulas allow us to filter noisy data and detect borders of obstacles, where *obst* is a propositional variable satisfied in cells where an obstacle was detected:

$$\text{noise} \leftrightarrow \text{obst} \wedge (\neg\oplus\text{filt-obst} \wedge \neg\ominus\text{filt-obst} \wedge \neg\ominus\text{filt-obst} \wedge \neg\oplus\text{filt-obst}) \quad (1)$$

$$\text{filt-obst} \leftrightarrow \text{obst} \wedge \neg\text{noise} \quad (2)$$

$$\text{bord-obst} \leftrightarrow \text{filt-obst} \wedge (\oplus\neg\text{filt-obst} \vee \ominus\neg\text{filt-obst} \vee \ominus\neg\text{filt-obst} \vee \oplus\neg\text{filt-obst}) \quad (3)$$

The formula (1) marks as *noise* each cell recognized as an obstacle, for which there are no adjacent cells which are also recognized as obstacles. In (2), the input data is filtered by marking all non-noise cells which were recognized as obstacles with *filt-obst*. Then, (3) determines the borders of obstacles by marking with *bord-obst* the cells which are marked as *filt-obst* and have at least one adjacent cell which is not marked as *filt-obst*.

**Combining qualitative and quantitative reasoning.** The modal operators in SOSL allow us to express qualitative as well as quantitative relations, and use both to perform reasoning. Indeed, for any  $d, d1, d2 \in \mathbb{R}$ , any nominals *objA*, *objB*, *objC*, and *subject* being a nominal for the subject, the following formulas are tautologies in SOSL:

$$\text{subject} \wedge \blacklozenge_{=d}\text{objA} \wedge @_{\text{objA}}\blacklozenge\text{objB} \rightarrow \blacklozenge_{>d}\text{objB} \quad (4)$$

$$\text{subject} \wedge \blacklozenge_{=d}\text{objA} \wedge \blacklozenge_{=d}\text{objB} \rightarrow @_{\text{objA}}(\text{objB} \vee \blacklozenge\text{objB} \vee \blacklozenge\text{objB}) \quad (5)$$

$$@_{\text{objA}}\blacklozenge_{=d1}\text{objB} \wedge @_{\text{objB}}\blacklozenge_{=d2}\text{objC} \rightarrow @_{\text{objA}}\blacklozenge_{\leq d1+d2}\text{objC} \quad (6)$$

The formula (4) states that if  $\text{objB}$  is behind  $\text{objA}$ , then  $\text{objB}$  is further away from the subject than  $\text{objA}$ , (5) states that if  $\text{objA}$  and  $\text{objB}$  are situated in the same distance from the subject, then they occupy the same cell, or one of them is to the left of the other, and (6) expresses the (purely quantitative) triangle inequality theorem.

The remaining part of the paper is structured as follows. In Section 2 we define the syntax and semantics of SOSL, and in Section 3 we compare SOSL with other modal logics for spatial reasoning. Then, in Section 4, we determine the operators that are expressible in SOSL, and in Section 5 we prove PSPACE-completeness of the satisfiability problem in the logic. Finally, we conclude the paper in Section 6.

## 2 Syntax and Semantics

In this section we present formally the syntax and semantics of SOSL. The language of the logic includes the following elements:

- PROP – a countably infinite set of propositional variables;
- $\{\neg, \vee\}$  – a set of logical connectives;
- $\{\blacklozenge, \blacktriangleright, \blacktriangleleft, \blacktriangle, \blacklozenge, \blacktriangleright, \blacktriangleleft, \blacktriangle, \oplus, \ominus, \ominus, \oplus\}$  – a set of SOSL modal operators;
- $\{(, )\}$  – a set of parentheses.

► **Definition 1.** *The set of SOSL-formulas is generated by the following grammar:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \blacklozenge\varphi,$$

where  $p \in \text{PROP}$  and  $\blacklozenge$  is an SOSL modal operator.

The logical constants and other connectives are defined in a standard way:  $\top = p \vee \neg p$ ,  $\perp = \neg\top$ ,  $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$ ,  $\varphi \rightarrow \psi = \neg\varphi \vee \psi$ ,  $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ , where  $p \in \text{PROP}$  and  $\varphi, \psi$  are SOSL-formulas. Moreover, for  $\blacklozenge \in \{\blacklozenge, \blacktriangleright, \blacktriangleleft, \blacktriangle, \blacklozenge, \blacktriangleright, \blacktriangleleft, \blacktriangle\}$  we define a dual box operator  $\blacksquare\varphi = \neg\blacklozenge\neg\varphi$ . For any SOSL-formula  $\varphi$ , we will use  $\oplus^n\varphi$  as an abbreviation of  $\overbrace{\oplus \dots \oplus}^{n \text{ times}}\varphi$ , and we adopt the analogous notation for iterations of  $\ominus$ ,  $\oplus$ , and  $\ominus$ . Furthermore, for  $n \in \mathbb{N}$ , we identify  $\oplus^{-n}$  with  $\ominus^n$ ; and  $\ominus^{-n}$  with  $\oplus^n$ .

The universe of an SOSL-model is an infinite plane partitioned into two-dimensional cells, each of length 1 and angular width of 1 degree, and an additional cell representing the subject, located at the center of the plane – see Figure 2. Each cell is identified with the pair of polar coordinates of its centroid, where the subject is denoted by  $\langle 0, 0 \rangle$  and every non-subject cell by  $\langle r, \theta \rangle$ , for  $r \in \mathbb{N}_+$  (we use  $\mathbb{N}_+$  for the set of positive natural numbers and  $\mathbb{N}$  for natural numbers with 0) and  $\theta$  belonging to the set of angle-values, defined as follows:

$$\text{ang} = \{(0.5 + i) \mid i \in \{-180, \dots, 179\}\}.$$

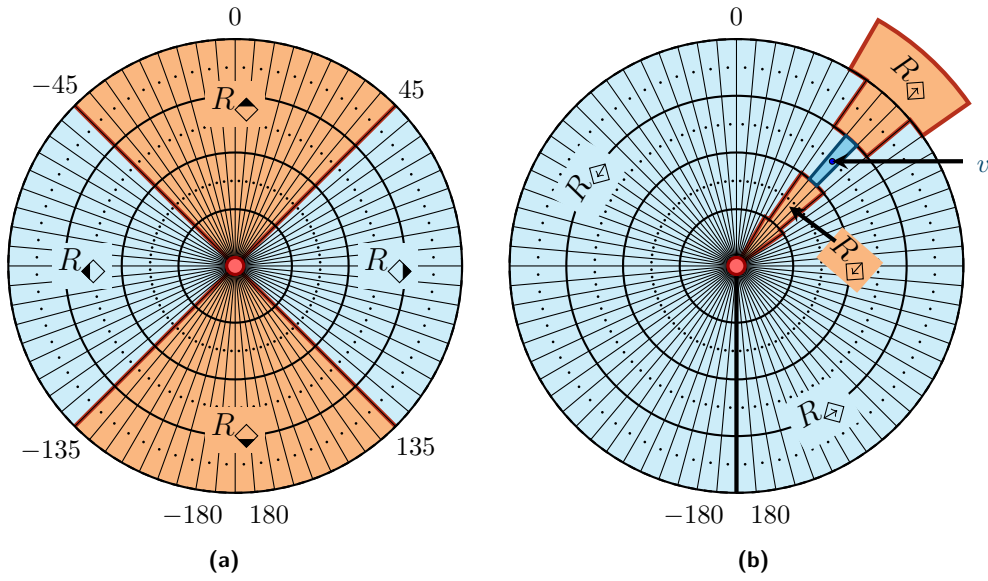
Note that  $\text{ang}$  contains 360 elements. We define the set  $\mathcal{C}$  of all cells as follows:

$$\mathcal{C} = \{\langle 0, 0 \rangle\} \cup \{\langle r, \theta \rangle \mid r \in \mathbb{N}_+ \text{ and } \theta \in \text{ang}\}. \quad (7)$$

We introduce 12 binary relations between cells in  $\mathcal{C}$ . First, we define 4 subject-orientation relations: *in front of the subject* ( $R_{\blacklozenge}$ ), *behind the subject* ( $R_{\blacktriangleright}$ ), *to the left of the subject* ( $R_{\blacktriangleleft}$ ), and *to the right of the subject* ( $R_{\blacktriangle}$ ) – for a pictorial representation see Figure 2(a):

$$\begin{aligned} R_{\blacklozenge} &= \{(\langle 0, 0 \rangle, \langle r, \theta \rangle) \mid -45 < \theta < 45\}; & R_{\blacktriangleright} &= \{(\langle 0, 0 \rangle, \langle r, \theta \rangle) \mid \theta < -135 \text{ or } \theta > 135\}; \\ R_{\blacktriangleleft} &= \{(\langle 0, 0 \rangle, \langle r, \theta \rangle) \mid 45 < \theta < 135\}; & R_{\blacktriangle} &= \{(\langle 0, 0 \rangle, \langle r, \theta \rangle) \mid -135 < \theta < -45\}, \end{aligned}$$

where  $\langle r, \theta \rangle \in \mathcal{C}$  and  $r \neq 0$ . The angle coordinates of cells cannot have integer values, so  $-135, -45, 45, 135 \notin \text{ang}$ , thus  $R_{\blacklozenge}, R_{\blacktriangleright}, R_{\blacktriangleleft}$ , and  $R_{\blacktriangle}$  partition the set of non-subject cells.



■ **Figure 2** Relations in intrinsic (a) and relative (b) frames of reference. For a better readability we present 72 instead of all 360 cells in every ring around the subject.

The next 4 relations represent the location of one cell with respect to another (reference) cell (denoted by  $v$  in Figure 2(b)), from the subject's perspective: *behind a cell* ( $R_{\diamond}$ ), *between a cell and the subject* ( $R_{\diamond}$ ), *to the left of a cell* ( $R_{\diamond}$ ), and *to the right of a cell* ( $R_{\diamond}$ ), which are defined as follows:

$$\begin{aligned}
 R_{\diamond} &= \{(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) \mid r_1 < r_2 \text{ and } |\theta_1 - \theta_2| \leq |r_1 - r_2|\}; \\
 R_{\diamond} &= \{(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) \mid r_2 < r_1 \text{ and } |\theta_1 - \theta_2| \leq |r_1 - r_2|\}; \\
 R_{\diamond} &= \{(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) \mid \theta_2 < \theta_1 \text{ and } |r_1 - r_2| < |\theta_1 - \theta_2|\}; \\
 R_{\diamond} &= \{(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) \mid \theta_1 < \theta_2 \text{ and } |r_1 - r_2| < |\theta_1 - \theta_2|\},
 \end{aligned}$$

where  $\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \in \mathcal{C}$ , and  $r_1, r_2 \neq 0$ . The relations defined in this manner allow us to capture the subject's perspective. Observe that for each non-subject cell  $\langle r_1, \theta_1 \rangle$  the relations  $R_{\diamond}$ ,  $R_{\diamond}$ ,  $R_{\diamond}$ , and  $R_{\diamond}$  partition the set of all non-subject cells distinct from  $\langle r_1, \theta_1 \rangle$ .

The remaining 4 relations hold between adjacent cells: *away from the subject* ( $R_{\oplus}$ ), *towards the subject* ( $R_{\oplus}$ ), *counter-clockwise* ( $R_{\ominus}$ ), and *clockwise* ( $R_{\ominus}$ ). We define:

$$\begin{aligned}
 R_{\oplus} &= \{(\langle r_1, \theta \rangle, \langle r_2, \theta \rangle) \mid r_2 = r_1 + 1\} \cup \{(\langle 0, 0 \rangle, \langle 1, \theta \rangle) \mid \theta \in \text{ang}\}; \\
 R_{\ominus} &= \{(\langle r, \theta_1 \rangle, \langle r, \theta_2 \rangle) \mid \theta_2 = \theta_1 - 1\},
 \end{aligned}$$

where  $\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \in \mathcal{C}$ . We define  $R_{\downarrow}$  and  $R_{\uparrow}$  as the converse relations of  $R_{\oplus}$  and  $R_{\ominus}$ .

