Latticepathology and Symmetric Functions (Extended Abstract)

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Abstract

In this article, we revisit and extend a list of formulas based on lattice path surgery: cut-and-paste methods, factorizations, the kernel method, etc. For this purpose, we focus on the natural model of directed lattice paths (also called generalized Dyck paths). We introduce the notion of prime walks, which appear to be the key structure to get natural decompositions of excursions, meanders, bridges, directly leading to the associated context-free grammars. This allows us to give bijective proofs of bivariate versions of Spitzer/Sparre Andersen/Wiener–Hopf formulas, thus capturing joint distributions. We also show that each of the fundamental families of symmetric polynomials corresponds to a lattice path generating function, and that these symmetric polynomials are accordingly needed to express the asymptotic enumeration of these paths and some parameters of limit laws. En passant, we give two other small results which have their own interest for folklore conjectures of lattice paths (non-analyticity of the small roots in the kernel method, and universal positivity of the variability condition occurring in many Gaussian limit law schemes).

2012 ACM Subject Classification Mathematics of computing → Generating functions; Mathematics of computing → Distribution functions; Theory of computation → Random walks and Markov chains; Theory of computation → Grammars and context-free languages

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1 Introduction and definitions

The recursive nature of lattice paths makes them amenable to context-free grammar techniques; their geometric nature makes them amenable to cut-and-paste bijections; their step-by-step nature makes them amenable to functional equations solvable by the kernel method (see e.g. [3–5, 8–11, 16, 30, 32, 35] for many applications of these ideas). We present in a unified way some consequences of these observations in Section 2 on context-free grammars (where we introduce the fruitful notion of prime walks) and in Section 3 on Spitzer and Wiener–Hopf identities. Additionally, we give new connections with symmetric functions in Section 4, see Table 2. All of this allows us to greatly extend the enumerative formulas and asymptotics given in [4], and gives us access to some limit laws, as shown in Section 5.

▶ **Definition 1** (Jumps and lattice paths). *A step set $\mathcal{S}$ is a finite subset of $\mathbb{Z}$. The elements of $\mathcal{S}$ are called steps or jumps. An n-step lattice path or walk ω is a sequence $(j_1, \ldots, j_n) \in \mathcal{S}^n$. The length $|\omega|$ of this lattice path is its number n of jumps.*

Such sequences are one-dimensional objects. Geometrically, they can be interpreted as two-dimensional objects which justifies the name *lattice path*. Indeed, $(j_1, \ldots, j_n)$ may be seen as a sequence of points $(\omega_0, \omega_1, \ldots, \omega_n)$, where ω_0 is the starting point and $\omega_i - \omega_{i-1} = (1, j_i)$ for $i = 1, \ldots, n$. Except when mentioned differently, the starting point ω_0 of these lattice paths is $(0, 0)$.

Let $\sigma_k := \sum_{i=1}^{k} j_i$ be the partial sum of the first k steps of the walk ω. We define the *height* or *maximum of* ω as $\max_k \sigma_k$, and the *final altitude of* ω as σ_n. For example, the first walk in Table 1 has height 3 and final altitude 1. Table 1 and Figure 1 are also illustrating the four following classical types of paths:

▶ **Definition 2** (Excursions, arches, meanders, bridges).

- *Excursions are paths never going below the x-axis and ending on the x-axis;*
- *Arches are excursions that only touch the x-axis twice: at the beginning and at the end;*
- *Meanders are prefixes of excursions, i.e., paths never going below the x-axis;*
- *Bridges are paths ending on the x-axis (allowed to cross the x-axis any number of times).*

Let $c := -\min \mathcal{S}$ be the maximal negative step, and let $d := \max \mathcal{S}$ be the maximal positive step. To avoid trivial cases we assume $\min \mathcal{S} < 0 < \max \mathcal{S}$. Furthermore we associate to each step $i \in \mathcal{S}$ a weight s_i. These weights s_i are typically real numbers, like probabilities or non-negative integers encoding the multiplicity of each jump. The weight of a lattice path is the product of the weights of its steps. Then we associate to this set of steps the following *step polynomial*:

