Perfect Resolution of Conflict-Free Colouring of Interval Hypergraphs

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Abstract

Given a hypergraph H, the conflict-free colouring problem is to colour vertices of H using minimum colours so that in every hyperedge e of H, there is a vertex whose colour is different from that of all other vertices in e. Our results are on a variant of the conflict-free colouring problem considered by Cheilaris et al.[4], known as the 1-Strong Conflict-Free (1-SCF) colouring problem, for which they presented a polynomial time 2-approximation algorithm for interval hypergraphs. We show that an optimum 1-SCF colouring for interval hypergraphs can be computed in polynomial time. Our results are obtained by considering a different view of conflict-free colouring which we believe could be useful in general. For interval hypergraphs, this different view brings a connection to the theory of perfect graphs which is useful in coming up with an LP formulation to select the vertices that could be coloured to obtain an optimum conflict-free colouring. The perfect graph connection again plays a crucial role in finding a minimum colouring for the vertices selected by the LP formulation.

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1 Introduction

A vertex colouring function $C : \mathcal{V} \rightarrow \{0, 1, 2, \ldots, k\}$ of a hypergraph $H = (\mathcal{V}, \mathcal{E})$ using k non-zero colours is a 1-*SCF colouring* of H, if for every hyperedge $e \in \mathcal{E}$ there exists a non-zero colour $j \in \{1, 2, \cdots, k\}$ such that $|e \cap C^{-1}(j)| = 1$. This problem was first studied by Cheilaris et al. [4] and is a variant of a well-studied hypergraph colouring problem known as the *Conflict-Free colouring* problem. A Conflict-Free (CF, in short) colouring is a vertex colouring of a hypergraph that colours every vertex of the hypergraph such that every hyperedge e has at least one vertex whose colour is different from that of every other vertex in e. The CF colouring problem seeks to find a CF colouring using minimum number of colours. The number of colours used in any optimum CF colouring of a hypergraph H is called the *CF colouring number* of H. In the 1-SCF colouring, the algorithm is presented with an input in which all vertices are initially coloured with colour 0, and the goal is to modify the colour of some vertices to a non-zero colour such that the resulting colouring is a 1-SCF colouring. We observe that a 1-SCF colouring can be used to find a CF colouring of a given hypergraph by adding one interval of length 1 for each vertex in H (this simple transformation ensures that each vertex must be given a non-zero colour). We refer to the number of non-zero colours used in any optimum 1-SCF colouring of a hypergraph H as the 1-*SCF colouring number* of H. Observe that the CF colouring number of a hypergraph

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H is at most one more than the 1-SCF colouring number of H. Our main result is an algorithm that solves the 1-SCF colouring problem optimally in polynomial time for interval hypergraphs, thus solving one part of an open problem posed by Cheilaris et al. [4].

▶ **Theorem 1.** *The* 1*-SCF colouring problem in interval hypergraphs can be optimally solved in polynomial time.*

As a corollary the 1-SCF colouring number of a given interval hypergraph can be computed in polynomial time.

Past Work in CF colouring. The survey due to Smorodinsky [17] presents a general framework for CF colouring that involves finding a proper colouring in every iteration and giving the largest colour class a new colour. Smorodinsky [17] showed that if for every induced sub-hypergraph $H' \subseteq H$, the chromatic number of H' is at most k, then $\chi_{cf}(H) \leq \log_{1+\frac{1}{k-1}} n = O(k \log n)$, where $n = |\mathcal{V}|$. Pach and Tardos [15] showed that if $|\mathcal{E}(H)| < \binom{s}{2}$ for some positive integer s, and Δ is the maximum degree of a vertex in H, then $\chi_{cf}(H) < s$ and $\chi_{cf}(H) \leq \Delta + 1$. The CF colouring problem has also been studied on different types of hypergraphs. Even et al. [8] have studied a number of hypergraphs induced by geometric regions on the plane including discs, axis-parallel rectangles, regular hexagons, and general congruent centrally symmetric convex regions in the plane. Let $\mathcal{D}$ be a set of n finite discs in $\mathbb{R}^2$. For a point $p \in \mathbb{R}^2$, define $r(p) = \{D \in \mathcal{D} : p \in D\}$. The hypergraph $(\mathcal{D}, \{r(p)\}_{p \in \mathbb{R}^2})$, denoted by $H(\mathcal{D})$, is called the hypergraph induced by $\mathcal{D}$. Smorodinsky showed that $\chi_{cf}(H(\mathcal{D})) \leq \log_{4/3} n$ [16, 17]. Similarly, if $\mathcal{R}$ is a set of n axis-parallel rectangles in the plane, then, $\chi_{cf}(H(\mathcal{R})) = O(\log^2 n)$. There have been many studies on hypergraphs induced by neighbourhoods in simple graphs. Given a simple graph $G = (V, E)$, the *open neighbourhood* (or simply neighbourhood) of a vertex $v \in V$ is defined as follows: $N(v) = \{u \in V \mid uv \in E\}$. The set $N(v) \cup v$ is known as the *closed neighbourhood* of v. Pach and Tardos [15] have shown that the vertices of a graph G with maximum degree Δ can be coloured with $O(\log^{2+\epsilon} \Delta)$ colours, so that the closed neighbourhood of every vertex in G is CF coloured. They also showed that if the minimum degree of vertices in G is $\Omega(\log \Delta)$, then the open neighbourhood can be CF coloured with at most $O(\log^2 \Delta)$ colours. Abel et al. [1] gave the following tight worst-case bound for neighbourhoods in planar graphs: three colours are sometimes necessary and always sufficient. Keller and Smorodinsky [14] studied conflict-colourings of intersection graphs of geometric objects. They showed that the intersection graph of n pseudo-discs in the plane admits a CF colouring with $O(\log n)$ colours, with respect to both closed and open neighbourhoods. Ashok et al. [2] studied an optimization variant of the CF colouring problem, namely MAX-CFC. Given a hypergraph $H = (\mathcal{V}, \mathcal{E})$ and integer $r \geq 2$, the problem is to find a maximum-sized subfamily of hyperedges that can be CF coloured with r colours. They have given an exact algorithm running in $O(2^{n+m})$ time. The paper also studies the problem in the parametrized setting where one must find if there exists a subfamily of at least k hyperedges that can be CF coloured using r colours. They showed that the problem is FPT and gave an algorithm with running time $2^{O(k \log \log k + k \log r)}(n + m)^{O(1)}$.

CF colouring in Interval Hypergraphs. A hypergraph $H_n = ([n], \mathcal{I}_n)$, where $[n] = \{1, \ldots, n\}$ and $\mathcal{I}_n = \big\{\{i, i+1, \ldots, , j\} \mid i \leq j \text{ and } i, j \in [n]\big\}$ is known as a *complete interval hypergraph* [4]. A hypergraph such that the set of hyperedges is a family of intervals $\mathcal{I} \subseteq \mathcal{I}_n$ is known as an *interval hypergraph.* One can view the CF colouring problem in interval hypergraphs as modelling the frequency assignment problem in a chain of units discs

[4, 8]. In the case of hypergraphs induced by arbitrary unit discs in the plane, the problem is known to be NP-complete [13]. It was shown in [8] that a complete interval hypergraph can be CF coloured using $\Theta(\log n)$ colours. Chen et al.[5] presented results on an online variant of CF colouring problem in complete interval hypergraphs. In the online variant, points arrive online and a point has to be assigned a colour upon its arrival such that the resulting colouring is conflict-free with respect to all intervals. Chen et al.[5] gave a greedy algorithm that uses $\Omega(\sqrt{n})$ colours, a deterministic algorithm that uses $\Theta(\log^2 n)$ colours and a randomized algorithm that uses $O(\log n)$ colours. There have been some studies [4, 13] on CF colouring in interval hypergraphs (instead of complete interval hypergraphs), in which case a subset of intervals in $\mathcal{I}_n$ is given as part of the input.