► **Definition 2.** An SOSL-frame is a pair  $\mathfrak{F} = (\mathcal{C}, \mathcal{R})$ , where  $\mathcal{C}$  is defined as in (7) and  $\mathcal{R} = \{R_\diamond \mid \diamond \text{ is an SOSL-operator}\}$ . An SOSL-model is a pair  $\mathfrak{M} = (\mathfrak{F}, V)$ , where  $\mathfrak{F} = (\mathcal{C}, \mathcal{R})$  is an SOSL-frame and  $V : \text{PROP} \rightarrow \mathcal{P}(\mathcal{C})$ . The forcing relation  $\Vdash$  is defined as follows:

$$\begin{aligned} \mathfrak{M}, \langle r, \theta \rangle \Vdash p & \quad \text{iff } \langle r, \theta \rangle \in V(p), \text{ for } p \in \text{PROP} \\ \mathfrak{M}, \langle r, \theta \rangle \Vdash \neg\varphi & \quad \text{iff } \mathfrak{M}, \langle r, \theta \rangle \not\Vdash \varphi \\ \mathfrak{M}, \langle r, \theta \rangle \Vdash \varphi \vee \psi & \quad \text{iff } \mathfrak{M}, \langle r, \theta \rangle \Vdash \varphi \text{ or } \mathfrak{M}, \langle r, \theta \rangle \Vdash \psi \\ \mathfrak{M}, \langle r, \theta \rangle \Vdash \diamond\varphi & \quad \text{iff there exists } \langle r', \theta' \rangle \text{ such that } \langle r, \theta \rangle R_\diamond \langle r', \theta' \rangle \text{ and } \mathfrak{M}, \langle r', \theta' \rangle \Vdash \varphi, \end{aligned}$$

where  $\langle r, \theta \rangle, \langle r', \theta' \rangle \in \mathcal{C}$ ,  $\varphi, \psi$  are SOSL-formulas, and  $\diamond$  is an SOSL-operator.

We will denote an SOSL-model of the form  $((\mathcal{C}, \mathcal{R}), V)$  by  $(\mathcal{C}, \mathcal{R}, V)$ . An SOSL-formula  $\varphi$  is satisfiable if there exists an SOSL-model  $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$  and  $\langle r, \theta \rangle \in \mathcal{C}$  such that  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \varphi$ . Observe that  $\langle 0, 0 \rangle$  is a distinguished cell in every SOSL-model. In particular, a formula of the form  $\diamond\varphi$ , for  $\diamond \in \{\blacktriangleleft, \blacktriangleright, \blacktriangle, \blacktriangleright\}$ , can be satisfied only in  $\langle 0, 0 \rangle$ , and no formula of the form  $\diamond\varphi$ , for  $\diamond \in \{\blacktriangleleft, \blacktriangleright, \blacktriangle, \blacktriangleright\}$ , can be satisfied in  $\langle 0, 0 \rangle$ .

### 3 Related Work

In what follows we compare SOSL with some other modal logics for spatial reasoning. One of the main lines of distinction we consider is between types of reference systems used to describe the spatial location of an object in different logics. Usually, three main types of frames of reference are distinguished, namely intrinsic, relative, and absolute [14, 18, 15].

In the intrinsic frame of reference, the location of an object is determined by means of a binary relation between this object and a landmark object. The landmark object is “parsed” into its major parts, e.g., its front, back, left side, and right side, and then a named facet of the landmark object is used to describe the location of the first object, e.g., “the flower is in front of the door” [14]. The subject itself can be treated as the landmark object, and so “the flower is in front of me” is also a description in the intrinsic frame of reference [18].

In the relative frame of reference the position of an object is described by a ternary relation involving this object, a landmark object, and the subject (say myself), e.g., “the flower is between the door and myself” [18]. When describing the location of an object in the relative frame of reference, one also needs to perform “parsing” of objects and so it seems that the relative frame of reference cannot occur without the intrinsic one [26, 18].

In the absolute frame of reference, one uses fixed bearings or absolute coordinates such as north, south, west, and east, e.g., “the flower is to the south of the door”. In the absolute frame of reference, in contrast to the intrinsic and relative ones, a spatial description does not change when the subject, not directly involved in the setting, changes their position [18].

In what follows we will compare SOSL with Compass Logic [32], Spatial Propositional Neighborhood Logic (SpPNL) [21], and Cone Logic [20]. Although spatial objects (constituting the universe of a model) and relations between them (interpreting modal operators) differ in these logics, the location of an object is always described in the absolute frame of reference. On the other hand, in SOSL we use both the intrinsic and relative frame of reference. Another difference is that in SOSL we adopt the polar coordinate system, whereas in the other logics the Cartesian system is used instead – see Table 1 for a cumulative comparison.

In Compass Logic [32, 19] spatial objects are points in the two-dimensional space, identified with their Cartesian coordinates. Modal operators express the following relations between points: “on the same horizontal line and above”, “on the same horizontal line and below”,



■ **Table 1** Comparison of modal logics for spatial reasoning.

logic:	spatial object:	coordinates:	frame of reference:	complexity:
Compass	point	Cartesian	absolute	undecidable
SpPNL	rectangle			undecidable
Cone	set of points			PSPACE
SOSL	cell	polar	intrinsic, relative	PSPACE

“on the same vertical line and to the left”, and “on the same vertical line and to the right”. It is shown that the halting problem of a Turing machine reduces to the satisfiability problem of Compass Logic formulas, which makes the latter problem undecidable [19].

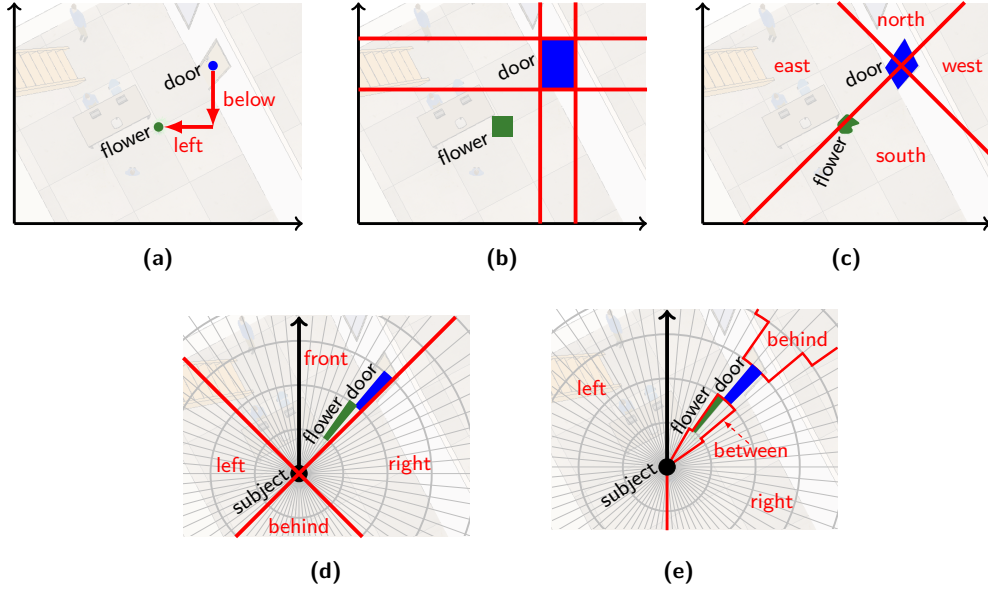
Similarly to Compass Logic, SpPNL uses the two-dimensional space with Cartesian coordinates. However, formulas are evaluated over rectangles rather than points. Intuitively, each spatial object is approximated with its minimum bounding rectangle, and modal operators express the following relations between these rectangles: “adjacent from above”, “adjacent from below”, “adjacent to the left”, and “adjacent to the right”. It is worth mentioning that similar relations are used in Rectangle Algebra [24]. Every formula from (undecidable) Compass Logic can be translated into an equisatisfiable SpPNL-formula, and so the satisfiability problem in SpPNL is also undecidable.

In Cone Logic [20] a spatial object is an arbitrary subset of the two-dimensional space. Each point is denoted by a pair of Cartesian coordinates and modal operators express the following cone-shaped relations between points: “to the north”, “to the south”, “to the west”, and “to the east”. The satisfiability problem in Cone Logic over the rational plane  $\mathbb{Q} \times \mathbb{Q}$  is shown to be PSPACE-complete using the tree (pseudo) model property [20].

To see how spatial representations differ in the above-discussed logics let us consider the configuration from Figure 1. As depicted in Figure 3, in Compass Logic “the flower is below and to the left of the door” (Figure 3a); in SpPNL “the minimal rectangle containing flower is (completely) to the left and (completely) below the minimal rectangle containing the door” (Figure 3b); in Cone Logic “the flower is to the south of the door” (Figure 3c); in SOSL “the flower is in front of the subject” (Figure 3d); and “the flower is between the door and the subject” (Figure 3e).

Observe that in Compass Logic, SpPNL, and Cone Logic the subject is not distinguished, and so their position has no influence on the spatial representation of a scene. On the other hand, the position of the subject is crucial when determining the relation between objects in SOSL. For instance, if in the scene from Figure 3 the subject was turned towards the west (instead of north), we would have “the flower is to the right of the subject” and if the subject was standing on the stairs, we would have “the flower is to the right of the door”.

There is a number of other modal logics for spatial reasoning which, however, are less related to SOSL. For instance,  $PDL_M^F$  [22] is an extension of Propositional Dynamic Logic [13] in which relative movement of one object with respect to another is represented by means of a tuple of parameters. There are also logics based on topological spaces [5, 17, 25], where closure and interior operators are used to express topological relations such as “disconnected”, “overlaps”, and “inside” known from Region Connection Calculus [28]. Furthermore, there exist modal logics over affine spaces (allowing to express, e.g., the betweenness operator), logics of nearness, and logics of convexity, among others [1].



■ **Figure 3** Spatial configuration from Figure 1 represented in Compass Logic (a), SpPNL (b), Cone Logic (c), SOSL (intrinsic frame of reference) (d), and SOSL (relative frame of reference) (e).

#### 4 Expressive Power

As we have mentioned in Section 1, the language of SOSL is expressive enough to define a number of useful operators. First, we define the *difference* operator ( $D$ ), where for an arbitrary SOSL-formula  $\varphi$ ,  $D\varphi$  states that  $\varphi$  holds in some cell distinct from the current cell:

$$D\varphi = \diamond\varphi \vee \diamond\varphi \vee \diamond\varphi \vee \diamond\varphi \vee \\ \diamond\varphi \vee \diamond\varphi \vee \diamond\varphi \vee \diamond\varphi \vee \diamond\varphi \vee \diamond\varphi.$$

The first line of the definition expresses  $D\varphi$  if the current cell is the subject itself, namely the formula states that  $\varphi$  holds in some cell which is in front, behind, to the left, or to the right of the subject. The second line allows us to express  $D\varphi$  if the current cell is not the subject. In particular, it states that  $\varphi$  holds in some cell which is behind the current cell, between this cell and the subject, to the left or to the right of this cell, one cell towards the subject wrt to the current cell, or one cell towards the subject wrt to a cell which is between the current cell and the subject. The last two disjuncts are necessary to express the possibility that  $\varphi$  is satisfied in the subject. Then, the disjunction of the first and the second line expresses  $D\varphi$  in any cell. Observe that irreflexivity of relations interpreting SOSL modal operators is crucial for the definition presented above.

