Report from Dagstuhl Seminar 19271

Graph Colouring: from Structure to Algorithms

Edited by
Maria Chudnovsky[1], Daniel Paulusma[2], and Oliver Schaudt[3]

1 **Princeton University, US,** mchudnov@math.princeton.edu
2 **Durham University, GB,** daniel.paulusma@durham.ac.uk
3 **RWTH Aachen, DE,** schaudt@mathc.rwth-aachen.de

Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 19271 "Graph Colouring: from Structure to Algorithm", which was held from 30 June to 5 July 2019. The report contains abstracts for presentations about recent structural and algorithmic developments for the Graph Colouring problem and variants of it. It also contains a collection of open problems on graph colouring which were posed during the seminar.

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1 Executive Summary

Maria Chudnovsky (Princeton University, US)
Daniel Paulusma (Durham University, GB)
Oliver Schaudt (RWTH Aachen, DE)

The Graph Colouring problem is to label the vertices of a graph with the smallest possible number of colours in such a way that no two neighbouring vertices are identically coloured. Graph Colouring has been extensively studied in Computer Science and Mathematics due to its many application areas crossing disciplinary boundaries. Well-known applications of Graph Colouring include map colouring, job or timetable scheduling, register allocation, colliding data or traffic streams, frequency assignment and pattern matching. However, Graph Colouring is known to be computationally hard even if the number of available colours is limited to 3.

The central research aim of our seminar was to increase our understanding of the computational complexity of the Graph Colouring problem and related NP-complete colouring problems, such as Precolouring Extension, List Colouring and H-Colouring. The approach followed at the seminar for achieving this aim was to restrict the input of a colouring problem to some special graph class and to determine wether such a restriction could make the problem tractable.

As input restriction, the main focus was to consider hereditary graph classes, which are those classes of graphs that are closed under vertex deletion. Hereditary graph classes provide a unified framework for a large collection of well-known graph classes. The reason for this is that a graph class is hereditary if and only if it can be characterized by a (unique) set $\mathcal{H}$ of minimal forbidden induced subgraphs. This property enables a *systematic* study into the computational complexity of a graph problem under input restrictions. For instance, one can first restrict the input to some hereditary graph class for which $\mathcal{H}$ is small, say $\mathcal{H}$ has size 1 or 2, or for which $\mathcal{H}$ consists of small graphs only.

In line with the seminar's research aim, the seminar brought together researchers from Discrete Mathematics, working in structural graph theory, and researchers from Theoretical Computer Science, working in algorithmic graph theory. In total, 45 participants participated from 14 different countries.

The scientific program of the seminar consisted of 23 sessions: 4 one-hour survey talks, 17 contributed talks of at most thirty minutes and 2 open problem sessions. This left ample time for discussions and problem solving.

Each of the four survey talks covered a particular structural or algorithmic key aspect of the seminar to enable collaborations of researchers with different backgrounds. On Monday, Sophie Sprikl presented a state-of–the-art summary of the Graph Colouring problem for H-free graphs and gave the main ideas and techniques behind an important, recent result in the area, namely a polynomial-time algorithm for colouring P_6-free graphs with at most four colours. On Tuesday, Marcin Pilipczuk gave a tutorial on the framework of minimal chordal completions and potential maximal cliques. This technique plays a crucial role for solving the Maximum Independent Set problem on some hereditary graph classes, but has a much wider applicability. On Wednesday, Bart Jansen gave a presentation on the parameterized complexity of the Graph Colouring problem and related colouring problems. Due to a large variety of possible parameterizatons, Jansen's talk covered a wide range of open problems. On Thursday, Konrad Dabrowski gave an introduction to the clique-width of hereditary graph classes. If a graph class has bounded clique-width, then Graph Colouring and many other NP-hard problems become polynomial-time solvable. Hence, as a first step in the design of a polynomial-time algorithm, one may first want to verify if the clique-width (or any equivalent width parameter) of the graph class under consideration is bounded.

The two general open problem sessions took place on Monday and Tuesday afternoon. Details of the presented problems can be found in the report, together with abstracts of all the talks.