1-SCF colouring in Interval Hypergraphs. Katz et al.[13] gave a polynomial-time approximation algorithm for 1-SCF colouring an interval hypergraph with approximation ratio 4. Cheilaris et al.[4] improved this result in their paper on *k-Strong CF colouring* (k-*SCF*) problem. The k-*SCF* problem seeks to find a vertex colouring of the hypergraph such that in every hyperedge e, there are at least $\min\{|e|, k\}$ vertices that are uniquely coloured. Cheilaris et al.[4] gave a polynomial-time approximation algorithm with approximation ratio 2 for $k = 1$ and $5 - \frac{2}{k}$ for $k \geq 2$. Further, they presented a quasi-polynomial time algorithm for the decision version of the k-*SCF* problem. This clearly ruled out the possibility of the decision version being NP-complete, unless NP-complete problems have quasi-polynomial time algorithms. The main result in this paper is a polynomial time optimum 1-SCF colouring algorithm for interval hypergraphs. We achieve this by observing a natural connection between the 1-SCF colouring problem and the problem of solving an exact hitting set problem with some constraints. We then formulate this exact hitting set problem as a linear program (LP) for interval hypergraphs, and show that this LP can be solved in polynomial time, and the LP solution can be rounded to an integer solution in polynomial time. For the rest of the paper, since we work entirely on 1-SCF colouring, we refer to non-zero colours as simply "colours".

Our Approach. We outline the approach towards proving our main result which is Theorem 1. The initial steps of our approach are simple observations regarding a different view of 1-SCF colouring which we present as Theorem 2 and Theorem 3. We present these observations as theorems because they bring a different perspective on 1-SCF colouring which turns out to be surprisingly useful for interval hypergraphs. However, in spite of the positive result with interval hypergraphs, we do not know of other hypergraphs for which our approach yields a better understanding of the 1-SCF colouring problem and the 1-SCF colouring number.
Theorem 2 brings into our perspective that 1-SCF colouring is actually the problem of computing a special type of colouring of the vertices of an exact hitting set. We observe that 1-SCF colouring of a hypergraph can be naturally seen as the proper colouring of a related simple graph which we call a *co-occurrence graph*. A co-occurrence graph is obtained from a 1-SCF colouring based on a function defined on $\mathcal{E}$, and we call this function a *representative function*. For a representative function t, the co-occurrence graph is denoted by Γ_t. We show that the search for the co-occurrence graph with the minimum chromatic number is equivalent to the search for the corresponding representative function.

▶ **Theorem 2.** *Let $H = (\mathcal{V}, \mathcal{E})$ be a hypergraph. Let $\chi_{cf}(H)$ be the number of colours used in any optimal 1-SCF colouring of H. Let $\chi_{min}(H)$ be the minimum chromatic number over all possible co-occurrence graphs of H. Then, $\chi_{cf}(H) = \chi_{min}(H)$.*

We then show that the search for an appropriate representative function is answered by finding an exact hitting set of some cliques in a graph $\hat{G}$ called the conflict graph of H. We identify two sets of cliques in $\hat{G}$, namely *hyperedge cliques* denoted by $\mathcal{Q}_1$ and *colour cliques* denoted by $\mathcal{Q}_2$. These cliques are defined in Definition 7. The conflict graph is designed in such a way that there exists an exact hitting set S of cliques in $\mathcal{Q}_1$ that hits every maximal clique in $\mathcal{Q}_2$ at most q times if and only if there exists a 1-SCF colouring of H with q colours (Lemma 9). We establish the relation between the conflict graph and a co-occurrence graph of a hypergraph in the theorem below. Theorem 3 gives us a framework for searching for an appropriate exact hitting set of the given hypergraph.

▶ **Theorem 3.** *Let $H = (\mathcal{V}, \mathcal{E})$ be a hypergraph. Let t be a representative function of H and let $\hat{G}$ be the conflict graph of H. Then, the set $S = \{(e, u) \mid (e, u) \in \hat{G}, t(e) = u\}$ is an exact hitting set of cliques in $\mathcal{Q}_1$, $\omega(\hat{G}[S]) \leq \omega(\Gamma_t)$, and $\chi(\hat{G}[S]) \leq \chi(\Gamma_t)$.*

We next state the following important structural properties of co-occurrence graphs and conflict graphs specific to interval hypergraphs in Section 3.

▶ **Theorem 4.** *Each co-occurrence graph of an interval hypergraph is perfect.*

▶ **Theorem 5.** *Let $H = (\mathcal{V}, \mathcal{I})$ be an interval hypergraph. Then the conflict graph $\hat{G}(H)$ is perfect.*

In order to find an optimal co-occurrence graph for interval hypergraphs, we formulate a Linear Program (LP) for a constrained exact hitting set of a set of cliques of $\hat{G}$ in Section 4. We show that the LP can be solved in polynomial time using the ellipsoid method. The ellipsoid method uses a separation oracle that we design specifically for interval hypergraphs, and this oracle crucially relies on the fact that the conflict graph is perfect. Further, an optimum fractional solution obtained from the LP is rounded to give an integer feasible solution. This integer feasible solution naturally gives a representative function for the given interval hypergraph, and we show that the corresponding co-occurrence graph has the minimum chromatic number over all co-occurrence graphs of the given interval hypergraph. The minimum colouring of the co-occurrence graph can also be computed in polynomial time using known algorithms for minimum colouring of perfect graphs. We believe that this technique can be used to design an optimal polynomial time algorithm for the k-SCF problem of Cheilaris et al.[4]. The perfectness of co-occurrence graphs and conflict graphs in the 1-SCF colouring are not dependent on k and hence we conjecture that the same approach will work for the k-SCF problem with minimal tuning to the rounding procedure for the LP. Finally, we have not been able to prove any other computationally useful structure on the co-occurrence graphs and conflict graphs of interval hypergraphs. In particular, we know that both these graphs can have induced cycles of length 4, and thus we cannot use techniques from chordal graph colouring or interval graph hitting set (which uses the consecutive ones property of the clique-vertex incidence matrix) algorithms.

1.1 Preliminaries

In an interval $I = \{i, i+1, \ldots, j\}$, i and j are the *left* and *right endpoints* of I respectively, denoted by $l(I)$ and $r(I)$, respectively. Since an interval is a finite set of consecutive integers, it follows that $|I|$ is well-defined. Throughout the paper, we assume that the hypergraph H has n vertices and m hyperedges.

If vertex $v \in e$ has been assigned a colour c that is different from the colour of all other vertices in e, then we say that e *is* 1-*SCF coloured* by vertex v and by colour c. Note that in our convention, we use colour 0 to indicate that a vertex with colour 0 does not 1-SCF colour any hyperedge.

For a set S of vertices in a simple graph G, $G[S]$ denotes the induced subgraph of G on S. Perfect graphs [9] are very well-studied and many hard problems are tractable on perfect graphs. We use four well known properties of perfect graphs.

P1 Let $G = (V, E)$ be a perfect graph. For a given subset $V' \subseteq V$, let $G[V'] = (V', E_{V'})$ be the subgraph induced by V', where $E_{V'} = \{uv \in E \mid u, v \in V'\}$. Then, it is known from [9] that $\omega(G[V']) = \chi(G[V'])$, where $\omega(G[V'])$ and $\chi(G[V'])$ are, respectively, the clique number and the chromatic number of $G[V']$. Recall that the clique number of a simple graph G is the size of its largest clique and the chromatic number of G is the number of colours needed in an optimal proper colouring of G.

P2 A *Berge graph* is a simple graph that has neither an *odd hole* nor an *odd anti-hole* as an induced subgraph [3, 6, 7, 9]. An odd hole is an induced cycle of odd length that has at least 5 vertices and an odd anti-hole is the complement of an odd hole. It is known from Theorem 1.2 in [7] that a graph is perfect if and only if it is Berge.