It is well-known that  $D$  allows us to express the *somewhere* operator ( $E$ ) and its dual operator, the *everywhere* operator ( $A$ ), as follows:

$$E\varphi = \varphi \vee D\varphi; \quad A\varphi = \neg E\neg\varphi,$$

where  $E\varphi$  states that  $\varphi$  holds in some cell and  $A\varphi$  states that  $\varphi$  holds in all cells [2]. These operators can be used to simulate a *nominal*  $i$  with a propositional variable  $p_i$ , by expressing that  $p_i$  holds in exactly one cell [2]:

$$Ep_i \wedge A(p_i \rightarrow \neg Dp_i).$$



Then, we can define a *satisfaction operator*  $@_i$  (recall that  $@_i\varphi$  states that  $\varphi$  holds in the cell in which  $i$  holds) as follows [2]:

$$@_i\varphi = \mathbf{E}(p_i \wedge \varphi).$$

Another interesting observation is that we can define a nominal subject which holds exactly in the cell  $\langle 0, 0 \rangle$ :

$$\text{subject} = \blacklozenge\top.$$

Indeed,  $\langle 0, 0 \rangle$  is the only cell which has  $R_{\blacklozenge}$ -successors, and so  $\blacklozenge\top$  holds only there. Furthermore, for any non-subject cell  $\langle r, \theta \rangle$ , we can define a formula  $\text{cell}_{\langle r, \theta \rangle}$  which holds exactly in this cell:

$$\text{cell}_{\langle r, \theta \rangle} = \oplus^r \ominus\top \wedge \ominus^{\theta+179.5} \ominus\top.$$

The formula states that the current cell has exactly  $r$   $R_{\oplus}$ -successors and exactly  $\theta + 179,5$   $R_{\ominus}$ -successors, which is the case only if the current cell is  $\langle r, \theta \rangle$ .

Next, we show how to define operators expressing (a restricted form of) the distance between the central points of two cells. It is well-known that for two points with the polar coordinates,  $\langle r_1, \theta_1 \rangle$  and  $\langle r_2, \theta_2 \rangle$ , we can compute the distance between them, denoted here by  $\text{dist}(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle)$ , as follows:

$$\text{dist}(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) = \sqrt{r_1^2 + r_2^2 - 2 \cdot r_1 \cdot r_2 \cdot \cos(\theta_1 - \theta_2)}.$$

Therefore, to compute the distance between  $\langle r_1, \theta_1 \rangle$  and  $\langle r_2, \theta_2 \rangle$ , we need to determine the values of  $r_1$ ,  $r_2$ , and  $\theta_1 - \theta_2$ , and then use them to perform arithmetic operations. As we show below, it can be done (in a restricted way) by means of an SOSL-formula. For any  $d \in \mathbb{R}_+$  (where  $\mathbb{R}_+$  is the set of positive real numbers) we can define an operator  $\blacklozenge_{=d}$ , such that  $\blacklozenge_{=d}\varphi$  holds in  $\langle r, \theta \rangle$  if  $\varphi$  holds in some  $\langle r', \theta' \rangle$  such that  $r \leq r'$  and  $\text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) = d$ :

$$\blacklozenge_{=d}\varphi = \oplus^d \vee \tag{8}$$

$$\bigvee_{\langle r, \theta \rangle \in \mathcal{C} \mid r < \text{last}(d)} \left( \text{cell}_{\langle r, \theta \rangle} \wedge \tag{9}$$

$$\bigvee_{\langle r', \theta' \rangle \in \mathcal{C} \mid r \leq r' \text{ and } \text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) = d} \oplus^{r'-r} \ominus^{\theta'-\theta} \varphi \right), \tag{10}$$

where  $\text{last}(d) = \min\{n \in \mathbb{N}_+ \mid \text{dist}(\langle n, 0.5 \rangle, \langle n, 1.5 \rangle) > d\}$  and we assume that  $\oplus^d\varphi$  is false if  $d$  is not an integer.

To capture the intuitions behind the definition above, assume that  $\varphi$  holds in  $\langle r', \theta' \rangle$  which is in the distance  $d$  from  $\langle r, \theta \rangle$  and such that  $r \leq r'$ . Clearly, we have either  $\theta' = \theta$  or  $\theta' \neq \theta$ . In the first case, since the distance between  $\langle r, \theta \rangle$  and  $\langle r', \theta' \rangle$  equals  $d$ , we have  $r' = r + d$ , which is expressed by (8).

In the second case, we have  $\theta' \neq \theta$ . Suppose that  $r \geq \text{last}(d)$ . Then by the definition of  $\text{last}(d)$  we can show that  $\text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) > d$ , which gives a contradiction with the assumption that  $\text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) = d$ . Therefore,  $r < \text{last}(d)$ , and so to determine the coordinates of  $\langle r, \theta \rangle$  it suffices to check for what values of  $r$  and  $\theta$  (where  $r < \text{last}(d)$ ) the formula  $\text{cell}_{\langle r, \theta \rangle}$  is satisfied in the current cell, which is expressed by (9). Importantly, by the condition  $r < \text{last}(d)$ , the number of candidate values for  $r$  is bounded and so the disjunction in (9) is not infinite.

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Assuming that we know the values of  $r$  and  $\theta$ , it suffices to check whether there are values for  $r'$  and  $\theta'$  such that  $r \leq r'$ ,  $\text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) = d$ , and  $\varphi$  holds in  $\langle r', \theta' \rangle$ . Observe that for any values of  $r$ ,  $\theta$ , and  $d$  there is a bounded number of values of  $r'$  and  $\theta'$  such that  $r \leq r'$  and  $\text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) = d$ , because non-subject cells cannot have arbitrarily small dimensions (we can compute the size of the smallest non-subject cell). Hence, the size of the disjunction from (10) is bounded. Now, for all values of  $r'$  and  $\theta'$  such that  $r \leq r'$  and  $\text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) = d$  it suffices to check whether  $\varphi$  holds in  $\langle r', \theta' \rangle$ , which is equivalent to checking if the formula  $\oplus^{r'-r} \ominus^{\theta'-\theta} \varphi$  is satisfied in  $\langle r, \theta \rangle$ , and which is expressed by (10).

In a similar way we can define operators  $\diamond_{<d}$  and  $\diamond_{\leq d}$ :

$$\begin{aligned} \diamond_{<d} \varphi &= \bigvee_{k < d} \oplus^k \varphi \vee \bigvee_{\langle r, \theta \rangle | r < \text{last}(d)} \left( \text{cell}_{\langle r, \theta \rangle} \wedge \bigvee_{\langle r', \theta' \rangle | r \leq r' \text{ and } \text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) < d} \oplus^{r'-r} \ominus^{\theta'-\theta} \varphi \right); \\ \diamond_{\leq d} \varphi &= \bigvee_{k \leq d} \oplus^k \varphi \vee \bigvee_{\langle r, \theta \rangle | r < \text{last}(d)} \left( \text{cell}_{\langle r, \theta \rangle} \wedge \bigvee_{\langle r', \theta' \rangle | r \leq r' \text{ and } \text{dist}(\langle r, \theta \rangle, \langle r', \theta' \rangle) \leq d} \oplus^{r'-r} \ominus^{\theta'-\theta} \varphi \right), \end{aligned}$$

where  $k \in \mathbb{N}$  and  $\langle r, \theta \rangle, \langle r', \theta' \rangle \in \mathcal{C}$ .

We can also define operators  $\diamond_{>d}$  and  $\diamond_{\geq d}$ , however, due to the limitation of space and relatively complex combinatorial properties which are encoded by these modalities, we omit their definitions.

The reader might have noticed that our distance operators are not symmetrical, i.e., they only define the distance between a cell  $\langle r, \theta \rangle$  and another cell  $\langle r', \theta' \rangle$  with  $r \leq r'$ . We can define, to a certain extent, symmetrical versions of these operators using hybrid machinery. By way of example, we can express that the cells in which nominals  $i$  and  $j$  are satisfied, are in distance  $d$ , as follows:

$$@_i \diamond_{=d} j \vee @_j \diamond_{=d} i.$$

We finish this section by presenting several tautologies of SOSL (formulas that are satisfied in every cell of every model) which demonstrate what kind of reasoning can be captured in this logic:

$$\diamond \diamond p \rightarrow \diamond p; \quad p \rightarrow \uparrow \diamond p; \quad \diamond \uparrow p \rightarrow \uparrow (\diamond \top \rightarrow \diamond p); \quad \diamond (i \wedge \diamond j) \rightarrow \neg \diamond j,$$

where  $p$  is a propositional variable, and  $i, j$ , and  $k$  are nominals. The first formula informs that  $R_{\diamond}$  is a transitive relation (the analogous formulas for  $\diamond$ ,  $\diamond$ , and  $\diamond$  are also tautologies); the second states that  $R_{\diamond}$  is the converse relation for  $R_{\diamond}$  (similarly to the fact that  $R_{\diamond}$  is the converse for  $R_{\diamond}$ ); the third encodes a restricted form of confluence of  $R_{\diamond}$  and  $R_{\diamond}$ ; and the fourth claims that whenever  $i$  is situated in front of the subject, and  $j$  is to the right of  $i$ , it cannot be the case that  $j$  is located to left of the subject. For more tautologies of SOSL see (4)–(6) in Section 1.

## 5 Computational Complexity

In this section, we will show that SOSL-satisfiability is PSPACE-complete. We will prove the lower bound by reducing the satisfiability problem of Linear Temporal Logic and the upper bound by constructing a generalized nondeterministic Büchi automaton.

For the lower bound we reduce the satisfiability problem of Linear Temporal Logic whose language contains the following operators: *in the next time point* ( $X$ ), *in the previous time point* ( $Y$ ), *everywhere in the future* ( $G$ ), and *everywhere in the past* ( $H$ ), to the

SOSL-satisfiability. Linear Temporal Logic with the operators  $X$ ,  $Y$ ,  $G$ , and  $H$  is often referred to as LTL( $X, Y, G, H$ ), but since we do not consider any other linear temporal logic in this paper, we denote it simply by LTL [11].

The set of LTL-formulas is generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid X\varphi \mid Y\varphi \mid G\varphi \mid H\varphi,$$

where  $p \in \text{PROP}$ .

An LTL-model is a pair  $\mathcal{M} = (\mathbb{N}, V)$ , where  $\mathbb{N}$  is the set of natural numbers strictly ordered by  $<$  and  $V : \text{PROP} \rightarrow \mathcal{P}(\mathbb{N})$ . The *forcing* relation  $\Vdash$  is defined in a standard way (see [11, Section 2.2]). An LTL-formula  $\varphi$  is satisfiable if there exists an LTL-model  $\mathcal{M}$  such that  $\mathcal{M}, 0 \Vdash \varphi$ . The satisfiability problem of LTL is known to be PSPACE-complete ([11, 3]).

► **Theorem 3.** *SOSL-satisfiability is PSPACE-hard.*

**Proof sketch.** Our reduction of LTL-satisfiability to SOSL-satisfiability is based on the idea that we can simulate an LTL-model with the left-most ray of an SOSL-model. To capture this correspondence formally, we will introduce a translation  $f$  of LTL-models into SOSL-models and a translation  $g$  of SOSL-models into LTL-models.

Let  $\mathcal{M} = (\mathbb{N}, V)$  be an LTL-model. We define  $f(\mathcal{M}) = (\mathcal{C}, \mathcal{R}, V')$  as an SOSL-model such that for all  $\langle r, \theta \rangle \in \mathcal{C}$  and  $p \in \text{PROP}$  (without loss of generality we assume that the sets of propositional variables in LTL and SOSL coincide):

$$\langle r, \theta \rangle \in V'(p) \quad \text{iff} \quad \theta = -179.5 \text{ and there is } n \in \mathbb{N} \text{ such that } n \in V(p) \text{ and } r = n + 1.$$

Hence, the valuation of propositional variables on the left-most ray of  $f(\mathcal{M})$  is the same as the valuation on the time line in  $\mathcal{M}$ , whereas in cells of  $f(\mathcal{M})$  which do not belong to the left-most ray, no propositional variable is satisfied.

On the other hand, for an SOSL-model  $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$ , we define  $g(\mathfrak{M}) = (\mathbb{N}, V')$  as an LTL-model such that for all  $n \in \mathbb{N}$  and  $p \in \text{PROP}$ :

$$n \in V'(p) \quad \text{iff} \quad \langle n + 1, -179.5 \rangle \in V(p).$$

Thus, the valuation of propositional variables in  $g(\mathfrak{M})$  coincides with the valuation in the left-most ray of  $\mathfrak{M}$ .

