Distance Between Mutually Reachable Petri Net Configurations

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Abstract

Petri nets are a classical model of concurrency widely used and studied in formal verification with many applications in modeling and analyzing hardware and software, data bases, and reactive systems. The reachability problem is central since many other problems reduce to reachability questions. In 2011, we proved that a variant of the reachability problem, called the reversible reachability problem is exponential-space complete. Recently, this problem found several unexpected applications in particular in the theory of population protocols. In this paper we revisit the reversible reachability problem in order to prove that the minimal distance in the reachability graph of two mutually reachable configurations is linear with respect to the Euclidean distance between those two configurations.

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1 Introduction

Petri nets are a classical model of concurrency widely used and studied in formal verification with many applications in modeling and analyzing hardware and software, data bases, and reactive systems. The reachability problem is central since many other problems reduce to reachability questions. Unfortunately, the reachability problem is difficult for several reasons. In fact, from a complexity point of view, we recently proved that the problem is non-elementary [6] by observing that the worst case complexity in space is at least a tower of exponential with height growing linearly in the dimension of the Petri nets. Moreover, even in practice, the reachability problem is difficult. Nowadays, no tool exists for deciding that problem since the known algorithms are difficult to be implemented and require many enumerations in exponentially large state spaces (see [13] for the state-of-the-art algorithm deciding the reachability problem).

Fortunately, easier natural variants of the reachability problems can be applied in various contexts. For instance, the coverability problem which consists in deciding if a configuration can be covered by a reachable one can be applied in the analysis of concurrent programs [1] (in that context, covered means component-wise smaller than or equal). The coverability problem is known to be exponential-space complete [16, 5], and efficient tools exist [4, 8]. Another variant is the reversible reachability problem. This problem consists in deciding if two configurations are mutually reachable one from the other. This problem was proved to be exponential-space complete in [11] and finds unexpected applications in population protocols [7], trace logics [12], universality problems related to structural liveness problems [10], and in solving the home state problem [2].

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Contribution. The exponential-space complexity upper-bound of the reversible reachability problem proved in [11] is obtained by observing that if two configurations are mutually reachable, then the two configurations belong to a cycle of the (infinite) reachability graph with a length at most doubly-exponential with respect to the size in binary of the two configurations. In this paper, we focus on the minimal length of such a cycle (called the distance in the sequel) with respect to the Euclidean distance between those two configurations. We prove that the distance is linearly bounded by the Euclidean distance up-to a doubly-exponential constant that only depends on the Petri net. As a direct consequence, this result generalizes [11] and it shows that the distance between two nearby (for the Euclidean distance) mutually reachable configurations is small.

Outline. In Section 2 we introduce our main problem about the distance between mutually reachable Petri net configurations and we motivate the problem. In Section 3 we extend Petri nets with control states. A petri net with states is basically given as a finite state automaton with transitions labeled by Petri net actions. We also introduce the subclass of structurally reversible Petri nets. Intuitively a Petri net with states is structurally reversible if the effect of every transition can be reverted as soon as we execute that transition from a large enough configuration. We provide in that section a sufficient condition to decide the reachability problem for structurally reversible Petri nets between large configurations. In Section 4 and Section 5, we recall some techniques called *Rackoff's extraction* to extract components that are "very large" compared to others from executions. Those techniques are applied in Section 6 in order to extract from a strongly connected component of the reachability graph of a Petri net, a structurally reversible Petri net with states. Intuitively, this Petri net with states is obtained by projecting away components that can be large in the considered strongly connected component. From that Petri net with states, and thanks to the sufficient condition for reachability introduced in Section 3, we proved in Section 7 our main result about the distance between mutually reachable configurations.

In the paper, d is a positive natural number denoting the number of components of vectors. Given a vector $\mathbf{x}$ in the set of reals $\mathbb{R}^d$, we denote by $\mathbf{x}(1), \dots, \mathbf{x}(d)$ its components in such a way $\mathbf{x} = (\mathbf{x}(1), \dots, \mathbf{x}(d))$. Moreover, we introduce the norms $\|\mathbf{x}\| \stackrel{\text{def}}{=} \sum_{i=1}^{d} |\mathbf{x}(i)|$ and $\|\mathbf{x}\|_\infty \stackrel{\text{def}}{=} \max_{1 \leq i \leq d} |\mathbf{x}(i)|$. The set of integers and the set of non-negative natural numbers are denoted as $\mathbb{Z}$ and $\mathbb{N}$ respectively.

2 Petri Nets

A *Petri net* A (*PN* for short) is a finite set of pairs $(\mathbf{a}_-, \mathbf{a}_+)$ in $\mathbb{N}^d \times \mathbb{N}^d$ called *actions*. In the literature, vectors $\mathbf{a}_-$ and $\mathbf{a}_+$ are respectively usually called the *pre-condition* and the *post-condition* of a. A *configuration* is a vector in $\mathbb{N}^d$. We associate to an action $a = (\mathbf{a}_-, \mathbf{a}_+)$ the binary relation $\xrightarrow{a}$ over the configurations defined by $\mathbf{x} \xrightarrow{a} \mathbf{y}$ if for some configuration $\mathbf{c}$ we have $\mathbf{x} = \mathbf{a}_- + \mathbf{c}$ and $\mathbf{y} = \mathbf{a}_+ + \mathbf{c}$. Notice that $\mathbf{x} \xrightarrow{a} \mathbf{y}$ if, and only if, $\mathbf{x} \geq \mathbf{a}_-$ and $\mathbf{y} = \mathbf{x} - \mathbf{a}_- + \mathbf{a}_+$. We denote by $\xrightarrow{A}$ the one-step reachability relation of A defined by $\mathbf{x} \xrightarrow{A} \mathbf{y}$ if there exists an action a in A such that $\mathbf{x} \xrightarrow{a} \mathbf{y}$. A PN A defines an infinite graph $(\mathbb{N}^d, \xrightarrow{A})$ called the *reachability graph* of A.

A *σ-execution*, where $\sigma = a_1 \dots a_k$ is a word of actions, is a non-empty word of configurations $e = \mathbf{c}_0 \mathbf{c}_1 \dots \mathbf{c}_k$ such that the following relations hold:

$$\mathbf{c}_0 \xrightarrow{a_1} \mathbf{c}_1 \cdots \xrightarrow{a_k} \mathbf{c}_k$$

We denote by $\mathrm{src}(e)$ and $\mathrm{tgt}(e)$ the configurations $\mathbf{c}_0$ and $\mathbf{c}_k$ respectively. An A^*-*execution* is a σ-execution for some word σ over A. An A^ω-*execution* e is an infinite word of configurations such that every finite non-empty prefix is an A^*-execution. We associate to a word σ of actions the binary relation $\xrightarrow{\sigma}$ over the configurations defined by $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ if there exists a σ-execution from $\mathbf{x}$ to $\mathbf{y}$. The *displacement* of a word $\sigma = a_1 \ldots a_k$ is the vector $\Delta(\sigma) \stackrel{\text{def}}{=} \sum_{j=1}^{k} \Delta(a_j)$ where $\Delta(a) \stackrel{\text{def}}{=} \mathbf{a}_+ - \mathbf{a}_-$ for every action $a = (\mathbf{a}_-, \mathbf{a}_+)$. Notice that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ implies $\Delta(\sigma) = \mathbf{y} - \mathbf{x}$ but the converse is not true in general. We introduce the *reachability relation* $\xrightarrow{A^*}$ defined as the union of the relations $\xrightarrow{\sigma}$ where $\sigma \in A^*$. Notice that this relation is the reflexive and transitive closure of $\xrightarrow{A}$.