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3 Overview of Talks

3.1 Revisiting a theorem by Folkman on graph colouring

Marthe Bonamy (University of Bordeaux, FR)

Joint work of Marthe Bonamy, Pierre Charbit, Oscar Defrain, Gwenaël Joret, Aurélie Lagoutte, Vincent Limouzy, Lucas Pastor, Jean-Sébastien Sereni
Main reference Marthe Bonamy, Pierre Charbit, Oscar Defrain, Gwenaël Joret, Aurélie Lagoutte, Vincent Limouzy, Lucas Pastor, Jean-Sébastien Sereni: "Revisiting a theorem by Folkman on graph colouring", CoRR, Vol. arXiv:1907.11429, 2019.
URL https://arxiv.org/abs/1907.11429v1

We give a short proof of the following theorem due to Jon H. Folkman [1]: The chromatic number of any graph is at most 2 plus the maximumover all sub-graphs of the difference between half the number of vertices and the independencenumber.

References

1 Folkman, J. H.. *An upper bound on the chromatic number of a graph* (No. RM-5808-PR). RAND CORP SANTA MONICA CALIF, 1969.

3.2 On an augmenting graph approach for the maximum-weight independent set problem

Christoph Brause (TU Bergakademie Freiberg, DE)

The augmenting graph technique is an approach that solves the maximum independent set problem in various graph classes polynomially. Although we know a little about this technique, our knowledge about implementations for the maximum-weight independent set problem is very limited.

In this talk, we present a polynomial-time augmenting graph approach for the weighted version and some suitable graph classes, e.g. subclasses of $S_{1,k,k}$-free graphs, and consider a combination with decompositions by clique separators.

3.3 Introduction to Clique-width and Open Problems

Konrad Dabrowski (Durham University, GB)

Joint work of Konrad K. Dabrowski, Matthew Johnson, Daniël Paulusma
Main reference Konrad K. Dabrowski, Matthew Johnson, Daniël Paulusma: "Clique-Width for Hereditary Graph Classes," London Mathematical Society Lecture Note Series 456:1-56, Cambridge University Press, 2019.
URL https://doi.org/10.1017/9781108649094.002

Graphs classes of bounded clique-width are interesting from a computational perspective, because many problems, such as Colouring, are polynomial-time solvable on such classes. I will give an introduction to clique-width and explain some of the techniques at our disposal when dealing with this parameter. I will also present a number of open problems on boundedness of clique-width for various graph classes and some related problems on Colouring. See also our survey https://arxiv.org/abs/1901.00335

3.4 Coloring graphs by forbidden induced subgraphs

Chinh T. Hoàng (Wilfrid Laurier University – Waterloo, CA)

Joint work of Yingjun Dai, Dallas J. Fraser, Angèle M. Hamel, Chính T. Hoàng, Frédéric Maffray
Main reference Dallas J. Fraser, Angèle M. Hamel, Chính T. Hoàng, Frédéric Maffray: "A coloring algorithm for -free line graphs", Discrete Applied Mathematics, Vol. 234, pp. 76–85, 2018.
URL https://doi.org/10.1016/j.dam.2017.06.006

Let F_4 be a set of four-vertex graphs. For any set F_4, it is known that COLORING F_4-free graphs is NP-hard or solvable in polynomial time, except when F_4 is one of the following three sets: {claw, $4K_1$}, {claw, $4K_1$, co-diamond}, $\{4K_1, C_4\}$. In this talk, we survey recent advances on these three open problems. We will discuss the two tools that have been proved to be useful in attacking the problems: perfect graph theory, and the theory of clique width.

3.5 Shitov's Counterexample to Hedetniemi Conjecture

Shenwei Huang (Nankai University – Tianjin, CN)

Hedetniemi conjecture is a well-known conjecture in the study of graph coloring. It remained open for 53 years until two months ago Shitov came up with a counterexample and hence disproved the conjecture. The proof is basic but elegant. In this talk, we will present the proof of Shitov's counterexample.

3.6 Parameterized Complexity of Graph Coloring Problems

Bart Jansen (TU Eindhoven, NL)

This talks surveys various aspects of the parameterized complexity of graph coloring problems. The goal is to understand how certain complexity parameters contribute to the difficulty of finding exact solutions to such problems. We discuss results in various parameterized algorithmic regimes, and point out open problems wherever possible. The regimes we consider are:

- Fixed-parameter tractable algorithms, for parameterizations that capture the structural complexity of the input graph. We will look at questions such as: if graph G is only k vertex deletions away from belonging to a graph class where coloring is easy, then can the coloring problem on G by solved in $f(k)n^c$ time for some function f and constant c?
- Fixed-parameter tractable algorithms that work on a decomposition of the input graph. Given a graph G and a tree decomposition of width w, one can test the q-colorability of G in time $O^*(q^w)$, which is essentially optimal assuming the Strong Exponential Time Hypothesis. We will see how working over a linear layout of cutwidth w allows the problem to be solved much faster, by exploiting an interesting connection to the rank a matrix that encodes the compatibility of colorings on two sides of small edge cut.