Now, let  $\varphi$  be an arbitrary LTL-formula. Based on the above-presented correspondence between LTL-models and SOSL-models, we will construct an SOSL-formula  $\varphi'$  which is equisatisfiable with  $\varphi$ . First, we define a translation  $\text{tr}$  of LTL-formulas into SOSL-formulas:

$$\begin{array}{ll} \text{tr}(p) & = p & \text{tr}(X\psi) & = \oplus\text{tr}(\psi) \\ \text{tr}(\neg\psi) & = \neg\text{tr}(\psi) & \text{tr}(Y\psi) & = \oplus\oplus\top \wedge \oplus\text{tr}(\psi) \\ \text{tr}(\psi \vee \chi) & = \text{tr}(\psi) \vee \text{tr}(\chi) & \text{tr}(G\psi) & = \boxplus(\boxplus\perp \rightarrow \text{tr}(\psi)) \\ & & \text{tr}(H\psi) & = \boxminus(\boxplus\perp \wedge \oplus\top \rightarrow \text{tr}(\psi)), \end{array}$$

where  $p \in \text{PROP}$  and  $\varphi, \psi$  are LTL-formulas. This translation allows us to simulate LTL-formulas with SOSL-formulas in the sense that the following implications hold for every LTL-formula  $\psi$ , LTL-model  $\mathcal{M}$ , SOSL-model  $\mathfrak{M}$ , and  $n \in \mathbb{N}$ :

- If  $\mathcal{M}, n \Vdash \psi$ , then  $f(\mathcal{M}), \langle n + 1, -179.5 \rangle \Vdash \text{tr}(\psi)$ ;
- If  $\mathfrak{M}, \langle n + 1, -179.5 \rangle \Vdash \text{tr}(\psi)$ , then  $g(\mathfrak{M}), n \Vdash \psi$ .

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Indeed, we can prove these implications by induction on the complexity of  $\psi$  (we omit the details due to space limitations). Then, we define  $\varphi'$  as follows:

$$\varphi' = \text{tr}(\varphi) \wedge \text{cell}_{\langle 1, -179, 5 \rangle},$$

where  $\text{cell}_{\langle 1, -179, 5 \rangle}$  is the formula which was defined formally in Section 4 and which is satisfied exactly in  $\langle 1, -179, 5 \rangle$ .

Assume that  $\varphi$  is LTL-satisfiable, so there is an LTL-model  $\mathcal{M}$  such that  $\mathcal{M}, 0 \models \varphi$ . Thus  $f(\mathcal{M}), \langle 1, -179, 5 \rangle \models \varphi'$ , and so  $\varphi'$  is SOSL-satisfiable. For the opposite direction, assume that  $\varphi'$  is SOSL-satisfiable. Then, there is an SOSL-model  $\mathfrak{M}$  such that  $\mathfrak{M}, \langle 1, -179, 5 \rangle \models \varphi'$  (note that  $\varphi'$  can only be satisfied in  $\langle 1, -179, 5 \rangle$  by the definition of  $\text{cell}_{\langle 1, -179, 5 \rangle}$ ). Hence,  $g(\mathfrak{M}), 0 \models \varphi$ , and so  $\varphi$  is LTL-satisfiable. It follows that  $\varphi$  is LTL-satisfiable if and only if  $\varphi'$  is SOSL-satisfiable. Since  $\varphi_{\langle 1, -179, 5 \rangle}$  is of a constant size and  $\text{tr}$  is a logarithmic-space translation, the whole reduction is in logarithmic space, which finishes the proof.  $\blacktriangleleft$

In the remainder of this section we show that SOSL-satisfiability is in PSPACE. The main part of the proof consists of checking in PSPACE whether there exists an SOSL-model such that a given SOSL-formula  $\varphi$  is satisfied in this model in  $\langle 0, 0 \rangle$ . To do it, we will first guess in NPSpace a set  $K$  (called a *kernel*) of formulas which are satisfied in  $\langle 0, 0 \rangle$  (thus  $\varphi \in K$ ). Then, we will construct a generalized Büchi automaton  $G_{\varphi, K}$  whose states determine formulas which are satisfied in cells sharing the same radius coordinate (a set of such cells forms a *ring* around  $\langle 0, 0 \rangle$ ); each symbol of the automaton's alphabet is a sequence of 360 sets of propositional variables (which encodes a valuation of propositional variables in cells located within one ring); the transition relation forces the consecutive rings to match each other; and the accepting sets assure that every “promise” (e.g., that  $p$  holds somewhere in front of the subject) will eventually be kept. As we will show, an infinite word  $w$  is accepted by  $G_{\varphi, K}$  (i.e., there is a run of  $G_{\varphi, K}$  on  $w$ , in which at least one state from each accepting set occurs infinitely many times during the run) if and only if  $w$  describes an SOSL-model in which  $\varphi$  is satisfied in  $\langle 0, 0 \rangle$ .

First, we define the closure of an SOSL-formula  $\varphi$ , denoted by  $\text{cl}(\varphi)$ , as the set of all subformulas of  $\varphi$  and their negations (where  $\psi$  and  $\neg\neg\psi$  are identified). The neighbourhood closure of  $\varphi$ , referred to as  $\text{ncl}(\varphi)$ , is the set of all formulas of the form  $\oplus^n\psi$ ,  $\ominus^n\psi$ , and their negations, for  $0 \leq n \leq 360$  and  $\psi \in \text{cl}(\varphi)$ . Moreover,  $\text{cl}_\circ(\varphi)$  is the set of subformulas of  $\varphi$  of the form  $\diamond\psi$ ,  $\blacklozenge\psi$ ,  $\blacktriangleright\psi$ , and  $\blacktriangleleft\psi$ , and their negations. A set  $X \subseteq \text{ncl}(\varphi)$  is:

(con) *consistent wrt propositional logic* if for all  $\psi, \varphi_1 \vee \varphi_2 \in \text{ncl}(\varphi)$  the following hold:

- if  $\psi \in X$ , then  $\neg\psi \notin X$ ;
- $\varphi_1 \vee \varphi_2 \in X$  iff  $\varphi_1 \in X$  or  $\varphi_2 \in X$ ;

(max) *maximal* if for all  $\psi \in \text{ncl}(\varphi)$ :

- if  $\psi \notin X$ , then  $\neg\psi \in X$ .

We will denote the set of propositional variables occurring in  $\varphi$  by  $\text{PROP}(\varphi)$ .

From now on, unless stated otherwise, we assume that  $\varphi$  is a fixed SOSL-formula. We define a *kernel* for  $\varphi$  as a set of formulas which are to be satisfied in  $\langle 0, 0 \rangle$  (consequently, a kernel contains  $\varphi$ ).

► **Definition 4.** A  $\varphi$ -kernel is a set  $K \subseteq \text{ncl}(\varphi)$  satisfying (con) and (max), and such that for all  $\psi \in \text{ncl}(\varphi)$  the following hold:

1.  $\blacklozenge\psi, \blacktriangleright\psi, \blacktriangleleft\psi, \oplus\psi, \ominus\psi, \oplus\psi, \ominus\psi \notin K$ ;
2.  $\varphi \in K$ .

A kernel contains formulas satisfied in  $\langle 0, 0 \rangle$ , whereas *rings* (which are to be states of the automaton) contain formulas satisfied in cells located in consecutive rings around  $\langle 0, 0 \rangle$ .

► **Definition 5.** A  $\varphi$ -ring is a function  $R$  whose domain is  $\text{ang} \cup \{\diamond\}$ , such that:

- $R(\theta) \subseteq \text{ncl}(\varphi)$  and satisfies (con) and (max), for all  $\theta \in \text{ang}$ ;
- $R(\diamond) \subseteq \text{cl}_\diamond(\varphi)$ .

Moreover, for all  $\psi \in \text{ncl}(\varphi)$ ,  $m, n \in \mathbb{N}$ , and  $\theta \in \text{ang}$ :

1.  $\ominus^n \psi \in R(\theta)$  iff  $\psi \in R(\theta - n)$ ;
2.  $\ominus^n \psi \in R(\theta)$  iff  $\psi \in R(\theta + n)$ ;
3.  $\diamond \psi \in R(\theta)$  iff there are  $n > 0$  and  $m < n$  such that  $\oplus^m \psi \in R(\theta - n)$  or  $\oplus^m \psi \in R(\theta - n)$ ;
4.  $\diamond \psi \in R(\theta)$  iff there are  $n > 0$  and  $m < n$  such that  $\oplus^m \psi \in R(\theta + n)$  or  $\oplus^m \psi \in R(\theta + n)$ ;
5.  $\diamond \psi, \diamond \psi, \diamond \psi, \diamond \psi \notin R(\theta)$ .

Intuitively, conditions 1-5 guarantee that a ring is locally consistent. By conditions 1-4, a formula of the form  $\ominus^n \psi$ ,  $\ominus^n \psi$ ,  $\diamond \psi$ , or  $\diamond \psi$ , belongs to  $R(\theta)$  if and only if it has a witness on the same ring; and by condition 5 formulas of the form  $\diamond \psi, \diamond \psi, \diamond \psi$ , and  $\diamond \psi$  cannot occur in any  $R(\theta)$  since such formulas can be satisfied only in  $\langle 0, 0 \rangle$ .

Next, we define conditions which need to be fulfilled by a ring succeeding a kernel, and by a ring succeeding another ring. First, we impose conditions on the formulas preceded with  $\diamond, \diamond, \diamond, \diamond$ , and their negations.

► **Definition 6.** Let  $X \subseteq \text{ncl}(\varphi)$  be a  $\varphi$ -kernel or  $R(\diamond)$  from a  $\varphi$ -ring  $R$ . The set  $X$   $\diamond$ -matches a  $\varphi$ -ring  $R'$  if the following conditions are met:

1. if  $\diamond \psi \in X$  and  $\psi \notin R'(\theta)$  for all  $\theta \in \text{ang}$  such that  $-45 < \theta < 45$ , then  $\diamond \psi \in R'(\diamond)$ ;
2. if  $\neg \diamond \psi \in X$ , then  $\neg \diamond \psi \in R'(\diamond)$  and  $\neg \psi \in R'(\theta)$ , for all  $\theta \in \text{ang}$  such that  $-45 < \theta < 45$ , and the analogous conditions (obtained by modifying the range of  $\theta$ ) for  $\diamond, \diamond, \diamond$ , and  $\diamond$ .

Intuitively, the first condition states that  $\diamond \psi$  needs to be present in consecutive rings until  $\psi$  is present in  $R(\theta)$  for some ring  $R$  and  $\theta$  such that  $-45 < \theta < 45$ . The second condition states that if  $\neg \diamond \psi$  is present in some ring, then it needs to be present in the consecutive ring.

► **Definition 7.** A  $\varphi$ -ring  $R'$  is a successor of a  $\varphi$ -kernel  $K$  if  $R'$   $\diamond$ -matches  $K$  and:

1. for all  $\oplus \psi \in \text{ncl}(\varphi)$ :  $\oplus \psi \in K$  iff there is  $\theta \in \text{ang}$  such that  $\psi \in R'(\theta)$ ;
2. for all  $\oplus \psi \in \text{ncl}(\varphi)$  the following are equivalent:  $\psi \in K$ ; there is  $\theta \in \text{ang}$  such that  $\oplus \psi \in R'(\theta)$ ; for all  $\theta \in \text{ang}$   $\oplus \psi \in R'(\theta)$ ;
3. for all  $\diamond \psi \in \text{ncl}(\varphi)$  and  $\theta \in \text{ang}$ :  $\diamond \psi \notin R'(\theta)$ .

