Unbounded Regions of High-Order Voronoi Diagrams of Lines and Segments in Higher Dimensions

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Abstract

We study the behavior at infinity of the farthest and the higher-order Voronoi diagram of n line segments or lines in a d-dimensional Euclidean space. The unbounded parts of these diagrams can be encoded by a *Gaussian map* on the sphere of directions $\mathbb{S}^{d-1}$. We show that the combinatorial complexity of the Gaussian map for the order-k Voronoi diagram of n line segments or lines is $O(\min\{k, n-k\}n^{d-1})$, which is tight for $n-k = O(1)$. All the d-dimensional cells of the farthest Voronoi diagram are unbounded, its $(d-1)$-skeleton is connected, and it does not have tunnels. A d-cell of the Voronoi diagram is called a tunnel if the set of its unbounded directions, represented as points on its Gaussian map, is not connected. In a three-dimensional space, the farthest Voronoi diagram of lines has exactly n^2-n three-dimensional cells, when $n \geq 2$. The Gaussian map of the farthest Voronoi diagram of line segments or lines can be constructed in $O(n^{d-1}\alpha(n))$ time, while if $d = 3$, the time drops to worst-case optimal $O(n^2)$.

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1 Introduction

The Voronoi diagram of a set of n geometric objects, called sites, is a well-known space-partitioning structure with numerous applications in diverse fields of science. The *nearest* variant partitions the underlying space into maximal regions such that all points within one region have the same nearest site. The Euclidean Voronoi diagram of points in $\mathbb{R}^d$ has been studied thoroughly, see, e.g., [7, 9, 13, 17]. This not the case, however, for non-point sites, which have been much less considered.

In the plane, many algorithmic paradigms, such as plane sweep, incremental construction, and divide-and-conquer have been applied to construct the Voronoi diagram of line segments in the plane [7]. However, in higher-dimensional spaces, results are quite sparse. Already in a three-dimensional space, the algebraic description of the features, such as the edges, of the Voronoi diagram of lines become very complicated [14]. As a result, the combinatorial

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complexity of this diagram has been a major open problem in computational geometry [21]. There is a gap of an order of magnitude between the $\Omega(n^2)$ lower bound [3] and the only known upper bound of $O(n^{3+\varepsilon})$ [25], where n is the number of sites. The gap carries over (and expands) to the Voronoi diagram of lines in d-space, $d \geq 3$, where the known bounds are $\Omega(n^{\lfloor \frac{d}{2} \rfloor})$ [4] and $O(n^{d+\varepsilon})$ [25]. The lower bound is derived from n parallel lines whose Voronoi diagram has the same complexity as the Voronoi diagram of n points in $d-1$ dimensional space. For points in $\mathbb{R}^d$, the bound is $\Theta(n^{\lceil \frac{d}{2} \rceil})$ [7], and for $(d-2)$-dimensional hyperplanes, the lower bound is $\Omega(n^{d-1})$ [3]. To the best of our knowledge, no other lower bound, other than $\Omega(n^{\lceil \frac{d}{2} \rceil})$, is available for line segments in $\mathbb{R}^d$, $d > 3$. Better combinatorial bounds are known only for some restricted cases [6, 10, 18, 19]. A numerically robust algorithm for computing the Voronoi diagram of lines in 3D has been given by Hemmer et al. [16].

The order-k (resp., farthest) Voronoi diagram of a set of sites is a partition of the underlying space into regions, such that the points of one region have the same k nearest sites (resp., same farthest site). Seidel [24] derived exact bounds on the maximal complexity of the Euclidean farthest Voronoi diagram of points in $\mathbb{R}^d$. Asymptotically, the worst-case complexity of the latter diagram remains $\Theta(n^{\lceil \frac{d}{2} \rceil})$. Edelsbrunner and Seidel [13] pointed out that the order-k Voronoi diagram of points in $\mathbb{R}^d$ can be derived from the $\leq k$-level of an arrangement of hyperplanes in $\mathbb{R}^{d+1}$. Agarwal and Mulmuley provided an algorithm which computes the $\leq k$-level of n hyperplanes in $\mathbb{R}^d$ in expected $O(n^{\lfloor \frac{d}{2} \rfloor} k^{\lfloor \frac{d}{2} \rfloor})$ time [1, 22]. For non-point sites, the problems have been mostly considered in the plane.

The farthest Voronoi diagram of n segments in the plane was first studied by Aurenhammer et al. [5], who gave results on its structure and an algorithm to compute it in $O(n \log n)$ time. The order-k counterpart of this diagram was then considered by Papadopoulou and Zavershynskyi [23], who showed that its complexity is $O(k(n-k))$, if segments are disjoint or touch only at endpoints, and that it can be constructed iteratively in $O(k^2 n \log n)$ time. If segments intersect, then the number of intersections affects the complexity only if $k < \frac{n}{2}$ [23]. These diagrams illustrate fundamental structural differences from their counterparts of points, such as disconnected Voronoi regions and no relation to convex hulls. Naturally, these differences carry over to higher dimensions, which is the subject of study in this paper.

In three dimensions, the Euclidean farthest-site Voronoi diagram of lines or line segments has the property that all its three-dimensional cells are unbounded [8]. Barequet and Papadopoulou [8] used a structure on the sphere of directions, called the Gaussian map, which reflects the directions under which the cells of this diagram are unbounded.

In this paper, we study the Gaussian map of order-k and farthest Voronoi diagrams of n line segments and lines as sites in $\mathbb{R}^d$, and characterize the unbounded directions of the cells in these diagrams. The dimension d is assumed a constant. We derive the bound $O(\min\{k, n-k\} n^{d-1})$ on the complexity of the Gaussian map of order-k Voronoi diagrams for these sites. This implies the same upper bound on the complexity of the unbounded features of the corresponding order-k Voronoi diagrams. For the farthest-site diagram ($k = n-1$), this is $O(n^{d-1})$. For segments as sites, we prove that the complexity of the Gaussian map is $\Omega(k^{d-1})$, which is tight when $n - k = O(1)$. In fact, the complexity bound is derived by the number of vertices on the Gaussian map. This leads to a lower bound of $\Omega(k^{d-1})$ on the complexity of the entire order-k Voronoi diagram for line segments. For the farthest-site Voronoi diagram, this bound becomes $\Omega(n^{d-1})$, which also holds for lines as sites. As a byproduct, we derive a bound on the complexity of the arrangement of n great hyperspheres on $\mathbb{S}^{d-1}$.