- Fixed-parameter tractable algorithms for parameterizations that measure how far the input graph violates conditions that guarantee the existence of a good coloring. Brooks' theorem guarantees that any graph G that is not a clique or odd cycle, can be colored with $\Delta(G)$ colors. Hence it is easy to test if a graph whose vertices have degree at most q, can be q-colored. How hard is it to test if G has a coloring with q colors, when only k vertices of G have degree more than q?
- Kernelization algorithms. Let k be a parameter that captures the structural complexity of the input graph – for example, the size of a minimum vertex cover. Is it possible to preprocess an input G in polynomial time, obtaining a graph G' of size polynomial in k, so that G has a 3-coloring if and only if G' has one? What is the best upper-bound on the size of G' in terms of k?

3.7 Classes with no long cycle as a vertex-minor are polynomially chi-bounded

O-joung Kwon (Incheon National University, KR)

Joint work of O-joung Kwon, Ringi Kim, Sang-Il Oum, Vaidy Sivaraman

A class $\mathcal{G}$ of graphs is χ-bounded if there is a function f such that for every graph $G \in \mathcal{G}$ and every induced subgraph H of G, $\chi(H) \leq f(\omega(H))$. In addition, we say that $\mathcal{G}$ is *polynomially χ-bounded* if f can be taken as a polynomial function. We prove that for every integer $n \geq 3$, there exists a polynomial f such that $\chi(G) \leq f(\omega(G))$ for all graphs with no vertex-minor isomorphic to the cycle graph C_n. To prove this, we show that if $\mathcal{G}$ is polynomially χ-bounded, then so is the closure of $\mathcal{G}$ under taking the 1-join operation.

3.8 The size Ramsey number of graphs with bounded treewidth

Anita Liebenau (UNSW Sydney, AU)

Joint work of Anita Liebanau, Nina Kamcev, David R Wood, Liana Yepremyan
Main reference Nina Kamcev, Anita Liebenau, David Wood, Liana Yepremyan: "The size Ramsey number of graphs with bounded treewidth", CoRR, Vol. abs/1906.09185, 2019.
URL http://arxiv.org/abs/1906.09185

A graph G is Ramsey for a graph H if every 2-colouring of the edges of G contains a monochromatic copy of H (not necessarily induced). The size Ramsey number of H is the smallest number of edges of a graph G that is Ramsey for H. This parameter received a lot of attention, in particular for sparse graphs H. We generalise earlier work and show that if the maximum degree and treewidth of H are bounded, then the size Ramsey number is linear in $|V(H)|$.

3.9 Cliquewidth III: The Odd Case of Graph ColoringParameterized by Cliquewidth

Daniel Lokshtanov (University of California – Santa Barbara, US)

Joint work of Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov, Saket Saurabh, Meirav Zehaviv
Main reference Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov, Saket Saurabh, Meirav Zehavi: "Clique-width III: Hamiltonian Cycle and the Odd Case of Graph Coloring", ACM Trans. Algorithms, Vol. 15(1), pp. 9:1–9:27, 2019.
URL https://doi.org/10.1145/3280824

Max-Cut(MC), Edge Dominating Set(EDS), Graph Coloring(GC) and Hamiltonian Path(HP) on graphs of bounded cliquewidth have received significant attention, as they can be formulated in MSO2 (and therefore have linear-time algorithms on bounded treewidth graphs by the celebrated Courcelle's theorem), but cannot be formulated in MSO1 (which would have yielded linear-time algorithms on bounded cliquewidth graphs by a well-known theorem of Courcelle, Makowsky, and Rotics). Each of these problems can be solved in time $g(k)\ n^{f(k)}$ on graphs of cliquewidth k. Fomin et al. [Intractability of Clique-Width Parameterizations. SIAM J. Comput. 39(5): 1941-1956 (2010)] showed that the running times cannot be improved to $g(k)n^{O(1)}$ assuming $W[1] \neq FPT$. However, this does not rule out non-trivial improvements to the exponent $f(k)$ in the running times. In a follow-up paper, Fomin et al. [Almost Optimal Lower Bounds for Problems Parameterized by Clique-Width. SIAM J. Comput. 43(5): 1541-1563 (2014)] improved the running times for EDS and MC to $n^{O(k)}$, and proved $g(k)n^{o(k)}$ lower bounds for EDS, MC and HP assuming the ETH. Recently, Bergougnoux, Kante and Kwon [WADS 2017] gave an $n^{O(k)}$-time algorithm for HP. Thus, prior to this work, EDS, MC and HP were known to have tight $n^{\Theta(k)}$ algorithmic upper and lower bounds. In contrast, GC has an upper bound of $n^{O(2^k)}$ and a lower bound of merely $n^{o(k^{1/4})}$ (implicit from the W[1]-hardness proof). Here we close the gap for GC by proving a lower bound of $n^{2^{o(k)}}$. This shows that GC behaves qualitatively different from the other three problems. To the best of our knowledge, GC is the first natural problem known to require exponential dependence on the parameter in the exponent of n.

