Cubical Assemblies, a Univalent and Impredicative Universe and a Failure of Propositional Resizing

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Abstract

We construct a model of cubical type theory with a univalent and impredicative universe in a category of cubical assemblies. We show that this impredicative universe in the cubical assembly model does not satisfy a form of propositional resizing.

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1 Introduction

Homotopy type theory [33] is an extension of Martin-Löf's dependent type theory [29] with homotopy-theoretic ideas. The most important features are Voevodsky's *univalence axiom* and *higher inductive types* which provide a novel synthetic way of proving theorems of abstract homotopy theory and formalizing mathematics in computer proof assistants [4].

Ordinary homotopy type theory [33] uses a cumulative hierarchy of universes

$$\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \dots,$$

but there is another choice of universes: one *impredicative* universe in the style of the Calculus of Constructions [13]. Here we say a universe $\mathcal{U}$ is impredicative if it is closed under dependent products along any type family: for any type A and function $B : A \to \mathcal{U}$, the dependent product $\prod_{x:A} B(x)$ belongs to $\mathcal{U}$. An interesting use of such an impredicative universe in homotopy type theory is the *impredicative encoding* of higher inductive types, proposed by Shulman [35], as well as ordinary inductive types in polymorphic type theory [19]. For instance, the unit circle $\mathbb{S}^1$ is encoded as $\prod_{X:\mathcal{U}} \prod_{x:X} x = x \to X$ which has a base point and a loop on the point and satisfies the recursion principle in the sense of the HoTT book [33, Chapter 6]. Although the impredicative encoding of a higher inductive type does not satisfy the induction principle in general, some truncated higher inductive types have refinements of the encodings satisfying the induction principle [36, 2].

In this paper we construct a model of type theory with a univalent and impredicative universe to prove the consistency of that type theory. Impredicative universes are modeled in the category of *assemblies* or *ω-sets* [28, 32], while univalent universes are modeled in the categories of groupoids [21], simplicial sets [26] and cubical sets [5, 6]. Therefore, in order to construct a univalent and impredicative universe, it is natural to combine them and construct a model of type theory in the category of internal groupoids, simplicial or cubical objects in the category of assemblies. There has been an earlier attempt to obtain a univalent and impredicative universe by Stekelenburg [38] who took a simplicial approach. A difficulty with this approach is that the category of assemblies does not satisfy the axiom of choice or law of excluded middle, so it becomes harder to obtain a model structure on the category of simplicial objects. Another approach is taken by van den Berg [43] using groupoid-like objects, but his model has a dimension restriction. Our choice is the cubical objects in the category of assemblies, which we will call *cubical assemblies*. Since the model in cubical sets [5, 10] is expressed, informally, in a constructive metalogic, one would expect that their construction can be translated into the internal language of the category of assemblies. A similar approach is taken by Awodey, Frey and Hofstra [1, 15].

Instead of a model of homotopy type theory itself, we construct a model of a variant of *cubical type theory* [10] in which the univalence axiom is provable. Orton and Pitts [30] gave a sufficient condition for modeling cubical type theory without universes of fibrant types in an elementary topos equipped with an interval object $\mathbb{I}$. Although the category of cubical assemblies is not an elementary topos, most of their proofs work in our setting because they use a dependent type theory as an internal language of a topos and the category of cubical assemblies is rich enough to interpret the type theory. For construction of the universe of fibrant types, we can use the right adjoint to the exponential functor $(\mathbb{I} \to -)$ in the same way as Licata, Orton, Pitts and Spitters [27].

Voevodsky [45] has proposed the *propositional resizing axiom* [33, Section 3.5] which implies that every homotopy proposition is equivalent to some homotopy proposition in the smallest universe. The propositional resizing axiom can be seen as a form of impredicativity for homotopy propositions. Since the universe in the cubical assembly model is impredicative, one might expect that the cubical assembly model satisfies the propositional resizing axiom. Indeed, for a homotopy proposition A, we have an approximation A^* of A by a homotopy proposition in $\mathcal{U}$ defined as

$$A^* := \prod_{X:\mathsf{hProp}} (A \to X) \to X,$$

where hProp is the universe of homotopy propositions in $\mathcal{U}$, and A is equivalent to some homotopy proposition in $\mathcal{U}$ if and only if the function $\lambda aXh.ha : A \to A^*$ is an equivalence. However, the propositional resizing axiom fails in the cubical assembly model. We construct a homotopy proposition A such that the function $A \to A^*$ is not an equivalence.

We begin Section 2 by formulating the axioms for modeling cubical type theory given by Orton and Pitts [30, 31] in a weaker setting. In Section 3 we describe how to construct a model of cubical type theory under those axioms. In Section 4 we give a sufficient condition for presheaf models to satisfy those axioms. As an example of presheaf model we construct a model of cubical type theory in cubical assemblies in Section 5, and show that the cubical assembly model does not satisfy the propositional resizing axiom.

2 The Orton-Pitts Axioms

We will work in a model $\mathcal{E}$ of dependent type theory with

- dependent product types, dependent sum types, extensional identity types, unit type, disjoint finite coproducts and propositional truncation;
- a constant type $\vdash \mathbb{I}$, called an *interval*, with two constants $\vdash 0 : \mathbb{I}$ and $\vdash 1 : \mathbb{I}$ called *end-points* and two operators $i, j : \mathbb{I} \vdash i \sqcap j : \mathbb{I}$ and $i, j : \mathbb{I} \vdash i \sqcup j : \mathbb{I}$ called *connections*;

1. $\neg(0 = 1)$
2. $\forall_{i:\mathbb{I}} 0 \sqcap i = i \sqcap 0 = 0 \wedge 1 \sqcap i = i \sqcap 1 = i$
3. $\forall_{i:\mathbb{I}} 0 \sqcup i = i \sqcup 0 = i \wedge 1 \sqcup i = i \sqcup 1 = 1$
4. $i : \mathbb{I} \vdash i = 0 : \mathsf{Cof}$
5. $i : \mathbb{I} \vdash i = 1 : \mathsf{Cof}$
6. $\varphi, \psi : \mathsf{Cof} \vdash \varphi \vee \psi : \mathsf{Cof}$
7. $\varphi : \mathsf{Cof}, \psi : \varphi \to \mathsf{Cof} \vdash \sum_{u:\varphi} \psi u : \mathsf{Cof}$
8. $\varphi : \mathbb{I} \to \mathsf{Cof} \vdash \forall_{i:\mathbb{I}} \varphi i : \mathsf{Cof}$
9. $\forall_{\varphi,\psi:\mathsf{Cof}} (\varphi \leftrightarrow \psi) \to (\varphi = \psi)$
10. $\varphi : \mathsf{Cof}, A : \varphi \to \mathcal{U}, B : \mathcal{U}, f : \prod_{u:\varphi} Au \cong B \vdash \mathsf{iea}(\varphi, f) : \sum_{\bar{A}:\mathcal{U}} \{\bar{f} : \bar{A} \cong B \mid \forall_{u:\varphi}(Au, fu) = (\bar{A}, \bar{f})\}$

Figure 1 The Orton-Pitts Axioms.

- a dependent right adjoint to the exponential functor $(\mathbb{I} \to -) : \mathcal{E} \to \mathcal{E}$;
- a propositional universe $\vdash \mathsf{Cof}$ whose inhabitants are called *cofibrations*;
- an impredicative universe $\vdash \mathcal{U}$

satisfying the axioms listed in Figure 1. In the rest of the section we explain these conditions in more detail.

The dependent type theory we use is Martin-Löf's extensional type theory [29]. The notion of model of dependent type theory we have in mind is categories with families [14] equipped with certain algebraic operators corresponding to the type formers. A *category with families* $\mathcal{E}$ consists of:

- a category $\mathcal{E}$ of *contexts* with a terminal object denoted by $\cdot$;
- a presheaf $\Gamma \mapsto \mathcal{E}(\Gamma) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ of *types*;
- a presheaf $(\Gamma, A) \mapsto \mathcal{E}(\Gamma \vdash A) : \mathbf{El}(\mathcal{E}(-))^{\mathrm{op}} \to \mathbf{Set}$ of *terms*, where $\mathbf{El}(P)$ is the category of elements for a presheaf P

such that, for any context $\Gamma \in \mathcal{E}$ and type $A \in \mathcal{E}(\Gamma)$, the presheaf

$$(\mathcal{E}/\Gamma)^{\mathrm{op}} \ni (\sigma : \Delta \to \Gamma) \mapsto \mathcal{E}(\Delta \vdash A\sigma) \in \mathbf{Set}$$

is representable, where $A\sigma$ denotes the element $P(\sigma)(A) \in P(\Delta)$ for a presheaf P, a morphism $\sigma : \Delta \to \Gamma$ and an element $A \in P(\Gamma)$. We assume that any category with families $\mathcal{E}$ has a choice of a representing object for this presheaf denoted by $\pi_A : \Gamma.A \to \Gamma$ and called the *context extension* of A. We also require that, for every context $\Gamma \in \mathcal{E}$, there exist types $C_0 \in \mathcal{E}(\cdot), C_1 \in \mathcal{E}(\cdot.C_0), \dots, C_n \in \mathcal{E}(\cdot.C_0.\dots.C_{n-1})$ and an isomorphism $\cdot.C_0.\dots.C_n \cong \Gamma$. This means that, having dependent sum types, every context Γ can be thought of a closed type $\vdash \Gamma$. Type formers are modeled by algebraic operators. For example, to model dependent product types, $\mathcal{E}$ has an operator Π that carries triples (Γ, A, B) consisting of a context Γ and types $A \in \mathcal{E}(\Gamma)$ and $B \in \mathcal{E}(\Gamma.A)$ to types $\Pi(\Gamma, A, B) \in \mathcal{E}(\Gamma)$ and a bijection $l(\Gamma, A, B) : \mathcal{E}(\Gamma \vdash \Pi(\Gamma, A, B)) \cong \mathcal{E}(\Gamma.A \vdash B)$. These operators must be stable under base changes, that is, for any morphism $\sigma : \Delta \to \Gamma$, we have $\Pi(\Gamma, A, B)\sigma = \Pi(\Delta, A\sigma, B\sigma)$ and $l(\Gamma, A, B)\sigma = l(\Delta, A\sigma, B\sigma)$. All type-theoretic operations we introduce are required to be stable under base changes, unless otherwise stated. Note that there are alternative choices of notions of model of dependent type theory including categories with attributes [9] and split full comprehension categories [24]. Whichever model is chosen, we proceed entirely in its internal language.