Table 1 Worst-case complexities of structures induced by a set S of n lines or segments in $\mathbb{R}^d$.

Structure	Lower bound	Upper bound
GM(VD$_k(S)$)	$\Omega(k^{d-1})$*	$O(\min\{k, n-k\}n^{d-1})$
GM(FVD(S))	$\Omega(n^{d-1})$	$O(n^{d-1})$
VD$_k(S)$	$\Omega(k^{d-1})$*	$O(\min\{k, n-k\}n^{d+\varepsilon})$
FVD(S)	$\Omega(n^{d-1})$	$O(n^{d+\varepsilon})$

*Only for segments.

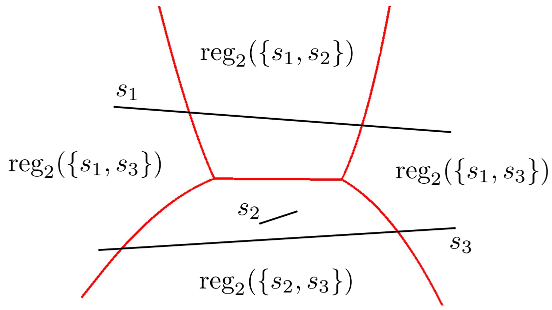


Figure 1 The order-2 Voronoi diagram (in red) of three segments s_1, s_2, s_3 in the plane.

Further, we describe a transformation that maps a set of lines to a set of segments, such that the two respective Gaussian maps of order-k Voronoi diagrams are identical. This transformation can be used to carry lower bounds from lines to segments and upper bounds from segments to lines. Table 1 summarizes most of the complexity results derived in this paper.

All the d-dimensional cells of the farthest Voronoi diagram of both lines and segments are unbounded, its $(d-1)$-skeleton is connected, and it does not have tunnels. In three dimensions, the farthest Voronoi diagram of lines has exactly n^2-n many 3-dimensional cells, when $n \geq 2$.

We show that we can compute the Gaussian map of this diagram in $O(n^{d-1}\alpha(n))$ time by using the algorithm of Edelsbunner et al. [12], which extends to higher dimensions [2, 15], for computing the envelope of piecewise-linear functions in $\mathbb{R}^d$. In fact, we conjecture that this bound can be improved to $O(n^{d-1})$. In three dimensions, we can compute the Gaussian map of the farthest Voronoi diagram of lines or segments in $O(n^2)$ time, which is optimal in the worst-case.

2 Preliminaries

2.1 Order-k Voronoi Diagrams

Let S be a set of sites in $\mathbb{R}^d$. In this paper, we consider as sites n (possibly intersecting) line segments or n lines in $\mathbb{R}^d$. The dimension d is considered constant. We denote by $d(x, y)$ the Euclidean distance between two points $x, y \in \mathbb{R}^d$. The distance $d(x, s)$ from a point $x \in \mathbb{R}^d$ to a site $s \in S$ is defined as $d(x, s) = min\{d(x, s)|y \in s\}$.

▶ **Definition 1.** *For a subset of sites $H \subset S$ of cardinality $|H| = k$, the* order-k region *of H is the set of points in $\mathbb{R}^d$ whose distance to any site in H is smaller than to any site not in H. It is denoted as $reg_k(H) = \{p \in \mathbb{R}^d \mid \forall h \in H\ \forall s \in S \setminus H :\ d(p, h) \leq d(p, s)\}$.*

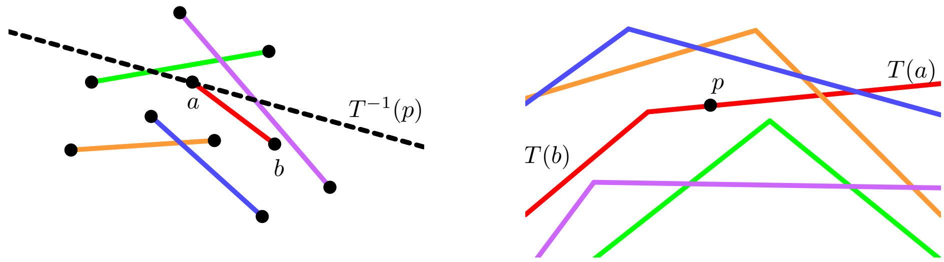


Figure 2 Point-hyperplane duality applied to segments: (left) Segments in primal space; and (right) their corresponding wedges in dual space.

The order-k regions of S induce a subdivision in $\mathbb{R}^d$. The induced cell complex is called the *order-k Voronoi diagram* of S, denoted by $\mathrm{VD}_k(S)$. A maximally connected i-dimensional set of points, which is on the boundary of the same set of order-k regions, is called an i-dimensional cell of the cell complex. We call the i-dimensional cells of the order-k Voronoi diagram "i-cells."

When $k = 1$, this diagram is the well-known nearest-neighbor Voronoi diagram, denoted by $\mathrm{VD}(S)$. For $k = n - 1$, it is the *farthest site Voronoi diagram*, denoted by $\mathrm{FVD}(S)$. Its *farthest regions* can also be defined directly as $\mathrm{freg}(h) = \{p \in \mathbb{R}^d \mid s \in S \setminus \{h\} : \ d(p, h) \geq d(p, s)\}$.

2.2 Point-Hyperplane Duality

Under the well-known point-hyperplane duality T in $\mathbb{R}^d$, a point $p \in \mathbb{R}^d$ is transformed to a non-vertical hyperplane $T(p)$, and vice versa. The transformation maps a point with coordinates $(p_1, p_2, ..., p_d)$ to the hyperplane $T(p)$ which satisfies the equation $x_d = -p_d + \sum_{i=1}^{d-1} p_i x_i$. The transformation is an involution, i.e., $T = T^{-1}$.

For a segment $s = uv$, the hyperplanes $T(u)$ and $T(v)$ partition the dual space into four *wedges*, among which the *lower wedge* (resp., the *upper wedge*) is the one that lies below (resp., above) both $T(u)$ and $T(v)$. The apex of the wedge is the intersection of $T(u)$ and $T(v)$.

Let S be a set of n segments, which in dual space corresponds to an arrangement of lower wedges. Let L_k be the k-th level of that arrangement. Let p be a point on L_k, which touches the dual wedge of segment $s = ab$, and let H be the set of segments whose wedges are below p, see Figure 2. Then, the point p corresponds to a hyperplane $T^{-1}(p)$ which touches the segment s. The closed halfspace above $T^{-1}(p)$ has a non-empty intersection with the segments in H. The open halfspace above $T^{-1}(p)$ does not intersect any segment in $S \setminus H$. We will use this property when we study the Gaussian map, which is defined in the next section. Symmetrically for the arrangement of upper wedges.