3.10 Flexibility of Planar Graphs

Tomáš Masařík (Charles University – Prague, CZ)

Joint work of Zdenek Dvorák, Tomás Masarík, Jan Musílek, Ondrej Pangrác
Main reference Zdenek Dvorák, Tomás Masarík, Jan Musílek, Ondrej Pangrác: "Flexibility of triangle-free planar graphs", CoRR, Vol. abs/1902.02971, 2019.
URL https://arxiv.org/abs/1902.02971

Proper graph coloring assigns different colors to adjacent vertices of the graph. Usually, the number of colors is fixed or as small as possible. Consider applications (e.g. variants of scheduling) where colors represent limited resources and graph represents conflicts, i.e., two adjacent vertices cannot obtain the same resource. In such applications, it is common that some vertices have preferred resource(s). However, unfortunately, it is not usually possible to satisfy all such preferences. The notion called flexibility was recently defined by Dvořák, Norin, and Postle [1]. There instead of satisfying all the preferences the aim is to satisfy at least a constant fraction of any request.

We introduce main technical tools in the area and we present a structural statement for:
- Planar graphs without 4-cycles and with lists of size at least five [4].
- Planar graphs without triangles and with lists of size at least four [3].
- Planar graphs of girth at least 6 and with lists of size at least three [2].

We derive the following statement for all of them. Let G be an above-defined graph with a list assignment L. There exists an absolute constant such that for any (weighted) choice of preferred colors for some of the vertices, there is an L-coloring respecting at least a constant fraction of the preferences.

References

1 Z. Dvořák, S. Norin, and L. Postle, *List coloring with requests*, Journal of Graph Theory, (2019).

2 Z. Dvořák, T. Masařík, J. Musílek, and O. Pangrác, *Flexibility of planar graphs of girth at least six*, arXiv:1902.04069 (2019).

3 Z. Dvořák, T. Masařík, J. Musílek, and O. Pangrác, *Flexibility of triangle-free planar graphs*, arXiv:1902.02971 (2019).

4 T. Masařík, *Flexibility of planar graphs without 4-cycles*, Accepted to Eurocomb 2019, arXiv:1903.01460 (2019).

3.11 The Erdős-Hajnal property for graphs with no fixed cycle as a pivot-minor

Sang-il Oum (IBS – Daejeon, KR)

Joint work of Sang-il Oum, Jaehoon Kim

We prove that for every integer k, there exists $\varepsilon > 0$ such that every n-vertex graph with no pivot-minors isomorphic to C_k, the cycle graph on k vertices, has a pair of disjoint sets A, B of vertices such that $|A|, |B| \geq \varepsilon n$ and A is complete or anticomplete to B. This proves the analog of the Erdős-Hajnal conjecture for the class of graphs with no pivot-minors isomorphic to C_k.

3.12 Computing the chromatic number of a ring

Irena Penev (Charles University – Prague, CZ)

Joint work of Frédéric Maffray, Irena Penev, Kristina Vušković
Main reference Frédéric Maffray, Irena Penev, Kristina Vušković: "Coloring rings", CoRR, Vol. arXiv:1907.11905, 2019
URL https://arxiv.org/abs/1907.11905

A *ring* is a graph R whose vertex set can be partitioned into $k \geq 4$ nonempty sets $X_1, \dots, X_k$ such that for all $i \in \{1, \dots, k\}$ the set X_i can be ordered as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that

$$X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \cdots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1},$$

with subscripts taken modulo k. Under such circumstances, we say that the ring R is of *length* k. An *odd* (resp. *even*) ring is a ring of odd (resp. even) length.

Truemper configurations are prisms, pyramids, thetas, and wheels. Rings have played an important role in the study of a couple of classes defined by excluding certain Truemper configurations as induced subgraphs. A maximum clique and a maximum stable set of a ring can be computed in polynomial time, as can an optimal vertex-coloring of an even ring. However, odd rings present obstacles for coloring.

