Report from Dagstuhl Seminar 19401

Comparative Theory for Graph Polynomials

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Abstract

This report documents the programme and outcomes of Dagstuhl Seminar 19401 "Comparative Theory for Graph Polynomials".

The study of graph polynomials has become increasingly active, with new applications and new graph polynomials being discovered each year. The genera of graph polynomials are diverse, and their interconnections are rich. Experts in the field are finding that proof techniques and results established in one area can be successfully extended to others. From this a general theory is emerging that encapsulates the deeper interconnections between families of graph polynomials and the various techniques, computational approaches, and methodologies applied to them.

The overarching aim of this Seminar was to exploit commonalities among polynomial invariants of graphs, matroids, and related combinatorial structures. Model-theoretic, computational and other methods were used in order to initiate a comparative theory that collects the current state of knowledge into a more cohesive and powerful framework.

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1 Executive Summary

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This 5-day Seminar built on the previous Dagstuhl Seminar 16241 together with several intervening workshops on graph polynomials, particularly those associated with William Tutte's Centenary, to advance an emerging comparative theory for graph polynomials. Graph polynomials have played a key role in combinatorics and its applications, having effected breakthroughs in conceptual understanding and brought together different strands of scientific thought. For example, the characteristic and matching polynomials advanced graph-theoretical techniques in chemistry; and the Tutte polynomial married combinatorics and statistical physics, and helped resolve long-standing problems in knot theory. The area of graph polynomials is incredibly active, with new applications and new graph polynomials being discovered each year. However, the resulting plethora of techniques and results urgently requires synthesis. Beyond catalogues and classifications we need a comparative theory and unified approaches to streamline proofs and deepen understanding.

The Seminar provided a space for the cross-fertilization of ideas among researchers in graph theory, algebraic graph theory, topological graph theory, computational complexity, logic and finite model theory, and biocomputing and statistical mechanics applications. There is a long history in this area of results in one field leading to breakthroughs in another when techniques are transferred, and this workshop leveraged that paradigm. More critically, experts in the field have recently begun noticing strong resonances in both results and proof techniques among the various polynomials. The species and genera of graph polynomials are diverse, but there are strong interconnections: in this seminar we worked towards a general theory that brings them together under one family. The process of developing such a theory of graph polynomials exposes deeper connections, giving great impetus to both theory and applications. This has immense and exciting potential for all those fields of science where combinatorial information needs to be extracted and interpreted.

The seminar was roughly organized according to the following themes:

- **Unification:** General frameworks for graph polynomials including meta-problems, K-theory, Second Order Logic, and Hopf algebras.
- **Generalizations:** Polynomial invariants for graphs with added structure (e.g. digraphs, ribbon graphs) or more general "underlying" combinatorial structures (e.g. matroids, Δ-matroids).
- **Distinction:** Distinguishing power of graph invariants (equivalence and uniqueness up to isomorphism with respect to a given graph polynomial, interrelations among graph polynomials, properties of graph polynomials).
- **Applications:** Applications of graph polynomials in other disciplines (e.g. self-assembly, sequencing, quantum walks, statistical mechanics, knot theory, quantum Ising model).
- **Conjectures:** Breakthrough conjectures (outstanding open problems whose resolution would have a broad impact on the understanding of graph polynomials).
- **Complexity:** Computational complexity and computational methods.

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3 Structure of the Workshop

Beyond simple information exchange, the goal of this workshop was to make concrete progress towards identified problems that would move the field forward. Thus, we adopted the following 'working group' format for the week. This format optimized collaboration time, but also included a number of plenary talks. We particularly note that the social spaces were fully crowded every night of the Seminar, with nearly all the participants reconvening after dinner to continue a freely flowing exchange of ideas about the themes of the Seminar.

- **Monday:**
  - The workshop began with an opening plenary talk, touching on the foundations of the field with emphasis on untapped open areas from seminal papers and long-standing conjectures.
  - After the plenary talk, the organizers gave an overview of open problems collected from other events, organized by the broad themes of the workshop: unification, generalizations, distinction, complexity, applications and conjectures.
  - This was followed by 5-minute presentations of new open problems, again grouped according to the workshop themes.
  - Presentations of open problems were completed shortly after lunch, after which participants self-selected into the six working groups following the broad themes of the workshop. The working groups adjourned to their breakout rooms to begin work on the identified open problems, and in many cases new problems were generated by further discussion within the working groups.
- **Tuesday:**
  - The day began with short invited talks by early career researchers.
  - The remainder of the day was dedicated to intensive collaborative work in working groups. This extended period of concentrated time led to meaningful progress in collaborative efforts.
- **Wednesday:**
  - The day began with short interim progress reports from the working groups and any subgroups that formed within them. Interim reports served to establish working notes that increase the likelihood of research continuance after workshop. At this juncture, there was also some movement of participants among groups, and one or two subgroups moved on to other projects.
  - The rest of the morning was spent collaborating in the working groups.
  - In the afternoon about half the participants continued their investigations while the other half went on an excursion to Völklinger Hütte.
- **Thursday:**
  - The entire day was devoted to working groups. Some groups reported that this concentrated, uninterrupted time was a significant factor in resolving their selected problems and framing out papers for publication.
  - On Thursday night at dinner we solicited written responses from each table as to where the field should go from here, what overarching questions would drive it there, and how we might continue the efforts of this workshop.
- **Friday:**
  - In the first morning session, the working groups consolidated their working notes, prepared their final reports, and made concrete plans for continuing their efforts to bring their work at the workshop to fulfilment.