2.3 Levels in an arrangement of hyperplanes

This section reviews the definition of levels of an arrangement of surfaces, where those surfaces satisfy some *mildness* conditions (A1)-(A3) as given in [2]. We will use Theorem 2 by Clarkson and Shor several times in this paper.

The level of a point $p \in \mathbb{R}^d$ in an arrangement $\mathcal{A}(\Gamma)$ of a set Γ of surface patches is the number of surfaces of Γ lying vertically below p. For $0 \leq k < n$, the k-level (resp. $\leq k$-level), denoted by $\mathcal{A}_k(\Gamma)$ (resp. $\mathcal{A}_{\leq k}(\Gamma)$), is the closure of all points on the surface of Γ whose level is k (resp. at most k). A face of $\mathcal{A}_k(\Gamma)$ or $\mathcal{A}_{\leq k}(\Gamma)$ is a maximal connected portion of a face of $\mathcal{A}(\Gamma)$ consisting of points having a fixed subset of surfaces lying below them. Let $\psi_k(\Gamma)$ (resp. $\psi_{\leq k}(\Gamma)$) be the total number of faces in $\mathcal{A}_k(\Gamma)$ (resp. $\mathcal{A}_{\leq k}(\Gamma)$). [2]

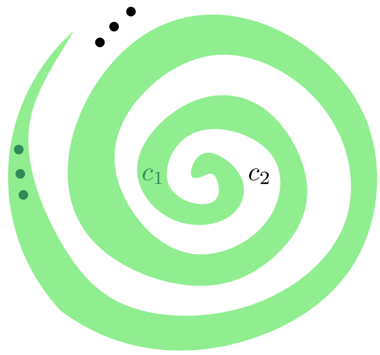


Figure 3 A cell complex in which none of the cells is unbounded in a specific direction.

▶ **Theorem 2** (Clarkson and Shor [11]). *Let $\mathcal{G}$ be an infinite family of surfaces satisfying some mildness assumptions (A1)-(A3) described in [2]. Then for any $0 \le k < n - d$,*

$$\psi_{\le k}(n, d, \mathcal{G}) = O\left((k+1)^d \kappa\left(\frac{n}{k+1}, d, \mathcal{G}\right)\right),$$

where $\kappa(n, d, \mathcal{G})$ is the maximum complexity of the lower envelope of n surfaces in $\mathcal{G}$.

It obviously holds that $\psi_k(\Gamma) \le \psi_{\le k}(\Gamma)$.

2.4 Defining the Gaussian Map

Let M be a cell complex in $\mathbb{R}^d$. The *complexity* of M is the total number of its cells of all dimensions. The *Gaussian map* of M encodes information about the unbounded cells of M. This structure is of particular interest when all cells of M are unbounded. For example, all the d-dimensional cells of the farthest Voronoi diagram of segments or lines are unbounded.

▶ **Definition 3.** *A cell c of M is* unbounded *in direction $\vec{v}$ if in the limit $\lambda \to 0$, the intersection of the scaled cell $\lambda \cdot c$ and the unit sphere $\mathbb{S}^{d-1}$ is non-empty in direction $\vec{v}$.*

The scaling of cell c can be done with an arbitrary center. The limit $\lim_{\lambda\to 0}\left(\lambda c \cap \mathbb{S}^{d-1}\right)$ should be understood with the concept of the Kuratowski convergence [20], which we briefly review. For any point $x \in \mathbb{R}^d$ and subset $\mathcal{S} \subset \mathbb{R}^d$ let $d(x, \mathcal{S}) = \inf\{d(x, s) | s \in \mathcal{S}\}$ be the distance between x and $\mathcal{S}$. Let $\mathcal{S}_\lambda \subset \mathbb{R}^d$ be a sequence of compact sets. We say that $\mathcal{S}_\lambda$ converges to $\mathcal{S}$ for $\lambda \to 0$ iff $\mathcal{S} = \{x \in \mathbb{R}^d | \limsup_{\lambda\to 0} d(x, \mathcal{S}_\lambda) = 0\} = \{x \in \mathbb{R}^d | \liminf_{\lambda\to 0} d(x, \mathcal{S}_\lambda) = 0\}$.

Note that the Kuratowski limit does not always need to exist [20]. Consider a cell complex consisting of 2 cells circling around each other, see Figure 3. The unbounded directions of the cells of this cell complex would not be defined in this case, because for any cell $c \in \{c_1, c_2\}$ the sets $\{x \in \mathbb{R}^d | \limsup_{\lambda\to 0} d(x, \lambda c \cap \mathbb{S}^1)\} = \emptyset$ and $\{x \in \mathbb{R}^d | \liminf_{\lambda\to 0} d(x, \lambda c \cap \mathbb{S}^1) = 0\} = \mathbb{S}^1$ are not the same. In this paper we only consider cell complexes, where the unbounded directions of cells are well defined.

It might be tempting to use an alternative simpler definition: A cell c of M is unbounded in direction $\vec{v}$ if it contains a ray with direction $\vec{v}$. However, this could cause problems for cells of dimension $< d$. For example, the trisector of three lines is in general a non-linear curve [14], containing no ray, therefore, it would not be unbounded in any direction. Thus Definition 3 is stronger in that sense and also defines unbounded directions for smaller dimensional cells.

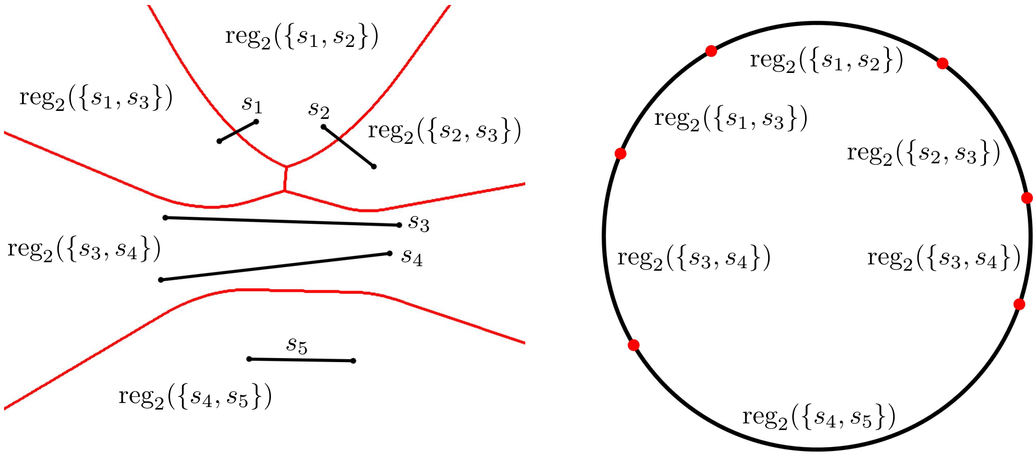


Figure 4 An order-2 Voronoi diagram $\mathrm{VD}_2(\{s_1, s_2, ..., s_5\})$ (left) and its Gaussian map (right).