In dependent type theory, a type $\Gamma \vdash \varphi$ is said to be a *proposition*, written $\Gamma \vdash \varphi\ \mathsf{Prop}$, if $\Gamma, u_1, u_2 : \varphi \vdash u_1 = u_2$ holds. For a proposition $\Gamma \vdash \varphi$, we say φ *holds* if there exists a (unique) inhabitant of φ. For a type $\Gamma \vdash A$, its *propositional truncation* [3] is a proposition $\Gamma \vdash \|A\|$ equipped with a constructor $\Gamma, a : A \vdash |a| : \|A\|$ such that, for every proposition $\Gamma \vdash \varphi$, the function $\Gamma \vdash \lambda fa.f(|a|) : (\|A\| \to \varphi) \to (A \to \varphi)$ is an isomorphism. Propositions are closed under empty type, cartesian products and dependent products along arbitrary types, and we write $\bot, \top, \varphi \land \psi, \forall_{x:A}\varphi(x)$ for $\mathbf{0}, \mathbf{1}, \varphi \times \psi, \prod_{x:A} \varphi(x)$, respectively, when emphasizing that they are propositions. Also the identity type $\mathsf{Id}(A, a_0, a_1)$ is a proposition because it is extensional, and often written $a_0 = a_1$. The other logical operators are defined using propositional truncation as $\varphi \lor \psi := \|\varphi + \psi\|$ and $\exists_{x:A}\varphi(x) := \| \sum_{x:A} \psi(x)\|$. One can show that these logical operations satisfy the derivation rules of first-order intuitionistic logic. Moreover, the type theory admits subset comprehension defined as

$$\Gamma \vdash \{x : A \mid \varphi(x)\} := \sum_{x:A} \varphi(x)$$

for a proposition $\Gamma, x : A \vdash \varphi(x)$.

A finite coproduct $A + B$ is said to be *disjoint* if the inclusions $\mathsf{inl} : A \to A + B$ and $\mathsf{inr} : B \to A + B$ are monic and $\forall_{a:A}\forall_{b:B}\mathsf{inl}(a) \neq \mathsf{inr}(b)$ holds. A proposition $\Gamma \vdash \varphi$ is said to be *decidable* if $\Gamma \vdash \varphi \lor \neg\varphi$ holds. If the coproduct $\mathbf{2} := \mathbf{1} + \mathbf{1}$ of two copies of the unit type is disjoint, then it is a *decidable subobject classifier*: for every decidable proposition $\Gamma \vdash \varphi$, there exists a unique term $\Gamma \vdash b : \mathbf{2}$ such that $\Gamma \vdash \varphi \leftrightarrow (b = 1)$ holds. For readability we identify a boolean value $b : \mathbf{2}$ with the proposition $b = 1$.

For a functor $H : \mathcal{E} \to \mathcal{F}$ between the underlying categories of categories with families $\mathcal{E}$ and $\mathcal{F}$, a *dependent right adjoint* [7] to H consists of, for each context $\Gamma \in \mathcal{E}$ and type $A \in \mathcal{F}(H\Gamma)$, a type $G_\Gamma A \in \mathcal{E}(\Gamma)$ and an isomorphism $\varphi_A : \mathcal{F}(H\Gamma \vdash A) \cong \mathcal{E}(\Gamma \vdash G_\Gamma A)$ that are stable under reindexing in the sense that, for any morphism $\sigma : \Delta \to \Gamma$, we have $(G_\Gamma A)\sigma = G_\Delta(A\sigma)$ and $(\varphi_A a)\sigma = \varphi_{A\sigma}(a\sigma)$ for any $a \in \mathcal{F}(H\Gamma \vdash A)$. One can show that H preserves all colimits whenever it has a dependent right adjoint. As a consequence, assuming the exponential functor $(\mathbb{I} \to -)$ has a dependent right adjoint, the interval $\mathbb{I}$ is connected

$$\forall_{\varphi:\mathbb{I}\to\mathbf{2}}(\forall_{i:\mathbb{I}}\varphi i) \lor (\forall_{i:\mathbb{I}}\neg\varphi i),$$

which is postulated in [30] as an axiom.

A *universe* (à la Tarski) is a type $\vdash U$ equipped with a type $U \vdash \mathsf{el}_U$. We often omit the subscript $_U$ and simply write el for el_U if the universe is clear from the context. The universe U is said to be *propositional* if $U \vdash \mathsf{el}_U$ is a proposition. An *impredicative universe* is a universe U equipped with the following operations.

- A term $A : U, B : \mathsf{el}(A) \to U \vdash \sum^U(A, B) : U$ equipped with an isomorphism $A : U, B : \mathsf{el}(A) \to U \vdash e : \mathsf{el}(\sum^U(A, B)) \cong \sum_{x:\mathsf{el}(A)} \mathsf{el}(Bx)$.
- A term $A : U, a_0, a_1 : \mathsf{el}(A) \vdash \mathsf{Id}^U(A, a_0, a_1) : U$ equipped with an isomorphism $A : U, a_0, a_1 : \mathsf{el}(A) \vdash e : \mathsf{el}(\mathsf{Id}^U(A, a_0, a_1)) \cong (a_0 = a_1)$.
- For every type $\Gamma \vdash A$, a term $\Gamma, B : \mathsf{el}(A) \to U \vdash \prod^U(A, B) : U$ equipped with an isomorphism $\Gamma, B : \mathsf{el}(A) \to U \vdash e : \mathsf{el}(\prod^U(A, B)) \cong \prod_{x:A} \mathsf{el}(Bx)$.

One might want to require that $\mathsf{el}(\sum^U(A, B))$ is equal to $\sum_{x:\mathsf{el}(A)} \mathsf{el}(Bx)$ on the nose rather than up to isomorphism, but in the category of assemblies described in Section 5, the impredicative universe of partial equivalence relations does not satisfy this equation. For this reason, the distinction between terms $A : U$ and types $\mathsf{el}(A)$ is necessary, but for readability we often identify a term $A : U$ with the type $\mathsf{el}(A)$. For example, in Axiom 10 some el's should be inserted formally. Also Axiom 6 formally means that there exists a term $\varphi, \psi : \mathsf{Cof} \vdash \vee^{\mathsf{Cof}}(\varphi, \psi) : \mathsf{Cof}$ such that $\varphi, \psi : \mathsf{Cof} \vdash \mathsf{el}(\vee^{\mathsf{Cof}}(\varphi, \psi)) \leftrightarrow (\mathsf{el}(\varphi) \lor \mathsf{el}(\psi))$ holds.

Almost all the axioms in Figure 1 are direct translations of those in [30, 31]. Strictly speaking, Axioms 4 to 8 are part of structures rather than axioms in our setting, because Cof is no longer a subobject of the subobject classifier. Also Axiom 10, called the *isomorphism extension axiom*, is part of structures. As already mentioned, the connectedness of the interval $\mathbb{I}$ follows from the existence of the right adjoint to the exponential functor $(\mathbb{I} \to -)$. We need Axiom 9, which asserts the extensionality of the propositional universe Cof, for fibration structures on identity types. This axiom trivially holds in case that Cof is a subobject of the subobject classifier in an elementary topos. We also note that Cof is closed under $\bot$, $\top$ and $\wedge$ using Axioms 1, 5 and 7.

3 Modeling Cubical Type Theory

We describe how to construct a model of a variant of cubical type theory in our setting following Orton and Pitts [30]. Throughout the section $\mathcal{E}$ will be a model of dependent type theory satisfying the conditions explained in Section 2. Type-theoretic notations in this section are understood in the internal language of $\mathcal{E}$.

Cubical type theory is an extension of dependent type theory with an *interval object* [10, Section 3], the *face lattice* [10, Section 4.1], *systems* [10, Section 4.2], *composition operations* [10, Section 4.3] and the *gluing operation* [10, Section 6]. It also has several type formers including dependent product types, dependent sum types, *path types* [10, Section 3] and, optionally, *identity types* [10, Section 9.1]. We make some modifications to the original cubical type theory [10] in the same way as Orton and Pitts [30]. Major differences are as follows.

1. In [10] the interval object $\mathbb{I}$ is a de Morgan algebra, while we only require that $\mathbb{I}$ is a path connection algebra.
2. Due to the lack of de Morgan involution, we need composition operations in both directions "from 0 to 1" and "from 1 to 0".

In this section we will construct from $\mathcal{E}$ a new model of dependent type theory $\mathcal{E}^F$ that supports all operations of cubical type theory.

3.1 The Face Lattice and Systems

The *face lattice* [10, Section 4.1] is modeled by the propositional universe Cof. Note that in [10] quantification $\forall_{i:\mathbb{I}}\varphi$ is not part of syntax and written as a disjunction of irreducible elements, and plays a crucial role for defining composition operation for gluing. Since Cof need not admit quantifier elimination, we explicitly require Axiom 8.

We use the following operation for modeling *systems* [10, Section 4.2] which allows one to amalgamate compatible partial functions.

▶ **Proposition 1.** *One can derive an operation*

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash \varphi_i\ \mathsf{Prop} \qquad \Gamma, u_i : \varphi_i \vdash a_i(u_i) : A \qquad \Gamma, u : \varphi_i, u' : \varphi_j \vdash a_i(u) = a_j(u') \qquad (i \text{ and } j \text{ run over } \{1,\dots,n\})}{\Gamma \vdash [(u_1 : \varphi_1) \mapsto a_1(u_1), \dots, (u_n : \varphi_n) \mapsto a_n(u_n)] : \varphi_1 \vee \dots \vee \varphi_n \to A}$$

such that $\Gamma, v : \varphi_i \vdash [(u_1 : \varphi_1) \mapsto a_1(u_1), \dots, (u_n : \varphi_n) \mapsto a_n(u_n)]v = a_i(v)$ *for* $i = 1, \dots, n$.