- After the coffee break, each working group and subgroup presented their outcomes to the whole group, and shared plans for continuing their work. These plans ranged from simply completing papers now well-underway to planning follow up workshops, and most included agreeing mechanisms for continuing collaboration (Dropbox, Skype, etc.).
- After the reports, we requested and received verbal feedback from the participants on what worked well for them at the workshop and what might improve it. The workshop concluded with a whole-group discussion about the next steps that we should take as a community to best enhance the subject.

4 Overview of Plenary Talks

4.1 Tutte–Whitney polynomials: some history and problems

Graham Farr (Monash University – Clayton, AU)

Tutte-Whitney polynomials are algebraic data structures that contain a lot of important information about graphs. Their evaluations and specialisations include, for example, the chromatic, flow and reliability polynomials of a graph, the number of spanning trees, numbers of acyclic and totally cyclic orientations, the Ising and Potts model partition functions of statistical physics, the weight enumerator of a linear code, and the Jones polynomial of an alternating link.

We describe the history of Tutte-Whitney polynomials, especially the contributions of Hassler Whitney and W. T. (Bill) Tutte, and with some emphasis on Tutte's 1947 paper, 'A ring in graph theory', parts of which are still not well known. We find unexpectedly early occurrences in the literature of several important results, concepts and questions, and take several opportunities to comment on computational aspects of the theory.

This history sets the scene for a number of open problems and proposed future research directions. These include: a question of Whitney about a possible strengthening of the Four-Colour Theorem; some questions about equivalence of graphs with respect to particular graph polynomials (e.g., chromatic equivalence), in particular some questions about certificates of chromatic equivalence; some questions about factorisation of graph polynomials (e.g., chromatic factorisation) and certificates of chromatic factorisation; trying to link these theories of certificates to work on rewriting systems, word problems, and computational algebra.

4.2 Transversal polynomial of covers of graphs

Krystal Guo (UL – Brussels, BE)

Joint work of Chris Godsil, Krystal Guo, Gordon Royle

We study a polynomial with connections to correspondence colouring, a recent generalization of list colouring, and the Unique Games Conjecture. Given a graph G and an assignment of elements of the symmetric group S_r to the edge of G, we define a cover graph: there are sets of r vertices corresponding to each vertex of G, called fibres, and for each edge uv, we add a perfect matching between the fibres corresponding to u and v. A transversal subgraph of the cover is an induced subgraph which has exactly one vertex in each fibre. In this setting, we can associate correspondence colourings with transversal cocliques and unique label covers with transversal copies of G.

We define a polynomial which enumerates the transversal subgraphs of G with k edges. We show that this polynomial satisfies a contraction deletion formula and use this to study the evaluation of this polynomial at $-r+1$.

4.3 Interpretations of the Tutte Polynomials of Regular Matroids

Martin Kochol (Bratislava, SK)

A regular chain group N is the set of integral vectors orthogonal with rows of a matrix representing a regular matroid M, i.e., a totaly unimodular matrix. N corresponds to the set of flows and tensions if M is a graphic and cographic matroid, respectively. We evaluate the Tutte polynomial of M as number of pairs of specified elements of N.

4.4 Matching polynomials

Bodo Lass (University Claude Bernard – Lyon, FR)

We use the commutative algebra of set functions to prove results about matching polynomials.

4.5 Complexity of evaluation of graph polynomials

Johann A. Makowsky (Technion – Haifa, IL)

Joint work of Andrew Goodall, Miki Hermann, Tomer Kotek, Johann A. Makowsky, Steven Noble

We define the complexity spectrum of graph polynomials and collect results from the literature, including our own, which completely describe the complexity spectrum of classical graph polynomials, such as the Tutte polynomial and its many variations and instantiations, the matching polynomial, the interlace polynomial and the cover polynomial. We then concentrate on univariate graph polynomials, especially Harary polynomials which count the number of various colorings with k colors. Besides exhibiting possible behaviour of complexity spectra we also formulate open problems, solutions of which we think should be in the range of our current capabilities.