▶ **Definition 4.** *The* Gaussian map *of M, denoted by* $\mathrm{GM}(M)$*, maps each cell in M to its unbounded directions, which are encoded on the unit sphere $\mathbb{S}^{d-1}$, see Figure 4. Let c be a cell of M; the set of directions, in which c is unbounded, is called the* region of c *on* $\mathrm{GM}(M)$*. The part of* $\mathrm{GM}(M)$ *where the d-th coordinate is ≥ 0 (resp., ≤ 0) is called the* upper *(resp.,* lower*) Gaussian map.*

The Kuratowski limit is a closed set, if it exists, and therefore, cells of the Gaussian map are closed. In this paper, we focus on cell complexes, such as the farthest Voronoi diagram and the order-k Voronoi diagram of lines and segments, where cells have unbounded directions and the Gaussian map is the respective partition of $\mathbb{S}^{d-1}$. This partition induces a cell complex on $\mathbb{S}^{d-1}$. The collection of cells on the Gaussian map of a Voronoi diagram $\mathrm{VD}_k(S)$, which correspond to the same set of sites $H \subset S$, is called the *region of H* on $\mathrm{GM}(\mathrm{VD}_k(S))$.

A Gaussian map region of a set of sites may consist of several $(d-1)$-cells for two reasons: A region of a set of sites of VD_k may split into many d-cells, which all have unbounded directions on the Gaussian map. Moreover the Gaussian map region of just one d-cell of VD_k can consist of several cells, e.g., $\mathrm{reg}_2(\{s_3, s_4\})$ in Fig. 4.

▶ **Definition 5.** *A d-cell of the order-k Voronoi diagram is called a tunnel if its set of unbounded directions, represented as points on its Gaussian map, is not connected.*

In Figure 4, one cell forms a *tunnel* in $\mathrm{VD}_2(S)$.

The Gaussian map essentially replaces the role of the convex hull in characterizing the unbounded regions the higher-order Voronoi diagram of $\mathrm{VD}_k(S)$), for $k > 1$.

3 Properties of the Farthest and Order-k Voronoi Diagram

3.1 Combinatorial Properties

It has already been stated [8] that the complexity of the farthest Voronoi diagram is $O(n^{3+\varepsilon})$ by following the general bound of Sharir [25]. This bound generalizes for the order-k Voronoi diagram in $\mathbb{R}^d$.

▶ **Theorem 6.** *The order-k Voronoi diagram of segments and lines in $\mathbb{R}^d$ has complexity $O(\min\{k, n-k\}n^{d+\varepsilon})$.*

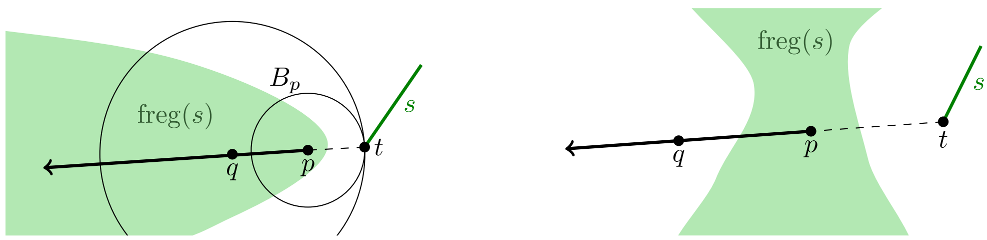


Figure 5 The farthest regions contain rays (left) and no farthest region can split the $d-1$ skeleton of the farthest Voronoi diagram into 2 parts (right).

Proof. Each site induces a distance function, which maps every point in $\mathbb{R}^d$ to its distance to that site. The general framework of Sharir [25] shows that the complexity of the 0-level (resp., $(n-1)$-level) of those distance functions is $O(n^{d+\varepsilon})$. Applying Theorem 2 by Clarkson and Shor [11], the complexity of the $\leq k$-level is $O(kn^{d+\varepsilon})$ and $O((n-k)n^{d+\varepsilon})$. ◀

In Section 4 we will prove the following lower bounds. These bounds are meaningful when k is comparable to n.

▶ **Theorem 7.** *The complexity of the order-k Voronoi diagram of segments in $\mathbb{R}^d$ is $\Omega(k^{d-1})$ in the worst case. For the farthest Voronoi diagram ($k = n-1$), this is $\Omega(n^{d-1})$.*

3.2 Structural Properties

▶ **Lemma 8.** *Let S be a set of lines or segments, and let $p \in \text{freg}(s)$ be a point in the farthest region of site s. Let t be the point on s, which realizes the distance between s and p. Then, the entire ray $\vec{r}$, which emanates from p with direction $\overrightarrow{tp}$, is contained in $\text{freg}(s)$.*

Proof. The ball B_p, centered at p and of radius $|pt|$, touches s. Its interior intersects all other sites in S. In addition, any hyperball centered at any point $q \neq p$ along $\vec{r}$ and of radius $|qt|$ must be properly enclosing B_p while touching s at t, see Figure 5. Thus, it must also intersect all sites in S except s. Therefore, $\text{freg}(s)$ must contain the entire ray $\vec{r}$. ◀

▶ **Corollary 9.** *Let S be a set of lines and segments. All d-cells of $\text{FVD}(S)$ are unbounded.*

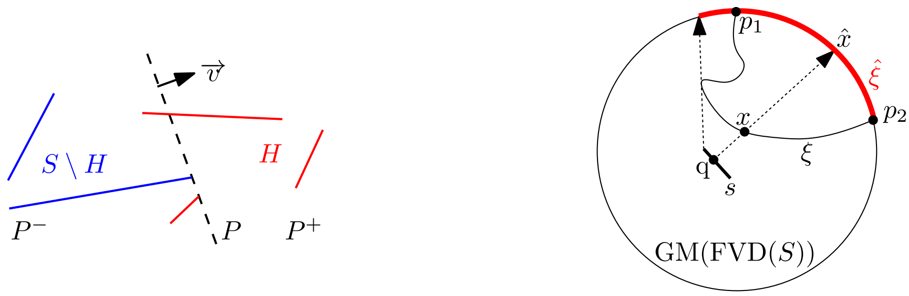
▶ Remark 10. The VD_k of segments can have bounded regions if $d \leq k \leq n-2$.