Similarly to the case of the 1st ring which is the successor of a kernel, the  $(n + 1)$ th ring needs to fulfil particular conditions to be the successor of the  $n$ th ring. In the following definition we present the conditions which have to be satisfied by consecutive rings:

► **Definition 8.** A  $\varphi$ -ring  $R'$  is a successor of a  $\varphi$ -ring  $R$  if  $R'$   $\diamond$ -matches  $R(\diamond)$  and for all  $\theta \in \text{ang}$  the following hold:

1. for all  $\oplus \psi \in \text{ncl}(\varphi)$ :  $\oplus \psi \in R(\theta)$  iff  $\psi \in R'(\theta)$ ;
2. for all  $\oplus \psi \in \text{ncl}(\varphi)$ :  $\oplus \psi \in R'(\theta)$  iff  $\psi \in R(\theta)$ ;
3. for all  $\diamond \psi \in \text{ncl}(\varphi)$ :  $\diamond \psi \in R(\theta)$  iff at least one of the following holds:  $\psi \in R'(\theta - 1)$ ,  $\psi \in R'(\theta)$ ,  $\psi \in R'(\theta + 1)$ ,  $\diamond \psi \in R'(\theta - 1)$ ,  $\diamond \psi \in R'(\theta)$ , or  $\diamond \psi \in R'(\theta + 1)$ ;
4. for all  $\diamond \psi \in \text{ncl}(\varphi)$ :  $\diamond \psi \in R'_\theta$  iff at least one of the following holds:  $\psi \in R(\theta - 1)$ ,  $\psi \in R(\theta)$ ,  $\psi \in R(\theta + 1)$ ,  $\diamond \psi \in R(\theta - 1)$ ,  $\diamond \psi \in R(\theta)$ , or  $\diamond \psi \in R(\theta + 1)$ .

Now, we are ready to define a generalized Büchi automaton for an  $\mathcal{SOSL}$ -formula  $\varphi$  and a  $\varphi$ -kernel  $K$ :

► **Definition 9.** For a  $\varphi$ -kernel  $K$  we define a generalized nondeterministic Büchi automaton  $G_{\varphi,K} = (\Sigma, Q, Q_0, \delta, \mathcal{F})$ , where:

- $\Sigma$  is the set of all functions  $\sigma : \text{ang} \longrightarrow \mathcal{P}(\text{PROP}(\varphi))$ ;
- $Q$  is the set of all  $\varphi$ -rings;
- $Q_0 = \{R \in Q \mid R \text{ is a successor of } K\}$ ;
- $\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$  is such that for all  $R \in Q$  and  $\sigma \in \Sigma$ :

$$\delta(R, \sigma) = \{R' \in Q \mid R' \text{ is a successor of } R \text{ and } R'(\theta) \cap \text{PROP}(\varphi) = \sigma(\theta), \text{ for all } \theta \in \text{ang}\};$$

- $\mathcal{F} = \{\mathcal{F}_{\diamond\psi}^\theta \mid \theta \in \text{ang} \text{ and } \diamond\psi \in \text{ncl}(\varphi)\} \cup \{\mathcal{F}_{\blacklozenge\psi} \mid \blacklozenge \in \{\blacklozenge, \blacklozenge, \blacklozenge, \blacklozenge\} \text{ and } \blacklozenge\psi \in \text{cl}_\diamond(\varphi)\}$ , where:
  - $\mathcal{F}_{\diamond\psi}^\theta = \{R \in Q \mid \diamond\psi \notin R(\theta) \text{ or } \psi \in R(\gamma), \text{ for some } \gamma \in \text{ang}\}$ ;
  - $\mathcal{F}_{\blacklozenge\psi} = \{R \in Q \mid \blacklozenge\psi \notin R(\diamond)\}$ .

Intuitively, the definition of a state of the automaton (i.e., the Definition 5 of a ring) is responsible for capturing the meaning of formulas preceded with  $\ominus, \oplus, \diamond$ , and  $\blacklozenge$ ; the transition relation (based on the Definition 7 of a successor ring) captures the meaning of formulas preceded with  $\oplus, \ominus$ , and  $\blacklozenge$ ; whereas the accepting sets allow us to capture the meaning of formulas preceded with  $\blacklozenge, \blacklozenge, \blacklozenge, \blacklozenge$ , and  $\blacklozenge$ . As we show below, the automaton accepts only those words that represent SOSL-models in which  $\varphi$  is satisfied in  $\langle 0, 0 \rangle$ .

► **Lemma 10.** For every SOSL-formula  $\varphi$  the following statements are equivalent:

1. There exists a  $\varphi$ -kernel  $K$  such that the language of  $G_{\varphi,K}$  is non-empty;
2. There exists an SOSL-model  $\mathfrak{M}$  such that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \varphi$ .

**Proof sketch.**

(1  $\Rightarrow$  2) Let  $w = \sigma^2\sigma^3 \dots$  be an infinite word accepted by  $G_{\varphi,K}$  and let  $R^1, R^2, \dots$  be an infinite sequence of states in an accepting run of  $G_{\varphi,K}$  on  $w$ . We define an SOSL-model  $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$  such that for all  $p \in \text{PROP}(\varphi)$ :

- $\langle 0, 0 \rangle \in V(p)$  iff  $p \in K$ ;
- for all  $\langle r, \theta \rangle \in \mathcal{C}$  distinct from  $\langle 0, 0 \rangle$  we have  $\langle r, \theta \rangle \in V(p)$  iff  $p \in R^r(\theta)$ .

We can show that for each  $\psi \in \text{ncl}(\varphi)$  the following implications hold:

$$\text{If } \psi \in K, \text{ then } \mathfrak{M}, \langle 0, 0 \rangle \Vdash \psi; \quad \text{If } \psi \in R^r(\theta), \text{ then } \mathfrak{M}, \langle r, \theta \rangle \Vdash \psi.$$

We prove these implications by simultaneous induction on the complexity of  $\psi$ . The proof has a great number of cases (as the language of SOSL contains 12 modal operators) and uses numerous conditions from Definitions 4–9. Due to space limitations we omit the details.

By Definition 4, we have  $\varphi \in K$ , and so, by the first of the proved implications, we obtain  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \varphi$ .

(1  $\Leftarrow$  2) Let  $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$  be an SOSL-model such that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \varphi$ . We define:

$$K = \{\psi \in \text{ncl}(\varphi) \mid \mathfrak{M}, \langle 0, 0 \rangle \Vdash \psi\}.$$

Since  $\mathfrak{M}$  is an SOSL-model, the set  $K$  satisfies the conditions from Definition 4, so  $K$  is a  $\varphi$ -kernel. Next, we will show that the language of  $G_{\varphi,K} = (\Sigma, Q, Q_0, \delta, \mathcal{F})$  is non-empty. Let  $w = \sigma^2\sigma^3 \dots$  be an infinite word such that for all  $r \in \mathbb{N}_+$  and  $\theta \in \text{ang}$  we have:

$$\sigma^r(\theta) = \{p \in \text{PROP}(\varphi) \mid \mathfrak{M}, \langle r, \theta \rangle \Vdash p\}.$$



Next, let  $\mathfrak{R} = R^1, R^2, \dots$  be an infinite sequence of functions  $R^r$  over the domain  $\text{ang} \cup \{\diamond\}$ , such that for all  $r \in \mathbb{N}_+$ ,  $\theta \in \text{ang}$ , and  $\blacklozenge \in \{\blacklozenge, \blacklozenge, \blacklozenge, \blacklozenge\}$  we have:

$$\begin{aligned} R^r(\diamond) &= \{\blacklozenge\psi \in K \mid \psi \notin R^l(\gamma), \text{ for all } \langle l, \gamma \rangle \in \mathcal{C} \text{ such that } \langle 0, 0 \rangle R_{\blacklozenge} \langle l, \gamma \rangle \text{ and } l \leq r\} \cup \\ &\quad \{\neg\blacklozenge\psi \mid \neg\blacklozenge\psi \in K\}; \\ R^r(\theta) &= \{\psi \in \text{ncl}(\varphi) \mid \mathfrak{M}, \langle r, \theta \rangle \Vdash \psi\}. \end{aligned}$$

Each  $R^r$  satisfies the conditions from Definition 5, so  $\mathfrak{R}$  is a sequence of  $\varphi$ -rings, i.e., a sequence of states of  $G_{\varphi, K}$ . It remains to show that  $\mathfrak{R}$  is an accepting run of  $G_{\varphi, K}$  on  $w$ , i.e., that the following hold (since the proof of these conditions is long and standard, we omit it):

- $R^1 \in Q_0$ ;
- for all  $r \in \mathbb{N}_+$  we have  $R^{r+1} \in \delta(R^r, \sigma^{r+1})$ ;
- for every set in  $\mathcal{F}$  there are infinitely many elements of  $\mathfrak{R}$  belonging to this set.

As a result, the language of  $G_{\varphi, K}$  is non-empty (as it contains  $w$ ), which finishes the proof.  $\blacktriangleleft$

Using Lemma 10 we can show the upper bound for the complexity of SOSL-satisfiability.

► **Theorem 11.** *SOSL-satisfiability is in PSPACE.*

**Proof.** To check whether an SOSL-formula  $\varphi$  is SOSL-satisfiable, we define:

$$\varphi' = \varphi \vee \blacklozenge\varphi \vee \blacklozenge\varphi \vee \blacklozenge\varphi \vee \blacklozenge\varphi.$$

Clearly,  $\varphi'$  is of polynomial size wrt the size of  $\varphi$  (denoted by  $|\varphi|$ ) and  $\varphi$  is satisfiable if and only if there exists an SOSL-model in which  $\varphi'$  is satisfied in  $\langle 0, 0 \rangle$ , which by Lemma 10 is equivalent to the existence of a  $\varphi'$ -kernel  $K$  such that the language of  $G_{\varphi', K}$  is non-empty.

To check this condition we nondeterministically guess a  $\varphi'$ -kernel  $K$ . By Definition 4 we have  $K \subseteq \text{ncl}(\varphi')$  and since  $\text{ncl}(\varphi')$  is of a polynomial size wrt  $|\varphi'|$ , the kernel  $K$  is also of a polynomial size wrt  $|\varphi'|$ , and thus we can guess it in NPSpace (note that  $\text{NPSpace} = \text{PSPACE}$  [30]). Although we cannot construct  $G_{\varphi', K}$  in PSPACE (as the number of states of the automaton is exponential wrt  $|\varphi'|$ ), we can examine emptiness of its language by guessing “on-the-fly” (one-by-one) the successor states in an accepting run [4]. Indeed, each state of  $G_{\varphi', K}$  is of polynomial size wrt  $|\varphi'|$ , so the guessing can be done in NPSpace. To confirm that this run is accepting it suffices to check if the first guessed state is an initial state, every next state matches the previous state wrt to the transition relation, and the accepting states are visited infinitely often, which can be done in PSPACE. Hence, the whole procedure is in PSPACE.  $\blacktriangleleft$

As a result of Theorem 3 and Theorem 11 we obtain the following tight complexity bound for the satisfiability problem:

► **Corollary 12.** *SOSL-satisfiability is PSPACE-complete.*

It is worth noticing that, in general, the satisfiability problem in two-dimensional logics in which each dimension is a linear order with an infinite ascending chain is undecidable [29, 12]. Decidability (and in particular membership in PSPACE) of the satisfiability problem of SOSL is obtained mainly as a result of using the polar coordinate system in which one dimension (corresponding to angular coordinates) is finite.

## 6 Conclusions

We have constructed a two-dimensional modal logic for subject-oriented spatial reasoning, denoted by SOSL. The approach employs the polar coordinate system to divide an infinite plane into cells of constant length and constant angle-width. Spatial representation reflects the subject’s perspective and exploits the relations “in front”, “behind”, “to the left”, and “to the right” of the subject; and “behind an object”, “between the subject and an object”, “to the left of an object”, and “to the right of an object”. Such relations are commonly used in most Indo-European languages, where the intrinsic frame of reference (the first 4 relations) and the relative frame of reference (the remaining 4 relations) are most often used to represent and reason about space. We have shown that the language of SOSL allows us to express, e.g., nominals, satisfaction operators, and modal operators for distance. We have proved decidability and PSPACE-completeness of the satisfiability problem in SOSL. In the future we plan to search for low-complexity fragments and modifications of this logic.