4.6 Acyclic orientation polynomials and the sink theorem for the chromatic symmetric function

Jaeseong Oh (Seoul National University, KR)

Joint work of Byung-Hak Hwang, Woo-Seok Jung, Kang-Ju Lee, Jaeseong Oh, Sang-Hoon Yu

We define the acyclic orientation polynomial of a graph to be the generating function for the sinks of its acyclic orientations, which is a refinement of the number of acyclic orientations. We show that our acyclic orientation polynomial satisfies a deletion-contraction recurrence with a change of variables. As the main application, we provide a new proof for Stanley's sink theorem for the chromatic symmetric function X_G, which gives a relation between the number of acyclic orientations of G with a fixed number of sinks and the coefficients in the expansion of X_G with respect to elementary symmetric functions.

5 A Selection of Open Problems and Questions Presented at the Seminar

Some three dozen problems were presented at the start of the Seminar, with more emerging during the breakout sessions. Some were solved during during the Seminar with papers drafted, while significant progress was made towards others. This section contains a small selection of the problems presented.

5.1 Graph polynomials through combinatorial Hopf algebra

Nantel Bergeron (York University – Toronto, CA)

We can construct the Stanley Chromatic Symmetric function using combinatorial Hopf algebras. In fact, for a graph G on vertex set $[n] = \{1, 2, \ldots n\}$, we get a quasisymmetric function (that is symmetric since the Hopf algebra is cocommutative):

$$\Psi(G) = \sum_{A \models [n]} \zeta(G|_{A_1}) \cdots \zeta(G|_{A_k}) M_{\alpha(A)},$$

where the sum is over all set composition $A = (A_1, A_2, \ldots, A_k)$ of the set $[n]$, ζ is 1 if the input graph has no edges and 0 otherwise, and $M_{\alpha(A)}$ is the monomial quasisymmetric function indexed by the integer composition $\alpha(A) = (|A_1|, |A_2|, \ldots, |A_k|)$. Stanley and Stembridge have conjectured that $\Psi(G)$ is e-positive if G is the incomparable graph of a $3 + 1$ avoiding poset, and that it is Schur positive if G is claw free. We have recently shown that $\Psi(G)$ is always positively h-alternating (the coefficient of $(-1)^{n-\ell(\lambda)} h_\lambda$ is always positive). This last result is a refinement of the alternation of the chromatic polynomial.

Question 1

We can choose different characters (analogue of *zeta* above) and produce different invariants for graphs. Can we find analogue of the Stanley and Stembridge conjecture?

Question 2

We can do this in larger Hopf algebras (For examples hypergraphs). For an arbitrary hypergraph H, it is not true that $\Psi(H)$ is positively h-alternating. Can we characterize families of hypergraph with the property that $\Psi(H)$ is positively h-alternating. Maybe something like for any edge $E \in H$ and any contraction H/S (with respect to a subset of edges S), the cardinality of E/S is even or 1? What can we say for matroids?

5.2 Recognition

Anna De Mier (UPC – Barcelona, ES)

A short statement of the problem is "Does the Tutte polynomial recognize (matroid) parallel connection?"

This question is motivated by the fact that to my knowledge we still do not know whether the Tutte polynomial can tell apart 3-connected graphs (that is, whether any graph Tutte-equivalent to a 3-connected one is also 3-connected). It is well known, though, that a 3-connected matroid can have the same Tutte polynomial as a non-3-connected one. By going through examples of Tutte-equivalent matroids I have not found any pair where one is 3-connected and the other one is a parallel connection (in the known examples mentioned above, then non-3-connected one is a 2-sum, but not a parallel connection). So this raises the question of whether the Tutte polynomial "sees" parallel connections, somehow in the same way that it does "see" direct sums. That is, we would like to know if there is a pair of Tutte-equivalent matroids M and N such that M can be written as a parallel connection but N cannot.

I would be also interested in the answer of the previous question for the G-invariant, or for any invariant stronger than the Tutte polynomial.

5.3 Some overarching 'meta-problems' for unification

Jo Ellis-Monaghan (Saint Michael's College – Colchester, US)

What attributes might a unified theory for graph polynomials have?

- Some desirable properties:
  - a common framework encompassing existing graph polynomials,
  - underlying (algebraic?) structure that reveals their relations,
  - a hierarchy or partial ordering with respect to the distinguishing power of the various polynomials,
  - common tools, theorems, or conditions for establishing common properties (recursion, generating functions, universality, etc.), and
  - generic computational complexity results.
- Some possible directions:

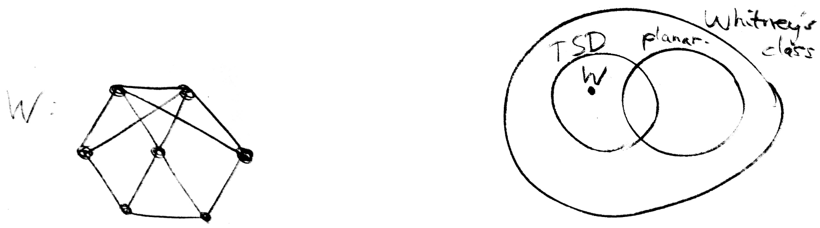

Figure 1 Whitney's example of a non-planar Tutte self-dual graph.