Our main result is that every ring R satisfies

$$\chi(R) = \max\{\chi(H) \mid H \text{ is a hyperhole in } R\}.$$

We present several corollaries of this result. One corollary is that the chromatic number of a ring can be computed in polynomial time.

3.13 Tutorial on Potential Maximal Cliques

Marcin Pilipczuk (University of Warsaw, PL)

In the tutorial, I presented the framework of finding maximum (weighted) independent set via minimal chordal completions and potential maximal cliques. The talk contained most of the details of the polynomial-time algorithm in the class of P_5-free graphs (Lokshtanov, Vatshelle, Villanger [2]). I also highlighted the main contribution of the seminal work of Bouchitté and Todinca [1] and difficulties in generalizing from P_5-free graphs to P_6-free graphs [3].

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1 V. Bouchitté and I. Todinca. Treewidth and Minimum Fill-in: Grouping the minimal separators. *SIAM J. Comput.*, 31(1):212–232, 2001.
2 D. Lokshtanov, M. Vatshelle, and Y. Villanger. Independent set in P_5-free graphs in polynomial time. In C. Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*, pages 570–581. SIAM, 2014.
3 A. Grzesik, T. Klimošová, M. Pilipczuk, and M. Pilipczuk. Polynomial-time algorithm for maximum weight independent set on P_6-free graphs. In T. M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 1257–1271. SIAM, 2019.

3.14 C_k-coloring of F-free graphs

Paweł Rzążewski (Warsaw University of Technology, PL)

Joint work of Maria Chudnovsky, Shenwei Huang, Paweł Rzążewski, Sophie Spirkl, Mingxian Zhong
Main reference Maria Chudnovsky, Shenwei Huang, Paweł Rzążewski, Sophie Spirkl, Mingxian Zhong: "Complexity of C_k-Coloring in Hereditary Classes of Graphs", in Proc. of the 27th Annual European Symposium on Algorithms (ESA 2019), LIPIcs, Vol. 144, pp. 31:1–31:15, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2019.
URL http://dx.doi.org/10.4230/LIPIcs.ESA.2019.31

For a graph F, a graph G is F*-free* if it does not contain an induced subgraph isomorphic to F. For two graphs G and H, an H*-coloring* of G is a mapping $f : V(G) \to V(H)$ such that for every edge $uv \in E(G)$ it holds that $f(u)f(v) \in E(H)$. We are interested in the complexity of the problem H-Coloring, which asks for the existence of an H-coloring of an input graph G. In particular, we consider H-Coloring of F-free graphs, where F is a fixed graph and H is an odd cycle of length at least 5. This problem is closely related to the well known open problem of determining the complexity of 3-Coloring of P_t-free graphs.

We show that for every odd $k \geq 5$ the C_k-Coloring problem, even in the precoloring-extension variant, can be solved in polynomial time in P_9-free graphs. On the other hand, we prove that the extension version of C_k-Coloring is NP-complete for F-free graphs whenever some component of F is not a subgraph of a subdivided claw.

References

1 M.Chudnovsky, S. Huang, P. Rzążewski, S. Spirkl, M. Zhong, Complexity of C_k-coloring in hereditary classes of graphs, to appear in ESA 2019 Proc.

3.15 Polynomial Chi-binding functions and forbidden induced subgraphs: A survey

Ingo Schiermeyer (TU Bergakademie Freiberg, DE)

A graph G with clique number $\omega(G)$ and chromatic number $\chi(G)$ is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. A family $\mathcal{G}$ of graphs is called χ-bounded with binding function f if $\chi(G') \leq f(\omega(G'))$ holds whenever $G \in \mathcal{G}$ and G' is an induced subgraph of G. In this talk we will present a survey on polynomial χ-binding functions. Especially we will address perfect graphs, hereditary graphs satisfying the Vizing bound ($\chi \leq \omega + 1$), graphs having linear χ-binding functions and graphs having non-linear polynomial χ-binding functions. Thereby we also survey polynomial χ-binding functions for several graph classes defined in terms of forbidden induced subgraphs, among them $2K_2$-free graphs, P_k-free graphs, *claw*-free graphs, and *diamond*-free graphs.

3.16 Detecting an odd hole

Paul Seymour (Princeton University, US)

Joint work of Maria Chudnovsky, Alex Scott, Paul Seymour, Sophie Spirkl

A *hole* is a graph is an induced subgraph of length at least four, and an *antihole* is a hole in the complement. Odd holes are of particular interest, because of the strong perfect graph theorem, that says a graph is perfect if and only if it has no odd hole or odd antihole. A poly-time algorithm to test if a graph has an odd hole or odd antihole was found in 2006 [1], but detecting an odd hole, without stopping on discovery of an odd antihole, has remained open. We have now found a poly-time algorithm to test for odd holes [2]. Its running time is the same as the old algorithm, but in fact the details are much simpler.