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## A Appendix (proofs)

► **Theorem 3.** *SOSL-satisfiability is PSPACE-hard.*

**Proof.** To finish the proof we need to show that the following implications hold for any LTL-formula  $\psi$  and any  $n \in \mathbb{N}$ :

$$\text{If } \mathcal{M}, n \Vdash \psi, \quad \text{then } f(\mathcal{M}), \langle n+1, -179, 5 \rangle \Vdash \text{tr}(\psi); \quad (11)$$

$$\text{If } \mathfrak{M}, \langle n+1, -179.5 \rangle \Vdash \text{tr}(\psi), \quad \text{then } g(\mathfrak{M}), n \Vdash \psi. \quad (12)$$

We conduct the proofs of both implications by induction on the complexity of  $\psi$ . Throughout the proof we use the definition for the  $\Vdash$  relation for LTL taken from [11, Section 2.2].

Proof of (11). We omit the boolean cases since they are straightforward.

**case 1:**  $\psi = X\chi$  Before we prove the implication in question, note that by the definition of  $R_{\oplus}$  we have that for each  $n \in \mathbb{N}$   $R_{\oplus}(\langle n+1, -179.5 \rangle) = \langle n+2, -179.5 \rangle$ . Now, assume that  $\mathcal{M}, n \Vdash X\chi$ . By the definition of  $\Vdash$  for the formulas preceded by the  $X$ -operator, we know that  $\mathcal{M}, n+1 \Vdash \chi$ . By the inductive hypothesis it means that  $f(\mathcal{M}), \langle n+2, -179.5 \rangle \Vdash \text{tr}(\chi)$ , which, in turn, yields  $f(\mathcal{M}), \langle n+1, -179.5 \rangle \Vdash \oplus\text{tr}(\chi)$ . Finally, by the definition of  $\text{tr}$  we obtain  $f(\mathcal{M}), \langle n+1, -179.5 \rangle \Vdash \text{tr}(X\chi)$ .

**case 2:**  $\psi = Y\chi$  This case is analogous to the previous one with the proviso that  $n$  has to be greater than 0.

**case 3:**  $\psi = G\chi$  Assume that  $\mathcal{M}, n \Vdash G\chi$ . By the definition of  $\Vdash$ , for all  $m$  such that  $m > n$  we have that  $\mathcal{M}, m \Vdash \chi$ . By the inductive hypothesis, for every such  $m$  we obtain  $f(\mathcal{M}), \langle m+1, -179.5 \rangle \Vdash \text{tr}(\chi)$ . Since for all  $k \in \mathbb{N}$  we have  $f(\mathcal{M}), \langle k+1, -179.5 \rangle \Vdash \boxminus\perp$  and, at the same time,  $f(\mathcal{M}), \langle k+1, \theta \rangle \not\Vdash \boxminus\perp$  for any  $\theta \neq -179.5$ , then, *a fortiori*, for each  $m > n$  we have  $f(\mathcal{M}), \langle m+1, -179.5 \rangle \Vdash \boxminus\perp \wedge \text{tr}(\chi)$ . Consequently,  $f(\mathcal{M}), \langle n+1, -179.5 \rangle \Vdash \boxplus(\boxminus\perp \rightarrow \text{tr}(\chi))$ , hence  $f(\mathcal{M}), \langle n+1, -179.5 \rangle \Vdash \text{tr}(G\chi)$ .

**case 4:**  $\psi = H\chi$  We handle this case in an analogical manner to the previous one with the proviso that for  $n = 0$  all  $H$ -formulas are vacuously satisfied in 0 and all  $\boxminus$ -formulas are vacuously satisfied in  $\langle 1, -179.5 \rangle$ , which automatically yields the proof for  $n = 0$ .  $\triangleleft$

Proof of (12). We omit the boolean cases since they are straightforward.

**case 1:**  $\text{tr}(\psi) = \text{tr}(X\chi)$  Assume that  $\mathfrak{M}, \langle n+1, -179.5 \rangle \Vdash \text{tr}(X\chi)$ . By the definition of  $\text{tr}$ ,  $\mathfrak{M}, \langle n+1, -179.5 \rangle \Vdash \oplus\text{tr}(\chi)$  and further, by the definition of  $\Vdash$ ,  $\mathfrak{M}, \langle n+2, -179.5 \rangle \Vdash \oplus\text{tr}(\chi)$ . By the inductive hypothesis we get  $g(\mathfrak{M}), n+1 \Vdash \chi$ , which, by the semantics of the  $X$  operator, yields  $g(\mathfrak{M}), n \Vdash X\chi$ .

**case 2:**  $\text{tr}(\psi) = \text{tr}(Y\chi)$  This case is analogous to the previous one with the proviso that  $n$  has to be greater than 0.

**case 3:**  $\text{tr}(\psi) = \text{tr}(G\chi)$  Assume that  $\mathfrak{M}, \langle n+1, -179.5 \rangle \Vdash \text{tr}(G\chi)$ . By the definition of  $\text{tr}$ ,  $\mathfrak{M}, \langle n+1, -179.5 \rangle \Vdash \boxplus(\boxminus\perp \rightarrow \text{tr}(\chi))$ . Since for all  $k \in \mathbb{N}$  it holds that  $\mathfrak{M}, \langle k+1, -179.5 \rangle \Vdash \boxminus\perp$ , then by the definition of  $R_{\diamond}$  we obtain  $\mathfrak{M}, \langle m+1, -179.5 \rangle \Vdash \text{tr}(\chi)$ , for each  $m > n$ . By the inductive hypothesis it follows that for each  $m > n$   $g(\mathfrak{M}), m \Vdash \chi$ , whence we get  $g(\mathfrak{M}), n \Vdash G\chi$ .

**case 4:**  $\text{tr}(\psi) = \text{tr}(H\chi)$  We handle this case in an analogical manner to the previous one with the proviso that for  $n = 0$  all  $H$ -formulas are vacuously satisfied in 0 and all  $\boxminus$ -formulas are vacuously satisfied in  $\langle 1, -179.5 \rangle$ , which automatically yields the proof for  $n = 0$ .  $\triangleleft$

► **Lemma 10.** For every SOSL-formula  $\varphi$  the following statements are equivalent:

1. There exists a  $\varphi$ -kernel  $K$  such that the language of  $G_{\varphi, K}$  is non-empty;
2. There exists an SOSL-model  $\mathfrak{M}$  such that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \varphi$ .

**Proof.**

(1  $\Rightarrow$  2) To finish the proof of the left-to-right implication, we now show that for each  $\psi \in \text{ncl}(\varphi)$  the following implications hold:

- (K) If  $\psi \in K$ , then  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \psi$ ;
- (R) If  $\psi \in R^r(\theta)$ , then  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \psi$ .

We prove them by simultaneous induction on the complexity of  $\psi$ . In the proof we leave aside the most straightforward cases, such as, e.g., the base case which follows directly from the definition of  $V$ .

**case 1:**  $\psi = \neg\chi$

**subcase 1(a):**  $\psi = \neg\blacklozenge\chi$  or  $\psi = \neg\blacklozenge\chi$  or  $\psi = \neg\blacklozenge\chi$  or  $\psi = \neg\blacklozenge\chi$  **(K)** Suppose, for the sake of contradiction, that  $\neg\blacklozenge\chi \in K$  and  $\mathfrak{M}, \langle 0, 0 \rangle \not\models \neg\blacklozenge\chi$ . By Definition 2 it means that  $\mathfrak{M}, \langle 0, 0 \rangle \models \blacklozenge\chi$ , thus for some  $\langle r, \theta \rangle \in \mathcal{C}$  such that  $\langle 0, 0 \rangle R_{\blacklozenge} \langle r, \theta \rangle$  it holds that  $\mathfrak{M}, \langle r, \theta \rangle \models \chi$ , and further, by Definition 2, that  $\mathfrak{M}, \langle r, \theta \rangle \not\models \neg\chi$ . By the inductive hypothesis it follows that  $\neg\chi \notin R^r(\theta)$  and by the fact that  $R^r(\theta)$  satisfies (max) we have  $\chi \in R^r(\theta)$ . Since  $\mathfrak{R}$  is an accepting run, by Definition 6 (2) we know that for every  $n \in \mathbb{N}_+$ ,  $\neg\blacklozenge\chi \in R^n(\diamond)$ . In particular,  $\neg\blacklozenge\chi \in R^r(\diamond)$ . By Definition 6 (2) it follows that  $\neg\chi \in R^r(\theta)$ , which gives a contradiction with the fact that  $R^r(\theta)$  satisfies (con). The proof of the remaining cases proceeds analogically.

**(R)** Assume that  $\neg\blacklozenge\chi \in R^r(\theta)$ . By the definition of  $R_{\blacklozenge}$ ,  $\langle r, \theta \rangle$  has no  $R_{\blacklozenge}$ -successors, hence, by Definition 2,  $\mathfrak{M}, \langle r, \theta \rangle \models \neg\blacklozenge\chi$ . The proof of the remaining cases proceeds analogically.

**subcase 1(b):**  $\psi = \neg\blacklozenge\chi$  **(K)** Assume that  $\neg\blacklozenge\chi \in K$ . By the definition of  $R_{\blacklozenge}$ ,  $\langle 0, 0 \rangle$  has no  $R_{\blacklozenge}$ -successors, thus by Definition 2,  $\mathfrak{M}, \langle 0, 0 \rangle \models \neg\blacklozenge\chi$ .

**(R)** Assume that  $\neg\blacklozenge\chi \in R^r(\theta)$ . By Definition 5,  $R^r(\theta)$  satisfies (con), i.e.,  $\blacklozenge\chi \notin R^r(\theta)$ . Since  $\mathfrak{R}$  is an accepting run,  $R^r$  and  $R^{r+1}$  satisfy the conditions from Definition 8. By the third condition of this definition we obtain  $\chi \notin R^{r+1}(\theta - 1)$  and  $\chi \notin R^{r+1}(\theta)$  and  $\chi \notin R^{r+1}(\theta + 1)$  and  $\blacklozenge\chi \notin R^{r+1}(\theta - 1)$  and  $\blacklozenge\chi \notin R^{r+1}(\theta)$ , and  $\blacklozenge\chi \notin R^{r+1}(\theta + 1)$ . By the fact that  $R^{r+1}$  satisfies (max) we further get  $\neg\chi \in R^{r+1}(\theta - 1)$  and  $\neg\chi \in R^{r+1}(\theta)$  and  $\neg\chi \in R^{r+1}(\theta + 1)$  and  $\neg\blacklozenge\chi \in R^{r+1}(\theta - 1)$  and  $\neg\blacklozenge\chi \in R^{r+1}(\theta)$ , and  $\neg\blacklozenge\chi \in R^{r+1}(\theta + 1)$ . By the inductive hypothesis it follows that  $\mathfrak{M}, \langle r + 1, \theta - 1 \rangle \models \neg\chi$  and  $\mathfrak{M}, \langle r + 1, \theta \rangle \models \neg\chi$ , and  $\mathfrak{M}, \langle r + 1, \theta + 1 \rangle \models \neg\chi$ . With  $\neg\blacklozenge\chi \in R^{r+1}(\theta - 1)$  and  $\neg\blacklozenge\chi \in R^{r+1}(\theta)$ , and  $\neg\blacklozenge\chi \in R^{r+1}(\theta + 1)$  we repeat the reasoning for this case. It follows that for all  $n > 0$  and all  $\gamma \in \text{ang}$  such that  $\gamma = \theta \pm m$ , where  $0 \leq m \leq n$ , we have  $\neg\chi \in R^{r+n}(\gamma)$ . From the definition of  $R_{\blacklozenge}$  we know that  $\langle r, \theta \rangle R_{\blacklozenge} \langle r', \theta' \rangle$  for all  $r' \in \mathbb{N}_+$  and  $\theta' \in \text{ang}$  such that  $r < r'$  and  $|\theta - \theta'| \leq |r - r'|$ . We have just shown that for all such  $\langle r', \theta' \rangle$  it holds that  $\mathfrak{M}, \langle r', \theta' \rangle \models \neg\chi$ , and so by Definition 2 we get  $\mathfrak{M}, \langle r, \theta \rangle \models \neg\blacklozenge\chi$ .