- Characterize graph reductions that give well-defined linear recurrence relations and hence lead to graph polynomials.
- What are the possible mechanisms for comparing graph polynomials? (e.g. via SOL or kernel containments). What partial orders for a class of graph polynomials arise from these? What can we learn from the structure of such a poset?
- A graph polynomial creates equivalence classes of graphs for which it returns the same polynomial. When is it possible to decide for a given equivalence relation of graphs whether there is a graph polynomial that realises it?
- Graph polynomials build bridges between algebraic/enumerative properties and structural/combinatorial properties. Can we make these bridges precise in the field?

5.4 Long-standing problems

Graham Farr (Monash University – Clayton, AU)

An old question of Hassler Whitney (1932) about an elegant extension of the Four Colour Theorem using duality and Tutte–Whitney polynomials.

Let G be a graph such that there exists a graph H with the property that $T(G^*; x, y) = T(H; x, y)$. In other words, G^* is Tutte-equivalent to a graph, even though it might not be a graph itself. Here, G^* is the matroid dual of G, so it is a cographic matroid, but is only graphic if G is planar.

Question: Are such graphs G 4-colourable?

Whitney observed that this was a strengthening of the Four Colour Problem.

- If G is planar then G^* is also a planar graph and we may take $H = G^*$. In this case, the answer to the question is Yes: it's just the Four Colour Theorem.
- A graph G is *Tutte self-dual* (TSD) if $T(G^*; x, y) = T(G; x, y)$, in which case we may take $H = G$. Whitney gave an example of a non-planar Tutte self-dual graph: see W in Figure 1. This graph is 4-colourable.

5.5 Some problems coming from algebra

Alex Fink (Queen Mary University of London, GB)

The one I have the strongest philosophical commitment to:

Problem 1

Consider the Tutte polynomial, say. It's known exactly which of its evaluations are efficiently computable. Which *partial derivatives* of each evaluation are efficiently computable?

Problem 2

Once we know this, we know the largest quotient of $Z[x, y]$ in which the image of Tutte is efficiently computable. Now look at the structure of that quotient ring and see if this evaluation can be characterised as the universal invariant satisfying meaningful recurrences vel sim. (The philosophy is stop thinking only of maps to polynomial rings: there's lots more rings out there.)

The obvious one: Milk more invariants of interest out of the machinery of arXiv:1711.09028 and arXiv:1508.00814.

Problem 3

One from last time: Construct a meaningful trivariate polynomial of a three-coloured triangulation of the sphere which reflects the triality of Kalman and Posnikovos univariate polynomial under permuting colours, and specialises to my and Amanda Cameron's bivariate polynomial.

Problem from last time

Let G be a connected bipartite graph and $V_{\mathrm{e}} \amalg V_{\mathrm{v}}$ its vertex set. A *hypertree* for G is the degree sequence in $Z^{|V_{\mathrm{e}}|}$ of some spanning tree of G (these form a *hypergraphic polymatroid*). Define the bivariate polynomial $Q(G; t, u)$ so that, when t and u are naturals,

$$\begin{aligned} Q(G;t,u) = \#\{p \in Z^{|V_{\mathrm{e}}|} : p = a + b + c, \qquad & a \text{ is a hypertree of } G, \\ b_i \in Z_{\leq 0},\ \sum_i b_i = -t, \text{ and} \qquad & c_i \in Z_{\geq 0},\ \sum_i c_i = u\}. \end{aligned}$$

Ehrhart theory guarantees the existence of this polynomial. When all vertices in V_{e} are bipartite, then G is the barycentric subdivision of a graph H; in this case, hypertrees for G are in bijection with spanning trees for H, and $Q(G; t, u)$ contains the same information as $T(H; x, y)$. To wit, with Amanda Cameron we've shown that

$$\sum_{t,u \geq 0} Q(G;t,u)\alpha^t \beta^u = \frac{T\left(H; \dfrac{1-\alpha\beta}{1-\beta}, \dfrac{1-\alpha\beta}{1-\alpha}\right)}{(1-\alpha)^{|V(H)-1|}(1-\beta)^{|E(H)-V(H)+1|}(1-\alpha\beta)}.$$

Now let Δ be a three-coloured triangulation of the sphere. Then there are six ways to delete one colour class from Δ, leaving a bipartite graph G, and label the other two colour classes V_{e} and V_{v}. If V_{e} is colour i, V_{v} is colour j, and the deleted colour is k, let $Q_{ijk}(\Delta; t, u) = Q(G; t, u)$. These are interrelated. Firstly,

$$Q_{ijk}(\Delta; t, u) = Q_{ikj}(\Delta; u, t).$$

In the case where all vertices of colour i have degree 4, this is plane graph duality (in general, it's a polymatroid duality). Secondly, Kálmán and Postnikov [1] have shown that

$$Q_{ijk}(\Delta; t, 0) = Q_{jik}(\Delta; t, 0).$$

This is all compatible with the existence of a trivariate polynomial $\widehat{Q}(\Delta; x_i, x_j, x_k)$ such that

$$\widehat{Q}(\Delta; 0, x_j, x_k) = Q_{ijk}(\Delta; x_k, x_j)$$

and such that permuting the colour classes of Δ permutes the variables of $\widehat{Q}(\Delta)$ in the corresponding way.
Problem: Construct a nice such $\widehat{Q}(\Delta)$.