The Petri net reachability problem consists in deciding given a PN A and two configurations $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$. In [6], we provided a non-elementary complexity lower-bound for the PN reachability problem. Moreover, we prove that for every natural number h, there exists a PN A_h such that the reachability problem for that PN is at least h-exponential space hard. It means that the minimal length of a word $\sigma \in A_h^*$ satisfying $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ is at least $(h+1)$-exponential with respect to $\|\mathbf{x}\| + \|\mathbf{y}\|$. This huge lower bound is no longer valid for mutually reachable configurations.

Two configurations $\mathbf{x}$ and $\mathbf{y}$ are said to be *mutually reachable* for a PN A if $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ and $\mathbf{y} \xrightarrow{A^*} \mathbf{x}$. The PN reversible reachability problem consists in deciding given a PN A and two configurations $\mathbf{x}$ and $\mathbf{y}$ if they are mutually reachable for A. In [11], we proved that the PN reversible reachability problem is decidable in exponential-space by proving that there exists at most doubly-exponential long words σ and w in A^* such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ and $\mathbf{y} \xrightarrow{w} \mathbf{x}$. This result can be refined by introducing the notion of distance. The *distance* between two mutually reachable configurations $\mathbf{x}$ and $\mathbf{y}$ for a PN A is formally defined as follows:

$$\mathrm{dist}_A(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \min_{\sigma, w \in A^*} \{|\sigma w| \mid \mathbf{x} \xrightarrow{\sigma} \mathbf{y} \xrightarrow{w} \mathbf{x}\}$$

A simple lower bound on the distance can be obtained by observing that configurations along an execution are relatively close one from each other as shown in the proof of the following lemma.

▶ **Lemma 1.** *Let us consider a PN $A \subseteq \{0, \ldots, m\}^d \times \{0, \ldots, m\}^d$ for some positive natural number m. For every mutually reachable configurations $\mathbf{x}$ and $\mathbf{y}$, we have:*

$$\mathrm{dist}_A(\mathbf{x}, \mathbf{y}) \geq \|\mathbf{y} - \mathbf{x}\| \frac{2}{dm}$$

Proof. Let σ be a word in A^* such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ and let us prove that $\|\mathbf{y} - \mathbf{x}\| \leq m|\sigma|$. Assume that $\sigma = a_1 \ldots a_k$. Since $\Delta(a_j) \in \{-m, \ldots, m\}^d$, it follows that $\Delta(\sigma) \in \{-mk, \ldots, mk\}^d$. In particular $\|\Delta(\sigma)\| \leq dmk$. As $\Delta(\sigma) = \mathbf{y} - \mathbf{x}$ and $k = |\sigma|$, we deduce the relation $\|\mathbf{y} - \mathbf{x}\| \leq md|\sigma|$. Now, let us consider a word w in $\mathbf{A}^*$ such that $\mathbf{y} \xrightarrow{w} \mathbf{x}$ and observe that we have $\|\mathbf{x} - \mathbf{y}\| \leq dm|w|$ by symmetry. It follows that $|\sigma w| \geq \|\mathbf{y} - \mathbf{x}\| \frac{2}{dm}$ and we have proved the lemma. ◀

This paper focus on an upper-bound of the form $\mathrm{dist}_A(\mathbf{x}, \mathbf{y}) \leq f_A(\|\mathbf{y} - \mathbf{x}\|)$ where f_A is a function that only depends on the PN A and not on the two mutually reachable configurations $\mathbf{x}$ and $\mathbf{y}$. Such a bound cannot be derived from [11]. In fact, the best upper bound that can be derived from that paper is the following one:

$$\mathrm{dist}_A(\mathbf{x}, \mathbf{y}) \leq 34 d^2 n^{15 d^{d+2}}$$

where $n = (1 + 2m)(1 + 2\max\{\|\mathbf{x}\|, \|\mathbf{y}\|\})$.

In this paper we prove that such a function f_A exists. Moreover a linear one exists as shown by the following theorem.

▶ **Theorem 2.** *Let us consider a PN $A \subseteq \{0,\dots,m\}^d \times \{0,\dots,m\}^d$ for some positive natural number m. For every mutually reachable configurations $\mathbf{x}$ and $\mathbf{y}$, we have:*

$$\text{dist}_A(\mathbf{x},\mathbf{y}) \leq \|\mathbf{y}-\mathbf{x}\| c_{d,m}$$

where:

$$c_{d,m} \leq (3dm)^{(d+1)^{2d+4}}$$

▶ Remark 3. The previous theorem provides as a corollary a new proof that the reversible reachability problem is solvable in exponential space. It also provides a bound on the minimal elements defining the *domain of reversibility* (introduced in [11]) of an action a in A defined as $\mathbf{D}_{a,A} \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{N}^d \mid \exists \mathbf{y}\ \mathbf{x} \xrightarrow{a} \mathbf{y} \xrightarrow{A^*} \mathbf{x}\}$. In fact, this set is *upward closed* and if $\mathbf{x}$ is a minimal element for $\leq$ in $\mathbf{D}_{a,A}$ then the vector $\mathbf{y}$ satisfying $\mathbf{x} \xrightarrow{a} \mathbf{y}$ is such that $\text{dist}_A(\mathbf{x},\mathbf{y}) \leq dmc_{d,m}$ since $\|\mathbf{y}-\mathbf{x}\| = \|\Delta(a)\| \leq dm$. We deduce that there exists a word σ of actions in A such that $\mathbf{y} \xrightarrow{\sigma} \mathbf{x}$ with a length bounded by $dmc_{d,m}$. If a component of $\mathbf{x}$ is larger than $m|\sigma|$, the vector $\mathbf{x}$ cannot be minimal since the vector $\mathbf{x}'$ obtained from $\mathbf{x}$ by replacing that coordinate by $m|\sigma|$ satisfies $\mathbf{x}' \xrightarrow{a} \mathbf{y}' \xrightarrow{\sigma} \mathbf{x}'$ where $\mathbf{y}' \stackrel{\text{def}}{=} \mathbf{x}' + \Delta(a)$. Hence $\|\mathbf{x}\| \leq dm^2 c_{d,m}$.

3 Structurally Reversible Petri Nets With States

A *Petri net with states* (*PNS* for short) is a tuple $\langle Q, A, T\rangle$ where Q is a non empty finite set of elements called *states*, A is a Petri net, and T is a set of triples in $Q \times A \times Q$ called *transitions*. A *path* π from a state p to a state q labeled by a word σ of actions is a word of transitions of the form $(q_0, a_1, q_1)\dots(q_{k-1}, a_k, q_k)$ for some states $q_0,\dots,q_k$ satisfying $q_0 = p$ and $q_k = q$, and for some actions $a_1,\dots,a_k$ satisfying $\sigma = a_1 \dots a_k$. The *displacement* of π is the vector $\Delta(\pi) \stackrel{\text{def}}{=} \Delta(\sigma)$. A path is said to be *elementary* if $q_i = q_j$ implies $i = j$. A path such that $q_0 = q_k$ is called a *cycle* on q_0. A cycle is said to be *simple* if $q_i = q_j$ with $i < j$ implies $i = 0$ and $j = k$. A pair $(q, \mathbf{x})$ in $Q \times \mathbb{N}^d$ is called a *state-configuration* and it is denoted as $q(\mathbf{x})$ in the sequel. We associate to a path π the binary relation $\xrightarrow{\pi}$ over the state-configurations defined by $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ if π is a path from p to q labeled by a word σ of actions such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$.