**case 2:**  $\psi = \blacklozenge\chi$  or  $\psi = \blacklozenge\chi$  or  $\psi = \blacklozenge\chi$  or  $\psi = \blacklozenge\chi$  **(K)** Suppose towards a contradiction that  $\blacklozenge\chi \in K$  and  $\mathfrak{M}, \langle 0, 0 \rangle \not\models \blacklozenge\chi$ , which, by Definition 2, means that  $\mathfrak{M}, \langle 0, 0 \rangle \models \neg\blacklozenge\chi$ . It follows that there is no  $\langle r, \theta \rangle \in \mathcal{C}$  such that  $\langle 0, 0 \rangle R_{\blacklozenge} \langle r, \theta \rangle$  and  $\mathfrak{M}, \langle r, \theta \rangle \models \chi$ , so for all  $\langle r, \theta \rangle$  such that  $\langle 0, 0 \rangle R_{\blacklozenge} \langle r, \theta \rangle$  it holds that  $\mathfrak{M}, \langle r, \theta \rangle \not\models \chi$ . By the inductive hypothesis we obtain that for all  $n \in \mathbb{N}_+$  and all  $\theta \in \text{ang}$ ,  $\chi \notin R^n(\theta)$ . By Definition 6 (1) it means that for all  $n \in \mathbb{N}_+$ ,  $\blacklozenge\chi \in R^n(\diamond)$ . Thus, no element from  $\mathcal{F}_{\blacklozenge\psi}$  ever occurs during the run  $\mathfrak{R}$ , and so this is not an accepting run, which yields a contradiction. The remaining cases are proven analogously.

**(R)** By Definition 5 (5) neither of these formulas:  $\blacklozenge\chi$ ,  $\blacklozenge\chi$ ,  $\blacklozenge\chi$ ,  $\blacklozenge\chi$  is included in  $R^r(\theta)$ , so the implication is vacuously satisfied for this case.

**case 3:**  $\psi = \ominus\chi$  or  $\psi = \ominus\chi$  **(K)** By Definition 4 (1) neither of these formulas:  $\ominus\chi$ ,  $\ominus\chi$  is included in  $K$ , so the implication is vacuously satisfied for this case.

**(R)** Assume that  $\ominus\chi \in R^r(\theta)$ . By Definition 5 (1),  $\chi \in R^r(\theta - 1)$ . By the inductive hypothesis we get  $\mathfrak{M}, \langle r, \theta - 1 \rangle \models \chi$ . From the definition of  $R_{\ominus}$  it follows that  $\langle r, \theta \rangle R_{\ominus} \langle r, \theta - 1 \rangle$ , hence by Definition 2 we get  $\mathfrak{M}, \langle r, \theta \rangle \models \ominus\chi$ . The proof of the other case proceeds analogically.

**case 4:**  $\psi = \oplus\chi$  **(K)** By Definition 4 (1)  $\oplus\chi \notin K$ , so the implication is vacuously satisfied for this case.

**(R)** Assume that  $\oplus\chi \in R^r(\theta)$ . Since  $\mathfrak{R}$  is an accepting run,  $R^r$  and  $R^{r+1}$  satisfy the conditions from Definition 8. By the first condition of this definition we obtain  $\chi \in R^{r+1}(\theta)$ . By the inductive hypothesis it follows that  $\mathfrak{M}, \langle r+1, \theta \rangle \Vdash \chi$ . From the definition of  $R_{\oplus}$  we know that  $\langle r, \theta \rangle R_{\oplus} \langle r+1, \theta \rangle$ , and so by Definition 2 we get  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \oplus\chi$ .

**case 5:**  $\psi = \oplus\chi$  **(K)** By Definition 4 (1),  $\oplus\chi \notin K$ , so the implication is vacuously satisfied for this case.

**(R)** Assume that  $\oplus\chi \in R^r(\theta)$ . Two cases need to be considered. 1)  $r = 1$  Since  $\mathfrak{R}$  is an accepting run, the kernel  $K$  and  $R^1$  satisfy the conditions from Definition 7. By the second condition of this definition we obtain  $\chi \in K$ . By the inductive hypothesis it follows that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \chi$ . From the definition of  $R_{\oplus}$  we know that  $\langle 1, \theta \rangle R_{\oplus} \langle 0, 0 \rangle$ , hence, Definition 2 we get  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \oplus\chi$ . 2)  $r > 1$  Since  $\mathfrak{R}$  is an accepting run,  $R^{r-1}$  and  $R^r$  satisfy the conditions from Definition 8. By the second condition of this definition we obtain  $\chi \in R^{r-1}(\theta)$ . By the inductive hypothesis it follows that  $\mathfrak{M}, \langle r-1, \theta \rangle \Vdash \chi$ . From the definition of  $R_{\oplus}$  we know that  $\langle r, \theta \rangle R_{\oplus} \langle r-1, \theta \rangle$ , and so by Definition 2 we get  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \oplus\chi$ .

**case 6:**  $\psi = \diamond\chi$  or  $\psi = \heartsuit\chi$  **(K)** By Definition 4 (1),  $\diamond\chi \notin K$  and  $\heartsuit\chi \notin K$ , so the implication is vacuously satisfied for this case.

**(R)** Assume that  $\diamond\chi \in R^r(\theta)$ . By Definition 5 (3) there exist  $n > 0$  and  $m < n$  such that  $\oplus^m\chi \in R(\theta - n)$  or  $\oplus^m\chi \in R(\theta + n)$ . Without loss of generality suppose that the former is the case. By applying Definition 8 (1)  $m$  times we obtain  $\chi \in R^{r+m}(\theta)$ . By the inductive hypothesis we obtain  $\mathfrak{M}, \langle r+m, \theta - n \rangle \Vdash \chi$ . By the definition of  $R_{\diamond}$  it holds that  $\langle r, \theta \rangle R_{\diamond} \langle r+m, \theta - n \rangle$ , so by Definition 2 we obtain  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \diamond\chi$ . The proof for the other case proceeds analogically.

**case 7:**  $\psi = \heartsuit\chi$  **(K)** By Definition 4 (1)  $\heartsuit\chi \notin K$ , so the implication is vacuously satisfied for this case.

**(R)** Assume that  $\heartsuit\chi \in R^r(\theta)$ . Two cases need to be considered. 1)  $r = 1$  Since  $\mathfrak{R}$  is an accepting run, then  $K$  and  $R^1(\theta)$  satisfy the conditions from Definition 7. By the third condition of this definition  $\heartsuit\chi \notin R^1(\theta)$ , so the implication is vacuously satisfied for this case. 2)  $r > 1$ . Since  $\mathfrak{R}$  is an accepting run,  $R^{r-1}$  and  $R^r$  satisfy the conditions from Definition 8. By the fourth condition of this definition we obtain  $\chi \in R^{r-1}(\theta - 1)$  or  $\chi \in R^{r-1}(\theta)$  or  $\chi \in R^{r-1}(\theta + 1)$  or  $\diamond\chi \in R^{r-1}(\theta - 1)$  or  $\diamond\chi \in R^{r-1}(\theta)$ , or  $\diamond\chi \in R^{r-1}(\theta + 1)$ . If the first disjunct holds, then by the inductive hypothesis we have  $\mathfrak{M}, \langle r-1, \theta - 1 \rangle \Vdash \chi$ . By the definition of  $R_{\heartsuit}$  it follows that  $\langle r, \theta \rangle R_{\heartsuit} \langle r-1, \theta - 1 \rangle$ , so by Definition 2 would get  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \heartsuit\chi$ . The argument is similar if the second or the third disjunct is true. If the fourth disjunct holds, i.e.,  $\diamond\chi \in R^{r-1}(\theta - 1)$ , we proceed in the same way as for the case  $\diamond\chi \in R^r(\theta)$ . Since by Definition 7 (3), for all  $\theta \in \text{ang}$ ,  $\diamond\chi \notin R^1(\theta)$ , it follows that there must exist  $0 < n < r$  and  $0 < m < n$ , such that  $\theta - m \in \text{ang}$  and  $\chi \in R^{r-n}(\theta - m)$  or  $\theta + m \in \text{ang}$  and  $\chi \in R^{r-n}(\theta + m)$ . Without loss of generality assume that the former is the case. By the inductive hypothesis,  $\mathfrak{M}, \langle r-n, \theta - m \rangle \Vdash \chi$ . From the definition of  $R_{\heartsuit}$  we know that  $\langle r, \theta \rangle R_{\heartsuit} \langle r-n, \theta - m \rangle$ , hence, by Definition 2, we get  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \heartsuit\chi$ .

**case 8:**  $\psi = \heartsuit\chi$  **(K)** By Definition 4 (1)  $\heartsuit\chi \notin K$ , so the implication is vacuously satisfied for this case.

**(R)** For the sake of contradiction suppose that  $\heartsuit\chi \in R^r(\theta)$  and  $\mathfrak{M}, \langle r, \theta \rangle \not\Vdash \heartsuit\chi$ . By Definition 2 it means that for all  $\langle r', \theta' \rangle$  such that  $\langle r, \theta \rangle R_{\heartsuit} \langle r', \theta' \rangle$  it holds that  $\mathfrak{M}, \langle r', \theta' \rangle \not\Vdash \chi$ . By the inductive hypothesis for all such  $r'$  and  $\theta'$  we have  $\chi \notin R^{r'}(\theta')$ . (Note that by the definition of  $R_{\heartsuit}$  there exists  $n_0 \in \mathbb{N}_+$ ,  $n_0 \leq 360$ , such that for any  $m \geq n_0$  and any  $\theta' \in \text{ang}$  we have  $\langle r, \theta \rangle R_{\heartsuit} \langle r+m, \theta' \rangle$ , and thus  $\psi \notin R^{r+m}(\theta')$ .)



Since  $\mathfrak{R}$  is an accepting run, then by Definition 8 (3), for each  $n \in \mathbb{N}_+$  there exists  $\theta \in \text{ang}$  such that  $\diamond\chi \in R^{r+n}(\theta)$ . Indeed, since by the assumption neither of the first three cases of Definition 8 (3) holds, then one of the last three cases must hold. This observation applies to subsequent rings. Observe also that by Definition 8 (3), for each two consecutive  $R^n, R^{n+1}$  in  $\mathfrak{R}$  if  $\diamond\chi \in R^{n+1}(\theta)$  for some  $\theta \in \text{ang}$ , then  $\diamond\chi \in R^n(\theta-1)$ ,  $\diamond\chi \in R^n(\theta)$ , and  $\diamond\chi \in R^n(\theta+1)$ . Let  $\text{card}(R_\xi^n) = |\{\theta \in \text{ang} \mid \xi \in R^n(\theta)\}|$ . Note that for any  $n \in \mathbb{N}_+$ , if  $1 \leq \text{card}(R_{\diamond\chi}^{n+1}) < 360$ , then  $\text{card}(R_{\diamond\chi}^{n+1}) < \text{card}(R_{\diamond\chi}^n)$ . A direct consequence of this fact is that if for some  $n \in \mathbb{N}_+$  and  $\theta \in \text{ang}$ ,  $\diamond\chi \in R^{n+360}(\theta)$ , then for all  $\gamma \in \text{ang}$   $\diamond\chi \in R^n(\gamma)$ . Recall that by the assumption, for each  $n \geq r$  there exists  $\theta \in \text{ang}$  such that  $\diamond\chi \in R^n(\theta)$ , which means that for all  $n \in \mathbb{N}_+$  there exists  $\theta \in \text{ang}$  such that  $\diamond\chi \in R^{n+360}(\theta)$ , and thus for all  $\theta \in \text{ang}$  it holds that  $\diamond\chi \in R^n(\theta)$ . Consequently, for all  $m \geq n_0$  and all  $\theta \in \text{ang}$  we have  $\diamond\chi \in R^m(\theta)$  and  $\chi \notin R^{n_0}(\theta)$ . It means that for each  $\theta \in \text{ang}$ , each final state from  $\mathcal{F}_{\diamond\chi}^\theta$  occurs at most finitely many times during  $\mathfrak{R}$  (precisely, at most  $n_0$  times), which contradicts the assumption that  $\mathfrak{R}$  is an accepting run.