▶ **Definition 11.** *The i-skeleton of a cell complex M is the union of all j-cells in M with dimension $j \leq i$.*

▶ **Theorem 12.** *Let S be a set of lines or segments in $\mathbb{R}^d$. The $(d-1)$-skeleton of $\text{FVD}(S)$ is connected.*

Proof. Assume, for the sake of contradiction, that the diagram is not connected. Then, there exists a d-cell c that splits the $(d-1)$-skeleton into at least two parts. Let s be the farthest site corresponding to c. The site s does not touch $\text{freg}(s)$. Let q be a point, which is separated from s by c. Let t be a point on s, which realizes the distance between q and s. Let p be a point on the segment $\overline{qt}$ in $\text{freg}(s)$, see Fig. 5. Then, by Lemma 8, the entire ray $\vec{r}$, emanating from p in direction $\overrightarrow{pq}$, is contained in $\text{freg}(s)$. In particular, $q \in \text{freg}(s)$, which is a contradiction. ◀

▶ Remark 13. The $(d-1)$-skeleton of $\text{VD}_k(S)$ need not be connected for $k \leq n-2$ and $S \subset \mathbb{R}^2$.



(a) A supporting hyperplane P (in dashed black) of sites H (in red) in direction $\overrightarrow{v}$.

(b) Construction of the path $\hat{\xi}$.

Figure 6

4 Line Segments as Sites

Let S be a set of line segments in $\mathbb{R}^d$. We assume that the segments are in general position, i.e. no $(d+1)$ segment endpoints lie on the same hyperplane. First, we characterize the segments that induce unbounded regions in the order-k Voronoi diagram in a given direction $\overrightarrow{v}$.

▶ **Definition 14.** *Let S be a set of segments, and let H be a subset of S. A hyperplane P is called a* supporting hyperplane *of H in direction $\overrightarrow{v}$ if*

1. *P is orthogonal to $\overrightarrow{v}$;*
2. *The closed halfspace P^+, bounded by P and unbounded in direction $\overrightarrow{v}$, intersects each of the sites in H; and*
3. *The sites in $S \setminus H$ do not intersect the interior of P^+, and at least one site in $S \setminus H$ touches P.*

Figure 6a illustrates a hyperplane supporting three segments.

The following theorem is a generalization of results for the plane [5, 23].

▶ **Theorem 15.** *A set of segments H, with $|H| = k$, induces an unbounded region in direction $\overrightarrow{v}$ in the order-k Voronoi diagram of segments S, if and only if there exists a supporting hyperplane of H in direction $\overrightarrow{v}$.*

Proof. Let H be a set of k segments, which has an unbounded d-cell c in direction $\overrightarrow{v}$ in the order-k Voronoi diagram of a set of segments S. Each point in the cell corresponds to the center of a closed ball which has non-empty intersection with the segments in H, and does not intersect any of the other segments. By definition, there exists a curve unbounded in direction $\overrightarrow{v}$, which is contained in c. Any point p on that curve is the center of a closed ball, which has a non-empty intersection with the segments in H and does not intersect any of the other segments in its interior. When p moves along the curve to infinity, the ball around p becomes a halfspace which is orthogonal to $\overrightarrow{v}$. By moving the bounding hyperplane in direction $-\overrightarrow{v}$ until it hits a segment in $S \setminus H$, we can make it a supporting hyperplane.

Let P be a supporting hyperplane of segments H in direction $\overrightarrow{v}$. Let $H' \subseteq H$ be the subset of segments in H, which touch P. Let x be a point on P, which is closer to all endpoints of segments in H' than those which belong to other segments. Consider the ray r which emanates from x and is unbounded in direction $\overrightarrow{v}$. On that ray, we find a point y, which is the center of a closed ball, which touches x and intersects only the segments in H. Every point z on r beyond the point y has the same properties because the ball keeps growing on the side P^+ and shrinks on the other side. This means that all those points on r beyond y belong to the order-k region of the set H. ◀

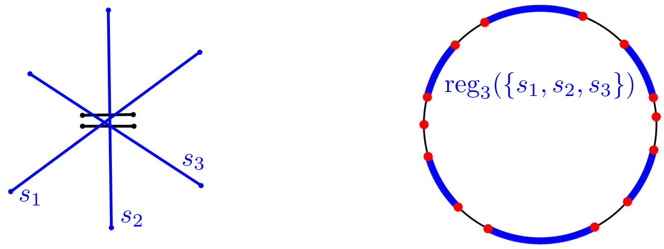


Figure 7 An instance of 5 segments (left), which has one region $\mathrm{reg}_3(\{s_1, s_2, s_3\})$, shown in blue, on the Gaussian map of the order-3 Voronoi diagram (right) with high complexity.

▶ **Corollary 16.** *A supporting hyperplane of H in direction $\overrightarrow{v}$, which touches i segments (at least one of which is in H), corresponds to a $(d{-}i{+}1)$-cell in $VD_k(S)$, which is unbounded in direction $\overrightarrow{v}$, and to a $(d{-}i)$-cell in $\mathrm{GM}(VD_k(S))$.*

▶ **Theorem 17.** *Let S be a set of segments. Then, $\mathrm{FVD}(S)$ does not have tunnels.*

Proof. Let $p_1, p_2 \in \mathrm{GM}(\mathrm{FVD}(S))$ be two points representing unbounded directions of a farthest cell of segment s. These two points represent directions $\overrightarrow{r_1}, \overrightarrow{r_2}$ along which there exist points $x_1, x_2 \in \mathrm{freg}(s)$, for which $\overrightarrow{r_i} = \overrightarrow{q_i x_i}$ with $(i = 1, 2)$, where q_i is the point on s realizing the distance between x_i and s. Since x_1 and x_2 are contained in the same cell $\mathrm{freg}(s)$, there exists a continuous path ξ connecting the points and being fully contained in $\mathrm{freg}(s)$. We can map every point $x \in \xi$ to the direction $\overrightarrow{r} = \overrightarrow{qx}$, with q realizing the distance between x and s. We represent direction $\overrightarrow{r}$ as a point $p \in \mathrm{GM}(\mathrm{FVD}(S))$. Note that p is contained in a farthest cell of the Gaussian map corresponding to segment s. By continuity, mapping the whole path ξ to $\mathrm{GM}(\mathrm{FVD}(S))$ draws a continuous path $\hat{\xi}$ between p_1 and p_2 consisting solely of points that belong to s. Therefore, the points p_1 and p_2 belong to the same cell of the Gaussian map. ◀

▶ Remark 18. The order-k Voronoi diagram of segments S can have tunnels, for $k \leq n - d$.