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2 M. Chudnovsky, A. Scott. P. Seymour and S. Spirkl, "Detecting an odd hole", submitted for publication, `arXiv 1903.00208`

3.17 4-coloring P_6-free graphs

Sophie Spirkl (Rutgers University – Piscataway, US)

Joint work of Maria Chudnovsky, Mingxian Zhong, Sophie Spirkl

I talked about a recent polynomial-time algorithm for deciding if a given graph with no induced six-vertex path is four-colorable, and I discussed some of the methods used in the proof.

This is joint work with Maria Chudnovsky and Mingxian Zhong.

3.18 3-coloring with forbidden paths and cycles

Maya Jakobine Stein (University of Chile – Santiago de Chile, CL)

Joint work of Maya Jakobine Stein, Alberto Rojas

Graph coloring is hard, even if the number of colors is fixed. Therefore much effort has gone into determining the complexity of k-coloring special classes of graphs, in particular H-free graphs, where H is a fixed graph (and k is also fixed). It turns out that the problem remains NP-complete whenever H is not a linear forest, and for $k \geq 4$, the complexity of k-coloring P_t-free graphs has been determined for all values of t. There are polynomial time algorithms for 3-coloring P_t-free graphs for $t \leq 7$, but it is not known if such algorithms exist for $t = 8, 9, 10, ...$ The algorithm given in [1] for P_7-free graphs was found by improving an earlier version which only worked for (P_7, C_3)-free graphs, so it seems natural to attack the problem by excluding one or more cycles in addition to the path P_t. We found a polynomial time algorithm for 3-coloring (P_9, C_5, C_3)-graphs. A variation of this algorithm works for all graphs having no induced P_{2t+1}, no induced odd cycle of length up to $2t-1$, and no induced C_8.

References

1 Bonomo, Flavia; Chudnovsky, Maria; Maceli, Peter; Schaudt, Oliver; Stein, Maya; Zhong, Mingxian *Three-Coloring and List Three-Coloring of Graphs Without Induced Paths on Seven Vertices.*Combinatorica (2018) 38: 779. https://doi.org/10.1007/s00493-017-3553-8

3.19 Layered wheels

Nicolas Trotignon (ENS – Lyon, FR)

Joint work of Ni Luh Dewi Sintiari, Nicolas Trotignon
Main reference Ni Luh Dewi Sintiari, Nicolas Trotignon: "(Theta, triangle)-free and (even hole, K_4)-free graphs. Part 1 : Layered wheels", CoRR, Vol. abs/1906.10998, 2019.
URL https://arxiv.org/abs/1906.10998

We present a construction called layered wheel. Layered wheels are graphs of arbitrarily large treewidth and girth. They might be an outcome for a possible theorem characterizing graphs with large treewidth in term of their induced subgraphs (while such a characterization is well understood in term of minors). They also provide examples of graphs of large treewidth and large rankwidth in well studied classes, such as (theta, triangle)-free graphs and even-hole-free graphs with no K_4 (where a hole is a chordless cycle of length at least 4, a theta is a graph made of three internally vertex disjoint paths of length at least 2 linking two vertices, and K_4 is the complete graph on 4 vertices).

3.20 Vertex colorings of interval hypergraphs

Zsolt Tuza (Alfréd Rényi Institute of Mathematics – Budapest, HU)

Joint work of Csilla Bujtás, Zsolt Tuza
Main reference Csilla Bujtás, Zsolt Tuza: "Color-bounded hypergraphs, VI: Structural and functional jumps in complexity", Discrete Mathematics, Vol. 313(19), pp. 1965–1977, 2013.
URL https://doi.org/10.1016/j.disc.2012.09.020

The classical notion of proper coloring requires a color assignment to the vertices in such a way that no hyperedge is monochromatic (Erdős & Hajnal, mid-1960's). Equivalently this means at least two colors in each hyperedge. In C-coloring the restriction is put from the other side, namely that every hyperedge e is allowed to contain at most $|e|-1$ colors (Berge / Sterboul, early 1970's). In the more complex model of mixed hypergraps both types of hyperedges may occur (Voloshin, early 1990's). A generalization of this structure class is obtained by putting lower and/or upper bounds on the largest cardinality of monochromatic subsets of – and/or on the number of colors occurring in – each hyperedge (Bujtás & Tuza, mid-2000's).