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5.6 Filtrations and decompositions

Emeric Gioan (University of Montpellier & CNRS, FR)

Let $G = (V, E)$ be a graph and suppose that E is linearly ordered. A *connected filtration* of G consists of a sequence of sets

$$\emptyset = F'_j \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_i = E$$

such that:

1. the sequence $min(F_k \setminus F_{k-1})$, $1 \leq k \leq i$, is increasing with k;
2. the sequence $min(F'_{k-1} \setminus F'_k)$, $1 \leq k \leq j$, is increasing with k;
3. for every $1 \leq k \leq i$, the minor $G(F_k)/F_{k-1}$ is either a single isthmus (if it has one edge) or is loopless and 2-connected (otherwise);
4. for every $1 \leq k \leq j$, the minor $G(F'_{k-1})/F'_k$ is either a single loop (if it has one edge) or is loopless and 2-connected (otherwise).

When the two last conditions are omitted, we define a *filtration* of G.

Consider the following result.
Theorem [G. & Las Vergnas, 2002, 2019]
With $\beta^*(G) = \beta(G)$ if $|E| > 1$, $\beta^*(\text{isthmus}) = 0$, $\beta^*(\text{loop}) = 1$, we have

$$t(G; x, y) = \sum_{\substack{\text{(connected)}\\ \text{filtrations}}} \Big(\prod_{1 \leq k \leq i} \beta\big(G(F_k)/F_{k-1}\big) \Big) \Big(\prod_{1 \leq k \leq j} \beta^*\big(G(F'_{k-1})/F'_k\big) \Big) x^i y^j.$$

Note that this result "contains" the spanning tree activity formula, the orientation activity formula, the duality formula, and the convolution formula. It is also related to a unique (connected) filtration canonically decomposing a spanning tree or a directed graph. It is probably related to Hopf algebras (Krajewski, Moffatt, Tanasa, Dupont, Fink, Moci) and to Bergman/conormal fans in matroid geometry (Ardila et al., Rincon et al.), and it can be seen in terms of the algebra of set functions (Lass). Actually, this result is more generally available in matroids as well.

We ask the following questions. What are the structural or computational uses of the underlying decompositions? Are there related meta-results without fixing a linear ordering? Are there similar approaches and results for other graph polynomials?

5.7 Psicle polynomials

Krystal Guo (UL – Brussels, BE)

We define a polynomial which interpolates between the characteristic polynomial of a graph and the matching polynomial of a graph. The polynomial takes, as input, a set of cycles of the graph, which can be chosen to be the set of contractible cycles of a given embedding. There polynomials gives a natural walk-generating function for certain types of walks in the graph. There are interesting questions about when this polynomial has real roots, or if the roots lie in a restrict section of the plane. More interestingly, we can ask whether or not this property is related to the orientability of the surface.

A graph is *sesquivalent* if all of its components are 1- or 2-regular; that is to say, a graph is sesquivalent if it is a disjoint union of cycles and copies of K_2. Let X be a graph on n vertices and let $\Gamma_r(X)$ be the set of all sesquivalent subgraphs of X on r vertices, for a fixed $0 \leq r \leq n$. For a sesquilinear graph Y, we define $\overline{Y}$ to be the number of components of Y and $\langle Y \rangle$ to be the number of 2-regular components of Y. By expanding the determinant of $tI - A(X)$, we see that the coefficient of x^{n-r} in the characteristic polynomial $\phi(X,t)$ of X is as follows:

$$\sum_{Y \in \Gamma_r(X)} (-1)^{\overline{Y}} 2^{\langle Y \rangle}.$$

We explore a polynomial of X with a similar expansion, in terms of sesquivalent subgraphs of X, where we only sum over all sesquivalent subgraphs whose cycles are in a set of allowable cycles.

Let C be a subset of all cycles of X. For a fixed $0 \leq r \leq n$, let $\Gamma_r(X,C)$ be the set of all sesquivalent subgraphs Y of X on r vertices, such that each 2-regular component of Y is in C. We define the *psicle polynomial* of X with respect to C to be the polynomial $\psi(X,C,t)$ such that the coefficient of x^{n-r} is as follows:

$$\sum_{Y \in \Gamma_r(X,C)} (-1)^{\overline{Y}} 2^{\langle Y \rangle}.$$

Observe that if we choose C to be the set of all cycles of X, then $\psi(X,C,r) = \phi(X,t)$. If $C = \emptyset$, then $\psi(X,C,r) = \mu(X,t)$, where $\mu(X,t)$ denotes the matching polynomial of X.