A PNS is said to be *structurally reversible* if for every transition (p, a, q) there exists a path π from q to p such that $\Delta(a) + \Delta(\pi) = \mathbf{0}$. Structurally reversible PNSes are such that the displacement of any cycle can be canceled by the displacement of another cycle as shown by the following lemma.

▶ **Lemma 4.** *For every state q and for every cycle θ on q, there exists a cycle θ' on q such that $\Delta(\theta') = -\Delta(\theta)$.*

Proof. Assume that $\theta = (q_0, a_1, q_1)\dots(q_{k-1}, a_k, q_k)$ with $q_0 = q = q_k$. Since the PNS is structurally reversible, for every $j \in \{1,\dots,k\}$, there exists a path π_j from q_j to q_{j-1} such that $\Delta(a_j) + \Delta(\pi_j) = \mathbf{0}$. Now, observe that $\theta' \stackrel{\text{def}}{=} \pi_k \dots \pi_1$ is a cycle on q such that $\Delta(\theta') = -\Delta(\theta)$. ◀

A *partial configuration* is a vector $\mathbf{x} \in \mathbb{N}^I$ where $I \subseteq \{1, \dots, d\}$. We associate to a configuration $\mathbf{x} \in \mathbb{N}^d$ and a set $I \subseteq \{1, \dots, d\}$ the partial configuration $\mathbf{x}|_I$ in $\mathbb{N}^I$ defined by $\mathbf{x}|_I(i) = \mathbf{x}(i)$ for every $i \in I$. Given an action $a = (\mathbf{a}_-, \mathbf{a}_+)$ of a Petri net, we extend the binary relation $\xrightarrow{a}$ over the partial configurations by $\mathbf{x} \xrightarrow{a} \mathbf{y}$ if $\mathbf{x}, \mathbf{y}$ are two partial configurations in $\mathbb{N}^I$ such that there exists a partial configuration $\mathbf{c} \in \mathbb{N}^I$ satisfying $\mathbf{x} = \mathbf{a}_-|_I + \mathbf{c}$ and $\mathbf{y} = \mathbf{a}_+|_I + \mathbf{c}$.

A *flow function* is a function $F : Q \to \mathbb{N}^I$ for some subset $I \subseteq \{1, \dots, d\}$ such that $F(p) \xrightarrow{a} F(q)$ for every transition (p, a, q) in T. In this section we prove the following result.

▶ **Lemma 5.** *Let us consider a structurally reversible PNS with at most r states and with actions in $\{0, \dots, m\}^d \times \{0, \dots, m\}^d$ for some positive natural number m, let $p(\mathbf{x})$ and $q(\mathbf{y})$ be two state-configurations such that the following conditions hold for some flow function $F : Q \to \mathbb{N}^I$:*

- *$\mathbf{x}|_I = F(p)$ and $\mathbf{y}|_I = F(q)$,*
- *$\mathbf{x}(i), \mathbf{y}(i) \geq mr^3(3drm)^d$ for every $i \notin I$, and*
- *$\mathbf{y} - \mathbf{x}$ is the sum of the displacement of a path from p to q and a vector in the subgroup of $(\mathbb{Z}^d, +)$ generated by the displacements of the cycles.*

Then $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ for a path π such that $|\pi| \leq (\|\mathbf{y} - \mathbf{x}\| + drm)r^3(3drm)^{2d}$.

In this section, we fix such a structurally reversible PNS G. Since G is a disjoint union of strongly connected components, we can assume without loss of generality that G is strongly connected. The proof of Lemma 5 follows an extended form of the *zigzag-freeness* approach introduced in [14]. Intuitively, we fix an elementary path π_0 from p to q, and we prove that there exists a sequence $\theta_1, \dots, \theta_k$ of short cycles on q such that for every $n \in \{0, \dots, k\}$ the displacement of $\Delta(\theta_1 \dots \theta_n)$ is almost the vector $\frac{n-d}{k}\mathbf{z}$ where $\mathbf{z} \stackrel{\text{def}}{=} \mathbf{y} - \mathbf{x} - \Delta(\pi_0)$. This result is based on the following lemma.

▶ **Lemma 6** ([9]). *Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a non-empty sequence of vectors in $\mathbb{R}^d$ such that $\|\mathbf{v}_j\|_\infty \leq 1$ for every $1 \leq j \leq k$ and let $\mathbf{v} = \sum_{j=1}^k \mathbf{v}_j$. There exists a permutation σ of $\{1, \dots, k\}$ such that for every $n \in \{d, \dots, k\}$, we have:*

$$\|\sum_{j=1}^{n} \mathbf{v}_{\sigma(j)} - \frac{n-d}{k}\mathbf{v}\|_\infty \leq d$$

From the previous lemma we deduce the following two corollaries.

▶ **Corollary 7.** *Let $\mathbf{Z}$ be a set of vectors in $\{-m, \dots, m\}^d$ for some positive natural number m, and assume that $\mathbf{z}$ is a finite sum of vectors in $\mathbf{Z}$. Then $\mathbf{z}$ is a finite sum of at most $(\|\mathbf{z}\| + 1)(3dm)^d$ vectors in $\mathbf{Z}$.*

Proof. By symmetry, we can assume without loss of generality that $\mathbf{z} \geq \mathbf{0}$. Let k be the minimal natural number such that there exists a sequence $\mathbf{z}_1, \dots, \mathbf{z}_k$ of vectors in $\mathbf{Z}$ such that $\mathbf{z} = \mathbf{z}_1 + \dots + \mathbf{z}_k$. If $k = 0$ the lemma is proved, so let us assume that $k \geq 1$. Observe that there exists a sequence $\mathbf{e}_1, \dots, \mathbf{e}_k$ of vectors in $\mathbb{N}^d$ such that $\mathbf{z} = \sum_{j=1}^k \mathbf{e}_j$ and such that $\mathbf{e}_j(i) \leq \max\{0, \mathbf{z}_j(i)\}$ for every $1 \leq i \leq d$ and every $1 \leq j \leq k$. We introduce the sequence $\mathbf{v}_1, \dots, \mathbf{v}_k$ defined by $\mathbf{v}_j \stackrel{\text{def}}{=} \mathbf{z}_j - \mathbf{e}_j$. Notice that $\|\mathbf{v}_j\|_\infty \leq m$ and $\sum_{j=1}^k \mathbf{v}_j = \mathbf{0}$. We introduce $\mathbf{x}_n \stackrel{\text{def}}{=} \sum_{j=1}^n \mathbf{v}_j$. By applying a permutation, Lemma 6 applied on the sequence $(\frac{1}{m}\mathbf{v}_j)_{1 \leq j \leq n}$ shows that we can assume without loss of generality that $\mathbf{x}_n \in \mathbf{X}$ for every $d \leq n \leq k$ where $\mathbf{X}$ is the set of vectors $\mathbf{x} \in \mathbb{Z}^d$ such that $\|\mathbf{x}\|_\infty \leq md$. Notice that if $n \in \{0, \dots, d\}$, we also have $\mathbf{x}_n \in \mathbf{X}$ since $\mathbf{x}_n$ is a sum of at most d vectors with a nom bounded by m.