The next theorem provides a lower bound on the complexity of the Gaussian map of order-k Voronoi diagrams. This bound is meaningful if k is comparable to n.

▶ **Theorem 19.** *Let S be a set of n line segments in $\mathbb{R}^d$. A single region of the Gaussian map of the order-k Voronoi diagram of S can have $\Omega(k^{d-1})$ many vertices. In particular, the $\mathrm{GM}(VD_k(S))$ has $\Omega(k^{d-1})$ complexity in the worst-case.*

Proof. The bound is shown by a generalization of examples provided for $\mathbb{R}^2$ [5, 23]. Place k long segments connecting almost antipodal points on a $(d{-}1)$-dimensional hypersphere and $n{-}k$ additional short segments near the center of the hypersphere, see Figure 7. Any $(d{-}1)$-tuple of long segments, together with one specific short segment, define a supporting hyperplane corresponding to an unbounded edge of the order-k Voronoi diagram of S. The supporting hyperplane is spanned by an endpoint of each of the d segments. An unbounded edge of the diagram manifests itself as a vertex in $\mathrm{GM}(\mathrm{VD}_k(S))$. All these vertices are on the boundary of the Gaussian map region of the long segments. ◀

We can now prove Theorem 7.

Proof of Thm. 7. Let S be a set of n line segments in $\mathbb{R}^d$. In Theorem 19, it was stated that there can be $\Omega(k^{d-1})$ vertices in $\mathrm{GM}(\mathrm{VD}_k(S))$ in the worst-case. Each vertex of the Gaussian map corresponds to an edge in the $\mathrm{VD}_k(S)$. On the other hand, an edge of the diagram corresponds to at most two vertices in the Gaussian map. Therefore, the diagram contains $\Omega(k^{d-1})$ edges. ◀

▶ **Theorem 20.** *The complexity of the Gaussian map of the order-k Voronoi diagram of n segments in $\mathbb{R}^d$ is $O(\min\{k, n-k\}n^{d-1})$.*

Proof. We use the point-hyperplane duality transformation T, which establishes a 1-1 correspondence between the upper Gaussian map of the order-k Voronoi diagram and the k-th level of the arrangement of d-dimensional wedges. (The lower Gaussian map is constructed in the same manner.) Each segment is mapped to a lower wedge in the dual space, which is bounded by two half-hyperplanes. Let p be a point in dual space. Each wedge below p corresponds to a segment in primal space, which has a non-empty intersection with the open halfspace above $T^{-1}(p)$. Each wedge touching p corresponds to a segment in primal space, which is touching the closed halfspace above $T^{-1}(p)$. Each wedge above p corresponds to a segment in primal space, whose intersection with the closed halfspace above $T(p)$ is empty. Therefore, every point on the k-th level of the arrangement of the lower wedges corresponds to a hyperplane in primal space, which supports k segments. The upper or lower envelope of those wedges, each composed of two half-hyperplanes, has complexity $O(n^{d-1})$ [12].

Using the bound on the lower envelope, we can now also bound the complexity of the $\leq k$-level of the arrangement of lower wedges. We apply Theorem 2 by Clarkson and Shor [11] to derive a complexity of $O((k+1)^d(\frac{n}{k+1})^{d-1}) = O(kn^{d-1})$. We can derive a similar upper bound of $O((n-k)n^{d-1})$ by using the complexity of the upper envelope of lower wedges as a basis. The upper Gaussian map of the order-k Voronoi diagram corresponds to the k-level of the lower wedges. Combining the two bounds completes the proof. ◀

The bounds in Theorems 19 and 20 are tight for $n-k = O(1)$. In this case, the complexity of the Gaussian map of VD_k of n segments is $\Theta(n^{d-1})$ in the worst case.

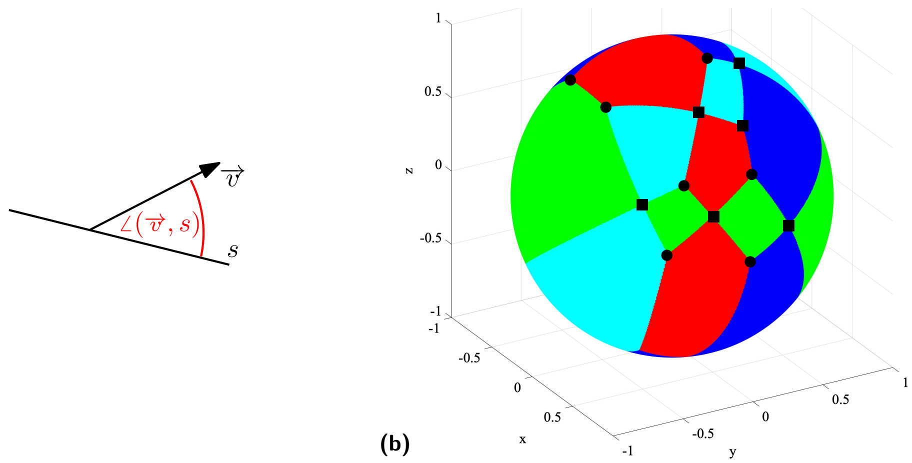
▶ **Theorem 21.** *Let S be a set of n line segments in $\mathbb{R}^3$. Then, $\mathrm{GM}(\mathrm{FVD}(S))$ can be constructed in worst-case optimal $O(n^2)$ time.*

Proof. We dualize the segments into lower wedges. The upper Gaussian map of the segments corresponds to the upper envelope of the lower wedges in dual space (recall the proof of Thm. 20). The upper envelope of those wedges, each composed of two halfplanes, is constructed in $O(n^2)$ time [12]. The lower Gaussian map is constructed in the same way. ◀

The algorithm of Edelsbrunner et al. [12] for piecewise-linear functions can be extended to higher dimensions, running in $O(\alpha(n)\, n^{d-1})$ time [2, 15]. In fact, the complexity of the upper envelope of half-hyperplanes is only $O(n^{d-1})$ [12]. We suspect that the same algorithm runs in $O(n^{d-1})$ time when it computes the upper envelope of half-hyperplanes, as in $\mathbb{R}^3$, since the complexity of the envelope does not contain the $\alpha(n)$ factor. If so, the Gaussian map of the farthest Voronoi diagram can be constructed in $O(n^{d-1})$ time.