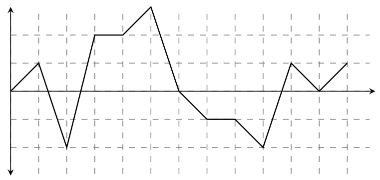
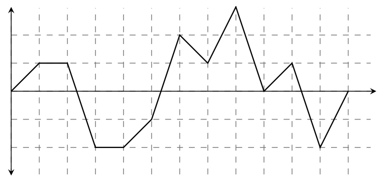
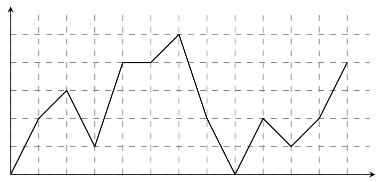
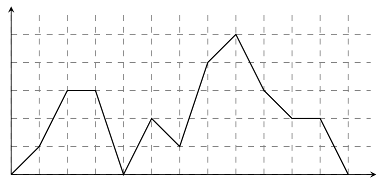
$$S(u) = \sum_{i=-c}^{d} s_i u^i.$$

The generating functions of directed lattice paths can be expressed in terms of the roots of the *kernel equation*

$$1 - zS(u) = 0. \tag{1}$$

More precisely, this equation has $c + d$ solutions in u. The *small roots* $u_i(z)$, for $i = 1, \ldots, c$, are the c solutions with the property $u_i(z) \sim 0$ for $z \sim 0$. The remaining d solutions are called *large roots* as they satisfy $|v_i(z)| \sim +\infty$ for $z \sim 0$. The generating functions of the four classical types of lattice paths introduced above are shown in Table 1.

Table 1 The four types of paths: walks, bridges, meanders and excursions, and the corresponding generating functions for directed lattice paths. The functions $u_i(z)$ for $i = 1, \dots, c$ are the roots of the kernel equation $1 - zS(u) = 0$ such that $\lim_{z=0} u_i(z) = 0$.

	ending anywhere	ending at 0
unconstrained (on $\mathbb{Z}$)	 walk/path ($\mathcal{W}$) $W(z) = \frac{1}{1-zS(1)}$	 bridge ($\mathcal{B}$) $B(z) = z \sum_{i=1}^{c} \frac{u_i'(z)}{u_i(z)}$
constrained (on $\mathbb{Z}$)	 meander ($\mathcal{M}$) $M(z) = \frac{1}{1-zS(1)} \prod_{i=1}^{c} (1 - u_i(z))$	 excursion ($\mathcal{E}$) $E(z) = \frac{(-1)^{c-1}}{s_{-c}z} \prod_{i=1}^{c} u_i(z)$

These results follow from the expression for the bivariate generating function $M(z,u)$ of meanders. Indeed, let $m_{n,k}$ be the number of meanders of length n going from altitude 0 to altitude k, then we have

$$M(z,u) = \sum_k M_k(z)u^k = \sum_{n,k \geqslant 0} m_{n,k} z^n u^k = \frac{\prod_{i=1}^{c}(u - u_i(z))}{u^c(1 - zS(u))}. \tag{2}$$

This last formula is obtained by the kernel method: this method starts with the functional equation which mimics the recursive definition of meanders, namely $M(z,u) = 1 + zS(u)M(z,u) - \{u^{<0}\}zS(u)M(z,u)$ (where $\{u^{<0}\}$ extracts the monomials of negative degree in u, as one does not want to allow a jump going below the x-axis). Note that $\{u^{<0}\}S(u)M(z,u)$ is a linear combination (with coefficients in u and z) of c unknowns, namely $M_0(z), \dots, M_{c-1}(z)$. Then, substituting $u = u_i(z)$ (each of the c small roots of (1)) into this system leads to the closed form (2). This also directly gives the generating function of excursions $E(z) := M(z,0)$ and meanders $M(z) := M(z,1)$. The generating function for bridges follows from the link given in Theorem 8 hereafter. See [4, 10] for more details.

It should be stressed that the closed forms of Table 1 grant easy access to the asymptotics of all these classes of paths after the localization of the dominant singularities:

▶ **Theorem 3** (Radius of convergence of excursions, bridges, and meanders [4])**.** *The radius of convergence of excursions $E(z) := M(z,0)$ and of bridges $B(z)$ is given by $\rho = 1/S(\tau)$, where τ is the smallest positive real number such that $S'(\tau) = 0$. For meanders $M(z) := M(z,1)$, the radius depends on the drift $\delta := S'(1)$: It is ρ if $\delta < 0$ and it is $1/S(1)$ if $\delta \geqslant 0$.*

We shall make use of all these facts in Section 5 on asympotitcs and limit laws, but, before to do so, we now present several combinatorial decompositions which will be the key to get these new asymptotic results.