The cardinal of $\mathbf{X}$ is bounded by $(1+2dm)^d \leq (3dm)^d$. Now, assume by contradiction that there exists $\ell \in \{0, \ldots, k-(3dm)^d\}$ satisfying $\mathbf{e}_j = \mathbf{0}$ for every $j \in \{\ell+1, \ldots, \ell+(3dm)^d\}$. Notice that there exists $p < q$ in $\{\ell, \ldots, \ell+(3dm)^d\}$ such that $\mathbf{x}_p = \mathbf{x}_q$ since the cardinal of $\mathbf{X}$ is bounded by $(3dm)^d$. It follows that $\sum_{j=p+1}^{q} \mathbf{v}_j = \mathbf{0}$. From $\mathbf{e}_j = \mathbf{0}$ for every $j \in \{\ell+1, \ldots, \ell+(3dm)^d\}$ it follows that $\mathbf{v}_j = \mathbf{z}_j$ for every $j \in \{p+1, \ldots, q\}$. In particular $\sum_{j=p+1}^{q} \mathbf{z}_j = \mathbf{0}$. Hence k is not minimal since we can remove the vectors $\mathbf{z}_{p+1}, \ldots, \mathbf{z}_q$ from the sequence $\mathbf{z}_1, \ldots, \mathbf{z}_k$, and we get a contradiction. It follows that for every $\ell \in \{0, \ldots, k-(3dm)^d\}$ there exists $j \in \{\ell+1, \ldots, \ell+(3dm)^d\}$ such that $\mathbf{e}_j \neq \mathbf{0}$. From $\|\mathbf{z}\| = \sum_{j=1}^{k} \|\mathbf{e}_j\|$, it follows that $\|\mathbf{z}\| \geq \frac{k}{(3dm)^d} - 1$. Hence $k \leq (\|\mathbf{z}\|+1)(3dm)^d$. ◀

▶ Remark 8. The bound $(\|\mathbf{z}\|+1)(3dm)^d$ provided by the previous lemma is better than the bound $(\|\mathbf{z}\|+1)(2+(3m+1)^d)^d$ that one can derive from [15] by introducing the linear system $\mathbf{z} = \sum_{j=1}^{k} n_j \mathbf{z}_j$ over the free variables $\mathbf{z}, n_1, \ldots, n_k$, where k is the cardinal of $\mathbf{Z}$, and $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\} = \mathbf{Z}$.

▶ **Corollary 9.** *Assume that $\mathbf{z} = \mathbf{z}_1 + \cdots + \mathbf{z}_k$ where $\mathbf{z}_1, \ldots, \mathbf{z}_k$ are vectors in $\{-m, \ldots, m\}^d$ for some positive natural number $m \geq 1$. There exists a permutation σ of $\{1, \ldots, k\}$ such that for every $n \in \{0, \ldots, k\}$ and for every $i \in \{1, \ldots, d\}$, we have:*

$$\sum_{j=1}^{n} \mathbf{z}_{\sigma(j)}(i) \geq \min\{\mathbf{z}(i), 0\} - md$$

Proof. If $k = 0$ the lemma is proved. So, we can assume that $k \geq 1$. By applying a permutation, Lemma 6 on the sequence $(\frac{1}{m}\mathbf{z}_j)_{1 \leq j \leq k}$ shows that we can assume without loss of generality that for every $n \in \{0, \ldots, k\}$, there exists a vector $\mathbf{e}_n \in \mathbb{R}^d$ such that $\|\mathbf{e}_n\|_\infty \leq md$ and such that $\mathbf{x}_n = \frac{n-d}{k}\mathbf{z} + \mathbf{e}_n$ where $\mathbf{x}_n \stackrel{\text{def}}{=} \sum_{j=1}^{n} \mathbf{z}_j$. Let $i \in \{1, \ldots, d\}$ and let us prove that $\mathbf{x}_n(i) \geq \min\{\mathbf{z}(i), 0\} - md$. Observe that if $n \in \{0, \ldots, d\}$ then the property is immediate since $\mathbf{x}_n(i) \geq -md$. So, let us assume that $n > d$. If $\mathbf{z}(i) \geq 0$ then $\frac{n-d}{k}\mathbf{z}(i) \geq 0$ and we get $\mathbf{x}_n(i) \geq \mathbf{e}_n(i) \geq -md$. If $\mathbf{z}(i) \leq 0$ then $\frac{n-d}{k}\mathbf{z}(i) \geq \mathbf{z}(i)$. In particular $\mathbf{x}_n(i) \geq \min\{\mathbf{z}(i), 0\} - md$ also in that case. ◀

A cycle is said to be *full-state* if every state occurs in the cycle. We first prove that there exists a "short" full-state cycle with a zero displacement thanks to the following lemma.

▶ **Lemma 10.** *Every transition occurs on a finite sequence $\theta_1, \ldots, \theta_n$ of (eventually disjoint) simple cycles such that $\Delta(\theta_1) + \cdots + \Delta(\theta_n) = \mathbf{0}$ and such that $n \leq (3drm)^d$.*

Proof. Let t be a transition. Since G is strongly connected, the transition t occurs in a simple cycle θ_0. Lemma 4 shows that $-\Delta(\theta_0)$ is a finite sum of displacements of simple cycles. In particular $-\Delta(\theta_0)$ is in the cone generated by the displacements of simple cycles, i.e. the finite sums of displacements of simple cycles multiplied by non-negative rational numbers. From Carathéodory theorem, there exists d simple cycles $r_1, \ldots, r_d$ and d non-negative rational numbers $\lambda_1, \ldots, \lambda_d$ such that $-\Delta(\theta_0) = \sum_{j=1}^{d} r_j \Delta(\theta_j)$. By introducing a positive integer β_0 such that $\beta_j \stackrel{\text{def}}{=} \beta_0 r_j$ is a natural number for every j, we derive that the following linear system over the sequences $(\beta_j)_{0 \leq j \leq d}$ of natural numbers

$$\sum_{j=0}^{d} \beta_j \mathbf{v}_j = \mathbf{0}$$

admits a solution satisfying $\beta_0 > 0$ where $\mathbf{v}_j \stackrel{\text{def}}{=} \Delta(\theta_j)$.

From [15], it follows that solutions of that system can be decomposed as finite sums of "minimal" solutions $(\beta_j)_{1\leq j\leq k}$ of the same system satisfying additionally the following constraint:

$$\sum_{j=0}^{d}\beta_j \leq (1+(d+1)rm)^d$$

From $1+(d+1)rm \leq (3drm)$, we derive $(1+(d+1)rm)^d \leq (3drm)^d$. Since there exist solutions of that system with $\beta_0 > 0$, there exists at least a minimal one satisfying the same constraint. We have proved the lemma. ◀

▶ **Lemma 11.** *There exists a full-state cycle with a zero displacement with a length bounded by $r^2(r-1)(3drm)^d$.*

Proof. Let us consider the set H of pairs $(p,q)\in Q\times Q$ such that there exists a transition from p to q with $p\neq q$. For every $h\in H$ of the form (p,q), we select a transition $t_h\in T$ from p to q. Lemma 10 shows that for every $h\in H$, there exists a sequence of at most $(3drm)^d$ simple cycles with a zero total displacement. It follows that there exists a sequence of at most $|H|(3drm)^d$ simple cycles with a zero total displacement that contains all the transitions t_h with $h\in H$. Since the set of transitions that occurs in that sequence is strongly connected, Euler's Lemma shows that there exists a cycle θ with the same Parikh image as the sum of the Parikh images of the cycles occurring in the sequence. It follows that $|\theta|\leq r|H|(3rdm)^d$. Notice that $\Delta(\theta)=\mathbf{0}$ and θ is a full-state cycle. From $|H|\leq r(r-1)$ we are done. ◀

Now, let us prove Lemma 5. Let π_0 be an elementary path from p to q, and let $\mathbf{z}\stackrel{\text{def}}{=}\mathbf{y}-\mathbf{x}-\Delta(\pi_0)$.