Given a cellular embedding of X on a surface Σ, a sensible choice for a subset of cycles C_c would be the set of contractible cycles of X. We define the *psicle polynomial of* X *with respect to an embedding* Π to be $\psi(X,C_c,t)$, where C_c is the set of contractible cycles of X with respect to Π. We will abuse notation and denote this as $\psi(X,\Pi,t)$.

We would like to investigate the location of the roots of $\psi(X,\Pi,t)$ in a complex plane.

1. For a given graph X, what are some properties of embeddings Π such that $\psi(X, \Pi, t)$ has real roots? Does orientability matter?
2. Is it true that every complex root of $\psi(X, \Pi, t)$ has negative real part? (Or a similar statement?)

5.8 Some problems coming from knot theory

Louis H. Kauffman (Novosibirsk State University, RU)

Loops (even with virtual crossings) receive the value d (the third variable). S. Baldridge, L. Kauffman and W. Rushworth are studying a mapping from virtual 3-valent graphs G with perfect matching M to virtual link diagrams. This mapping takes a natural perfect matching polynomial for graphs (generalizing the Penrose evaluation) to the three variable bracket, and hence to the Jones polynomial by specialization.

This is a new relationship between graph polynomials and link invariants, and it goes both ways, allowing essentially all invariants of virtual knots to be viewed as invariants of perfect matching graphs up to a pullback to graphs of knot theoretic isotopy. More generally, I am working on the interface between virtual knot theory, graph theory and statistical mechanics and quantum theory via the above construction and via generalizations of the medial constructions that have a long history in this subject, via surfaces and ribbon graphs and via the category of Morse diagrams, quantum link invariants and link homology. The purpose of this five-minute talk is to give a very quick introduction to the new mapping from perfect matching graphs to virtual links and to indicate the many problems and constructions that this entails.

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5.9 Graph polynomials and (n, r)-matroids

Joseph Kung (University of North Texas – Denton, US)

The Tutte polynomial of a rank r matroid on n elements has degree r as a polynomial in x and degree $n-r$ as a polynomial in y. The Tutte polynomials of such matroids span a subspace of $\mathbb{C}[x, y]$ and an upper bound for

$$\dim\langle\, T(M; x, y) : M \text{ an } (n, r)\text{-matroid}\,\rangle$$

is $(r+1)(n-r+1)$.

The dimension is in fact equal to $r(n-r)+1$.

Problem: Answer the same question for other graph polynomials.
The dimension of the subspace spanned by the graph polynomial for graphs of given order and size serves as a measure of information contained in a graph polynomial: how useful is this way of measuring the combinatorial information contained in a given polynomial graph invariant?

5.10 Graph polynomial problems arising from Vassiliev and quantum knot invariants

Sergei Lando (NRU Higher School of Economics – Moscow, RU)

Problem 1: The existence of the Lie algebra polynomial graph invariant

A weight system is a function on chord diagrams (circles with a tuple of chords without common ends) satisfying certain conditions called 4-term relations. In particular, Bar-Natan and Kontsevich associated a weight system to a semi-simple Lie algebra endowed with an invariant nondegenerate scalar product. To a chord diagram, its intersection graph can be associated: the vertices of the graph are the chords of the diagram, and two vertices are connected by an edge if and only if the corresponding chords intersect one another. Not every graph arises as the intersection graph of a chord diagram, and certain graphs are intersection graphs of several chord diagrams.

In [1], it is shown that for the Lie algebra sl_2 the value of the corresponding weight system depends on the intersection graph of the chord diagram rather than on the diagram itself. As a result, we obtain a partially defined graph invariant taking values in the ring of polynomials in a single variable (which is the Casimir element of the Lie algebra sl_2).

Question: is there is an extension of this partially defined polynomial graph invariant to a completely defined polynomial graph invariant satisfying 4-term relations for graphs?

The 4-term relations for graphs were introduced in [3]. E. Krasilnikov constructed an extension of this graph invariant to all graphs with up to 8 vertices and proved uniqueness of the extension.

Problem 2: The value of the Lie algebra sl_2 weight system on complete graphs

The graph invariant from the previous problem is well-defined for complete graphs since they are intersection graphs. However, known methods of computation of its value on a complete graph with a given number of vertices, are laborious, and I know the answer only for complete graphs with up to 14 vertices. These results allow me to formulate a conjecture representing the generating function for these polynomials as a continuous fraction.