Proof. Let B denote the union of images of a_i's:

$$\Gamma \vdash B := \{a : A \mid (\exists_{u_1:\varphi_1} a_1(u_1) = a) \vee \dots \vee (\exists_{u_n:\varphi_n} a_n(u_n) = a)\}.$$

Then $\Gamma \vdash B$ is a proposition because $\Gamma, u : \varphi_i, u' : \varphi_j \vdash a_i(u) = a_j(u')$ for all i and j. Hence the function $[a_1, \dots, a_n] : \varphi_1 + \dots + \varphi_n \to B$ induces a function $\|\varphi_1 + \dots + \varphi_n\| \to B$. ◀

3.2 Fibrations

We regard the type of Boolean values $\mathbf{2}$ as a subtype of the interval $\mathbb{I}$ via the end-point inclusion $[0,1] : \mathbf{2} \cong \mathbf{1}+\mathbf{1} \to \mathbb{I}$. We define a term $e : \mathbf{2} \vdash \bar{e} : \mathbf{2}$ as $\bar{0} = 1$ and $\bar{1} = 0$.

▶ **Definition 2.** *For a type $\Gamma, i : \mathbb{I} \vdash A(i)$, we define a type of* composition structures *as*

$$\Gamma \vdash \mathsf{Comp}^i(A(i)) := \prod_{e:\mathbf{2}} \prod_{\varphi:\mathsf{Cof}} \prod_{f:\varphi \to \prod_{i:\mathbb{I}} A(i)} \prod_{a:A(e)} (\forall_{u:\varphi} fue = a) \to \{a' : A(\bar{e}) \mid \forall_{u:\varphi} fu\bar{e} = a'\}.$$

In this notation, the variable i is considered to be bound.

▶ **Definition 3.** *For a type $\gamma : \Gamma \vdash A(\gamma)$, we define a type of* fibration structures *as*

$$\vdash \mathsf{Fib}(A) := \prod_{p:\mathbb{I}\to\Gamma} \mathsf{Comp}^i(A(pi)).$$

A fibration *is a type $\Gamma \vdash A$ equipped with a global section $\vdash \alpha : \mathsf{Fib}(A)$.*

For a fibration structure $\alpha : \mathsf{Fib}(A)$ on a type $\gamma : \Gamma \vdash A(\gamma)$ and a morphism $\sigma : \Delta \to \Gamma$, we define a fibration structure $\alpha\sigma : \mathsf{Fib}(A\sigma)$ on $\delta : \Delta \vdash A(\sigma(\delta))$ as

$$\alpha\sigma = \lambda p.\alpha(\sigma \circ p) : \prod_{p:\mathbb{I}\to\Delta} \mathsf{Comp}^i(A(\sigma(pi))).$$

Thus, for a fibration (A, α) on Γ, we have its *base change* $(A\sigma, \alpha\sigma)$ along a morphism $\sigma : \Delta \to \Gamma$. With this base change operation we get a model $\mathcal{E}^F$ of dependent type theory where

- the contexts are those of $\mathcal{E}$;
- the types over Γ are fibrations over Γ;
- the terms of a fibration $\Gamma \vdash A$ are terms of the underlying type $\Gamma \vdash A$ in $\mathcal{E}$

together with a forgetful map $\mathcal{E}^F \to \mathcal{E}$. In the same way as Orton and Pitts [30], one can show the following.

▶ **Theorem 4.** *The model of dependent type theory $\mathcal{E}^F$ supports:*

- *composition operations, path types and identity types; and*
- *dependent product types, dependent sum types, unit type and finite coproducts preserved by the forgetful map $\mathcal{E}^F \to \mathcal{E}$.*

We also introduce a class of objects that automatically carry fibration structures.

▶ **Definition 5.** *A type $\vdash A$ is said to be* discrete *if $\forall_{f:\mathbb{I}\to A}\forall_{i:\mathbb{I}} fi = f0$ holds.*

▶ **Proposition 6.** *If $\vdash A$ is a discrete type, then it has a fibration structure.*

Proof. Let $e : \mathbf{2}$, $\varphi : \mathsf{Cof}$, $f : \varphi \to \mathbb{I} \to A$ and $a : A$ such that $\forall_{u:\varphi} fue = a$. Then $a' := a : A$ satisfies $\forall_{u:\varphi} fu\bar{e} = a'$ by the discreteness. ◀

3.3 Path Types and Identity Types

For a type $\Gamma \vdash A$ and terms $\Gamma \vdash a_0 : A$ and $\Gamma \vdash a_1 : A$, we define the *path type* $\Gamma \vdash \mathsf{Path}(A, a_0, a_1)$ to be

$$\Gamma \vdash \{p : \mathbb{I} \to A \mid p0 = a_0 \wedge p1 = a_1\}.$$

We also define the *identity type* $\Gamma \vdash \mathsf{Id}(A, a_0, a_1)$ to be

$$\Gamma \vdash \sum_{p:\mathsf{Path}(A,a_0,a_1)} \{\varphi : \mathsf{Cof} \mid \varphi \to \forall_{i:\mathbb{I}} pi = a_0\}$$

which is a variant of Swan's construction [39]. Theorem 4 says that, if A has a fibration structure, then so do $\mathsf{Path}(A, a_0, a_1)$ and $\mathsf{Id}(A, a_0, a_1)$.

In the model $\mathcal{E}^F$, both path types and identity types admit the following introduction and elimination operations:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}_a : P(A, a, a)}\ P\text{-intro}$$

$$\frac{\begin{array}{c}\Gamma, x_0 : A, x_1 : A, z : P(A, x_0, x_1) \vdash C(z) \\ \Gamma, x : A \vdash c(x) : C(\mathsf{refl}_x) \qquad \Gamma \vdash a_0 : A \qquad \Gamma \vdash a_1 : A \qquad \Gamma \vdash p : P(A, a_0, a_1)\end{array}}{\Gamma \vdash \mathsf{ind}_{P(A)}(C, c, p) : C(p)}\ P\text{-elim}$$

where P is either Path or Id. A difference between them is their computation rules. Identity types admit the *judgmental* computation rule like Martin-Löf's identity types:

$$\Gamma \vdash \mathsf{ind}_{\mathsf{Id}(A)}(C, c, \mathsf{refl}_a) = c(a)$$

for a term $\Gamma \vdash a : A$. On the other hand, path types only admit the *propositional* computation rule: for a term $\Gamma \vdash a : A$, one can find a term

$$\Gamma \vdash H(C, c, a) : \mathsf{Path}(C(a), \mathsf{ind}_{\mathsf{Path}(A)}(C, c, \mathsf{refl}_a), c(a)).$$

Therefore, when interpreting homotopy type theory, which is based on Martin-Löf's type theory, we use $\mathsf{Id}(A, a_0, a_1)$ rather than $\mathsf{Path}(A, a_0, a_1)$. However, it can be shown that $\mathsf{Id}(A, a_0, a_1)$ and $\mathsf{Path}(A, a_0, a_1)$ are equivalent, and thus we can replace $\mathsf{Id}(A, a_0, a_1)$ by simpler type $\mathsf{Path}(A, a_0, a_1)$ when analyzing the model $\mathcal{E}^F$ (see, for instance, the definition of homotopy proposition in Section 5.1).

3.4 Universes and Gluing

For a type $\gamma : \Gamma \vdash A(\gamma)$, a fibration structure on A corresponds to a term of the type $p : \mathbb{I} \to \Gamma \vdash \mathsf{C}(A)(p) := \mathsf{Comp}^i(A(pi))$. We define a type $\Gamma \vdash FA := \mathsf{C}(A)_{\mathbb{I}}$, using the dependent right adjoint $(-)_{\mathbb{I}}$ to the exponential functor $(\mathbb{I} \to -)$. By definition a morphism $\sigma : \Delta \to \sum_\Gamma FA$ corresponds to a pair (σ_0, α) consisting of a morphism $\sigma_0 : \Delta \to \Gamma$ and a fibration structure $\vdash \alpha : \prod_{p:\mathbb{I}\to\Delta} \mathsf{Comp}^i(A(\sigma_0(pi)))$.

Using this construction for the universe $\mathcal{U} \vdash \mathsf{el}_{\mathcal{U}}$, we have a new universe $\mathcal{U}^F := \sum_{\mathcal{U}} F(\mathsf{el})$ together with a fibration $(A, \alpha) : \mathcal{U}^F \vdash \mathsf{el}_{\mathcal{U}^F}(A, \alpha) := \mathsf{el}_{\mathcal{U}}(A)$. By definition $\mathcal{U}^F$ classifies fibrations whose underlying types belong to $\mathcal{U}$.

▶ **Theorem 7.** *The universe $\mathcal{U}^F$ is closed under dependent product types along arbitrary fibrations, dependent sum types and path types. If* Cof *belongs to $\mathcal{U}$, then $\mathcal{U}^F$ is closed under identity types.*

Proof. By Theorem 4, it suffices to show that $\mathcal{U}$ is closed under those type constructors, but this is clear by definition. ◀

We describe the *gluing operation* on the universe $\mathcal{U}^F$ following Orton and Pitts [30].

For a proposition $\Gamma \vdash \varphi$, types $\Gamma, u : \varphi \vdash A(u)$ and $\Gamma \vdash B$ and a function $\Gamma, u : \varphi \vdash f(u) : A(u) \to B$, we define a type $\mathsf{Glue}(\varphi, f)$ to be

$$\Gamma \vdash \mathsf{Glue}(\varphi, f) := \sum_{a:\prod_{u:\varphi} A(u)} \{b : B \mid \forall_{u:\varphi} f(u)(au) = b\}.$$

There is a canonical isomorphism $\Gamma, u : \varphi \vdash e(u) := \lambda(a, b).au : \mathsf{Glue}(\varphi, f) \cong A(u)$ with inverse $\lambda a.(\lambda v.a, f(u)a)$.