Let us first explain why $\mathbf{z}$ is a finite sum of displacements of simple cycles. By hypothesis, there exists a path π_1 from p to q such that $\mathbf{y}-\mathbf{x}-\Delta(\pi_1)$ is in the group generated by displacements of cycles. Let π' be a path from q to p and observe that $\mathbf{z}=(\mathbf{y}-\mathbf{x}-\Delta(\pi_1))-\Delta(\pi'\pi_0)+\Delta(\pi'\pi_1)$. Since $\pi'\pi_0$ and $\pi'\pi_1$ are two cycles, it follows that $\mathbf{z}$ is in the group generated by the displacements of the cycles. Lemma 4 shows that $\mathbf{z}$ is finite sum of displacements of simple cycles.

Corollary 7 and Corollary 9 shows that there exists a sequence $\mathbf{z}_1,\ldots,\mathbf{z}_k$ of displacements of simple cycles such that $\mathbf{z}=\sum_{j=1}^{k}\mathbf{z}_j$, $k\leq(1+\|\mathbf{z}\|)(3drm)^d$, and such that for every $n\in\{0,\ldots,k\}$, we have:

$$\sum_{j=1}^{n}\mathbf{z}_j(i)\geq \min\{0,\mathbf{z}(i)\}-drm$$

Lemma 11 shows that there exists a full-state cycle θ_0 with a zero displacement with a length bounded by $r^2(r-1)(3drm)^d$. Thanks to a rotation of θ_0, we can assume that θ_0 is a cycle on q. Now, observe that for every $1\leq j\leq k$, there exists a simple cycle θ'_j with a displacement equal to $\mathbf{z}_j$. By inserting θ'_j in the full-state cycle θ_0, we get a cycle θ_j on q. Notice that $\Delta(\theta_j)=\mathbf{z}_j$ and $|\theta_j|\leq r^2(r-1)(3drm)^d+r$. We introduce the path π defined as follows:

$$\pi\stackrel{\text{def}}{=}\pi_0\theta_1\ldots\theta_n$$

We are going to prove that $p(\mathbf{x})\xrightarrow{\pi}q(\mathbf{y})$. To do so, let u be a prefix of π and let $i\in\{1,\ldots,d\}$ and let us prove that $\mathbf{x}(i)+\Delta(u)(i)\geq 0$. Oberve that if $i\in I$, the flow function F shows that $\mathbf{x}(i)+\Delta(u)(i)=F(q_u)(i)\geq 0$ where q_u is a state reached from p by reading u. So, we can assume that $i \notin I$. Observe that if u is a prefix of π_0 the property is immediate since $\Delta(u)(i) \geq -m|u| \geq -mr$. In particular $\mathbf{x}(i) + \Delta(u)(i) \geq 0$. So, we can assume that there exists $n \in \{1, \dots, k\}$ and a prefix u' of θ_n such that $u = \pi_0\theta_1 \dots \theta_{n-1}u'$. It follows that $\Delta(u) = \Delta(\pi_0) + \Delta(u') + \sum_{j=1}^{n-1} \mathbf{z}_j(i)$. Moreover, notice that $\Delta(u')(i) \geq -m|u'| \geq -mr^2(r-1)(3drm)^d - mr$ for every $i \in \{1, \dots, d\}$.

We decompose the proof that $\mathbf{x}(i) + \Delta(u)(i) \geq 0$ in two cases following that $\mathbf{z}(i) \leq 0$ or $\mathbf{z}(i) \geq 0$.

- Assume first that $\mathbf{z}(i) \geq 0$. In that case $\sum_{j=1}^{n-1} \mathbf{z}_j(i) \geq -drm$. It follows that $\Delta(u)(i) \geq -mr - mr^2(r-1)(3drm)^d - mr - drm \geq -mr^3(3drm)^d$. Hence $\mathbf{x}(i) + \Delta(u)(i) \geq 0$.
- Now, assume that $\mathbf{z}(i) \leq 0$. In that case $\sum_{j=1}^{n-1} \mathbf{z}_j(i) \geq \mathbf{z}(i) - drm$. It follows that $\mathbf{x}(i) + \Delta(u)(i) \geq \mathbf{x}(i) + \Delta(\pi_0) + \mathbf{z}(i) + \Delta(u')(i) - drm = \mathbf{y}(i) - \Delta(u')(i) - drm \geq \mathbf{y}(i) - mr^2(r-1)(3drm)^d - mr \geq 0$.

We have proved that $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$. Now, observe that $|\pi| \leq r + k(r^2(r-1)(3drm)^d + r)$. From $k \leq (1+\|\mathbf{z}\|)(3drm)^d$ and $\|\mathbf{z}\| \leq \|\mathbf{y} - \mathbf{x}\| + d(r-1)m$, we get $|\pi| \leq (\|\mathbf{y} - \mathbf{x}\| + drm)r^3(3drm)^{2d}$. Lemma 5 is proved.

4 Extractors

The notion of extractors was first introduced in [11]. Intuitively, extractors provides a natural way to classify components of a vector of natural numbers into two categories: large ones and small ones. The notion is parametrized by a set $I \subseteq \{1, \dots, d\}$ that provides a way to focus only on components in I. More formally, a *d-dimensional extractor* λ is a non-decreasing sequence $(\lambda_0 \leq \dots \leq \lambda_{d+1})$ of positive natural numbers denoting some *thresholds*. Given a d-dimensional extractor λ and a set $I \subseteq \{1, \dots, d\}$, a *$(\lambda, I)$-small set* of a set $\mathbf{C} \subseteq \mathbb{N}^d$ is a subset $J \subseteq I$ such that $\mathbf{c}(j) < \lambda_{|J|}$ for every $j \in J$ and $\mathbf{c} \in \mathbf{C}$. The following lemma shows that there exists a unique maximal (λ, I)-small set w.r.t. inclusion. We denote by $\text{extract}_{\lambda,\mathbf{C}}(I)$ this set.

▶ **Lemma 12.** *The class of (λ, I)-small sets of a set $\mathbf{C} \subseteq \mathbb{N}^d$ is non empty and stable under union.*

Proof. We adapt the proof of [11, Section 8]. Since the class contains the empty set, it is nonempty. Now, let us prove the stability by union by considering two (λ, I)-small sets J_1 and J_2 of $\mathbf{C}$ and let us prove that $J \stackrel{\text{def}}{=} J_1 \cup J_2$ is a (λ, I)-small set of $\mathbf{C}$. Since $J_1, J_2 \subseteq I$, we derive $J \subseteq I$. Let $\mathbf{c} \in \mathbf{C}$ and $j \in J$. If $j \in J_1$ then $\mathbf{c}(j) < \lambda_{|J_1|} \leq \lambda_{|J|}$ since $|J_1| \leq |J|$. Symmetrically, if $j \in J_2$ we deduce that $\mathbf{c}(j) < \lambda_{|J_2|} \leq \lambda_{|J|}$. We have proved that J is a (λ, I)-small set of $\mathbf{C}$. ◀

▶ **Example 13.** Let us consider the 2-dimensional extractor $\lambda = (\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3)$ and assume that $I = \{1, 2\}$ and let $\mathbf{C} = \{(m, n)\}$ with $m, n \in \mathbb{N}$. We have:

$$\text{extract}_{\lambda,X}(I) = \begin{cases} \{1,2\} & \text{if } m, n < \lambda_2 \\ \emptyset & \text{if } (m \geq \lambda_2 \wedge n \geq \lambda_1) \vee (m \geq \lambda_1 \wedge n \geq \lambda_2) \\ \{1\} & \text{if } m < \lambda_1 \wedge n \geq \lambda_2 \\ \{2\} & \text{if } m \geq \lambda_2 \wedge n < \lambda_1 \end{cases}$$

▶ Remark 14. As shown by the previous example, the values λ_0 and λ_{d+1} of any d-dimensional extractor λ are not used directly by our definitions. Those extremal values are introduced to simplify some notations in the sequel.