P3 The chromatic number of a perfect graph can be found in polynomial time [11].

P4 The maximum weighted clique problem can be solved in polynomial time in perfect graphs [10],[11].

We use linear programming and combinatorial optimization concepts from [12] and [10]. All other definitions and notations used in this paper are from West [19] and Smorodinsky [17].

2 Co-occurrence Graphs, Conflict Graphs and 1-SCF colouring

We define two types of simple graphs associated with a hypergraph $H = (\mathcal{V}, \mathcal{E})$. We also establish the relationship between a proper colouring of these graphs and a 1-SCF colouring of H. For a 1-SCF colouring function C defined on $\mathcal{V}$, let $t : \mathcal{E} \to \mathcal{V}$ be a function such that for each $e \in \mathcal{E}$, $t(e)$ is that vertex v in e such that the colour given to v by C is not given to any other vertex in e. We refer to t as a representative function obtained from the colouring C. Further, each function $t : \mathcal{E} \to \mathcal{V}$ such that for each edge e, $t(e) \in e$ is referred to as a representative function of H. We now define the *Co-occurrence Graph*, $\Gamma_t(H)$, given a representative function t of H. Let $R \subseteq \mathcal{V}$ denote the image of $\mathcal{E}$ under the function t. The vertex set of the co-occurrence graph $\Gamma_t(H)$ is R, and for $u, v \in R$, uv is an edge in $\Gamma_t(H)$ if and only if for some $e \in \mathcal{E}$, $u \in e$ and $v \in e$ and $t(e)$ is either u or v. Further, given a representative function t of H, a proper colouring of the graph $\Gamma_t(H)$ defines a 1-SCF colouring of H for which t is a representative function obtained from the 1-SCF colouring. Note that in this 1-SCF colouring, the vertices which are not present in $\Gamma_t(H)$ get the 0 colour. Wherever H is implied, we use Γ_t to denote $\Gamma_t(H)$. Define $\chi_{min}(H) = \min_t \chi(\Gamma_t)$ where $\chi(\Gamma_t)$ is the chromatic number of the co-occurrence graph Γ_t and the minimum is taken over all representative functions t of the hypergraph H.

We prove the equivalence stated in Theorem 2.

Proof of Theorem 2. Let t be a representative function such that $\chi(\Gamma_t) = \chi_{min}(H)$. We extend a proper colouring C of Γ_t to a vertex colouring function C' of $\mathcal{V}(H)$ by assigning the colour 0 to those vertices in $\mathcal{V}(H) \setminus R$. C' is a 1-SCF colouring of H since for each $e \in \mathcal{E}$, the colour assigned to the vertex $t(e)$ by C' is different from the colour assigned to every other vertex in e. The reason for this is as follows: let $v \in e$ be a vertex different from $t(e)$. If $C'(v) = 0$, then definitely its colour is different from $C'(t(e))$. On the other hand, if $C'(v) \neq 0$, then it implies that there is an e' such that $v = t(e')$. Consequently, $v \in V(\Gamma_t)$, and since $v \in e$, $\{v, t(e)\}$ is an edge in Γ_t by the definition of Γ_t. Further, since C' is obtained from a proper colouring C of Γ_t it follows that $C'(v)$ is different from $C'(t(e))$.

Thus $\chi_{cf} \leq \chi_{min}(H)$. We prove that $\chi_{min}(H) \leq \chi_{cf}(H)$ as follows: since a minimum 1-SCF colouring of H gives a representative function t as defined above, it follows that $\chi_{cf}(H) \geq \chi(\Gamma_t) \geq \chi_{min}(H)$. Therefore, it follows that $\chi_{cf}(H) = \chi_{min}(H)$. ◀

As a consequence of Theorem 2, to find a 1-SCF colouring with the least number of colours, we need to find a co-occurrence graph for which the chromatic number is the least over all co-occurrence graphs. This entails first computing the representative function corresponding to the co-occurrence graph which has the minimum chromatic number, and then computing a minimum colouring of the co-occurrence graph. We pose the problem of finding the candidate representative function as a hitting set problem on a graph called the *Conflict Graph* associated with a hypergraph. Given a hypergraph $H = (\mathcal{V}, \mathcal{E})$, we use $\hat{G}(H) = (V, E)$ to denote the conflict graph of H. Wherever H is implied, we use $\hat{G}$ to denote $\hat{G}(H)$. The elements of $V(\hat{G})$ and $\mathcal{V}(H)$ are referred to as *nodes* and *vertices*, respectively. The node set of $\hat{G}(H)$ is $V = \{(e, v) \mid e \in \mathcal{E}, v \in e\}$. In a node (e, v), we refer to e as the hyperedge coordinate and v as the vertex coordinate. Conceptually, a node (e, v) in $\hat{G}$ represents the logical proposition that hyperedge e is 1-SCF coloured by vertex $v \in e$. $E(\hat{G})$ is defined such that each edge encodes a constraint to be satisfied by any 1-SCF colouring of H. The edge set of $\hat{G}$ is $E = E_{edge} \cup E_{colour}$, where E_{edge} and E_{colour} are defined as follows:

1. $E_{edge} = \{((e, v), (e, u)) \mid \{v, u\} \subseteq e, u \neq v\}$. For each hyperedge e in H, the nodes in $\hat{G}$ with e as the hyperedge coordinate form a clique.
2. $E_{colour} = \{((e, v), (g, u)) \mid \{v, u\} \subseteq e \text{ or } \{v, u\} \subseteq g, u \neq v, e \neq g\}$. These edges encode the proposition that if two vertices co-occur in a hyperedge, they must get two different colours, irrespective of other hyperedges to which any of these vertex may belong to.

The following structural property of a conflict graph is crucial in the proof of Lemma 11.

▶ **Observation 6.** *Given a hypergraph $H = (\mathcal{V}, \mathcal{E})$, for each vertex $v \in \mathcal{V}$, the set of nodes $\{(e, v) \mid e \in \mathcal{E}, v \in e\}$ in $\hat{G}$ forms an independent set.*

We identify the following sets of "useful" cliques in a conflict graph.

▶ **Definition 7** (Hyperedge Cliques and Colour Cliques). *Hyperedge Clique is a clique in a conflict graph formed by nodes having the same hyperedge coordinate. The set of hyperedge cliques in a conflict graph is denoted by $\mathcal{Q}_1$. Colour Clique is a maximal clique in a conflict graph that has at least one edge from E_{colour}. The set of colour cliques in a conflict graph is denoted by $\mathcal{Q}_2$.*

We now prove Theorem 3 that states the relationship between the clique sizes of the co-occurrence graph and the conflict graph.

Proof of Theorem 3. By our premise, t is a representative function and hence the set S, obtained from t as defined above, hits every hyperedge clique exactly once. Therefore, it follows from definition of an exact hitting set that S is indeed an exact hitting set of the set of hyperedge cliques. Now, we show that $\omega(\Gamma_t) \geq \omega(\hat{G}[S])$. Let $\{(e, u), (f, v)\}$ be an edge in $\hat{G}[S]$ such that $e \neq f$. By definition of set S, $t(e) = u$ and $t(f) = v$. Since $e \neq f$, the edge $\{(e, u), (f, v)\}$ belongs to E_{colour} of $\hat{G}$. Hence u and v are both present together in either e or f. Without loss of generality, let $u, v \in e$. Since $(e, u) \in \hat{G}[S]$, we have $t(e) = u$ by construction of S. Hence, $\{u, v\}$ is an edge in Γ_t.