▶ **Proposition 8.** *For $\gamma : \Gamma \vdash \varphi(\gamma) : \mathsf{Cof}$, $\gamma : \Gamma, u : \varphi(\gamma) \vdash A(u)$, $\gamma : \Gamma \vdash B(\gamma)$ and $\gamma : \Gamma, u : \varphi(\gamma) \vdash f(u) : A(u) \to B$, if A and B are fibrations and f is an equivalence, then $\gamma : \Gamma \vdash \mathsf{Glue}(\varphi(\gamma), f)$ has a fibration structure preserved by the canonical isomorphism $\Gamma, u : \varphi \vdash e(u) : \mathsf{Glue}(\varphi, f) \cong A(u)$.*

Proof. The construction is similar to the definition of the composition operation for glue types [10, Section 6.2]. ◀

Since the universe $\mathcal{U}$ is closed under type formers used in the definition of $\mathsf{Glue}(\varphi, f)$, we get a term

$$\varphi : \mathsf{Cof}, A : \varphi \to \mathcal{U}, B : \mathcal{U}, f : \prod_{u:\varphi} A(u) \to B \vdash \mathsf{Glue}(\varphi, f) : \mathcal{U}$$

such that $\prod_{u:\varphi} \mathsf{Glue}(\varphi, f) \cong A(u)$. However, the gluing operation in cubical type theory [10, Section 6] requires that, assuming $u : \varphi$, $\mathsf{Glue}(\varphi, f)$ is equal to $A(u)$ on the nose rather than up to isomorphism. So we use Axiom 10 and get a term

$$\varphi : \mathsf{Cof}, A : \varphi \to \mathcal{U}, B : \mathcal{U}, f : \prod_{u:\varphi} A(u) \to B \vdash \mathsf{SGlue}(\varphi, f) : \mathcal{U}$$

such that $\mathsf{SGlue}(\varphi, f) \cong \mathsf{Glue}(\varphi, f)$ and $\forall_{u:\varphi} \mathsf{SGlue}(\varphi, f) = A(u)$. By Proposition 8 we also have a term

$$\varphi : \mathsf{Cof}, A : \varphi \to \mathcal{U}^F, B : \mathcal{U}^F, f : \prod_{u:\varphi} A(u) \simeq B \vdash \mathsf{SGlue}(\varphi, f) : \mathcal{U}^F$$

such that $\mathsf{SGlue}(\varphi, f) \cong \mathsf{Glue}(\varphi, f)$ and $\forall_{u:\varphi} \mathsf{SGlue}(\varphi, f) = A(u)$. Hence the universe $\mathcal{U}^F$ in the model $\mathcal{E}^F$ supports the gluing operation. The composition operation for universes is defined using the gluing operation [10, Section 7.1], so we have the following proposition.

▶ **Proposition 9.** *$\vdash \mathcal{U}^F$ has a fibration structure.*

Since the univalence axiom can be derived from the gluing operation [10, Section 7], we conclude that $\mathcal{U}^F$ is a univalent and impredicative universe in the model of cubical type theory $\mathcal{E}^F$.

4 Presheaf Models

In this section we give a sufficient condition for a presheaf category to satisfy the conditions in Section 2. We will work in a model $\mathcal{S}$ of dependent type theory with dependent product types, dependent sum types, extensional identity types, unit type, disjoint finite coproducts and propositional truncation.

A *category* in $\mathcal{S}$ consists of:

- a type $\vdash \mathbf{C}_0$ of *objects*;
- a type $c_0, c_1 : \mathbf{C}_0 \vdash \mathbf{C}_1(c_0, c_1)$ of *morphisms*;
- a term $c : \mathbf{C}_0 \vdash \mathsf{id}_c : \mathbf{C}_1(c, c)$ called *identity*;
- a term $c_0, c_1, c_2 : \mathbf{C}_0, g : \mathbf{C}_1(c_1, c_2), f : \mathbf{C}_1(c_0, c_1) \vdash gf : \mathbf{C}_1(c_0, c_2)$ called *composition*

satisfying the standard axioms of category. We will simply write $\mathbf{C}$ and $\mathbf{C}(c_0, c_1)$ for $\mathbf{C}_0$ and $\mathbf{C}_1(c_0, c_1)$ respectively. The notions of functor and natural transformation in $\mathcal{S}$ are defined in the obvious way. For a category $\mathbf{C}$ in $\mathcal{S}$, a *presheaf* on $\mathbf{C}$ consists of:

- a type $c : \mathbf{C} \vdash A(c)$;
- a term $c_0, c_1 : \mathbf{C}, \sigma : \mathbf{C}(c_0, c_1), a : A(c_1) \vdash a\sigma : A(c_0)$ called *(right)* $\mathbf{C}$*-action*

satisfying $a\mathsf{id} = a$ and $a(\sigma\tau) = (a\sigma)\tau$. For presheaves A and B, a *morphism* $f : A \to B$ is a term $c : \mathbf{C}, a : A(c) \vdash f(a) : B(c)$ satisfying $c_0, c_1 : \mathbf{C}, \sigma : \mathbf{C}(c_0, c_1), a : A(c_1) \vdash f(a\sigma) = f(a)\sigma$. For a presheaf A, its *category of elements*, written $\mathbf{El}(A)$, is defined as

- $\vdash \mathbf{El}(A)_0 := \sum_{c:\mathbf{C}_0} A(c)$;
- $(c_0, a_0), (c_1, a_1) : \mathbf{El}(A)_0 \vdash \mathbf{El}(A)_1((c_0, a_0), (c_1, a_1)) := \{\sigma : \mathbf{C}_1(c_0, c_1) \mid a_1\sigma = a_0\}$.

There is a projection functor $\pi_A : \mathbf{El}(A) \to \mathbf{C}$.

For a category $\mathbf{C}$ in $\mathcal{S}$, we describe the presheaf model $\mathbf{PSh}(\mathbf{C})$ of dependent type theory. Contexts are interpreted as presheaves on $\mathbf{C}$. For a context Γ, types on Γ are interpreted as presheaves on $\mathbf{El}(\Gamma)$. For a type $\Gamma \vdash A$, terms of A are interpreted as sections of the projection $\pi_A : \mathbf{El}(A) \to \mathbf{El}(\Gamma)$. For a type $\Gamma \vdash A$, the context extension $\Gamma.A$ is interpreted as the presheaf $c : \mathbf{C} \vdash \sum_{\gamma:\Gamma(c)} A(c, \gamma)$. This construction is also used for dependent sum types. The dependent product for a type $\Gamma.A \vdash B$ is the presheaf

$$\begin{aligned}(c, \gamma) : \mathbf{El}(\Gamma) \vdash \{f : \prod_{c':\mathbf{C}} \prod_{\sigma:\mathbf{C}(c',c)} \prod_{a:A(c',\gamma\sigma)} B(c', a) \mid \\ \forall_{c',c'':\mathbf{C}} \forall_{\sigma:\mathbf{C}(c',c)} \forall_{\tau:\mathbf{C}(c'',c')} \forall_{a:A(c',\gamma\sigma)} (fc'\sigma a)\tau = fc''(\sigma\tau)(a\tau)\}.\end{aligned}$$

Extensional identity types, unit type, disjoint finite coproducts and propositional truncation are pointwise.

4.1 Lifting Universes

We describe the Hofmann-Streicher lifting of a universe [20]. Let $\mathbf{C}$ be a category in $\mathcal{S}$ and U a universe in $\mathcal{S}$. We define a universe $[\mathbf{C}^{\mathrm{op}}, U]$ in $\mathbf{PSh}(\mathbf{C})$ as follows. The universe U can be seen as a category whose type of objects is U and type of morphisms is $A, B : U \vdash \mathsf{el}_U(A) \to \mathsf{el}_U(B)$. For an object $c : \mathbf{C}$, we define $[\mathbf{C}^{\mathrm{op}}, U](c)$ to be the type of functors from $(\mathbf{C}/c)^{\mathrm{op}}$ to U. The $\mathbf{C}$-action on $[\mathbf{C}^{\mathrm{op}}, U]$ is given by precomposition. The type $[\mathbf{C}^{\mathrm{op}}, U] \vdash \mathsf{el}_{[\mathbf{C}^{\mathrm{op}},U]}$ in $\mathbf{PSh}(\mathbf{C})$ is defined as $(c, A) : \mathbf{El}([\mathbf{C}^{\mathrm{op}}, U]) \vdash \mathsf{el}_{[\mathbf{C}^{\mathrm{op}},U]}(c, A) := \mathsf{el}_U(A(\mathsf{id}_c))$.

It is easy to show that, if U is an impredicative universe, then dependent product types, dependent sum types and extensional identity types in U can be lifted to those in $[\mathbf{C}^{\mathrm{op}}, U]$ so that $[\mathbf{C}^{\mathrm{op}}, U]$ is an impredicative universe in $\mathbf{PSh}(\mathbf{C})$. If U is a propositional universe in $\mathcal{S}$, then $[\mathbf{C}^{\mathrm{op}}, U]$ is a propositional universe in $\mathbf{PSh}(\mathbf{C})$.

▶ **Proposition 10.** *Let $\mathcal{U}$ be an impredicative universe and* Cof *a propositional universe in $\mathcal{S}$. If they satisfy Axioms 6, 7, 9 and 10, then so do $[\mathbf{C}^{\mathrm{op}}, \mathcal{U}]$ and $[\mathbf{C}^{\mathrm{op}}, \mathsf{Cof}]$.*

Proof. We only check Axiom 10. The other axioms are easy to verify.

We have to define a term $\varphi : [\mathbf{C}^{\mathrm{op}}, \mathsf{Cof}], A : \varphi \to [\mathbf{C}^{\mathrm{op}}, \mathcal{U}], B : [\mathbf{C}^{\mathrm{op}}, \mathcal{U}], f : \prod_{u:\varphi} Au \cong B \vdash (D(\varphi, f), g(\varphi, f)) : \sum_{\bar{A}:[\mathbf{C}^{\mathrm{op}},\mathcal{U}]} \{\bar{f} : \bar{A} \cong B \mid \forall_{u:\varphi}(Au, fu) = (\bar{A}, \bar{f})\}$ in $\mathbf{PSh}(\mathbf{C})$. It corresponds to a natural transformation that takes an object $c : \mathbf{C}$, functors $\varphi : (\mathbf{C}/c)^{\mathrm{op}} \to \mathsf{Cof}$, $A :$