The following lemma shows that components that are not in $\text{extract}_{\lambda,\mathbf{C}}(I)$ are large for at least one vector in $\mathbf{C}$.

▶ **Lemma 15.** *Let* $J \stackrel{\text{def}}{=} \text{extract}_{\lambda,\mathbf{C}}(I)$*. For every* $i \in I \backslash J$ *there exists* $\mathbf{c} \in \mathbf{C}$ *such that:*

$$\mathbf{c}(i) \geq \lambda_{|J|+1}$$

Proof. Assume that for some $i \in I \backslash J$, we have $\mathbf{c}(i) < \lambda_{|J|+1}$ for every $\mathbf{c} \in \mathbf{C}$. Let $J' \stackrel{\text{def}}{=} J \cup \{i\}$ and observe that J' is a (λ, I)-small set of $\mathbf{C}$. In fact, for every $\mathbf{c} \in \mathbf{C}$ and for every $j \in J'$, we have $\mathbf{c}(j) < \lambda_{|J|} \leq \lambda_{|J'|}$ if $j \in J$, and $\mathbf{c}(j) < \lambda_{|J|+1} = \lambda_{|J'|}$ if $j = i$. We get a contradiction by maximality of $\text{extract}_{\lambda,\mathbf{C}}(I)$. We deduce the lemma. ◀

Given a set $I \subseteq \{1, \ldots, d\}$ we define $\text{extract}_{\lambda,e}(I)$ for a finite word e of configurations by $\text{extract}_{\lambda,\varepsilon}(I) \stackrel{\text{def}}{=} I$, and by $\text{extract}_{\lambda,e\mathbf{c}}(I) \stackrel{\text{def}}{=} \text{extract}_{\lambda,\{\mathbf{c}\}}(\text{extract}_{\lambda,e}(I))$ for every $\mathbf{c} \in \mathbb{N}^d$ and for every finite word e of configurations. Given an infinite word e of configurations, we observe that $(\text{extract}_{\lambda,e_n}(I))_{n\in\mathbb{N}}$ where e_n is the finite prefix of e of length n is a non-increasing sequence of sets in $\{1, \ldots, d\}$. It follows that this sequence is asymptotically constant and equals to a set included in $\{1, \ldots, d\}$. We denote $\text{extract}_{\lambda,e}(I)$ that set. The following lemma shows that extracting along a word of configurations in $\mathbf{C}$ asymptotically coincides with an extraction of $\mathbf{C}$.

▶ **Lemma 16.** *Let us consider a set* $I \subseteq \{1, \ldots, d\}$*, an extractor* λ*, a set* $\mathbf{C}$ *of configurations, and an infinite word* e *over* $\mathbf{C}$*. We have* $\text{extract}_{\lambda,\mathbf{C}}(I) \subseteq \text{extract}_{\lambda,e}(I)$*. Moreover,* $\text{extract}_{\lambda,\mathbf{C}}(I) = \text{extract}_{\lambda,e}(I)$ *if every configuration of* $\mathbf{C}$ *occurs infinitely often in* e*.*

Proof. We introduce $J \stackrel{\text{def}}{=} \text{extract}_{\lambda,\mathbf{C}}(I)$, $J_\infty \stackrel{\text{def}}{=} \text{extract}_{\lambda,e}(I)$, the prefix e_n of length n of e, and $J_n \stackrel{\text{def}}{=} \text{extract}_{\lambda,e_n}(I)$.

Let us prove that $J \subseteq J_n$ for every n. Since $J_0 = I$ the property is proved for $n = 0$. Assume that $J \subseteq J_{n-1}$ for some $n \geq 1$ and let us prove that $J \subseteq J_n$. There exists $\mathbf{c} \in \mathbf{C}$ such that $e_n = e_{n-1}\mathbf{c}$. Since $\mathbf{c} \in \mathbf{C}$, it follows that $\mathbf{c}(j) < \lambda_{|J|}$ for every $j \in J$. As $J \subseteq J_{n-1}$, we deduce that J is a (λ, J_{n-1})-small set of $\{\mathbf{c}\}$. Since J_n is the maximal set satisfying that property, we get $J \subseteq J_n$ and we have proved the induction. It follows that $J \subseteq J_n$ for every $n \in \mathbb{N}$. Moreover, since $J_\infty = \bigcap_{n\in\mathbb{N}} J_n$, we deduce the inclusion $J \subseteq J_\infty$.

Now, assume that every $\mathbf{c} \in \mathbf{C}$ occurs in e infinitely often. Since $(J_n)_{n\in\mathbb{N}}$ is a non increasing sequence of $\{1, \ldots, d\}$, there exists N such that $J_n = J_\infty$ for every $n \geq N$. Let $\mathbf{c} \in \mathbf{C}$. There exists $n > N$ such that $e_n = e_{n-1}\mathbf{c}$. From $J_n = \text{extract}_{\lambda,\{\mathbf{c}\}}(J_{n-1})$ and $J_n = J_{n-1} = J_\infty$, we derive $J_\infty = \text{extract}_{\lambda,\{\mathbf{c}\}}(J_\infty)$. In particular $\mathbf{c}(j) < \lambda_{|J_\infty|}$ for every $j \in J_\infty$. We have proved that $\mathbf{c}(j) < \lambda_{|J_\infty|}$ for every $j \in J_\infty$ and for every $\mathbf{c} \in \mathbf{C}$. As $J_\infty \subseteq I$, we deduce that J_∞ is a (λ, I)-small set of $\mathbf{C}$. Since J is the maximal set satisfying that property, we deduce that $J_\infty \subseteq J$. It follows that $J = J_\infty$. ◀

5 Rackoff Extraction

An A^*-execution e is said to be *I-cyclic* for some $I \subseteq \{1, \ldots, d\}$ if $\text{src}(e)|_I = \text{tgt}(e)|_I$. We say that a word $\sigma = \mathbf{a}_1 \ldots \mathbf{a}_k$ of actions in A is obtained from an A^*-execution e by removing I-cycles where $I \subseteq \{1, \ldots, d\}$, if there exists a decomposition of e into a concatenation $e_0 \ldots e_k$ of I-cyclic A^*-executions $e_0, \ldots, e_k$ such that $\text{tgt}(e_{j-1}) \xrightarrow{a_j} \text{src}(e_j)$ for every $1 \leq j \leq k$.