(1  $\Leftarrow$  2) To finish the proof of the right-to-left implication it remains to show that  $\mathfrak{R}$  is an accepting run of  $G_{\varphi, K}$  on  $w$ , i.e., that the following hold:

1.  $R^1 \in Q_0$ ;
2. for all  $r \in \mathbb{N}_+$  we have  $R^{r+1} \in \delta(R^r, \sigma^{r+1})$ ;
3. for every set in  $\mathcal{F}$  there are infinitely many elements of  $\mathfrak{R}$  belonging to this set.

**Ad. 1** We need to check whether  $R^1$  is a successor of the kernel  $K$ , i.e., whether the conditions from Definitions 6 and 7 are satisfied. We will start from the former.

**Condition 1.** Assume that  $\diamond\psi \in K$ . By the definition of  $K$  it means that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \diamond\psi$ . If there exists no  $\theta \in \text{ang}$  such that  $\langle 0, 0 \rangle R_{\diamond} \langle 1, \theta \rangle$  and  $\mathfrak{M}, \langle 1, \theta \rangle \Vdash \psi$ , then by the definition of  $R^r(\diamond)$  we obtain that  $\diamond\psi \in R^1(\diamond)$ .

**Condition 2.** Assume that  $\neg\diamond\psi \in K$ . By the definition of  $K$  it means that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \neg\diamond\psi$ . Since  $\mathfrak{M}$  is a model, then by Definition 2 we know that for all  $\theta \in \text{ang}$  such that  $\langle 0, 0 \rangle R_{\diamond} \langle 1, \theta \rangle$  it holds that  $\mathfrak{M}, \langle 1, \theta \rangle \Vdash \neg\psi$ . Therefore, by the definition of  $R^r(\theta)$  we obtain that  $\neg\psi \in R^1(\theta)$  for all  $\theta \in \text{ang}$ . Moreover, by the definition of  $R^r(\diamond)$  we get  $\neg\diamond\psi \in R^1(\diamond)$ .

Now let's proceed to the conditions from Definition 7.

**Condition 1.** Assume that  $\oplus\psi \in K$ . By the definition of  $K$  it means that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \oplus\psi$ . Since  $\mathfrak{M}$  is a model, then by Definition 2 we know that there exists  $\theta \in \text{ang}$  such that  $\mathfrak{M}, \langle 1, \theta \rangle \Vdash \psi$ . By the definition of  $R^r(\theta)$  we infer that  $\psi \in R^1(\theta)$ .

**Condition 2.** Assume that  $\psi \in K$ . By the definition of  $K$  it means that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \psi$ . By the definition of  $R_{\oplus}$  we know that for all  $\theta \in \text{ang}$  it holds that  $\langle 1, \theta \rangle R_{\oplus} \langle 0, 0 \rangle$ . Since  $\mathfrak{M}$  is a model, then by Definition 2 we know that for all  $\theta \in \text{ang}$   $\mathfrak{M}, \langle 1, \theta \rangle \Vdash \oplus\psi$ . By the definition of  $R^r(\theta)$  we infer that for all  $\theta \in \text{ang}$   $\oplus\psi \in R^1(\theta)$ .

**Condition 3.** Observe that by the definition of  $R_{\diamond}$ , for all  $\theta \in \text{ang}$  cells of the form  $\langle 1, \theta \rangle$  have no  $R_{\diamond}$ -successors. Since  $\mathfrak{M}$  is a model, it follows that for all  $\theta \in \text{ang}$   $\mathfrak{M}, \langle 1, \theta \rangle \not\Vdash \diamond\psi$ . By the definition of  $R^r(\theta)$ , for all  $\theta \in \text{ang}$   $\diamond\psi \notin R^1(\theta)$ .

**Ad. 2** We need to check if  $R^{r+1}$  is indeed a  $\delta$ -successor of  $R^r$  on the word  $w$ . First observe that by the definition of  $\sigma^r(\theta)$  and  $R^r(\theta)$ , for all  $\theta \in \text{ang}$ ,  $\sigma^{r+1}(\theta) = R^{r+1}(\theta) \cap \text{PROP}(\varphi)$ . Next, we need to verify whether the conditions from Definitions 6 and 8 are satisfied. The former are proven analogously to the same conditions for  $K$ , so let's proceed to the conditions from Definition 8.

**Condition 1 and 2.** These conditions are proven analogously to condition 2 from the previous point.

**Condition 3.** Assume that  $\diamond\psi \in R^r(\theta)$  for some  $\theta \in \text{ang}$ . From the definition of  $R^r(\theta)$  we know that  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \diamond\psi$ . Since  $\mathfrak{M}$  is a model, then by Definition 2 there exist  $r' \in \mathbb{N}_+$  and  $\theta' \in \text{ang}$  such that  $\langle r, \theta \rangle R_{\diamond} \langle r', \theta' \rangle$  and  $\mathfrak{M}, \langle r', \theta' \rangle \Vdash \psi$ . By the definition of  $R_{\diamond}$  we have  $r' > r$  and  $|\theta - \theta'| \leq |r - r'|$ . If  $r' = r + 1$ , then  $|\theta - \theta'| \leq 1$ , so  $\theta' = \theta - 1$  or  $\theta' = \theta$  or  $\theta' = \theta + 1$ . In such a case, by the definition of  $R^r(\theta)$ ,  $\psi \in R^{r+1}(\theta - 1)$  or  $\psi \in R^{r+1}(\theta)$  or  $\psi \in R^{r+1}(\theta + 1)$ . If  $r' > r + 1$ , then we need to consider 3 cases. 1)  $\theta' - \theta = r' - r$ . It follows that  $\theta' - (\theta + 1) = r' - (r + 1)$ , so by the definition of  $R_{\diamond}$ ,  $\langle r + 1, \theta + 1 \rangle R_{\diamond} \langle r', \theta' \rangle$  and thus, by Definition 2,  $\mathfrak{M}, \langle r + 1, \theta + 1 \rangle \Vdash \diamond\psi$ . Consequently, by the definition of  $R^r(\theta)$ ,  $\psi \in R^{r+1}(\theta + 1)$ . 2)  $|\theta' - \theta| < r' - r$ . It follows that  $\theta' - \theta \leq r' - (r + 1)$ , so by the definition of  $R_{\diamond}$ ,  $\langle r + 1, \theta \rangle R_{\diamond} \langle r', \theta' \rangle$  and thus, by Definition 2,  $\mathfrak{M}, \langle r + 1, \theta \rangle \Vdash \diamond\psi$ . Consequently, by the definition of  $R^r(\theta)$ ,  $\psi \in R^{r+1}(\theta)$ . 3)  $\theta - \theta' = r' - r$ . It follows that  $(\theta - 1) - \theta' = r' - (r + 1)$ , so by the definition of  $R_{\diamond}$ ,  $\langle r + 1, \theta - 1 \rangle R_{\diamond} \langle r', \theta' \rangle$  and by Definition 2,  $\mathfrak{M}, \langle r + 1, \theta - 1 \rangle \Vdash \diamond\psi$ . Hence, by the definition of  $R^r(\theta)$ ,  $\psi \in R^{r+1}(\theta - 1)$ .

**Condition 4.** This condition is proven analogously to condition 3.

**Ad. 3** First, let's consider a set  $\mathcal{F}_{\diamond\psi}$  for some  $\diamond\psi \in \text{ncl}(\varphi)$ . If  $\diamond\psi \notin K$ , then, by the definition of  $R(\diamond)$ , for all  $n \in \mathbb{N}_+$  we have  $\diamond\psi \notin R^n(\diamond)$ . It means that for all  $n \in \mathbb{N}_+$ ,  $R^n \in \mathcal{F}_{\diamond\psi}$ , so elements of the set  $\mathcal{F}_{\diamond\psi}$  occur infinitely many times during the run  $\mathfrak{R}$ . Assume, then, that  $\diamond\psi \in K$ . By the definition of  $K$  it means that  $\mathfrak{M}, \langle 0, 0 \rangle \Vdash \diamond\psi$ . Since  $\mathfrak{M}$  is a model, then by Definition 2 there exist the least  $r \in \mathbb{N}_+$  and  $\theta \in \text{ang}$  such that  $\langle 0, 0 \rangle R_{\diamond} \langle r, \theta \rangle$  and  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \psi$ . By the definition of  $R^r(\theta)$  we obtain  $\psi \in R^r(\theta)$ . By the definition of  $R^r(\diamond)$  it follows that for all  $n \geq r$ ,  $\diamond\psi \notin R^n(\diamond)$ . Consequently, for all  $n \geq r$   $R^n \in \mathcal{F}_{\diamond}$ , so infinitely many elements of  $\mathfrak{R}$  belong to  $\mathcal{F}_{\diamond}$ .

Now, let's consider  $\mathcal{F}_{\diamond\psi}^{\theta}$ . If there exists no  $r \in \mathbb{N}_+$  such that  $\diamond\psi \in R^r(\theta)$ , then for all  $n \in \mathbb{N}_+$  we have  $\diamond\psi \notin R^n(\theta)$ , so for all  $n \in \mathbb{N}_+$ ,  $R^n \in \mathcal{F}_{\diamond\psi}^{\theta}$ . Consequently, elements of the set  $\mathcal{F}_{\diamond\psi}^{\theta}$  occur infinitely many times during the run. Assume, then, that there exists  $r \in \mathbb{N}_+$  such that  $\diamond\psi \in R^r(\theta)$ . By the definition of  $R^r(\theta)$  it means that  $\mathfrak{M}, \langle r, \theta \rangle \Vdash \diamond\psi$ . Since  $\mathfrak{M}$  is a model, then by Definition 2 there exist  $r' \in \mathbb{N}_+$  and  $\theta' \in \text{ang}$  such that  $\langle r, \theta \rangle R_{\diamond} \langle r', \theta' \rangle$  and  $\mathfrak{M}, \langle r', \theta' \rangle \Vdash \psi$ . By the definition of  $R^r(\theta)$  we obtain  $\psi \in R^{r'}(\theta')$ . There are either finitely, or infinitely many  $n \in \mathbb{N}_+$  such that  $\diamond\psi \in R^n(\theta)$ . If the former is the case, it means that  $|\{n \in \mathbb{N}_+ \mid \diamond\psi \notin R^n(\theta)\}| = \infty$ . Since  $\{R^n \mid \diamond\psi \notin R^n(\theta)\} \subseteq \mathcal{F}_{\diamond\psi}^{\theta}$ , infinitely many elements of  $\mathfrak{R}$  belong to  $\mathcal{F}_{\diamond\psi}^{\theta}$ . If, on the other hand, there are infinitely many  $n \in \mathbb{N}_+$  such that  $\diamond\psi \in R^n(\theta)$ , then by the earlier argument, for each  $n \in \mathbb{N}_+$  such that  $\diamond\psi \in R^n(\theta)$  there exists  $m > n$  and  $\theta' \in \text{ang}$  such that  $\psi \in R^m(\theta')$ . By the unboundedness of  $\mathbb{N}_+$  we get  $|\{m \in \mathbb{N}_+ \mid \psi \in R^m(\theta') \text{ for some } \theta' \in \text{ang}\}| = \infty$ . Since  $\{R^m \mid \psi \in R^m(\theta') \text{ for some } \theta' \in \text{ang}\} \subseteq \mathcal{F}_{\diamond\psi}^{\theta}$ , infinitely many elements of  $\mathfrak{R}$  are from  $\mathcal{F}_{\diamond\psi}^{\theta}$ .  $\blacktriangleleft$