Therefore, for every edge $\{(e, u), (f, v)\}$ in $\hat{G}[S]$, there exists an edge $\{u, v\}$ in Γ_t. It follows that for every clique in $\hat{G}[S]$, there exists a clique of same size in Γ_t. Hence, $\omega(\Gamma_t) \geq \omega(\hat{G}[S])$. Given a proper colouring of Γ_t, let the colour given to the node (e, u) be the colour given to vertex u in the proper colouring of Γ_t. From Observation 6, we know that there are no edges between two nodes with the same vertex coordinate. Further, for each edge $\{(e,u),(f,v)\}$ in $\hat{G}[S]$, the edge $\{u,v\}$ is in Γ_t, and hence it follows that the colouring of $\hat{G}[S]$ is a proper colouring. Thus $\chi(\Gamma_t) \geq \chi(\hat{G}[S])$. ◀

As a consequence of Theorem 3, it follows that a representative function for H could be computed by finding an exact hitting set S of hyperedge cliques in $\hat{G}$ such that the chromatic number of $\hat{G}[S]$ is the minimum over all such exact hitting sets. We apply this approach to find an optimum 1-SCF colouring in polynomial time for interval hypergraphs by showing that such a hitting set can be computed in polynomial time for intervals. We show that this hitting set indeed gives the representative function whose co-occurrence graph has the minimum chromatic number. Further for interval hypergraphs, we show that the minimum vertex colouring of the co-occurrence graph can be computed efficiently. These results rely on the results in Section 3 which show that the co-occurrence graphs and the conflict graph of interval hypergraphs are perfect graphs.

3 Perfectness of Co-occurrence graphs and Conflict graphs of Interval Hypergraphs

We now prove two perfectness results when the given hypergraph is an interval hypergraph. The perfectness of co-occurrence graphs proved in Theorem 4 enables us to find a proper colouring of Γ_t in polynomial time. The perfectness of conflict graphs proved in Theorem 5 is used to prove Lemma 10. Lemma 10 is crucial in finding a hitting set of hyperedge cliques in $\hat{G}$. We first prove Theorem 4.

Proof of Theorem 4. We use property P2 of perfect graphs given in Section 1.1 to prove this result. By property P2, we know that an induced odd cycle and its complement are forbidden induced subgraphs for perfect graphs. Given an interval hypergraph H, let t be a representative function and let Γ_t be the resulting co-occurrence graph. We first show that Γ_t does not have an induced cycle of length at least 5. Note that we prove a stronger statement than required by property P2 of perfect graphs which states that there are no induced odd cycles of length at least 5. Our proof is by contradiction. Assume that $F = \{p_1, p_2 \ldots p_r\}$ is an induced C_r-cycle for $r \geq 5$. Let the sequence of nodes in F be $p_1, p_2 \ldots p_r, p_1$. Let p_i be the rightmost point of F on the line. In what follows, the arithmetic among the indices of p is *mod* r. Observe that, due to cyclicity of C_r, if $i = 1$, then $i - 1 = r$. Similarly, if $i = r$, then $i + 1 = 1$ and $i + 2 = 2$. Without loss of generality, let us assume that $p_{i-1} < p_{i+1}$, which are the two neighbours of p_i in F. Therefore, $p_{i-1} < p_{i+1} < p_i$. Since edge $\{p_{i-1}, p_i\}$ is in F, it follows that there exists an interval I for which $t(I) = p_{i-1}$ or $t(I) = p_i$. We claim that $t(I)$ is p_i: if $t(I)$ is p_{i-1}, then $\{p_{i-1}, p_{i+1}\}$ is an edge in Γ_t by definition. Therefore, $\{p_{i-1}, p_{i+1}\}$ is a chord in F, a contradiction to the fact that F is an induced cycle. Therefore, $t(I) = p_i$. Further, we claim that the point $p_{i+2} < p_{i-1}$: if $p_{i+2} > p_{i-1}$, then p_{i+2} belongs to the interval I and by the definition of the edges in Γ_t, $\{p_i, p_{i+2}\}$ is an edge in Γ_t. Therefore, $\{p_i, p_{i+2}\}$ is a chord in F. This contradicts the fact that F is an induced cycle. Therefore, $p_{i+2} < p_{i-1}$. At this point in the proof we have concluded that $p_{i+2} < p_{i-1} < p_{i+1} < p_i$ and $t(I) = p_i$. Since $\{p_{i+1}, p_{i+2}\}$ is an edge in F, it follows that there exists an interval J such that both p_{i+1} and p_{i+2} belong to J and $t(J) = p_{i+1}$ or $t(J) = p_{i+2}$, and therefore $p_{i-1} \in J$. Since F is an induced cycle of length at least 5, $\{p_{i-1}, t(J)\}$ is an edge in Γ_t by definition. Therefore, $\{p_{i-1}, t(J)\}$ is a chord in either case, that is when $t(J) = p_{i+1}$ or $t(J) = p_{i+2}$. This contradicts the assumption that F is an induced cycle of length at least 5. Thus, Γ_t cannot have an induced cycle of size at least 5. Next, we show that Γ_t does not contain the complement of an induced cycle of length ≥ 5, ($\overline{C_r}$, $r \geq 5$) as an induced subgraph. Again, our proof is by contradiction. Assume that F is an induced $\overline{C_r}$, $r \geq 5$ in Γ_t. Let $q_1, q_2, \ldots, q_r$ be the nodes of F. Also, let $q_1 < q_2 < \ldots < q_r$ be the left to right ordering of points on the line corresponding to vertices of F. Since $deg(q_i) = r - 3$ for all q_i in F, it follows that no interval I, such that $t(I) \in F$, contains more than $r - 2$ vertices from F. Otherwise, if there exists an interval I such that $t(I) \in F$ contains more than $r - 2$ vertices from F, then $deg(t(I)) \geq r - 2$ in F which is a contradiction. Therefore, there does not exist any interval that contains both q_1 and q_r. Similarly, there does not exist any interval that contains both q_1 and q_{r-1} and any interval that contains both q_2 and q_r. Since $deg(q_1) = r - 3$, it follows that q_1 must be adjacent to all vertices in $\{q_2, q_3, \ldots, q_{r-2}\}$. Similarly, q_r must be adjacent to all vertices in $\{q_3, q_4, \ldots, q_{r-1}\}$. Next, we consider the degrees of vertices q_2 and q_{r-1} in F. Since they are in F, q_2 is adjacent to q_1 and q_{r-1} is adjacent to q_r. Now, q_2 must be adjacent to $r - 4$ more vertices. We show that q_2 is not adjacent to q_{r-1}. Suppose not, that is, if q_2 is adjacent to q_{r-1}, then there exists an interval I that contains both q_2 and q_{r-1} and $t(I) = q_2$ or $t(I) = q_{r-1}$. Then $t(I)$ is adjacent to all points in the set $\{\{q_2, q_3, \ldots, q_{r-1}\} \setminus t(I)\}$. Thus, by considering the one additional edge incident on $t(I)$ depending on whether $t(I) = q_2$ or q_{r-1}, it follows that $deg(t(I)) \geq r - 2$, a contradiction to the fact that the degree of each vertex inside F is $r - 3$. Therefore, the edge $\{q_2, q_{r-1}\}$ does not exist in F. It follows that in $\overline{F}$, which we know is an induced cycle of length at least 5, there is an induced cycle $q_1, q_{r-1}, q_2, q_r, q_1$ of length 4. This contradicts the structure of an induced cycle of length at least 5. Hence, we conclude that Γ_t does not have an induced cycle of length 5 or more or its complement. Therefore Γ_t is a perfect graph. ◀

We now prove that for an interval hypergraph H, $\hat{G}(H)$ is perfect. In this proof, $\mu(H)$ denotes the number of vertices in $\hat{G}$. Note that $\mu(H) = \sum_{I \in \mathcal{I}} |I|$.