An extractor $\lambda = (\lambda_0 \leq \cdots \leq \lambda_{d+1})$ is said to be *m-adapted* if for every $n \in \{0, \ldots, d\}$:

$$\lambda_{n+1} \geq \lambda_n + m\lambda_n^n$$

▶ **Lemma 17** (slight extension of [16]). *Let λ be an m-adapted extractor and e be an A^*-execution for a PN $A \subseteq \{0, \ldots, m\}^d \times \mathbb{N}^d$. Let $I \stackrel{\text{def}}{=} \text{extract}_{\lambda,e}(\{1, \ldots, d\})$. There exists a word σ that can be obtained from e by removing I-cycles such that*

$$|\sigma| \leq \sum_{j=1}^{d} \lambda_j^j$$

and such that $\text{src}(e) \xrightarrow{\sigma} \mathbf{c}$ for some configuration $\mathbf{c}$ satisfying $\mathbf{c}(i) = \text{tgt}(e)(i)$ for every $i \in I$, and such that for every $i \notin I$ we have:

$$\mathbf{c}(i) \geq \lambda_{|I|+1} - m \sum_{j=1}^{|I|} \lambda_j^j$$

Proof. The proof follows a similar approach to the original one from Rackoff [16]. A detailed proof is given in a long version of the paper available online. ◀

6 Strongly-Connected Components of Configurations

A *strongly-connected component of configurations* of a PN A (*SCCC* for short) is a strongly-connected component of the reachability graph $(\mathbb{N}^d, \xrightarrow{A})$.

We associate to an extractor λ and a SCCC $\mathbf{C}$ of a PN A, a PNS G defined as follows. We introduce the set $I \stackrel{\text{def}}{=} \text{extract}_{\lambda,\mathbf{C}}(\{1, \ldots, d\})$, the set of states $Q \stackrel{\text{def}}{=} \{\mathbf{c}|_I \mid \mathbf{c} \in \mathbf{C}\}$ and the set of transitions $T \stackrel{\text{def}}{=} \{(\mathbf{x}|_I, a, \mathbf{y}|_I) \mid (\mathbf{x}, a, \mathbf{y}) \in \mathbf{C} \times A \times \mathbf{C} \wedge \mathbf{x} \xrightarrow{a} \mathbf{y}\}$. Notice that Q is finite since it contains at most $\lambda_{|I|}^{|I|}$ elements. In particular T is finite as well. The PNS G is defined as the tuple $\langle Q, A, T \rangle$.

▶ **Lemma 18.** *The PNS G is structurally reversible.*

Proof. Let (p, a, q) be a transition in T. There exist $\mathbf{x}, \mathbf{y} \in \mathbf{C}$ such that $\mathbf{x} \xrightarrow{a} \mathbf{y}$ and such that $p = \mathbf{x}|_I$ and $q = \mathbf{y}|_I$. Moreover since $\mathbf{C}$ is a SCCC, there exists a word σ of actions in A such that $\mathbf{y} \xrightarrow{\sigma} \mathbf{x}$. We deduce that there exists a path in G from q to p labeled by σ. Notice that $\Delta(a) + \Delta(\sigma) = \mathbf{y} - \mathbf{x} + \mathbf{x} - \mathbf{y} = \mathbf{0}$. It follows that G is structurally reversible. ◀

Let us prove the following technical lemma.

▶ **Lemma 19.** *If $\mathbf{C}$ is not reduced to a singleton, there exists an A^ω-execution e of configurations in $\mathbf{C}$ such that every configuration of $\mathbf{C}$ occurs infinitely often in e.*

Proof. Since $\mathbf{C}$ is countable, there exists an infinite sequence $(\mathbf{c}_n)_{n \in \mathbb{N}}$ such that $\mathbf{C} = \{\mathbf{c}_n \mid n \in \mathbb{N}\}$. Moreover, by replacing that sequence by the sequence $s_0, s_1, \ldots$ where $s_n \stackrel{\text{def}}{=} \mathbf{c}_0, \ldots, \mathbf{c}_n$, we can assume without loss of generality that every configuration of $\mathbf{C}$ occurs infinitely often in the sequence $(\mathbf{c}_n)_{n \in \mathbb{N}}$. Since $\mathbf{C}$ is a SCCC, for every positive natural number n, there exists an A^*-execution from $\mathbf{c}_{n-1}$ to $\mathbf{c}_n$ of the form $e_n \mathbf{c}_n$. Let us introduce the word $e \stackrel{\text{def}}{=} e_1 e_2 \ldots$. Notice that since $\mathbf{C}$ is not reduced to a singleton, the word e is infinite. Moreover, notice that e is an A^ω-execution satisfying the lemma. ◀

Now, assume that λ is m-adapted for some positive natural number m.

▶ **Lemma 20.** *If $A \subseteq \{0,\dots,m\}^d \times \mathbb{N}^d$, for every $\mathbf{x} \in \mathbf{C}$, there exists a cycle in G on $\mathbf{x}|_I$ labeled by a word u such that:*

$$|u| \leq \sum_{j=1}^{d} \lambda_j^j$$

and a configuration $\mathbf{x}'$ such that $\mathbf{x} \xrightarrow{u} \mathbf{x}'$, $\mathbf{x}'|_I = \mathbf{x}|_I$ and such that $\mathbf{x}'(i) \geq \lambda_{|I|+1} - m\sum_{j=1}^{|I|} \lambda_j^j$ for every $i \notin I$.

Proof. Observe that if $\mathbf{C}$ is reduced to a singleton, the lemma is trivial with $u \stackrel{\text{def}}{=} \varepsilon$. So, we can assume that $\mathbf{C}$ is not a singleton. Lemma 19 shows that there exists an A^ω-execution $e = \mathbf{c}_0\mathbf{c}_1\dots$ of configurations in $\mathbf{C}$ such that every configuration of $\mathbf{C}$ occurs infinitely often. Without loss of generality, by replacing e by a suffix of e we can assume that $\mathbf{x} = \mathbf{c}_0$. Lemma 16 shows that $\text{extract}_{\lambda,e}(\{1,\dots,d\}) = I$. It follows that there exists $N \in \mathbb{N}$ such that for every $n \geq N$ the prefix e_n of e of length n satisfies $\text{extract}_{\lambda,e_n}(\{1,\dots,d\}) = I$. Since $\mathbf{x}$ occurs infinitely often in e, there exists $n \geq N$ such that $\mathbf{x}$ is the last configuration of e_n. Lemma 17 shows that there exists a word u that can be obtained from e_n by removing I-cycles such that

$$|u| \leq \sum_{j=1}^{d} \lambda_j^j$$

and such that $\mathbf{x} \xrightarrow{u} \mathbf{x}'$ for some configuration $\mathbf{x}'$ satisfying $\mathbf{x}'|_I = \mathbf{x}|_I$, and such that for every $i \notin I$ we have:

$$\mathbf{x}'(i) \geq \lambda_{|I|+1} - m\sum_{j=1}^{|I|} \lambda_j^j$$

Since u can be obtained from e_n by removing I-cycles, it follows that u is the label of a cycle on $\mathbf{x}|_I$ in the PNS G. ◀

Symmetrically, we deduce a similar backward property.

▶ **Lemma 21.** *If $A \subseteq \mathbb{N}^d \times \{0,\dots,m\}^d$, for every $\mathbf{y} \in \mathbf{C}$, there exists a cycle in G on $\mathbf{y}|_I$ labeled by a word v such that:*

$$|v| \leq \sum_{j=1}^{d} \lambda_j^j$$

and a configuration $\mathbf{y}'$ such that $\mathbf{y}' \xrightarrow{v} \mathbf{y}$, $\mathbf{y}'|_I = \mathbf{y}|_I$, and such that for every $i \notin I$: $\mathbf{y}'(i) \geq \lambda_{|I|+1} - m\sum_{j=1}^{|I|} \lambda_j^j$.