2 Prime walks and context-free grammars

Context-free grammars are a powerful tool to tackle problems related to directed lattice paths (we refer to [27] for a detailed presentation of grammar techniques). In this section, we introduce some key families of lattice paths (generalized arches, prime walks), which will also be used in the next section. Illustrating the philosophy of "latticepathology", these new families allow short concise visual proofs based on lattice path surgery: we give grammars generating the most fundamental classes of lattice paths (excursions, bridges, meanders); this generalizes and unifies results from [11, 16, 32, 35].

All our grammars are non-ambiguous: there is only one way to generate each lattice path. They require the introduction of two classes of paths: *generalized arches* and *prime walks*.

▶ **Definition 4** (Generalized arches). *An arch from i to j is a walk starting at altitude i ending at altitude j and staying always strictly above altitude $\max(i, j)$ except for its first and final position; see Figure 1.*

An important consequence of this definition is that generalized arches cannot have an excursion as left or right factor. Note that an arch from i to j can be considered as an arch from 0 to $j - i$. This justifies that we now focus on arches starting at 0. Let $\mathcal{A}_k$ be the class of arches from 0 to k; see Figure 1. Following the tradition of several authors, we refer to *arches* (omitting the start and end point) as arches from 0 to 0, see e.g. [4]. Thus, an excursion is clearly a sequence of arches.

▶ **Definition 5** (Prime walks). *Given a set of steps $\mathcal{S}$, with $d = \max \mathcal{S}$, the set $\mathcal{P}$ of prime walks is defined as the following sets of arches*

$$\mathcal{P} = \bigcup_{k=0}^{d} \mathcal{A}_k.$$

These prime walks are the key to get short proofs for the decomposition of several constrained classes of paths (Section 3) and for meanders (Theorem 6). Note that these decompositions hold for any set of jumps: it is straightforward to extend them to multiplicities (jumps with different colours) or even to an infinite set of jumps.

▶ **Theorem 6** (The universal context-free grammar for directed lattice paths). *Meanders and excursions are generated by the following grammar:*

$$\begin{aligned} \mathcal{M} &\to \varepsilon + \mathcal{P}\,\mathcal{M} && \textit{(meanders)},\\ \mathcal{E} &\to \varepsilon + \mathcal{A}_0\mathcal{E} && \textit{(excursions)}, \end{aligned}$$

which can be rephrased as "meanders are sequences of prime walks": $\mathcal{M} = \mathrm{Seq}\left(\sum_{k=0}^{d} \mathcal{A}_k\right)$ and "excursions are sequences of arches": $\mathcal{E} = \mathrm{Seq}(\mathcal{A}_0)$, where the arches $\mathcal{A}_k$ from 0 to k are generated by

$$\begin{aligned} \mathcal{A}_k &\to k + \sum_{j=k+1}^{d} \mathcal{A}_j\,\mathcal{E}\,\mathcal{A}_{k-j} && (\textit{arches for } k \geqslant 0),\\ \mathcal{A}_k &\to k + \sum_{j=-c}^{k-1} \mathcal{A}_{k-j}\,\mathcal{E}\,\mathcal{A}_j && (\textit{arches for } k < 0), \end{aligned}$$

with the convention that, in these two rules, the part $\mathcal{A}_k \to k$ is omitted whenever $k \notin \mathcal{S}$.

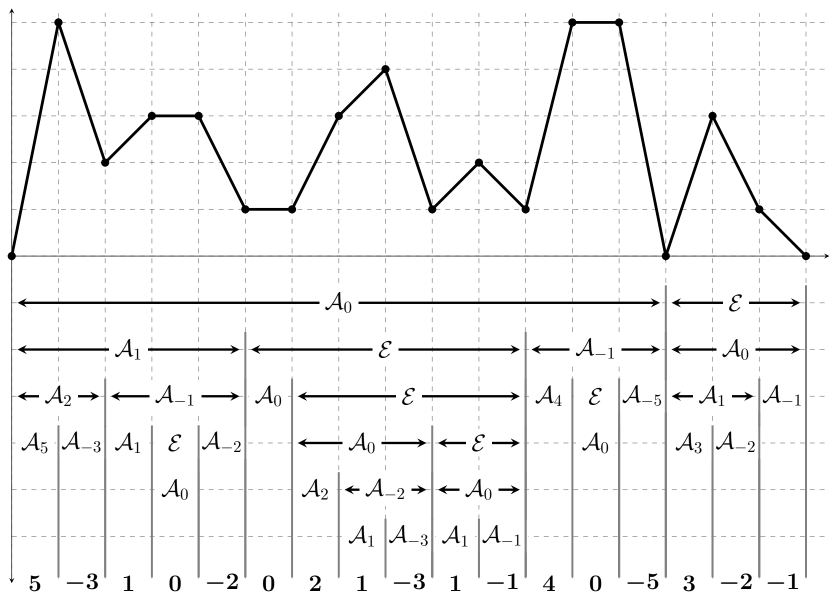


Figure 1 Example of our non-ambiguous decomposition of an excursion into generalized arches. Similar decompositions hold for the factorization of meanders into prime walks.