Problem: prove the conjectural formula for the values of the sl_2*-invariant on complete graphs.*

Problem 3: Umbral invariants for binary delta-matroids

An *umbral polynomial graph invariant* is a graded Hopf algebra homomorphism from the Hopf algebra of graphs to the Hopf algebra of polynomials in infinitely many variables. An example is given by the Stanley symmetrized chromatic polynomial. It is proved in [2] that the mean value of an umbral polynomial graph invariant is, essentially, a linear combination of one-part Schur polynomials. However, a similar statement is not valid for the Hopf algebra of binary delta-matroids.

Question: what should be a correct replacement of the statement for graphs for binary delta-matroids?

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5.11 Matchings and monotonicity

Bodo Lass (University Claude Bernard – Lyon, FR)

Let us consider bipartite graphs $G = (X, Y; E)$, where X is a fixed set of vertices, $|X| = n$, and Y is a variable set of vertices. For every subset X' of X, let us denote by $d(X')$ the number of neighbors (in Y) of X'. Hall's classical theorem on matchings affirms that there is an injective function $f : X \to Y$ such that $\{x, f(x)\}$ is an edge of G for every $x \in X$ if and only if $d(X') \geq |X'|$ for every nonempty subset X' of X.

But one may also want to count the number $m(X)$ of such injective functions (matchings covering X). I have shown that the $2^n - -1$ numbers $d(X')$ (X' nonempty subset of X) determine G up to isomorphism and allow to calculate $m(X)$ in the following way:

We must sum over all partitions of X into nonempty blocks X', where each block X' contributes the factor

$$(|X'| - 1)![d(X') - (n - 1)].$$

For example, if $X = \{1, 2, 3\}$, then

$$m = (d_1-2)(d_2-2)(d_3-2)+(d_{12}-2)(d_3-2)+(d_{13}-2)(d_2-2)+(d_{23}-2)(d_1-2)+2(d_{123}-2).$$

This formula shows that m, in general, depends monotonically on the set function d. Because of the substraction of $(n - 1)$, however, this formula does not show monotonicity in all cases. In particular, it does not prove the most extreme case : Hall's theorem. Eberhard Triesch conjectured that the monotonicity is always true, and it would be nice to find a proof or counterexample. More details (in latex and pdf) can be found on http://math.univ-lyon1.fr/~lass/articles/pub17triesch.html.

5.12 Complexity of the evaluation of graph polynomials

Johann A. Makowsky (Technion – Haifa, IL)

Let C be a set of clauses and (V, C, R) a directed graph in which arcs (R) join variables (V) to clauses (C) with direction according to whether the variable is negated or not in the clause. (See Figure 2.)

For $A \subseteq V$, define $\mathrm{SAT}(A) = \{c \in C : A \text{ satisfies } c\}$ and the SAT-polynomial in indeterminate X by

$$\sum_{A \subseteq V} \prod_{c \in \mathrm{SAT}(A)} X = \sum_{A \subseteq V} X^{|\mathrm{SAT}(A)|}.$$

Is this polynomial useful for studying the satisfiability problem SAT?

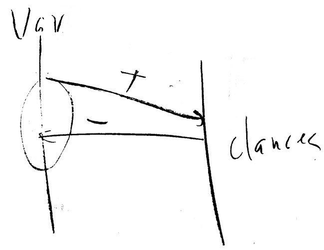

Figure 2 Assignments of values (V) to variables in clauses (C), negative arcs for negated variables.

5.13 Zero-free regions for the chromatic polynomial

Guus Regts (University of Amsterdam, NL)

Determine regions in the complex plane on which the chromatic polynomial does not vanish for interesting classes of (and possibly all) bounded degree graphs. In particular improve on the results from [1]

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5.14 Distinguishing power: Krushkal vs 4-variable from surface Tutte polynomial

Lluís Vena Cros (Charles University – Prague, CZ)

Goodall, Krajewski, Regts and Vena, and Goodall, Litjens, Regts and Vena recently introduced a polynomial for maps on orientable and non-orientable surfaces. These two polynomials can be seen as a common generalization of the flow polynomial (which counts nowhere-identity flows of a map in which edges are given values in a not necessarily abelian finite group G, and the Kirchhoff law equations are determined by the cyclic orientation of edges around each vertex of the embedding) and the local tension polynomial (which count nowhere-identity flows of the surface-dual map).

The polynomial for maps on orientable surfaces is defined by

$$\mathcal{T}(M;\mathbf{x},\mathbf{y}) = \sum_{A\subseteq E} x^{n^*(M/A)} y^{n(M\backslash A^c)} \prod_{\substack{\text{conn. cpts } M_i\\ \text{of } M/A}} x_{g(M_i)} \prod_{\substack{\text{conn. cpts } M_j\\ \text{of } M\backslash A^c}} y_{g(M_j)}, \tag{1}$$

where M is the given map on an orientable surface (a graph with a cyclic ordering of edges, actually half-edges, around each vertex), $M\backslash A^c$ is the map where the edges of the complement of A have been deleted (the relative cyclic order of the other half-edges is preserved), and M/A is the map where the edges in A have been map-contracted (map-contraction is the surface-dual operation to edge deletion), and g is the orientable genus of the map (the number of handles attached to a sphere, extended additively over connected components for non-connected maps), $n(M) = e(M) - v(M) + k(M)$ is the nullity of the map (the number of edges minus the number of vertices plus the number of connected components), and $n^*(M) = e(M) - f(M) + k(M)$ is the dual nullity ($f(M)$ is the number of faces of M).