Proof of Theorem 5. By property P2 of perfect graphs given in Section 1.1, we know that for each $p > 1$, induced odd cycle C_{2p+1} and its complement denoted by $\overline{C_{2p+1}}$ are forbidden induced subgraphs for perfect graphs. We now show that for an interval hypergraph, the graph $\hat{G}$ is perfect. Our proof is by starting with the hypothesis that the claim is false and deriving a contradiction. Let $H = (\mathcal{V}, \mathcal{J})$ be an interval hypergraph for which $\hat{G}$ is not perfect, and among all such interval hypergraphs, H minimizes $\mu(H)$. Since $\hat{G}$ is not perfect, let us consider a minimal induced subgraph of $\hat{G}$, denoted by, say F for which $\omega(F) \neq \chi(F)$. We claim that for every interval $I \in \mathcal{J}$ such that $|I| > 1$, both the nodes $(I, l(I))$ and $(I, r(I))$ belong to F. The proof of this claim is by contradiction to the fact that H is an interval hypergraph that minimizes $\mu(H)$ and for which $\hat{G}$ is not perfect. Let I be an interval in $\mathcal{J}$ such that $|I| > 1$ and the node $(I, r(I)) \notin V(F)$. Consider the hypergraph $H' = (\mathcal{V}, \mathcal{J}')$ where $\mathcal{J}' = (\mathcal{J} \setminus I) \cup (I \setminus r(I))$. Let $\hat{G}'$ denote the conflict graph of H'. Observe that $V(\hat{G}') = V(\hat{G}) \setminus \{(I, r(I))\}$. Since $(I, r(I)) \notin V(F)$ and $(I, r(I)) \notin V(\hat{G}')$, it follows that F is an induced subgraph of $\hat{G}'$ also. Hence it follows that $\hat{G}'$ is imperfect. Further, $\mu(H') < \mu(H)$. This contradicts the hypothesis that H is the interval hypergraph with minimum $\mu(H)$ for which $\hat{G}$ is imperfect. Therefore, it follows that for each interval $I \in \mathcal{J}$, $(I, r(I))$ is a node in F. An identical argument shows that for each interval $I \in \mathcal{J}$, $(I, l(I))$ is also a node in F. Hence it follows that $\forall I \in \mathcal{J}$ such that $|I| > 1$, both the nodes $(I, l(I))$ and $(I, r(I))$ belong to F. We now consider two exhaustive cases to obtain a contradiction to the known structure of F which we know is either a C_{2p+1} or a $\overline{C_{2p+1}}$ for some $p > 1$.
Case 1- When F is an induced odd cycle $C_j, j \geq 5$: In the following proof we consider different cases, and in each case we conclude that three nodes of C_j form a K_3 in $\hat{G}$. This is a contradiction to the fact that induced cycles of length at least 4 do not have a K_3, and we refer to this as *a contradiction.*

We know that all the intervals I such that $|I| > 1$ have both the nodes $(I, l(I))$ and $(I, r(I))$ in F. Therefore, if (I'', q) is a node such that for some I', q is in interval I' and q is different from $r(I')$ and $l(I')$, then from the definition of E_{edge} and E_{colour}, we know that the 3 nodes $(I', l(I')), (I', r(I')), (I'', q)$ form a K_3, a contradiction. As a consequence of this observation, it also follows that for any two nodes (I_1, q_1) and (I_2, q_2) in C_j for which $|I_1| > 1$ and $|I_2| > 1$, $l(I_1)$ and $l(I_2)$ are different, and $r(I_1)$ and $r(I_2)$ are different. Therefore, for each node (I, q) in C_j, q is either $l(I)$ or $r(I)$ or $|I| = 1$, and q is the left end point (right end point) of at most one interval, and for each interval I', q is not an element of $I' \setminus \{l(I'), r(I')\}$ (we call this set as the strict interior of I').

From the conclusions above, the intervals of length more than 1 contribute an even number of distinct nodes to the cycle C_j. Since C_j is an induced odd cycle, it follows that in C_j there is at least one more node (I'', q) for which $|I''| = 1$. It follows that I'' contains only the point q. Let (I_1, q_1) and (I_2, q_2) be the two neighbours of (I'', q) in C_j. From Observation 6, it follows that q is different from q_1 and q_2. From the analysis above, it follows that q and q_1 are end points of I_1, and q and q_2 are end points of I_2. Again from the conclusions above, since $l(I_1)$ and $l(I_2)$ are different, and since $r(I_1)$ and $r(I_2)$ are different, without loss of generality, let us consider $q = l(I_1) = r(I_2)$ and $l(I_2) = q_2 < q < q_1 = r(I_1)$. Therefore, the three nodes $(I_1, r(I_1)), (I'', q), (I_2, l(I_2))$ form a path in F. We know that $(I_1, l(I_1))$ and $(I_2, r(I_2))$ are also vertices in C_j which is an induced (that is, chordless) cycle. Therefore, $(I_1, l(I_1)), (I_1, r(I_1)), (I'', q), (I_2, l(I_2)), (I_2, r(I_2))$ is a path in C_j. In other words, $(I_1, q), (I_1, q_1), (I'', q), (I_2, q_2), (I_2, q)$ is an induced path of length 5 in C_j, since (I_1, q) and (I_2, q) are not adjacent, by Observation 6. Since C_j is a cycle, it has at least one another node, say (I_3, q_3), which is the second neighbour of $(I_1, l(I_1))$ in C_j. We now show that all the points in I_3 are at least $r(I_1)$, and thus they are all larger than q. By the definition of $E(\hat{G})$ we know that $I_3 \cap I_1 \neq \emptyset$. Further, q_3 is an endpoint of I_3, and q_3 is not in the strict interior of I_1, and since q is in I_1 and I_2, and as per our convention each interval corresponds to a single hyperedge in H, it follows that q_3 is different from q. Consequently, it follows that $l(I_3) = r(I_1)$ and q_3 is $r(I_3)$. Note that this argument includes the case when $|I_3| = 1$, in which case $r(I_1) = q_3$. It follows that I_3 is an interval such that each point in I_3 is at least $r(I_1)$ which is larger than q.

Therefore from the conclusions made thus far, each node in C_j is one of two types: either the hyperedge coordinate is such that all the points in the corresponding interval are at most q or the hyperedge coordinate is such that all the points in the corresponding interval are more than q. In particular, I_2 is such that all the points are at most q and I_3 is such that all points are more than q. Since C_j is an induced cycle, it follows that there are two adjacent nodes (I_l, q_l) and (I_r, q_r) such that all points in I_l are at most q, and all points in I_r are more than q. In other words, I_l and I_r are two disjoint intervals, and we have concluded that (I_l, q_l) and (I_r, q_r) are adjacent. This is a contradiction to the definition of $E(\hat{G}) = E_{colour} \cup E_{edge}$. This contradiction has been arrived at due to the assumption that there is a C_j of odd length at least 5. Hence, in this case our hypothesis that there is a minimal H for which $\hat{G}$ is not perfect is wrong.

Case 2- When F is the complement of an odd cycle, say $\overline{C_j}, j \geq 5$: Here we order the nodes in non-decreasing order of their vertex coordinate. Let the order be $(I_1, p_1), (I_2, p_2), \ldots, (I_j, p_j)$. Since each node is adjacent to exactly $j - 3$ vertices in F, it follows that (I_1, p_1) is not adjacent to (I_{j-1}, p_{j-1}) and (I_j, p_j) in $\hat{G}$. Similarly, (I_j, p_j) is not adjacent to (I_1, p_1) and (I_2, p_2). The reason is that if there is an edge between (I_1, p_1) and (I_{j-1}, p_{j-1}) then one of the two nodes is adjacent to all the nodes whose vertex coordinates are between p_1 and p_{j-1}. Such a node will have degree $j - 2$ which contradicts the fact that all the nodes in F have degree $j-3$. Since the degree of each vertex is $j-3$, it follows that (I_1, p_1) is adjacent to all nodes from (I_2, p_2) to (I_{j-2}, p_{j-2}). Similarly, (I_j, p_j) is adjacent to all nodes from (I_3, p_3) to (I_{j-1}, p_{j-1}). Now, let us consider (I_2, p_2) and (I_{j-1}, p_{j-1}). If these two nodes are adjacent in F, then one of the two will have degree at least $j-2$. Such a case cannot happen. Therefore, the nodes $(I_1, p_1), (I_j, p_j), (I_2, p_2), (I_{j-1}, p_{j-1}), (I_1, p_1)$ forms an induced 4-cycle in $\overline{F}$. $\overline{F}$ is an induced cycle $C_j, j \geq 5$, and by definition does not contain an induced cycle of length 4. Thus our hypothesis that $\hat{G}$ contains F is false.