Proof. Let us start with arches $\mathcal{A}_k$ from 0 to $k \geqslant 0$. (The results for $\mathcal{A}_{-k}$ follow analogously.) For such arches of length > 1, we cut them at the first and the last time their minimal altitude (not taking end points into account) is attained. The first factor goes from 0 to j and stays in-between always strictly above j, and therefore is given by $\mathcal{A}_j$. The second factor is a (possibly empty) excursion. The last factor is an arch from j to k given by $\mathcal{A}_{k-j}$. This gives $\mathcal{A}_k = \mathcal{A}_j \mathcal{E} \mathcal{A}_{k-j}$. From this, it is immediate to get the grammar for excursions, as they are a sequence of arches $\mathcal{A}_0$; thus $\mathcal{E} = \varepsilon + \mathcal{A}_0 \mathcal{E}$.

Now take any meander and cut it at the last time it touches altitude 0. The first part is a (possibly empty) sequence of arches. We cut the second part at the first point where its minimal altitude > 0 is attained. The remaining part is again a meander. This gives the factorization $\mathcal{M} = \mathcal{E} + \sum_{k=1}^{d} \mathcal{E}\, \mathcal{A}_k\, \mathcal{M}$, which is in turn equivalent to $\mathcal{M} = \operatorname{seq}(\mathcal{P})$.

All these decompositions are clearly 1-to-1 correspondences, as exemplified in Figure 1. ◀

We end this section with the grammar of bridges. It uses another class of walks: the negative arches from 0 to k, denoted by $\bar{\mathcal{A}}_k$. These stay always strictly below $\min(0, k)$. Their grammar is just the mirror of the one for $\mathcal{A}_k$ given in Theorem 6.

▶ **Theorem 7.** *Bridges $\mathcal{B} = \mathcal{B}_0$ are generated by the following grammar:*

$$\mathcal{B}_0 \quad \rightarrow \quad \varepsilon + \sum_{k \in \mathcal{S}} k \mathcal{B}_{-k},$$

where $\mathcal{B}_k$ stands for the "bridges ending at k", i.e. walks on $\mathbb{Z}$ from 0 to k, given by

$$\mathcal{B}_k \quad \rightarrow \quad \sum_{j=-c}^{0} \mathcal{A}_j\, \mathcal{B}_{k-j} \qquad (\textit{if } k > 0),$$

$$\mathcal{B}_k \quad \rightarrow \quad \sum_{j=0}^{d} \bar{\mathcal{A}}_j\, \mathcal{B}_{k-j} \qquad (\textit{if } k < 0).$$

In the next section we present some applications of our decompositions (obtained above in the framework of the non-commutative world of words) to famous identities from probability theory (stated below in the framework of the commutative world of generating functions).

3 Latticepathology and surgery of paths

The decompositions of lattice paths mentioned in the previous section find application in the bivariate versions of the Spitzer/Sparre Andersen[1]/Wiener–Hopf formulas [2,25,26,34,37]. It gives for free elegant short proofs for these fundamental results which were definitively missing in [4], neatly illustrating the latticepathology philosophy!

▶ **Theorem 8** (Bivariate version of Spitzer/Sparre Andersen's identities). *The generating function $W^+(z,u) = \sum_n w_n^+(u)z^n$ of walks on $\mathbb{Z}$ ending at an altitude $\geqslant 0$ and the generating function $M(z,u) = \sum_n m_n(u)z^n$ of meanders (where u encodes the final altitude and z encodes the length of the lattice path) are related by the formulas*

$$W^+(z,u) = 1 + z\frac{M'(z,u)}{M(z,u)} \qquad \text{or, equivalently,} \tag{3a}$$

$$M(z,u) = \exp\left(\int_0^z \frac{W^+(t,u)-1}{t}\,dt\right) = \exp\left(\sum_{n\geqslant 1} \frac{w_n^+(u)}{n} t^n\right). \tag{3b}$$