The polynomial is defined more generally for maps M embedded in (not necessarily orientable) surfaces by

$$\mathcal{T}(M; \mathbf{x}, \mathbf{y}) = \sum_{A \subseteq E} x^{n^*(M/A)} y^{n(M \setminus A^c)} \prod_{\substack{\text{conn. cpts } M_i \\ \text{of } M/A}} x_{\bar{g}(M_i)} \prod_{\substack{\text{conn. cpts } M_j \\ \text{of } M \setminus A^c}} y_{\bar{g}(M_j)}, \tag{2}$$

where $\bar{g}(M)$ is the "signed genus", equal to the genus when M is orientable, and equal to the negative of Euler's demigenus when M is non-orientable (equal to the number of cross-caps attached to a sphere).

Another polynomial defined for maps is the Krushkal polynomial (defined in a paper from 2011), which has four variables and that can be defined as follows:

$$\mathcal{K}(M; X, Y, C, D) = \sum_{A \subseteq E} X^{k(M \setminus A^c) - k(M)} Y^{k(M/A) - k(M)} C^{s(M \setminus A^c)/2} D^{s(M/A)/2}, \tag{3}$$

where $s(M) = 2k(M) - \chi(M)$ is the Euler genus, with $\chi(M)$ the Euler characteristic of M.

It can be checked that

$$\mathcal{K}(M; X, Y, C, D) = X^{-k(M)} Y^{-k(M)} \mathcal{T}(M; \mathbf{x}, \mathbf{y})$$

with $x = 1$, $y = 1$, $x_{\bar{g}} = Y D^{\frac{\bar{g}+3|\bar{g}|}{4}}$ and $x_{\bar{g}} = X C^{\frac{\bar{g}+3|\bar{g}|}{4}}$ for $\bar{g} \in \mathbb{Z}$.

There are two maps M_1 and M_2 for which (1) induces two different polynomials, yet they are the same polynomial for (3). Hence, (1) induces a strictly finer partition on the set of maps.

A 4-variable polynomial specialization of (1) and (2) is given by

$$Q(M, x, y, a, b) = \sum_{A \subseteq E} x^{n^*(M/A)} y^{n(M \setminus A^c)} a^{\bar{g}(M/A)} b^{\bar{g}(M \setminus A^c)}.$$

upon taking $x_{\bar{g}} = a^{\bar{g}}$ and $y_{\bar{g}} = b^{\bar{g}}$ for each $\bar{g} \in \mathbb{Z}$.

The question is: Does the polynomial Q have the same, more, less, or different distinguishing power than $\mathcal{K}$?

6 Heidelberg Laureate Forum Participant Talk

Animesh Chaturvedi (PhD student, IIT Indore) who was attending the Heidelberg Laureate Forum (2019) was invited by Schloss Dagstuhl – Leibniz Center to attend the seminar, and presented a short talk.

6.1 System evolution analytics: data mining and learning of evolving graphs representing evolving complex system

Animesh Chaturvedi (Indian Institute of Technology – Indore, IN)

Usually, real-world evolving systems have many entities (or components), which evolves over time. A technique can be used to analyze the evolving system; the technique is named as System Evolution Analytics. Such techniques can be applied on an evolving system represented as a set of temporal networks. These techniques fall in the categories of system learning and system mining. The state series of an evolving system is denoted as $SS = \{S1, S2, \ldots, SN\}$. Then, the connections (or relationships) between entities of each state are pre-processed to make a temporal network, and this resulted in a series of evolving networks $EN = \{EN1, EN2, \ldots, ENN\}$. These temporal networks can be merged to make an evolution representor, which is used with learning and mining techniques for system evolution analysis. This made us to analyze evolving inter-connected entities of a system state series. The system learning is performed by applying active learning and deep learning on the evolution representor. The system mining is performed by applying two proposed pattern-mining techniques: network rule mining and subgraph mining. Specifically, the publications describe the following proposed approaches: System State Complexity, Evolving System Complexity, System Evolution Recommender, Stable Network Evolution Rule, and System Changeability Metric. The proposed approaches are used to generate recommendation and evolution information to perform system evolution analysis. For example, a graph theory application of a service change classifier algorithm assigning change labels to a web service's call graph representing calls between operations and procedures, which helped to do Web Service Slicing by extracting a WSDL slice for Inter-operational analysis.

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