In either case the assumption of the existence of a minimal H for which $\hat{G}$ is not perfect leads to a contradiction to the known structure of perfect graphs. Hence, it follows that for an interval hypergraph $\hat{G}$ is perfect. ◀

4 Computing the Optimal Co-occurrence Graph of Interval Hypergraphs using Conflict Graphs

We present the algorithm to compute an optimal co-occurrence graph of a given interval hypergraph from a constrained exact hitting set of hyperedge cliques in the conflict graph.

4.1 Representative Function from a Hitting Set of Hyperedge Cliques

In Lemma 8 below, we strengthen Theorem 3 for interval hypergraphs by proving the equality of the chromatic number of the conflict graph and the co-occurrence graph. This plays a crucial role in formulating an LP relaxation to compute a constrained exact hitting set of hyperedge cliques. Let $H = (\mathcal{V}, \mathcal{I})$ be the given interval hypergraph and let $\hat{G}$ be its conflict graph. Let q_{min} be the smallest positive integer such that there exists an exact hitting set of hyperedge cliques that hits every maximal clique in the colour cliques at most q_{min} times. Let $S_{min} \subseteq V(\hat{G})$ be such an exact hitting set of hyperedge cliques that hits every maximal clique in the colour cliques at most q_{min} times. Clearly, $|S_{min}| = m$ since there are m hyperedge cliques, each corresponding to an interval. Define the representative function $t : \mathcal{I} \rightarrow \mathcal{V}$ as follows: $t(I) = u$ if $(I, u) \in S_{min}$. We show that the chromatic number of the co-occurrence graph Γ_t is upper bounded by q_{min}. Note that Theorem 3 proves the opposite inequality for all hypergraphs.

▶ **Lemma 8.** *Let $t : \mathcal{I} \rightarrow \mathcal{V}$ be the function as defined above. Then t is a representative function obtained from some 1-SCF colouring and $\chi(\Gamma_t) \leq q_{min}$.*

Proof. Since S_{min} is an exact hitting set of hyperedge cliques, it follows that for every hyperedge $I \in \mathcal{I}$, there exists exactly one node in S_{min} whose hyperedge coordinate is I. Hence, t is indeed a function. Since every interval is assigned a unique representative by t, it follows from the proof of Theorem 2 that any proper colouring of Γ_t is a 1-SCF colouring of H. Therefore, t is a representative function obtained from such a 1-SCF colouring of H. Now, we show that $\chi(\Gamma_t) \leq q_{min}$. In Theorem 4, we show that Γ_t is a perfect graph. It follows from property P1 of perfect graphs in Section 1.1 that the clique number ω and the chromatic number χ of every induced subgraph of Γ_t are equal. Hence it suffices to show that $\omega(\Gamma_t) \leq q_{min}$. To prove this, we show that $\omega(\Gamma_t)$ is at most the size of the maximum clique in $\hat{G}[S_{min}]$, and by assumption $\omega(\hat{G}[S_{min}]) \leq q_{min}$. In particular, for each clique in Γ_t we identify a clique of the same size in $\hat{G}[S_{min}]$. The proof is by induction on the size of a clique in Γ_t. The base case is for a clique of size 1 in Γ_t. Clearly, there is a clique of size 1 in $\hat{G}[S_{min}]$. By the induction hypothesis, corresponding to a clique comprising of $u_1, u_2, \ldots, u_{q-1}$ in Γ_t, there is a clique containing nodes $(I_1, u_1), (I_2, u_2), \ldots, (I_{q-1}, u_{q-1})$ in $\hat{G}[S_{min}]$. Now, we prove the claim when there are q vertices in a clique in Γ_t. Let $u_1, u_2, \ldots, u_q$ be the set of vertices in the clique. Without loss of generality, assume that $u_1, u_2, \ldots, u_{q-1}, u_q$ is the left to right ordering of points on the line. Since $\{u_1, u_q\} \in E(\Gamma_t)$, there exists an interval, say I' such that u_1 and u_q belong to I' and $t(I') \in \{u_1, u_q\}$. We prove the claim for the case when $t(I')$ is u_1. It follows that the node $(I', u_1) \in S_{min}$. Since both u_1 and u_q belong to the interval I', it follows that $u_2, \ldots, u_{q-1}$ also belong to interval I'. By the induction hypothesis, for the $q-1$ sized clique $u_2, u_3, \ldots, u_q$ in Γ_t, there is a clique containing the nodes $(I_2, u_2), (I_3, u_3), \ldots, (I_q, u_q)$ in $\hat{G}[S_{min}]$. Therefore, it follows that (I', u_1) is adjacent to all the nodes $(I_2, u_2), (I_3, u_3), \ldots, (I_q, u_q)$ in $\hat{G}$. It follows that corresponding to the clique $u_1, \ldots, u_q$ in Γ_t, there is a clique $(I', u_1), (I_2, u_2), \ldots, (I_q, u_q)$ in $\hat{G}[S_{min}]$. In case $t(I')$ is u_q, an identical argument is applied to the clique $u_1, \ldots, u_{q-1}$ in Γ_t to prove the claim. Hence the lemma is proved. ◀

We next show that finding a 1-SCF colouring using minimum colours is equivalent to finding an exact hitting set of hyperedge cliques such that colour cliques are hit as few times as possible.

▶ **Lemma 9.** *There exists a set $S \subseteq V(\hat{G})$ such that for each $Q \in \mathcal{Q}_1$, $|S \cap Q| = 1$ and for each $Q' \in \mathcal{Q}_2$, $|S \cap Q'| \leq q$ if and only if there is a 1-SCF colouring of H with q colours.*

Proof. Let S be a subset of $V(\hat{G})$ such that for each $Q \in \mathcal{Q}_1$, $|S \cap Q| = 1$ and for each $Q' \in \mathcal{Q}_2$, $|S \cap Q'| \leq q$. Then by Lemma 8, there exists a representative function t such that $\chi(\Gamma_t) \leq q$. It further follows from Theorem 2 that a proper colouring of Γ_t is a 1-SCF colouring of H using $\chi(\Gamma_t) \leq q$ colours. This completes the forward direction of the claim. Next we prove the reverse direction. Let C be a 1-SCF colouring of H using q colours. Then by Theorem 2, C gives a representative function t with the property $q \geq \chi(\Gamma_t)$. Since co-occurrence graphs are perfect by Theorem 4, it follows that $q \geq \chi(\Gamma_t) = \omega(\Gamma_t)$. Define $S \triangleq \{(I, u) \mid I \in \mathcal{I}, t(I) = u\}$. By Theorem 3, the set S is an exact hitting set of cliques in $\mathcal{Q}_1$ and $\omega(\hat{G}[S]) \leq \omega(\Gamma_t) \leq q$. Thus we conclude that if there is a 1-SCF colouring of H using q colours, then there exists an exact hitting set of cliques in $\mathcal{Q}_1$ that intersects every maximal clique in $\mathcal{Q}_2$ at most q times. ◀

4.2 Linear Program for Exact Hitting Sets of Hyperedge Cliques

From Lemma 9, it is clear that an optimal 1-SCF colouring of an interval hypergraph H can be found by computing an exact hitting set of hyperedge cliques of $\hat{G}(H)$ such that each colour clique is hit as few times as possible. We present an LP formulation for this exact hitting set problem. In this LP, there is one variable corresponding to each node of $\hat{G}$ and integer valued variable q. Define $X \triangleq \{x_{I,u} \mid (I, u) \in \hat{G}\}$ to be the set of variables in the LP, where

$$x_{I,u} = \begin{cases} 1, & \text{if node } (I,u) \text{ hits hyperedge clique corresponding to } I \\ 0, & \text{otherwise} \end{cases}$$