5 Lines as Sites

Let S be a set of lines in $\mathbb{R}^d$. We assume that the lines are in general position, i.e., the lines are non-intersecting and the directions of any d lines are linearly independent. In this section we derive similar conditions for the order-k Voronoi diagram of lines to have unbounded cells in some direction. We omit proofs whose principles are similar to the ones of segments.



(a) (b)

Figure 8 (a) The angular distance $\angle(\vec{v}, s)$ between line s and direction $\vec{v}$.
(b) GM(FVD) of four lines in $\mathbb{R}^3$. The farthest regions of the lines are colored in different colors. Vertices of anomaly are shown with squared boxes; proper vertices with disks.

▶ **Definition 22.** *For a line s and a direction $\vec{v}$, the* angular distance $\angle(\vec{v}, s)$ *is the smallest angle between $\vec{v}$ and the direction of s, see Figure 8a.*

▶ **Definition 23.** *Let S be a set of lines, and let H be a subset of S. An angle β is a* supporting angle *of H in direction $\vec{v}$ if*
1. *The angular distance between $\vec{v}$ and any of the lines in H is at most β; and*
2. *The angular distance between $\vec{v}$ and any of the lines in $S \setminus H$ is at least β, and at least one site in $S \setminus H$ realizes the angular distance β.*

▶ **Theorem 24.** *A set of lines H, with $|H| = k$, induces an unbounded region in direction $\vec{v}$ in $VD_k(S)$ if and only if there exists a supporting angle of H in direction $\vec{v}$.*

The proof of the above theorem is essentially the same as that of Theorem 15, with a supporting hyperplane replaced by a supporting angle, and intersections with a halfspace replaced by angular distances.

▶ **Corollary 25.** *A supporting angle of H, which is realized by i lines (at least one of which is in H), corresponds to an unbounded $(d{-}i{+}1)$-cell in the order-k Voronoi diagram of S.*

All d-cells, which are unbounded in the same direction $\vec{v}$, touch at a common cell. This cell is determined by the lines, which have the same angular distance to $\vec{v}$. A cell, which is equidistant to i lines, is $d{-}i{+}1$-dimensional.

▶ **Theorem 26.** *A supporting angle β of H in direction $\vec{v}$, which is realized by i lines (of which, at least one belongs to H), corresponds to a $(d{-}i)$-cell (resp., $(d{-}i{-}1)$-cell) in* GM($VD_k(S)$) *if $\beta < \pi/2$ (resp., $\beta = \pi/2$).*

Typically, i-cells of the Gaussian map correspond to $(i{+}1)$-cells of the corresponding Voronoi diagram. The only exceptions are cells whose supporting angle is $\pi/2$, and, thus, they correspond to $(i{+}2)$-cells of VD_k.

▶ **Definition 27.** *The i-cells of the Gaussian map, $i < d-1$, which correspond to a supporting angle of $\pi/2$, are called* cells of anomaly. *All other cells are called* proper.

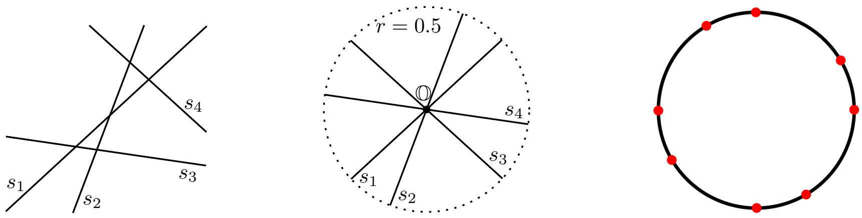


Figure 9 (Left) Lines S and their (center) transformed segments $\tau(S)$ have identical (right) Gaussian maps $\mathrm{GM}(\mathrm{VD}_2(S)) = \mathrm{GM}(\mathrm{VD}_2(\tau(S)))$.

In $\mathbb{R}^3$, the only cells of anomaly are vertices, see Figure 8b. Such a vertex corresponds to a direction in which the bisector of two lines seems to be self-intersecting. The bisector of two lines s, s' is a hyperbolic paraboloid. Seen "from infinity" this hyperbolic paraboloid looks like two intersecting planes. The intersection of those planes is a line l, which is unbounded in two antipodal directions $-\vec{v}, \vec{v}$, which are the vertices of anomaly on the Gaussian map. One of the lines s, s' is actually strictly closer to direction $\vec{v}$ than the other. Only "at infinity" both lines seem to have equal distance in direction $\vec{v}$.

In general space $\mathbb{R}^d$, the i-cells of anomaly on the Gaussian map correspond to $(i+2)$-cells in the order-k Voronoi diagram. Looking at the Gaussian map, these $(i+2)$-cells seem as if they intersect, however, they do not intersect in the actual diagram. Let $\vec{v}$ be the direction of a cell of anomaly. The lines, which are orthogonal to $\vec{v}$, can actually be ordered along direction $\vec{v}$. Let j be the number of lines that are not orthogonal to $\vec{v}$. The region of those j lines, together with the closest $k-j$ orthogonal lines, is unbounded in direction $\vec{v}$ and, moreover, is not split by an $(i+1)$-cell in direction $\vec{v}$.

We define a transformation τ that maps lines to segments. Each line ℓ is mapped to a unit segment $\tau(\ell)$ that has the same direction as the line and the origin O as midpoint, see Figure 9. When applied to a set of lines, the result of the transformation is a set of segments in non-general position, but this does not affect the upper bound on the complexity of the Gaussian map.

▶ **Theorem 28.** *Let S be a set of lines. Then,* $\mathrm{GM}(VD_k(S)) = \mathrm{GM}(VD_k(\tau(S)))$.

As a consequence, lower bounds on the worst-case complexity of the Gaussian map, derived for lines as sites, carry over to segments as sites. In the same manner, all upper bounds on the worst-case complexity on the Gaussian map for segments also apply to lines. In addition, the algorithm of Theorem 21 to construct the Gaussian map of the farthest Voronoi diagram extends to lines as sites. (Note that the algorithm does not require the segments to be in general position.)