$\mathbf{El}(\varphi)^{\mathrm{op}} \to \mathcal{U}$ and $B : (\mathbf{C}/c)^{\mathrm{op}} \to \mathcal{U}$ and an isomorphism $f : A \cong B\pi_\varphi$ of presheaves on $\mathbf{El}(\varphi)$ and returns a pair $(D(c,\varphi,f), g(c,\varphi,f))$ consisting of a functor $D(c,\varphi,f) : (\mathbf{C}/c)^{\mathrm{op}} \to \mathcal{U}$ and an isomorphism $g(c,\varphi,f) : A \cong B$ of presheaves on $(\mathbf{C}/c)^{\mathrm{op}}$ such that $D(c,\varphi,f)\pi_\varphi = A$ and $g(c,\varphi,f)\pi_\varphi = f$. Let $\sigma : \mathbf{C}(c',c)$ be a morphism. Then we have $\varphi(\sigma) : \mathsf{Cof}$, $\lambda u.A(\sigma,u) : \varphi(\sigma) \to \mathcal{U}$, $B(\sigma) : \mathcal{U}$ and an isomorphism $\lambda u.f(\sigma,u) : \prod_{u:\varphi(\sigma)} A(\sigma,u) \cong B(\sigma)$. By the isomorphism lifting on $\mathcal{U}$, we have $D(c,\varphi,f)(\sigma) : \mathcal{U}$ and an isomorphism $g(c,\varphi,f)(\sigma) : D(c,\varphi,f)(\sigma) \cong B(\sigma)$ such that $\forall_{u:\varphi(\sigma)}(A(\sigma,u), f(\sigma,u)) = (D(c,\varphi,f)(\sigma), g(c,\varphi,f)(\sigma))$. For the morphism part of the functor $D(c,\varphi,f)$, let $\tau : \mathbf{C}(c'',c')$ be another morphism. Then we define $\tau^* : D(c,\varphi,f)(\sigma) \to D(c,\varphi,f)(\sigma\tau)$ to be the composition

$$D(c,\varphi,f)(\sigma) \xrightarrow[\cong]{g(c,\varphi,f)(\sigma)} B(\sigma) \xrightarrow{\quad\tau^*\quad} B(\sigma\tau) \xrightarrow[\cong]{g(c,\varphi,f)(\sigma\tau)^{-1}} D(c,\varphi,f)(\sigma\tau).$$

By definition $g(c,\varphi,f)$ becomes a natural isomorphism and $(D(c,\varphi,f)\pi_\varphi, g(c,\varphi,f)\pi_\varphi) = (A,f)$. It is easy to see the naturality of $(c,\varphi,f) \mapsto (D(c,\varphi,f), g(c,\varphi,f))$. ◀

4.2 Intervals

Suppose a category $\mathbf{C}$ in $\mathcal{S}$ has finite products. A *path connection algebra* in $\mathbf{C}$ consists of an object $\mathbb{I} : \mathbf{C}$, morphisms $\delta_0, \delta_1 : \mathbf{C}(1,\mathbb{I})$ called *end-points* and morphisms $\mu_0, \mu_1 : \mathbf{C}(\mathbb{I}\times\mathbb{I},\mathbb{I})$ called *connections* satisfying $\mu_e(\delta_e \times \mathbb{I}) = \mu_e(\mathbb{I}\times\delta_e) = \delta_e$ and $\mu_e(\delta_{\bar{e}} \times \mathbb{I}) = \mu_e(\mathbb{I}\times\delta_{\bar{e}}) = \mathsf{id}$ for $e \in \{0,1\}$.

For a path connection algebra $\mathbb{I}$ in $\mathbf{C}$, we have a representable presheaf $\mathbf{y}\mathbb{I}$ on $\mathbf{C}$. Since the Yoneda embedding is fully faithful and preserves finite products, $\mathbf{y}\mathbb{I}$ has end-points and connections satisfying Axioms 2 and 3. The interval $\mathbf{y}\mathbb{I}$ satisfies Axiom 1 if and only if $\forall_{c:\mathbf{C}}\delta_0!_c \neq \delta_1!_c$ holds, where $!_c : \mathbf{C}(c,1)$ is the unique morphism into the terminal object.

▶ **Proposition 11.** *Let* Cof *be a propositional universe in* $\mathcal{S}$ *and suppose that, for every pair of objects* $c, c' : \mathbf{C}$*, the equality predicate on* $\mathbf{C}(c,c')$ *belongs to* Cof*. Then, for every object* $c : \mathbf{C}$*, the equality predicate on* $\mathbf{y}c$ *belongs to* $[\mathbf{C}^{\mathrm{op}}, \mathsf{Cof}]$*. In particular,* $\mathbf{y}\mathbb{I}$ *and* $[\mathbf{C}^{\mathrm{op}}, \mathsf{Cof}]$ *in* $\mathbf{PSh}(\mathbf{C})$ *satisfy Axioms 4 and 5.*

Proof. Because equality on a presheaf is pointwise. ◀

▶ **Proposition 12.** *For a functor* $f : \mathbf{C} \to \mathbf{D}$ *between categories in* $\mathcal{S}$*, the precomposition functor* $f^* : \mathbf{PSh}(\mathbf{D}) \to \mathbf{PSh}(\mathbf{C})$ *has a dependent right adjoint* f_**.*

Proof. For a context Γ in $\mathbf{PSh}(\mathbf{D})$ and a type $f^*\Gamma \vdash A$ in $\mathbf{PSh}(\mathbf{C})$, the type $\Gamma \vdash f_*A$ is given by the presheaf $(d,\gamma) : \mathbf{El}(\Gamma) \vdash \lim_{(c,\sigma):(f\downarrow d)} A(c,\gamma\sigma)$. ◀

▶ **Proposition 13.** *Suppose that a category* $\mathbf{C}$ *in* $\mathcal{S}$ *has finite products. For an object* $c : \mathbf{C}$*, the exponential functor* $(\mathbf{y}c \to -) : \mathbf{PSh}(\mathbf{C}) \to \mathbf{PSh}(\mathbf{C})$ *is isomorphic to* $(- \times c)^*$*.*

Proof. $(\mathbf{y}c \to A)(c') \cong \mathbf{PSh}(\mathbf{C})(\mathbf{y}c' \times \mathbf{y}c, A) \cong \mathbf{PSh}(\mathbf{C})(\mathbf{y}(c' \times c), A) \cong A(c' \times c)$. ◀

Hence the exponential functor $(\mathbf{y}\mathbb{I} \to -)$ has a dependent right adjoint. Proposition 13 also implies Axiom 8 for the propositional universe $[\mathbf{C}^{\mathrm{op}}, \mathsf{Cof}]$. Explicitly, $\forall_{\mathbf{y}\mathbb{I}} : (- \times \mathbf{y}\mathbb{I})^*[\mathbf{C}^{\mathrm{op}}, \mathsf{Cof}] \to [\mathbf{C}^{\mathrm{op}}, \mathsf{Cof}]$ is a natural transformation that carries a functor $\varphi : (\mathbf{C}/c \times \mathbb{I})^{\mathrm{op}} \to \mathsf{Cof}$ to $\lambda\sigma.\varphi(\sigma \times \mathbb{I}) : (\mathbf{C}/c)^{\mathrm{op}} \to \mathsf{Cof}$.

In summary, we have:

▶ **Theorem 14.** *Suppose:*

- $\mathcal{S}$ *is a model of dependent type theory with dependent product types, dependent sum types, extensional identity types, unit type, disjoint finite coproducts and propositional truncation;*
- Cof *is a propositional universe and* $\mathcal{U}$ *is an impredicative universe satisfying Axioms 6, 7, 9 and 10;*
- $\mathbf{C}$ *is a category in* $\mathcal{S}$ *with finite products and the equality on* $\mathbf{C}(c, c')$ *belongs to* Cof *for every pair of objects* $c, c' : \mathbf{C}$*;*
- $\mathbb{I}$ *is a path connection algebra in* $\mathbf{C}$*;*
- $\mathbf{y}\mathbb{I}$ *satisfies Axiom 1.*

Then the presheaf model $\mathbf{PSh}(\mathbf{C})$ *together with propositional universe* $[\mathbf{C}^{\mathrm{op}}, \mathsf{Cof}]$*, impredicative universe* $[\mathbf{C}^{\mathrm{op}}, \mathcal{U}]$ *and interval* $\mathbf{y}\mathbb{I}$ *satisfies all the axioms in Figure 1.*

4.3 Decidable Subobject Classifier

An example of the propositional universe Cof in Theorem 14 is the decidable subobject classifier $\mathbf{2}$ which always satisfies Axioms 6, 7 and 9.

▶ **Proposition 15.** *In a model of dependent type theory with dependent product types, dependent sum types, extensional identity types, unit type, disjoint finite coproducts and propositional truncation, any universe* $\mathcal{U}$ *satisfies Axiom 10 with* $\mathsf{Cof} = \mathbf{2}$*.*

Proof. Let $\varphi : \mathbf{2}, A : \varphi \to \mathcal{U}, B : \mathcal{U}, f : \prod_{u:\varphi} Au \cong B$. We define $\mathsf{iea}(\varphi, f)$ by case analysis on $\varphi : \mathbf{2}$ as $\mathsf{iea}(0, f) := (B, \mathsf{id})$ and $\mathsf{iea}(1, f) := (A*, f*)$ where $*$ is the unique element of a singleton type. ◀

4.4 Categories of Cubes

We present examples of internal categories $\mathbf{C}$ with a path connection algebra $\mathbb{I}$ satisfying the hypotheses of Theorem 14 with $\mathsf{Cof} = \mathbf{2}$. Obvious choices of $\mathbf{C}$ are the category of free de Morgan algebras [10] and various syntactic categories of the language $\{0, 1, \sqcap, \sqcup\}$ [8], but some inductive types and quotient types are required to construct these categories in dependent type theory. Although the motivating example of $\mathcal{S}$, the category of assemblies described in Section 5, has inductive types and finite colimits, quotients are not well-behaved in general and we need to be careful in using quotients. Instead, we give examples definable only using natural numbers.

Suppose $\mathcal{S}$ is a model of dependent type theory with dependent product types, dependent sum types, extensional identity types, unit type, disjoint finite coproducts, propositional truncation and natural numbers. We define a type of finite types $n : \mathbb{N} \vdash \mathsf{Fin}_n$ to be $\mathsf{Fin}_n = \{k : \mathbb{N} \mid k < n\}$. We define a category $\mathbf{B}$ as follows. Its object of objects is $\mathbb{N}$. The morphisms $m \to n$ are functions $(\mathsf{Fin}_m \to \mathbf{2}) \to (\mathsf{Fin}_n \to \mathbf{2})$. In the category $\mathbf{B}$, the terminal object is $0 : \mathbb{N}$ and the product of m and n is $m + n$. One can show, by induction, that every $\mathbf{B}(m, n)$ has decidable equality. $\mathbf{B}$ has a path connection algebra $1 : \mathbb{N}$ together with end-points $0, 1 : (\mathsf{Fin}_0 \to \mathbf{2}) \to (\mathsf{Fin}_1 \to \mathbf{2})$ and connections $\mathsf{min}, \mathsf{max} : (\mathsf{Fin}_1 \to \mathbf{2}) \times (\mathsf{Fin}_1 \to \mathbf{2}) \to (\mathsf{Fin}_1 \to \mathbf{2})$. One can show that the category $\mathbf{B}$ satisfies the hypotheses of Theorem 14. Moreover, any subcategory of $\mathbf{B}$ that has the same finite products and contains the path connection algebra 1 satisfies the same condition. An example is the wide subcategory $\mathbf{B}_{\mathsf{ord}}$ of $\mathbf{B}$ where the morphisms are order-preserving functions $(\mathsf{Fin}_m \to \mathbf{2}) \to (\mathsf{Fin}_n \to \mathbf{2})$.