Despite that lots of results are known, some simple questions are still open, even on interval hypergraphs. (An interval hypergraph is a collection of hyperedges $e_1, \dots, e_m$ whose underlying vertex set admits an ordering such that each hyperedge e_i consists of consecutive vertices without gap in that order.) As an example of problems unsolved for over a decade, assume that for each e_i there is a color which appears on at least a given number a_i of vertices inside e_i. The task is to determine the largest possible number of colors. Is this optimization problem polynomial-time solvable or NP-hard?

In the talk we mention open questions concerning the chromatic polynomial, too.

4 Open problems

4.1 Parameterized complexity of the coloring problems for H-free graphs

Petr A. Golovach (University of Bergen, NO)

Only very few parameterized results for Coloring on H-free graphs are known. We refer to [2] for the detailed survey of the known results and open problems. Here we underline two problems that we believe to be the most interesting.

It is a very long standing open problem whether the 3-Coloring problem for P_ℓ-free graphs admits a polynomial algorithm for every positive ℓ or it becomes NP-complete for some $\ell \geq 8$. Currently, it is known that the problem can be solved in polynomial time for $\ell \leq 7$ [1]. This leads to the following question.

- Is 3-Coloring W[1]-hard on P_ℓ-free graphs when parameterized by ℓ?

The next problem was first stated by Hoàng et al. [3]. They proved that that k-Coloring can be solved in polynomial time on P_5-free graphs for every positive integer k, that is the problem is in XP when parameterized by k, but left open the question whether there is a matching lower bound or their result may be improved.

- Is k-Coloring FPT on P_5-free graphs when parameterized by k?

The question whether k-Coloring parameterized by k is FPT is also open and interesting for $2P_2$-free graphs that compose a subclass of P_5-free graphs.

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4.2 The Dilworth number of a graphs

Chinh T. Hoàng (Wilfrid Laurier University – Waterloo, CA)

Given a graph G, a vertex x dominates a vertex y if every neighbor of y, different from x, is a neighbor of x. Vertex x is comparable to vertex y, if x dominates y, or y dominates x. The domination relation is a partial order. The Dilworth number of a graph G is the largest number of pairwise incomparable vertices in G.

Problem 1. Is it true that there is a polynomial time algorithm to optimally color all graphs with bounded Dilworth number?

Problem 2. Is it true that if a graph G has bounded Dilworth number then it has bounded clique width? After I posed this problem, it was pointed out to me that the answer is NO. The authors Korpelainen, Lozin, and Mayhill constructed a graph with Dilworth number two and with arbitrarily high clique width.
(https://link.springer.com/content/pdf/10.1007/s00373-013-1290-3.pdf)

4.3 Colouring Graphs of Bounded Diameter

Daniel Paulusma (Durham University, GB)

It is known that k-Colouring is NP-complete for graphs of diameter at most d for all pairs (k, d) with $k \geq 3$ and $d \geq 2$ except when $(k, d) = (3, 2)$: determining the computational complexity of 3-Colouring for graphs of diameter 2 is a long-standing open problem. The following related problems are also open:

Determine the computational complexity of 3-Colouring and Colouring (where k is part of the input) for triangle-free graphs of diameter 2.

It can be observed that for all integers $d, k, r \geq 1$, the k-Colouring problem is constant-time solvable for $K_{1,r}$-free graphs of diameter d and that Colouring is NP-complete for $K_{1,4}$-free graphs. However, the following problem is open:

Determine the computational complexity of Colouring restricted to $K_{1,3}$-free graphs of diameter d for every $d \geq 2$.

The above observations and open problems can all be found in [1].

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4.4 Odd cycle transversal in P_5-free graphs

Paweł Rzążewski (Warsaw University of Technology, PL)

Joint work of Konrad K. Dabrowski, Carl Feghali, Matthew Johnson, Giacomo Paesani, Daniël Paulusma, Paweł Rzążewski
Main reference Konrad K. Dabrowski, Carl Feghali, Matthew Johnson, Giacomo Paesani, Daniël Paulusma, Paweł Rzążewski: "On Cycle Transversals and Their Connected Variants in the Absence of a Small Linear Forest", CoRR, Vol. abs/1908.00491, 2019.
URL http://arxiv.org/abs/1908.00491

It is known that the Odd Cycle Transversal is polynomial-time solvable in P_4-free graphs, but is NP-complete in P_6-free graphs [1]. What is the complexity of the problem in P_5-free graphs?