▶ **Corollary 29.** *The Gaussian map of the order-k Voronoi diagram of n lines in $\mathbb{R}^d$ has $O(\min\{k, n-k\}n^{d-1})$ complexity. The Gaussian map can be constructed in $O(n^{d-1}\alpha(n))$ time, while if $d = 3$, the time drops to $O(n^2)$.*

▶ **Theorem 30.** *Let S be a set of lines. Then,* $\mathrm{FVD}(S)$ *does not have tunnels.*

A similar construction, as in Remark 18, can be used for showing that $\mathrm{VD}_k(S)$ can have tunnels for a set of lines S and $k \leq n - d$.

The following result stands by its own and will be used to analyze the number of d-cells in the farthest Voronoi diagram of lines and its Gaussian map. We look at an arrangement of great spheres with same center and radius on a $(d-1)$-sphere. For example, consider

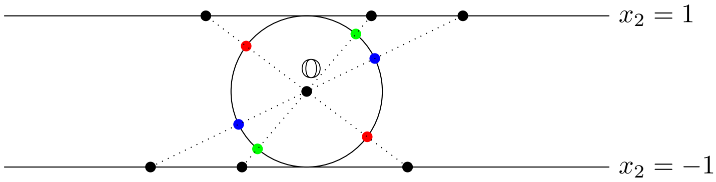


Figure 10 Example for $d = 2$ and $n = 3$: Three 0-dimensional unit spheres (blue, green, red) split the unit circle into 6 arcs.

the 2-dimensional unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ and n great circles on it. We answer the following question: "Into how many 2-dimensional faces the unit sphere is split by the great circles?" We assume that no d great spheres have a point in common.

▶ **Theorem 31.** *Let $\mathbb{S}$ be a set of n many $(d{-}2)$-dimensional unit hyperspheres in $\mathbb{R}^d$, centered at the origin. Then, the arrangement of $\mathbb{S}$ on the $(d{-}1)$-dimensional unit hypersphere $\mathbb{S}^{d-1}$ contains $\binom{n-1}{d-1} + \sum_{k=0}^{d-1}\binom{n}{k}$ many $(d{-}1)$-cells.*

Theorem 31 can be proven by bijectively mapping the upper and lower hemisphere of $\mathbb{S}^{d-1}$ to two parallel hyperplanes, see Figure 10. The $(d{-}2)$-dimensional hyperspheres become hyperplanes of the same dimension. We add the $(d{-}1)$-cells of each arrangement of $(d{-}2)$-dimensional hyperplanes, while making sure that we do not count any cell twice.

▶ **Theorem 32.** *Let S be a set of n lines. The Gaussian map of* FVD(S) *has $\Theta(n^{d-1})$ many $(d{-}1)$-cells.*

Proof. We consider, for each line, the orthogonal directions. We get n many $(d{-}2)$-dimensional hyperspheres in total. Each of those hyperspheres is partitioned into $\binom{n-2}{d-2} + \sum_{k=0}^{d-2}\binom{n-1}{k}$ parts by the other $(n{-}1)$-hyperspheres due to Theorem 31. A direction in one of those parts is orthogonal to exactly one line in S and, hence, is also part of the farthest Voronoi region of that line. In total, all n hyperspheres are split into $n\left(\binom{n-2}{d-2} + \sum_{k=0}^{d-2}\binom{n-1}{k}\right) = \Theta(n^{d-1})$ parts. Now, consider a direction $\vec{v}$ not on any hypersphere but in the farthest region of s. The shortest path on the Gaussian map from $\vec{v}$ to the hypersphere corresponding to line s contains only directions of freg(s). Therefore, there are no additional $(d{-}1)$-cells not containing a part of a hypersphere. ◀

It is easy to prove that all cells of GM(FVD(S)) are convex, in the sense that the shortest path between any two points of a cell is contained in that cell.

For a set of lines S in $\mathbb{R}^3$, we count the number of 2-cells of GM(FVD(S)) and subtract the number of vertices of anomaly to derive the exact number of 3-cells in FVD(S).

▶ **Theorem 33.** *Let S be any set of $n \geq 2$ lines in $\mathbb{R}^3$. Then,* FVD(S) *has exactly $(n^2{-}n)$ many 3-cells.*

An unbounded i-cell of a cell complex M may correspond to many $(i{-}1)$-cells in the Gaussian map of M. Therefore, we need to study carefully the Gaussian map in order to derive a lower bound on the complexity of M.

▶ **Theorem 34.** *The worst-case complexity of* FVD *of n lines is $\Omega(n^{d-1})$.*

Proof. We bound the number of proper vertices (not those of anomaly) of GM(FVD(S)) from below. Those vertices correspond to unbounded edges of the farthest Voronoi diagram. The set of orthogonal directions to a line is a hypersphere of dimension $d{-}2$ in GM(FVD(S)). By

Theorem 31, the hyperspheres of all lines partition GM(FVD(S)) into $\binom{n-1}{d-1} + \sum_{k=0}^{d-1} \binom{n}{k} = \Omega(n^{d-1})$ many (d–1)-dimensional parts. If $n \geq d$ (which is the case in the asymptotic analysis), each of those parts contains at least one proper vertex. Then, FVD(S) has an unbounded edge in that direction. Each edge is unbounded in at most two directions. Hence, the number of edges can be bounded from below by half of the number of proper vertices of the Gaussian map. Thus, the number of edges in FVD(S) is $\Omega(n^{d-1})$. ◀

6 Conclusion and Open Problems

We derived bounds on the complexity of the order-k Voronoi diagram and its Gaussian map, listed in Table 1. The results are tight for large values of k such as $k = n - 1$. Moreover we provided an algorithm to compute the Gaussian map of the farthest Voronoi diagram in three dimensional space in worst-case optimal time. It remains an open problem to determine whether or not the lower bounds on the complexity of VD_k and $\mathrm{GM}(\mathrm{VD}_k)$ for segments, as listed in Table 1, extend also to lines, when $k < n - 1$.

There is a gap between our lower and upper bounds on the complexity of the Gaussian map of the order-k Voronoi diagram. What is the correct bound and how can the diagram be constructed efficiently? This question is related to problem 3 in [21]: "What is the combinatorial complexity of the Voronoi diagram of a set of lines (or line segments) in three dimensions"?

We believe that knowing the structure of the Gaussian map of the order-k Voronoi diagram can help in analyzing the whole diagram. It may also be useful in constructing the full diagram. We leave this question for further research.

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