Proof. Let us introduce the PN $A' \stackrel{\text{def}}{=} \{(\mathbf{a}_+, \mathbf{a}_-) \mid (\mathbf{a}_-, \mathbf{a}_+) \in A\}$. Observe that $\mathbf{C}$ is a SCCC of A'. Let G' be the PNS associated to the extractor λ and the SCCC $\mathbf{C}$ of A'. Lemma 22 shows that there exists a cycle in G' on $\mathbf{y}|_I$ labeled by a word u such that:

$$|u| \leq \sum_{j=1}^{d} \lambda_j^j$$

and a configuration $\mathbf{y}'$ such that $\mathbf{y} \xrightarrow{u} \mathbf{y}'$, $\mathbf{y}|_I = \mathbf{y}'|_I$, and such that $\mathbf{y}'(i) \geq \lambda_{|I|+1} - m\sum_{j=1}^{|I|} \lambda_j^j$ for every $i \notin I$. Assume that $u = a_1' \dots a_n'$ with $a_j' = (\mathbf{x}_j, \mathbf{y}_j)$ and let $v \stackrel{\text{def}}{=} a_1 \dots a_n$ with $a_j \stackrel{\text{def}}{=} (\mathbf{y}_j, \mathbf{x}_j)$. Observe that since u is a cycle on $\mathbf{y}|_I$ in G', then v is a cycle on $\mathbf{y}|_I$ in G. Moreover, from $\mathbf{y} \xrightarrow{u} \mathbf{y}'$ we derive $\mathbf{y}' \xrightarrow{v} \mathbf{y}$. We have proved the lemma. ◀

7 Mutually Reachable Configurations

In this section, we prove Theorem 2. We consider a PN $A \subseteq \{0, \dots, m\}^d \times \{0, \dots, m\}^d$ for some positive natural number m. We consider two mutually reachable configurations $\mathbf{x}, \mathbf{y}$ for A. Since the theorem is trivial when $\mathbf{x} = \mathbf{y}$, we can assume that $\mathbf{x} \neq \mathbf{y}$. In particular $\|\mathbf{y} - \mathbf{x}\| \geq 1$.

We let $\mathbf{C}$ be the SCCC of A containing $\mathbf{x}$ and $\mathbf{y}$. We introduce the extractor λ satisfying $\lambda_0 = 1$, and for every $n \in \{0, \dots, d\}$:

$$\lambda_{n+1} \stackrel{\text{def}}{=} m \sum_{j=1}^{n} \lambda_j^j + m\lambda_n^{3n}(3d\lambda_n^n m)^d$$

Observe that λ is m-adapted. We introduce $I \stackrel{\text{def}}{=} \text{extract}_{\lambda,\mathbf{C}}(\{1, \dots, d\})$ and the structurally reversible PNS G associated to $\mathbf{C}$, λ and $\mathbf{A}$. Notice that the number of states of G is bounded by $r \stackrel{\text{def}}{=} \lambda_{|I|}^{|I|}$. We introduce the states p, q of G defined as $p \stackrel{\text{def}}{=} \mathbf{x}|_I$ and $q \stackrel{\text{def}}{=} \mathbf{y}|_I$. Observe that $\mathbf{y} - \mathbf{x}$ is the displacement of a path from p to q in G. We introduce the flow function $F : Q \to \mathbb{N}^I$ defined as the identity.

Let us observe that $\lambda_j \leq \lambda_d$ for every $j \in \{1, \dots, d\}$. In particular $r \leq \lambda_d^d$.

▶ **Lemma 22.** *The PNS G admits a cycle on p labeled by a word u and a cycle on q labeled by a word v such that:*

$$|u|, |v| \leq d\lambda_d^d$$

and such that there exist configurations $\mathbf{x}', \mathbf{y}'$ such that $\mathbf{x} \xrightarrow{u} \mathbf{x}'$, $\mathbf{y}' \xrightarrow{v} \mathbf{y}$, and such that for every $i \notin I$, we have:

$$\mathbf{x}'(i), \mathbf{y}'(i) \geq mr^3(3drm)^d$$

Proof. This lemma is a direct corollary of Lemma 20 and Lemma 21. ◀

From $\mathbf{y}' - \mathbf{x}' = \mathbf{y} - \mathbf{x} - \Delta(u) - \Delta(v)$, we deduce from Lemma 5 that there exists a word σ of actions in $\mathbf{A}$ such that $\mathbf{x}' \xrightarrow{\sigma} \mathbf{y}'$ and such that $|\sigma| \leq (\|\mathbf{y}' - \mathbf{x}'\| + drm)r^3(3drm)^{2d}$. Observe that we have:

$$\begin{aligned}
\|\mathbf{y}' - \mathbf{x}'\| &\leq \|\mathbf{y} - \mathbf{x}\| + \|\Delta(u)\| + \|\Delta(v)\| \\
&\leq \|\mathbf{y} - \mathbf{x}\| + dm(|u| + |v|) \\
&\leq \|\mathbf{y} - \mathbf{x}\| + 2d^2 m\lambda_d^d
\end{aligned}$$

Let $w = u\sigma v$. Observe that $\mathbf{x} \xrightarrow{w} \mathbf{y}$. We derive:

$$\begin{aligned}
|w| &\leq 2d\lambda_d^d + (\|\mathbf{y} - \mathbf{x}\| + 2d\lambda_d^d m + d\lambda_d^d m)\lambda_d^{3d}(3d\lambda_d^d m)^{2d} \\
&\leq \|\mathbf{y} - \mathbf{x}\| 8d\lambda_d^d m\lambda_d^{3d}(3d\lambda_d^d m)^{2d} \\
&\leq \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|(3d\lambda_d^d m)^{6d}
\end{aligned}$$

From the following Lemma 23 we derive:

$$|w| \leq \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|(3dm)^{(d+1)^{2d+4}}$$

We deduce Theorem 2.

▶ **Lemma 23.** *We have:*

$$(3d\lambda_d^d m)^{6d} \leq (3dm)^{(d+1)^{2d+4}}$$

Proof. Assume first that $d = 1$. In that case, the definiton of λ_{n+1} with $n = 0$ provides $\lambda_1 = 3m^2$ and the lemma is immediate. So, let us assume that $d \geq 2$. Notice that $\lambda_j^j \leq \lambda_n^n$ for every $j \in \{1, \dots, n\}$ for every $n \in \{0, \dots, d-1\}$. It follows that we have:

$$\begin{aligned}\lambda_{n+1} &\leq 2d\lambda_n^{3n} m(3d\lambda_n^n m)^d \\ &\leq (3d\lambda_n m)^{(d+1)^2-4}\end{aligned}$$

By induction, we deduce that for every $n \in \{0, \dots, d\}$, we have:

$$\lambda_n \leq (3dm)^{n((d+1)^2-4)^n}$$

In particular:

$$3d\lambda_d^d m \leq (3dm)^{d^2(d+1)^{2d}}$$

Hence

$$(3d\lambda_d^d m)^{6d} \leq (3dm)^{6d^3(d+1)^{2d}} \leq (3dm)^{(d+1)^{2d+4}}$$

where we use the inequality $6d^3 \leq (d+1)^4$. ◀

8 Conclusion

In this paper we proved that the distance in the reachability graph between two mutually reachable configurations is linear with respect to the Euclidean distance between those two configurations. As a future work, we would like to apply that result to provide lower bounds on the number of states of population protocols computing some predicates [3].

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