LP Formulation. Find values to variables $\{x_{I,u} \mid u \in I, I \in \mathcal{I}\}$ subject to

(P.1)

$$\sum_{u \in I} x_{I,u} = 1, \forall I \in \mathcal{I} \tag{1}$$

$$\sum_{(I,u) \in Q} x_{I,u} \leq q, \text{ for each maximal clique } Q \text{ in } \mathcal{Q}_2. \tag{2}$$

$$x_{I,u} \leq 1, q \geq 0$$

The LP relaxation has a set of equations, which are given in (P.1):(1) and a set of inequalities, which are given in (P.1):(2). Logically, an equation corresponds to choosing exactly one vertex per interval; that is, each equation corresponds to choosing exactly one node from one hyperedge clique. On the other hand, an inequality corresponds to a maximal clique in the set of colour cliques. Logically, the inequality means that we pick at most q nodes from every maximal clique in $\mathcal{Q}_2$. Together, any integral solution to the LP relaxation is an exact hitting set of hyperedge cliques such that each maximal clique in the set of colour cliques is hit at most q times. This LP relaxation is solved using the ellipsoid method which uses a polynomial time separation oracle that we next design. Let x denote an optimum solution to the LP relaxation. In Section 4.4 we present a rounding technique that converts the fractional solution x to a feasible integer solution for the LP in polynomial time.

4.3 Separation Oracle based LP Algorithm `SPAlg`

A separation oracle is a polyomial time algorithm that given a point in $\mathbb{R}^d$, where d is the number of variables in a linear program relaxation, either confirms that this point is a feasible solution, or produces a violated constraint (See Section 12.3.1 in[18]). A polynomial time separation oracle is used by the ellipsoid method to give a P-time algorithm for finding a feasible solution to the LP. In this section, we describe a polynomial time separation oracle `SPMaxWtClique` for our LP for a fixed positive integer value q. For each $(I, v) \in V(\hat{G})$, let $x_{I,v}$ be a real value assigned to the corresponding variable in the LP relaxation. Given this as an input, for a fixed positive integer value q, the separation oracle `SPMaxWtClique` considers the vertex-weighted graph $\hat{G}^w$ corresponding to $\hat{G}$, where the weight of node (I, v) is $x_{I,v}$ for all $(I, v) \in V(\hat{G})$. The oracle then computes the maximum weight clique of $\hat{G}^w$. If the weight of the maximum weight clique of $\hat{G}^w$ exceeds q, then it follows that there is some maximal clique Q' whose weight is more than q. This implies that the given point violates the inequality corresponding to Q', and this inequality is returned by the oracle as the violated inequality. If the weight of the maximum weight clique is at most q, then the oracle checks if all equalities hold. If any equality is violated, then we have found a violated constraint, and the oracle returns the appropriate inequality as the violated inequality. If all the constraints are satisfied, then the oracle reports that the given point is feasible. This completes the description of the separation oracle `SPMaxWtClique`. We show in Lemma 10 that `SPMaxWtClique` runs in polynomial time.

▶ **Lemma 10.** *For an input interval hypergraph and for each integer value $q \geq 0$ the separation oracle `SPMaxWtClique` runs in polynomial time.*

Proof. For an interval hypergraph we know that $\hat{G}$ is perfect by Theorem 5. From property P4 of perfect graphs listed in Section 1.1, it is known that the maximum weighted clique problem in perfect graphs can be solved in polynomial time. Thus we can find the maximum weighted clique in the graph vertex-weighted graph $\hat{G}^w$. Thus, finding an inequality in the LP corresponding to a maximal clique whose weight exceeds q can be done in polynomial time. Also, since there are only a polynomial number of hyperedge cliques, it follows that the check of whether there is a violated equation can also be done in polynomial time. It follows that `SPMaxWtClique` runs in polynomial time. ◀

Let $\mathcal{B}$ be an instance of the given LP. Algorithm `SPAlg` takes as inputs the LP instance $\mathcal{B}$. It uses the separation oracle `SPMaxWtClique` and iterates over integer values of q and outputs a solution x and a value q that satisfies the system of equations and inequalities. Otherwise, it reports that the system is infeasible. Let q_{min} be the smallest value of q for which Algorithm

SPAlg finds a feasible solution of the instance $\mathcal{B}$ and let B_{opt} be the solution returned by Algorithm SPAlg for the integer q_{min}. If B_{opt} is an integral solution, then we have an integer solution in polynomial time. If B_{opt} is not integral, then we present steps in Section 4.4 to round the fractional values in B_{opt} that results in a feasible integral solution for the value q_{min}.

▶ **Lemma 11.** *For an input interval hypergraph, the algorithm* SPAlg *runs in polynomial time.*

Proof. We have shown in Lemma 10 that the separation oracle in SPMaxWtClique runs in polynomial time. Since there is a polynomial time separation oracle, using the ellipsoid method, a feasible solution in the polytope of $\mathcal{B}$ can be found in polynomial time for each q. The number of values of q is at most the number of points in the interval hypergraph H. This is because, from Observation 6, for each vertex $u \in \mathcal{V}$ each clique in $\hat{G}$ can contain at most one node whose vertex coordinate is u. Hence the lemma is proved. ◀

4.4 Rounding the LP solution

RoundingFrac is a recursive function which takes as input a fractional feasible solution of the LP $\mathcal{B}$ for the value q_{min} on the intervals $\mathcal{I}$ and returns a feasible integer solution for $\mathcal{B}$ for the value q_{min}.
In every iteration of the *while* loop in Algorithm 1, at least one variable in X is rounded to

Algorithm 1 RoundingFrac($B_{opt}, \mathcal{I}'$).

Output: Fractional solution B_{opt} rounded to integer solution B_{optI}

$i \leftarrow 0$;
$B_{opt}(0) \leftarrow B_{opt}$;
while $\exists x_{I,v} \in B_{opt}(i)$ *that does not belong to* $\{0,1\}$ **do**
  $i \leftarrow i + 1$;
  $B_{opt}(i) \leftarrow B_{opt}(i-1)$;
  $I_i \leftarrow$ Longest Interval in $\mathcal{I}'$ with the smallest left endpoint ;
  $r \leftarrow r(I_i)$;
  $r - 1 \leftarrow$ vertex to the immediate left of $r(I_i)$ on the line ;
  for *each interval* I' *that contains* r *and* $r-1$ **do**
    $x_{I',r-1} \leftarrow x_{I',r-1} + x_{I_i,r}$;
    $x_{I',r} \leftarrow x_{I',r} - x_{I_i,r}$;
    Modify entries in $B_{opt}(i)$ corresponding to the values changed above ;
  end
  $\mathcal{I}' \leftarrow \mathcal{I}' \setminus I_i$;
  $I_i \leftarrow I_i \setminus r$;   ▷ *Remove right end point of* I_i;
  $\mathcal{I}' \leftarrow \mathcal{I}' \cup I_i$;
end
$B_{optI} \leftarrow B_{opt}(i)$;
return B_{optI} ;

an integer value. In iteration i, let I_i be the interval with the smallest left end point among all intervals of maximum length. Let $l(I_i)$ and $r(I_i)$ denote the left and right endpoints of interval I_i respectively. Since $r(I_i)$ is removed during iteration i, it follows that the total number of points (in all the intervals) in iteration $i+1$ is one less than the total number of points in iteration i. Hence the conflict graph corresponding to intervals in iteration $i+1$ has strictly fewer number of nodes than the conflict graph corresponding to intervals in iteration i. In Lemma 13, we show that for every $i \geq 0$, the solution $B_{opt}(i)$ is feasible for the linear program $\mathcal{B}$ for the value q_{min}. We show in Lemma 12 that for some positive integer j, $B_{opt}(j)$ will be an all integer solution for $\mathcal{B}$, at which time algorithm exits.