4.5 Constant and Codiscrete Presheaves

We show some properties of constant and codiscrete presheaves which will be used in Section 5. Let $\mathcal{S}$ be a model of dependent type theory satisfying the hypotheses of Theorem 14. For an object $A \in \mathcal{S}$, we define the *constant presheaf* ΔA to be $\Delta A(c) := A$ with the trivial $\mathbf{C}$-action.

▶ **Proposition 16.** *Every constant presheaf* ΔA *is discrete.*

Proof. For every $c : \mathbf{C}$, we have $(\mathbf{y}\mathbb{I} \to \Delta A)(c) \cong \Delta A(c \times \mathbb{I}) = A$ by Proposition 13. ◀

For a type $\Gamma \vdash A$ in $\mathcal{S}$, we define the *codiscrete presheaf* $\Delta\Gamma \vdash \nabla A$ to be $\nabla A(c, \gamma) := \mathbf{C}(1, c) \to A(\gamma)$ with composition as the $\mathbf{C}$-action.

▶ **Proposition 17.** *Suppose that* $\mathsf{Cof} = \mathbf{2}$*. Then for every type* $\Gamma \vdash A$ *in* $\mathcal{S}$*, the type* $\Delta\Gamma \vdash \nabla A$ *has a fibration structure.*

Proof. Since $\Delta\Gamma$ is discrete, it suffices to show that $\nabla A(\gamma)$ has a fibration structure for every $\gamma : \Gamma$. Thus we may assume that Γ is the empty context. We construct a term

$$\alpha : \prod_{e:\mathbf{2}} \prod_{\varphi:[\mathbf{C}^{\mathrm{op}},\mathbf{2}]} \prod_{f:\varphi \to \mathbb{I} \to \nabla A} \prod_{a:\nabla A} (\forall_{u:\varphi} fue = a) \to \{\bar{a} : \nabla A \mid \forall_{u:\varphi} fu\bar{e} = \bar{a}\}$$

in $\mathbf{PSh}(\mathbf{C})$. It corresponds to a natural transformation that takes an object $c : \mathbf{C}$, an element $e : \mathbf{2}$, a functor $\varphi : (\mathbf{C}/c)^{\mathrm{op}} \to \mathbf{2}$, a natural transformation $f : \int_{c' \in \mathbf{C}} (\sum_{\sigma:\mathbf{C}(c',c)} \varphi(\sigma)) \times \mathbf{C}(c', \mathbb{I}) \to \nabla A(c')$ and an element $a : \nabla A(c)$ such that $\forall_{c':\mathbf{C}} \forall_{\sigma:\mathbf{C}(c',c)} \forall_{u:\varphi(\sigma)} f(\sigma, u, e) = a\sigma$ and returns an element $\alpha(e, \varphi, f, a) : \nabla A(c)$ such that $\forall_{c':\mathbf{C}} \forall_{\sigma:\mathbf{C}(c',c)} \forall_{u:\varphi(\sigma)} f(\sigma, u, \bar{e}) = \alpha(e, \varphi, f, a)\sigma$. We define $\alpha(e, \varphi, f, a) : \mathbf{C}(1, c) \to A$ as

$$\alpha(e, \varphi, f, a)(\sigma) := \begin{cases} f(\sigma, u, \bar{e})(\mathsf{id}_1) & \text{if } u : \varphi(\sigma) \text{ is found} \\ a(\sigma) & \text{otherwise} \end{cases}$$

for $\sigma : \mathbf{C}(1, c)$. Then by definition $\forall_{c':\mathbf{C}} \forall_{\sigma:\mathbf{C}(c',c)} \forall_{u:\varphi(\sigma)} f(\sigma, u, \bar{e}) = \alpha(e, \varphi, f, a)\sigma$. ◀

▶ **Proposition 18.** *Suppose that* $\mathbf{C}(1, \mathbb{I})$ *only contains* 0 *and* 1*, namely* $\forall_{\sigma:\mathbf{C}(1,\mathbb{I})} \sigma = 0 \vee \sigma = 1$*. Then for every type* $\Gamma \vdash A$ *in* $\mathcal{S}$*, there exists a term*

$$\Delta\Gamma \vdash p : \prod_{a_0, a_1 : \nabla A} \mathsf{Path}(\nabla A, a_0, a_1)$$

in $\mathbf{PSh}(\mathbf{C})$.

Proof. We may assume that Γ is the empty context. The term p corresponds to a natural transformation that takes an object $c : \mathbf{C}$, elements $a_0, a_1 : \nabla A(c)$ and a morphism $i : \mathbf{C}(c, \mathbb{I})$ and returns an element $p(a_0, a_1, i) : \nabla A(c)$ such that $p(a_0, a_1, 0) = a_0$ and $p(a_0, a_1, 1) = a_1$. We define $p(a_0, a_1, i) : \mathbf{C}(1, c) \to A$ as

$$p(a_0, a_1, i)(\sigma) := \begin{cases} a_0(\sigma) & \text{if } i\sigma = 0 \\ a_1(\sigma) & \text{if } i\sigma = 1 \end{cases}$$

for $\sigma : \mathbf{C}(1, c)$. Then by definition $p(a_0, a_1, 0) = a_0$ and $p(a_0, a_1, 1) = a_1$. ◀

5 A Failure of Propositional Resizing in Cubical Assemblies

An *assembly*, also called an *ω-set*, is a set A equipped with a non-empty set $E_A(a)$ of natural numbers for every $a \in A$. When $n \in E_A(a)$, we say n is a *realizer* for a or n *realizes* a. A morphism $f : A \to B$ of assemblies is a function $f : A \to B$ between the underlying sets such that there exists a partial recursive function e such that, for any $a \in A$ and $n \in E_A(a)$, the application en is defined and belongs to $E_B(f(a))$. In that case we say f is *tracked* by e or e is a *tracker* of f. We shall denote by $\mathbf{Asm}$ the category of assemblies and morphisms of assemblies. Note that assemblies can be defined in terms of partial combinatory algebras instead of natural numbers and partial recursive functions [44], and that the rest of this section works for assemblies on any non-trivial partial combinatory algebra.

The category $\mathbf{Asm}$ is a model of dependent type theory. Contexts are interpreted as assemblies. Types $\Gamma \vdash A$ are interpreted as families of assemblies $(A(\gamma) \in \mathbf{Asm})_{\gamma \in \Gamma}$ indexed over the underlying set of Γ. Terms $\Gamma \vdash a : A$ are interpreted as sections $a \in \prod_{\gamma \in \Gamma} A(\gamma)$ such that there exists a partial recursive function e such that, for any $\gamma \in \Gamma$ and $n \in E_\Gamma(\gamma)$, the application en is defined and belongs to $E_{A(\gamma)}(a(\gamma))$. For a type $\Gamma \vdash A$, the context extension $\Gamma.A$ is interpreted as an assembly $(\sum_{\gamma \in \Gamma} A(\gamma), (\gamma, a) \mapsto \{\langle n, m\rangle \mid n \in E_\Gamma(\gamma), m \in E_{A(\gamma)}(a)\})$ where $\langle n, m\rangle$ is a fixed effective encoding of tuples of natural numbers. It is known that $\mathbf{Asm}$ supports dependent product types, dependent sum types, extensional identity types, unit type, disjoint finite coproducts and natural numbers. See, for example, [44, 28, 25]. For a family of assemblies A over Γ, the propositional truncation $\|A\|$ is the family

$$\|A\|(\gamma) = \begin{cases} \{*\} & \text{if } A(\gamma) \neq \emptyset \\ \emptyset & \text{if } A(\gamma) = \emptyset \end{cases}$$

with realizers $E_{\|A\|(\gamma)}(*) = \bigcup_{a \in A(\gamma)} E_{A(\gamma)}(a)$.

It is also well-known that $\mathbf{Asm}$ has an impredicative universe $\mathbf{PER}$. It is an assembly whose underlying set is the set of partial equivalence relations, namely symmetric and transitive relations, on $\mathbb{N}$ and the set of realizers of R is $E_{\mathbf{PER}}(R) = \{0\}$. The type $\mathbf{PER} \vdash \mathsf{el}_{\mathbf{PER}}$ is defined as $\mathsf{el}_{\mathbf{PER}}(R) = \mathbb{N}/R$, the set of R-equivalence classes on $\{n \in \mathbb{N} \mid R(n, n)\}$ with realizers $E_{\mathbb{N}/R}(\xi) = \xi$. The universe $\mathbf{PER}$ classifies *modest families*. An assembly A is said to be *modest* if $E_A(a)$ and $E_A(a')$ are disjoint for distinct $a, a' \in A$. By definition $\mathbb{N}/R$ is modest for every $R \in \mathbf{PER}$. Conversely, for a modest assembly A, one can define a partial equivalence relation R such that $A \cong \mathbb{N}/R$. For the impredicativity of $\mathbf{PER}$, see [23, 28, 25].

The category $\mathbf{Asm}$ satisfies the hypotheses of Theorem 14 with impredicative universe $\mathbf{PER}$, propositional universe $\mathbf{2}$ and the internal category $\mathbf{B}_{\mathsf{ord}}$ defined in Section 4.4. We will refer to the presheaf model of cubical type theory generated by these structures as the *cubical assembly model*.