References

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4.5 Computing disjoint paths

Nicolas Trotignon (ENS – Lyon, FR)

Consider the following problem:

Input: A graph G and a, b, c, d four vertices of G.

Question: Does there exist two paths, vertex disjoint, with no edges between them and such that their ends are all in $\{a, b, c, d\}$?

The complexity of this problem is not known.

Remarks:

- The problem is trivial when $\{a, b, c, d\}$ is not a stable set of size four. In a solution, each of a, b, c and d must be an end of exactly one of the paths.
- Deciding whether there exit a path from a to b and a path from c to d, vertex-disjoint and with edges between them is NP-complete as shown by Bienstock. So, to solve the problem, it is hopeless to try the three possible ways the paths may exist separately (a-b + c-d; a-c + b-d; a-d + b-c).

The problem is motivated by the detection of induced minors, see the references.

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4.6 Precoloring extension of graphs

Zsolt Tuza (Alfréd Rényi Institute of Mathematics – Budapest, HU)

The decision problem t-PrExt is the following subproblem of Precoloring Extension.

Input: Graph $G = (V, E)$, nonnegative integer k, proper coloring of an induced subgraph $H \subset G$, such that each color occurs at most t times in H.

Question: Can the coloring of H be extended to a proper k-coloring of the entire G?

Clearly, 0-PrExt means k-colorability, hence it is linear-time solvable on bipartite graphs. However, already 1-PrExt is NP-complete on bipartite graphs [2]. Further, on interval graphs 1-PrExt is solvable in polynomial time, but 2-PrExt is NP-complete [1].

PROBLEM 1. For $t > 1$, find graph classes in which t-PrExt is polynomial-time solvable and $(t + 1)$-PrExt is NP-complete.

PROBLEM 2. On interval graphs, design a linear-time algorithm for 1-PrExt, or prove that every 1-PrExt algorithm is superlinear in $|V| + |E|$.

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Participants

- Isolde Adler
University of Leeds, GB
- Marthe Bonamy
University of Bordeaux, FR
- Nicolas Bousquet
Grenoble INP, FR
- Christoph Brause
TU Bergakademie Freiberg, DE
- Kathie Cameron
Wilfrid Laurier University – Waterloo, CA
- Maria Chudnovsky
Princeton University, US
- Konrad Dabrowski
Durham University, GB
- Cemil Dibek
Princeton University, US
- Francois Dross
University of Montpellier & CNRS, FR
- Esther Galby
University of Fribourg, CH
- Petr A. Golovach
University of Bergen, NO
- Chinh T. Hoàng
Wilfrid Laurier University – Waterloo, CA
- Shenwei Huang
Nankai University – Tianjin, CN
- Bart Jansen
TU Eindhoven, NL
- Matthew Johnson
Durham University, GB
- Mamadou Moustapha Kanté
Université Clermont Auvergne – Aubiere, FR
- Tereza Klimosova
Charles University – Prague, CZ
- Stefan Kratsch
HU Berlin, DE
- O-joung Kwon
Incheon National University, KR
- Bernard Lidicky
Iowa State University, US
- Anita Liebenau
UNSW Sydney, AU
- Paloma Lima
University of Bergen, NO
- Daniel Lokshtanov
University of California – Santa Barbara, US
- Peter Maceli
Adelphi University – Garden City, US
- Tomáš Masařík
Charles University – Prague, CZ
- Sang-il Oum
IBS – Daejeon, KR
- Daniel Paulusma
Durham University, GB
- Irena Penev
Charles University – Prague, CZ
- Marcin Pilipczuk
University of Warsaw, PL
- Michal Pilipczuk
University of Warsaw, PL
- Bernard Ries
University of Fribourg, CH
- Paweł Rzążewski
Warsaw University of Technology, PL
- Oliver Schaudt
RWTH Aachen, DE
- Ingo Schiermeyer
TU Bergakademie Freiberg, DE
- Pascal Schweitzer
TU Kaiserslautern, DE
- Paul Seymour
Princeton University, US
- Sophie Spirkl
Rutgers University – Piscataway, US
- Juraj Stacho
Google Switzerland – Zürich, CH
- Maya Jakobine Stein
University of Chile – Santiago de Chile, CL
- Stéphan Thomassé
ENS – Lyon, FR
- Nicolas Trotignon
ENS – Lyon, FR
- Zsolt Tuza
Alfréd Rényi Institute of Mathematics – Budapest, HU
- Erik Jan van Leeuwen
Utrecht University, NL
- Gerhard J. Woeginger
RWTH Aachen, DE
- Victor Zamaraev
Durham University, GB