▶ **Lemma 12.** *Let B_{opt} be a fractional feasible solution returned by* `SPAlg`$(\mathcal{B}, q_{min})$*. Then,* `RoundingFrac` *returns an integer solution for $\mathcal{B}$ on the input B_{opt} in a polynomial number of steps.*

Proof. From the description of `RoundingFrac`, in each iteration i, $x_{I_i,r(I_i)}$ becomes zero and the variable $x_{I_i,r(I_i)}$ does not become non-zero in any subsequent iteration. Then the number of variables whose value is not 0 or 1 reduces in each iteration. Further, the rounding is such that if a variable $x_{I,r}$ is reduced by a certain value then $x_{I,r-1}$ is increased by the exact same value. This ensures that after each iteration the equations in (P.1):(1) are all satisfied, and in particular they add up to 1. Therefore, eventually in each equation there will be a variable which is 1 and all others are 0. Further, it follows from Lemma 13 that the inequalities in (P.1):(2) also hold after each iteration. It follows that the resulting values are indeed a solution of the given LP and will be integral in at most $\mu(H)$ iterations, where $\mu(H)$ is the number of nodes in $\hat{G}$. ◀

Let B_{optI} be the integer solution returned by `RoundingFrac`. We show in Lemma 13 that B_{optI} is feasible for the instance $\mathcal{B}$ for the value q_{min}. In other words, the values to the variables in each inequality corresponding to the colour cliques add up to at most the same value as it was adding up to in B_{opt}. We show that the solution returned on a smaller instance after every iteration is feasible for $\mathcal{B}$ for the value q_{min}. In the proof of Lemma 13 below, we use r to denote $r(I_i)$, where I_i is the longest interval with the smallest left endpoint in iteration i. Similarly, denote the point to the immediate left of r on the number line by $r-1$. For every other interval I', denote its right endpoint and the point immediately to the left of the right endpoint by $r(I')$ and $r(I')-1$, respectively.

▶ **Lemma 13.** *Let B_{opt} be a fractional feasible solution returned by* `SPAlg`$(\mathcal{B}, q_{min})$*. The solution B_{optI} returned by* `RoundingFrac` *is a feasible solution for the LP instance $\mathcal{B}$ for the value q_{min}.*

Proof. The proof of correctness is by induction on the iteration number. We know that B_{opt} is feasible for $\mathcal{B}$ for the value q_{min}. Let us assume that for an integer $i \geq 0$ $B_{opt}(i-1)$ is feasible for $\mathcal{B}$ for the value q_{min}. We show that $B_{opt}(i)$ is also feasible for $\mathcal{B}$ for the value q_{min}. From the description of the `RoundingFrac`, during iteration i, the value which is subtracted from one variable from $x_{I,r}$ is added to the variable $x_{I,r-1}$. This fact is crucial in the analysis below. Hence all equations in (P.1):(1) are satisfied by $B_{opt}(i)$. Now, we show that the inequalities in (P.1):(2) corresponding to the maximal cliques are also satisfied by $B_{opt}(i)$. Let I' be an interval that contains the point $r-1$ such that $x_{I',r-1}$ has increased due to step 10 in Algorithm 1. By the choice of I' for which $x_{I',r-1}$ is increased, it follows that $x_{I',r}$ is reduced and thus I' contains the point r. It follows from the definition of the edge set E_{colour} that there is an edge between $(I', r-1)$ and (I_i, r) in $\hat{G}$.

Let Q be a maximal clique that contains the node $(I', r-1)$. By Observation 6, all nodes with the same vertex coordinate form an independent set. Hence Q does not contain any node of the form $(I'', r-1)$, where $I'' \neq I'$. Further, for any clique Q, there is at most one node whose value increases. If Q contains the node (I', r), then $x_{I',r}$ has reduced and hence the inequality corresponding to Q is satisfied under $B_{opt}(i)$. If Q does not contain the node

(I', r) we now show that it must contain a node whose vertex coordinate is r. To prove this, among all nodes in Q, consider two nodes - one for which the vertex coordinate is leftmost and another for which the vertex coordinate is the rightmost on the line. We denote the leftmost coordinate by λ and the rightmost coordinate by ρ. Let (J, λ) and (J', ρ) be two nodes in Q.

First, we show that $\lambda \geq l(I_i)$. The proof is by contradiction. Suppose $\lambda < l(I_i)$. Due to the edge between nodes (J, λ) and $(I', r-1)$ in Q, it is clear that either J or I' contains both λ and $r-1$. Without loss of generality, assume that J contains both λ and $r-1$. Since by our assumption $\lambda < l(I_i)$, it follows that J is at least as long as I_i and $l(J) < l(I_i)$. This is a contradiction to our choice of I_i being the longest interval with the smallest left endpoint. It follows that $\lambda \geq l(I_i)$. We show using the following cases that the inequality corresponding to Q is still feasible.

1. Case $\rho < r-1$. We show that this case is not possible. Since $(I', r-1)$ belongs to Q, and ρ is the rightmost vertex coordinate among all nodes in Q, it follows that $\rho \geq r-1$.
2. Case $\rho = r-1$. Since $\lambda \geq l(I_i)$ and $\rho = r-1$, it follows that all points from λ to ρ belong to I_i. Therefore, by the definition of the edges of $\hat{G}$, (I_i, r) is adjacent to all the nodes of Q. This contradicts the premise that Q is a maximal clique that does not contain (I_i, r). Therefore $\rho = r-1$ is not possible.
3. Case $\rho = r$. Since (J', ρ), which is the same as (J', r) belongs to Q, it follows that the inequality corresponding to Q is still feasible. Since the decrease in $x_{J',r}$ is exactly the same as the increase in $x_{I_i,r-1}$.
4. Case $\rho > r$. Observe that there is an edge between nodes (J, λ) and (J', ρ) since they are both in Q. It follows that either J or J' both contain λ and ρ. Without loss of generality, let J be this interval. Since J contains all the points on the line from λ to ρ, both included, it follows that the interval J contains both points r and $r-1$. Further, by the definition of the graph $\hat{G}$, it follows that (J, r) is adjacent to all the nodes in Q whose vertex coordinates which are different from r and lie between λ and ρ, both included. Also, since there can be at most one node in a maximal clique with any particular vertex coordinate, and since Q is a maximal clique, it follows that either (J, r) belongs to Q or that Q contains a node (J'', r) where $J \neq J''$. Since $x_{I_i,r}$ is reduced in iteration i, follows that $x_{J,r}$ and $x_{J'',r}$ are also reduced. Therefore, in the maximal clique Q the increase in $x_{I',r-1}$ is compensated by a decrease in $x_{J,r}$ or $x_{J'',r}$ whichever is present in Q. Therefore, the inequality corresponding to Q is satisfied in by $B_{opt}(i)$.

Therefore, in all the cases we have concluded the $B_{opt}(i)$ satsfies $\mathcal{B}$. This completes the proof by induction. ◀

We show in Theorem 1 that the 1-SCF colouring problem in interval hypergraphs can be solved in polynomial time. Finally, we prove the main result in this paper.

Proof of Theorem 1. By Lemma 11 the LP returns a feasible solution in polynomial time using the separation oracle `SPMaxWtClique`. By Lemmas 12 and 13, a feasible integer solution can be obtained from the fractional feasible solution in polynomial time. Further, the representative function t and thereof, the co-occurrence graph Γ_t can also be obtained in polynomial time. By Theorem 4, the co-occurrence graph Γ_t is perfect. Since a proper colouring of a perfect graph can be found in polynomial time, it follows from Theorem 2 that an optimal 1-SCF colouring of an interval hypergraph can be found in polynomial time. ◀

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