5.1 Propositional Resizing

In cubical type theory, a type $\Gamma \vdash A$ is a *homotopy proposition* if the type $\Gamma, a_0, a_1 : A \vdash \mathsf{Path}(A, a_0, a_1)$ has an inhabitant. For a universe $\mathcal{U}$, we define the universe of homotopy propositions as

$$\mathsf{hProp}_{\mathcal{U}} := \sum_{A : \mathcal{U}} \prod_{a_0, a_1 : A} \mathsf{Path}(A, a_0, a_1).$$

Following the HoTT book [33], we regard $\mathsf{hProp}_{\mathcal{U}}$ as a subtype of $\mathcal{U}$.

The *propositional resizing axiom* [33, Section 3.5] asserts that, for nested universes $\mathcal{U} : \mathcal{U}'$, the inclusion $\mathsf{hProp}_{\mathcal{U}} \to \mathsf{hProp}_{\mathcal{U}'}$ is an equivalence. When $\mathcal{U}$ is an impredicative universe, we define

$$A : \mathsf{hProp}_{\mathcal{U}'} \vdash A^* := \prod_{X:\mathsf{hProp}_{\mathcal{U}}} (A \to X) \to X : \mathsf{hProp}_{\mathcal{U}}$$
$$A : \mathsf{hProp}_{\mathcal{U}'} \vdash \eta_A := \lambda a.\lambda X f.fa : A \to A^*.$$

If η_A is an equivalence for any $A : \mathsf{hProp}_{\mathcal{U}'}$, then the inclusion $\mathsf{hProp}_{\mathcal{U}} \to \mathsf{hProp}_{\mathcal{U}'}$ is an equivalence by univalence. Conversely, if the inclusion $\mathsf{hProp}_{\mathcal{U}} \to \mathsf{hProp}_{\mathcal{U}'}$ is an equivalence, then one can find $A' : \mathsf{hProp}_{\mathcal{U}}$ and $e : A \simeq A'$ from $A : \mathsf{hProp}_{\mathcal{U}'}$. Then we have a function $\lambda\alpha.e^{-1}(\alpha A' e) : A^* \to A$, and thus η_A is an equivalence because both A and A^* are homotopy propositions. Note that the construction $A \mapsto (A^*, \eta_A)$ works for any homotopy proposition A and is independent of the choice of the upper universe $\mathcal{U}'$. Therefore, we can formulate the propositional resizing axiom in cubical type theory with an impredicative universe as follows.

▶ **Axiom 19.** *For every homotopy proposition $\Gamma \vdash A$, the function $\Gamma \vdash \eta_A : A \to A^*$ is an equivalence.*

We will show that the cubical assembly model does not satisfy Axiom 19.

▶ Remark 20. We focus on resizing propositions into the impredicative universe. The cubical assembly model also has predicative universes, assuming the existence of Grothendieck universes in the metatheory. It remains an open question whether the predicative universes in the cubical assembly model satisfy the propositional resizing axiom.

5.2 Uniform Objects

The key idea to a counterexample to propositional resizing is the orthogonality of modest and *uniform* assemblies [44]: if X is modest and A is uniform and well-supported, then the map $\lambda xa.x : X \to (A \to X)$ is an isomorphism. Since the impredicative universe **PER** classifies modest assemblies, $\prod_{X:\mathbf{PER}}(A \to X) \to X$ is always inhabited for a uniform, well-supported assembly A. We extend the notion of uniformity for internal presheaves in **Asm**.

An assembly A is said to be *uniform* if $\bigcap_{a\in A} E_A(a)$ is non-empty. We say an internal presheaf A on an internal category $\mathbf{C}$ is *uniform* if every $A(c)$ is uniform. An internal presheaf A on $\mathbf{C}$ is said to be *well-supported* if the unique morphism into the terminal presheaf is regular epi. For an internal presheaf A, the following are equivalent:

- A is well-supported;
- $\|A\|$ is the terminal presheaf;
- there exists a partial recursive function e such that, for any $c \in \mathbf{C}_0$ and $n \in E_{\mathbf{C}_0}(c)$, there exists an $a \in A(c)$ such that en is defined and belongs to $E_A(a)$.

By definition a modest assembly cannot distinguish elements with a common realizer, while elements of a uniform assembly have a common realizer. Thus a modest assembly "believes a uniform assembly has at most one element". Formally, the following proposition holds.

▶ **Proposition 21.** *Let $\mathbf{C}$ be a category in* **Asm**. *For a uniform internal presheaf A on $\mathbf{C}$ and an internal functor $X : \mathbf{C}^{\mathrm{op}} \to \mathbf{PER}$, the precomposition function*

$$i^* : (\|A\| \to X) \to (A \to X)$$

is an isomorphism, where $i : A \to \|A\|$ is the constructor for propositional truncation. In particular, if, in addition, A is well-supported, then the function $\lambda xa.x : X \to (A \to X)$ is an isomorphism.

Proof. Since i is regular epi, i^* is a monomorphism. Hence it suffices to show that i^* is regular epi. Let k_c denote a common realizer of $A(c)$, namely $k_c \in \bigcap_{a \in A(c)} E(a)$. Let $c \in \mathbf{C}_0$ be an object and $x : \mathbf{y}c \times A \to X$ a morphism of presheaves tracked by e. We have to show that there exists a morphism $\hat{x} : \mathbf{y}c \times \|A\| \to X$ such that $\hat{x} \circ (\mathbf{y}c \times i) = x$ and that a tracker of $\hat{x}$ is computable from the code of e. For any $\sigma : c' \to c$ and $a, a' \in A(c')$, we have $enk_{c'} \in E(x(\sigma, a)) \cap E(x(\sigma, a'))$ for some $n \in E(\sigma)$. Since $X(c')$ is modest, we have $x(\sigma, a) = x(\sigma, a')$. Hence x induces a morphism of presheaves $\hat{x} : \mathbf{y}c \times \|A\| \to X$ tracked by e such that $\hat{x} \circ (\mathbf{y}c \times i) = x$. ◀

▶ **Theorem 22.** *Let $\Gamma \vdash A$ be a type in the cubical assembly model. Suppose that A is uniform and well-supported as an internal presheaf on $\mathbf{El}(\Gamma)$ and does not have a section. Then the function $\Gamma \vdash \eta : A \to A^*$ is not an equivalence.*

Proof. By Proposition 21, we see that $A^* = \prod_{X:\mathsf{hProp}}(A \to X) \to X$ has an inhabitant while A does not have an inhabitant by assumption. ◀

▶ **Theorem 23.** *Let $\Gamma \vdash A$ be a type in $\mathbf{Asm}$. Suppose that A is uniform and well-supported but does not have a section. Then the function $\Delta\Gamma \vdash \eta : \nabla A \to (\nabla A)^*$ is not an equivalence.*

Proof. By Theorem 22, it suffices to show that the type $\Delta\Gamma \vdash \nabla A$ is uniform and well-supported but does not have a section. For the uniformity, let k_γ be a common realizer of $A(\gamma)$ for $\gamma \in \Gamma$. For any object $c \in \mathbf{C}$ and element $\gamma \in \Gamma$, the code of the constant function $n \mapsto k_\gamma$ is a common realizer of $\nabla A(c, \gamma) = \mathbf{C}(1, c) \to A(\gamma)$.

For the well-supportedness, let e be a partial recursive function such that, for any γ and $n \in E_\Gamma(\gamma)$, there exists an $a \in A(\gamma)$ such that en is defined and belongs to $E_{A(\gamma)}(a)$. Then the function f mapping (n, x) to the code of the function $y \mapsto ex$ realizes that ∇A is well-supported. Indeed, for any $c \in \mathbf{C}$, $n \in E_{\mathbf{C}}(c)$, $\gamma \in \Gamma$ and $x \in E_\Gamma(\gamma)$, the code $f(n, x)$ realizes the constant function $\mathbf{C}(1, c) \ni \sigma \mapsto a \in A(\gamma)$ for some $a \in A(\gamma)$ such that $ex \in E_{A(\gamma)}(a)$.

Finally ∇A does not have a section because $\nabla A(1) \cong A$ and A does not have a section. ◀

5.3 The Counterexample

We define an assembly Γ to be $(\mathbb{N}, n \mapsto \{m \in \mathbb{N} \mid m > n\})$ and a family of assemblies A on Γ as $A(n) = (\{m \in \mathbb{N} \mid m > n\}, m \mapsto \{n, m\})$. Then A is uniform because every $A(n)$ has a common realizer n. The identity function realizes that A is well-supported. To see that A does not have a section, suppose that a section $f \in \prod_{n \in \Gamma} A(n)$ is tracked by a partial recursive function e. Then for any $m > n$, we have $em \in \{n, f(n)\}$. This implies that $m \leq e(m+1) \leq f(0)$ for any m, a contradiction. Note that this construction of $\Gamma \vdash A$ works for any non-trivial partial combinatory algebra C because natural numbers can be effectively encoded in C.

Since $\mathbf{B}_{\mathsf{ord}}(1, \mathbb{I}) \cong \mathbf{2}$ only contains end-points, the type $\Delta\Gamma \vdash \nabla A$ in the cubical assembly model is a fibration and homotopy proposition by Propositions 17 and 18, while by Theorem 23 the function $\Delta\Gamma \vdash \eta : \nabla A \to (\nabla A)^*$ is not an equivalence. Hence the propositional resizing axiom fails in the cubical assembly model.

6 Conclusion and Future Work

We have formulated the axioms for modeling cubical type theory in an elementary topos given by Orton and Pitts [30] in a weaker setting and explained how to construct a model of cubical type theory in a category satisfying those axioms. As a striking example, we have constructed a model of cubical type theory with an impredicative and univalent universe in the category of cubical assemblies which is not an elementary topos. It has turned out that this impredicative universe in the cubical assembly model does not satisfy the propositional resizing axiom.

There is a natural question: can we construct a model of type theory with a univalent and impredicative universe satisfying the propositional resizing axiom? One possible approach to this question is to consider a full subcategory of the category of cubical assemblies in which every homotopy proposition is equivalent to some modest family. Benno van den Berg [43] constructed a model of a variant of homotopy type theory with a univalent and impredicative universe of 0-types that satisfies the propositional resizing axiom. Roughly speaking he uses a category of degenerate trigroupoids in the category of *partitioned assemblies* [44], and thus the category of cubical partitioned assemblies is a candidate for such a full subcategory. However, the model given in [43] only supports weaker forms of identity types and dependent product types, and it is unclear whether it can be seen as a model of ordinary homotopy type theory.

Higher inductive types are another important feature of homotopy type theory. One can construct some higher inductive types including propositional truncation in the cubical assembly model [42], internalizing the construction of higher inductive types in cubical sets [12] using W-types with reductions [41]. An open question, raised by Steve Awodey, is whether these higher inductive types are equivalent to their impredicative encodings.

The cubical assembly model is a realizability-based model of type theory with higher dimensional structures, but it does not seem to be what should be called a *realizability ∞-topos*, a higher dimensional analogue of a realizability topos [44]. One problem is that, in the cubical assembly model, realizers seem to play no role in its internal cubical type theory, because the existence of a realizer of a homotopy proposition does not imply the existence of a section of it. Indeed, the cubical assembly model does not satisfy Church's Thesis [42] which holds in the effective topos [22]. One can nevertheless find a left exact localization of the cubical assembly model in which Church's Thesis holds [42].

Our construction of models of cubical type theory is a syntactic one following Orton and Pitts [30]. The original idea of using the internal language of a topos to construct models of cubical type theory was proposed by Coquand [11]. There are also semantic and categorical approaches. Frumin and van den Berg [16] presented a way of constructing a model structure on a full subcategory of an elementary topos with a path connection algebra, which is essentially same as the model structure on the category of fibrant cubical sets described by Spitters [37]. Since they make no essential use of subobject classifiers, we conjecture that one can construct a model structure on a full subcategory of a suitable locally cartesian closed category with a path connection algebra. Sattler [34], based on his earlier work with Gambino [17], gave a construction of a right proper combinatorial model structure on a suitable category with an interval object. Although Gambino and Sattler use Garner's small object argument [18] which requires the cocompleteness of underlying categories, their construction is expected to work for non-cocomplete categories such as the category of cubical assemblies using Swan's small object argument over codomain fibrations [40, 41].

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A Details of Composition for Gluing and Universe

We give explicit definitions of composition operations for gluing and universes described in Section 3.4.

Before that, we introduce some notations. for a fibration $\Gamma, i : \mathbb{I} \vdash A(i)$, one can derive the *composition operation*

$$\frac{\Gamma \vdash \varphi : \mathsf{Cof} \qquad \Gamma, i : \mathbb{I} \vdash f(i) : \varphi \to A(i) \qquad \begin{array}{c}\Gamma \vdash e : \mathbf{2} \\ \Gamma \vdash a : A(e)\end{array} \qquad \Gamma, u : \varphi \vdash f(e)u = a}{\Gamma \vdash \mathsf{comp}^i_e(A(i), f(i), a) : A(\bar{e})}$$

such that $\Gamma, u : \varphi \vdash f(\bar{e})u = \mathsf{comp}^i_e(A(i), f(i), a)$. Concretely, for a fibration structure $\alpha : \mathsf{Fib}(A)$, we define

$$\gamma : \Gamma \vdash \mathsf{comp}^i_e(A(i), f(i), a) := \alpha(\lambda i.(\gamma, i), e, \varphi, \lambda ui.f(i)u, a).$$

In the notation $\mathsf{comp}^i_e(A(i), f(i), a)$, the variable i is considered to be bound. Usually we use the composition operation in the form of

$$\mathsf{comp}^i_e(A(i), [(u_1 : \varphi_1) \mapsto g_1(u_1, i), \dots, (u_n : \varphi_n) \mapsto g_n(u_n, i)], a)$$

with a system $[(u_1 : \varphi_1) \mapsto g_1(u_1, i), \dots, (u_n : \varphi_n) \mapsto g_n(u_n, i)] : \varphi_1 \vee \cdots \vee \varphi_n \to A(i)$.

A.1 Some Derived Notions and Operations

We recall some notions and operations derivable in cubical type theory without gluing and universes.

Composition operations are preserved by function application [10, Section 5.2]: one can derive an operation

$$\frac{\Gamma \vdash \varphi : \mathsf{Cof} \qquad \begin{array}{c}\Gamma, i : \mathbb{I} \vdash h(i) : A(i) \to B(i) \\ \Gamma, i : \mathbb{I} \vdash f(i) : \varphi \to A(i)\end{array} \qquad \begin{array}{c}\Gamma \vdash e : \mathbf{2} \\ \Gamma \vdash a : A(e)\end{array} \qquad \Gamma, u : \varphi \vdash f(e)u = a}{\Gamma \vdash \mathsf{pres}^i_e(h(i), f(i), a) : \mathsf{Path}(B(\bar{e}), c_1, c_2)}$$

such that $\Gamma, u : \varphi, j : \mathbb{I} \vdash h(\bar{e})(f(\bar{e})u) = \mathsf{pres}^i_e(h(i), f(i), a)j$, where $c_1 = \mathsf{comp}^i_e(B(i), h(i) \circ f(i), h(e)a)$ and $c_2 = h(\bar{e})(\mathsf{comp}^i_e(A(i), f(i), a))$.

Equivalences are characterized by a kind of extension property [10, Section 5.3]: for fibrations $\Gamma \vdash A$ and $\Gamma \vdash B$, one can derive an operation

$$\frac{\Gamma \vdash e : \mathbf{2} \qquad \Gamma \vdash \varphi : \mathsf{Cof} \qquad \begin{array}{c}\Gamma \vdash f : A \simeq B \\ \Gamma \vdash b : B\end{array} \qquad \Gamma \vdash p : \varphi \to \sum_{a:A} \mathsf{Path}(B, b, fa)}{\Gamma \vdash \mathsf{equiv}(f, p, b) : \sum_{a:A} \mathsf{Path}(B, b, fa)}$$

such that $\Gamma, u : \varphi \vdash pu = \mathsf{equiv}(f, p, b)$.

For a fibration $\Gamma, i : \mathbb{I} \vdash A(i)$, we define a function called *transport* $\Gamma, e : \mathbf{2} \vdash \mathsf{tp}^i_e(A(i)) : A(e) \to A(\bar{e})$ to be $\mathsf{tp}^i_e(A(i))a = \mathsf{comp}^i_e(A(i), [], a)$. This function $\mathsf{tp}^i_e(A(i))$ is an equivalence [10, Section 7.1].

A.2 Gluing

Proof of Proposition 8. Let $p : \mathbb{I} \to \Gamma$, $e : \mathbf{2}$, $\psi : \mathsf{Cof}$, $g : \psi \to \prod_{i:\mathbb{I}} \prod_{u:\varphi(pi)} A(u)$, $h : \psi \to \prod_{i:\mathbb{I}} B(pi)$, $a : \prod_{u:\varphi(pe)} A(u)$ and $b : B(pe)$, and suppose $\forall_{v:\psi} \forall_{i:\mathbb{I}} \forall_{u:\varphi(pi)} f(u)(gviu) = hvi$, $\forall_{u:\varphi(pe)} f(u)(au) = b$ and $\forall_{v:\psi} gve = a \wedge hve = b$. We have to find elements $\bar{a} : \prod_{u:\varphi(p\bar{e})} A(u)$ and $\bar{b} : B(p\bar{e})$ such that $\forall_{u:\varphi(p\bar{e})} f(u)(\bar{a}u) = \bar{b}$ and $\forall_{v:\psi} gv\bar{e} = \bar{a} \wedge hv\bar{e} = \bar{b}$. We define

$$\begin{aligned}
\bar{b}_1 &:= \mathsf{comp}^i_e(B(pi), [(v : \psi) \mapsto hvi], b) : B(p\bar{e}) \\
\delta &:= \forall_{i:\mathbb{I}} \varphi(pi) : \mathsf{Cof} \\
\bar{a}_1 &:= \lambda w.\mathsf{comp}^i_e(A(wi), [(v : \psi) \mapsto gvi(wi)], a(we)) : \prod_{w:\delta} A(w\bar{e}) \\
q &: \prod_{w:\delta} \mathsf{Path}(\bar{b}_1, f(w\bar{e})(\bar{a}_1 w)) \\
qw &:= \mathsf{pres}^i_e(f(wi), [(v : \psi) \mapsto gvi(wi)], a(we)) \\
\bar{a} &: \prod_{u:\varphi(p\bar{e})} A(u) \\
q_2 &: \prod_{u:\varphi(p\bar{e})} \mathsf{Path}(\bar{b}_1, f(u)(\bar{a}u)) \\
(\bar{a}u, q_2 u) &:= \mathsf{equiv}(f(u), [(w : \delta) \mapsto (\bar{a}_1 w, qw), (v : \psi) \mapsto (gv\bar{e}u, \lambda i.\bar{b}_1)], \bar{b}_1) \\
\bar{b} &:= \mathsf{comp}^i_0(B(p\bar{e}), [(u : \varphi(p\bar{e})) \mapsto q_2 ui, (v : \psi) \mapsto hv\bar{e}], \bar{b}_1) : B(p\bar{e})
\end{aligned}$$

Then one can derive that $\bar{b} = q_2 u1 = f(u)(\bar{a}u)$ for $u : \varphi(p\bar{e})$ and that $\bar{a} = gv\bar{e}$ and $\bar{b} = hv\bar{e}$ for $v : \psi$. Moreover, for every $w : \prod_{i:\mathbb{I}} \varphi(pi)$, we have $\bar{a}(w\bar{e}) = \bar{a}_1 w = \mathsf{comp}^i_e(A(wi), [(v : \psi) \mapsto gvi(wi)], a(we))$ which means the preservation of fibration structure by the function $\Gamma, u : \varphi \vdash \lambda(a, b).au : \mathsf{Glue}(\varphi, f) \to A(u)$. ◀

A.3 Universes

Proof of Proposition 9. Let $e : \mathbf{2}$, $\varphi : \mathsf{Cof}$, $f : \varphi \to \mathbb{I} \to \mathcal{U}^F$ and $B : \mathcal{U}^F$ such that $\forall_{u:\varphi} fue = B$. We have to find a $\bar{B} : \mathcal{U}^F$ such that $\forall_{u:\varphi} fu\bar{e} = \bar{B}$. Let $A := \lambda u.fu\bar{e} : \varphi \to \mathcal{U}^F$. We have an equivalence $g := \lambda u.\mathsf{tp}^i_{\bar{e}}(fui) : \prod_{u:\varphi} Au \simeq B$. Let $\bar{B} := \mathsf{SGlue}(\varphi, g) : \mathcal{U}^F$, then $\forall_{u:\varphi} fu\bar{e} = Au = \bar{B}$. ◀