Proof (Sketch). We give a bijective proof. It consists in factorizing any non-empty walk ω ending at an altitude $\geqslant 0$ into 3 factors: $\omega = \phi_1.m.\phi_2$ where m is the longest meander starting at the first minimum of the walk and such that $\phi_2.\phi_1$ is a prime walk (pointed, in order to remember where to split it); see Figure 2. The fact that this factorization exists and is unique follows from the positivity of ω and from the grammar for meanders from Theorem 6. This decomposition directly keeps track of the last altitude of each of its factors:

$$W^+(z,u) - 1 = M(z,u)z\frac{\partial}{\partial z}\left(1 - \frac{1}{M(z,u)}\right). \qquad ◀$$

▶ Remark 9 (Spitzer/Sparre Andersen's identities for excursions and bridges). Extracting the constant coefficient with respect to u in the above identities leads to the following links between bridges and excursions (these specific identities were also proven in [4]).

$$B(z) = 1 + E(z)z\frac{\partial}{\partial z}\left(1 - \frac{1}{E(z)}\right) = 1 + z\frac{E'(z)}{E(z)} \qquad \text{or, equivalently,} \tag{4a}$$

$$E(z) = \exp\left(\int_0^z \frac{B(t)-1}{t}\,dt\right) = \exp\left(\sum_{n\geqslant 1} \frac{b_n}{n} t^n\right). \tag{4b}$$

Nota bene: Spitzer's formula is often given as a variant of Formula (3b), stated in terms of characteristic functions instead of generating functions, and also keeping track of the height of the path (see e.g. [37,39,42]). More generally, in Brownian motion theory, path decompositions are also useful for Vervaat transformations, quantile transforms [13,33,40], Ray–Knight theorems for local times and Lamperti, Jeulin, Bougerol, Donati-Martin identities [1,7,15,28].

We now illustrate such approaches with one more important surgery of lattice paths. (This requires the natural classes of positive and negative meanders, see Definition 12 hereafter.)

[1] Funnily, in the literature, this identity of Erik Albrecht Sparre Andersen (Andersen is the family name) is often called the "Sparre Andersen identity", probably as he was often signing E. Sparre Andersen.

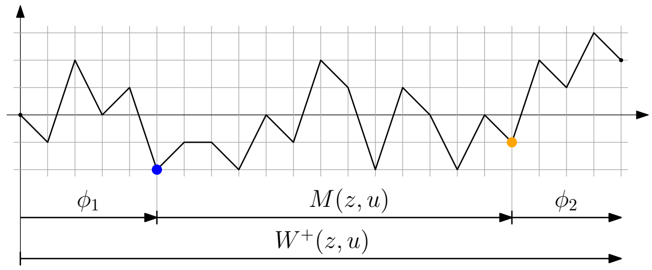


Figure 2 The bijection at the heart of Spitzer/Sparre Andersen identity decomposes a walk $\omega \in \mathcal{W}^+$ into $\omega = \phi_1.m.\phi_2$, where the meander $m \in \mathcal{M}$ starts at the first minimum of ω and ends at the rightmost point such that $\phi_2.\phi_1$ ends at altitude $\geqslant 0$ (and $\phi_2.\phi_1$ is thus a prime walk).

▶ **Theorem 10** (Bivariate version of Wiener–Hopf formula). *The bivariate generating functions $W_{+h}(z,u)$ and $W_{-h}(z,u)$ of walks on $\mathbb{Z}$ with u marking the positive and negative height (not the altitude!) are related to the bivariate generating functions $M^+(z,u)$ of positive meanders and $M^-(z,u)$ of negative meanders (with u marking the final altitude, see Figure 3):*

$$W_{+h}(z,u) = M^-(z)E(z)M^+(z,u) = -\frac{1}{s_d z}\left(\prod_{j=1}^{c}\frac{1}{1-u_j(z)}\right)\left(\prod_{\ell=1}^{d}\frac{1}{u-v_\ell(z)}\right),$$

$$W_{-h}(z,u) = M^-(z,u)E(z)M^+(z) = -\frac{1}{s_d z}\left(\prod_{j=1}^{c}\frac{1}{1-u_j(z)/u}\right)\left(\prod_{\ell=1}^{d}\frac{1}{1-v_\ell(z)}\right).$$

This Wiener–Hopf factorization $W = M^-EM^+$ thus gives

$$M^-(z) = \frac{W(z)}{M(z)} = \prod_{j=1}^{c}\frac{1}{1-u_j(z)} \qquad \textit{and} \qquad M^+(z) = \frac{M(z)}{E(z)} = \prod_{\ell=1}^{d}\frac{1}{1-1/v_\ell(z)}.$$

Proof (Sketch). The proof follows from the decomposition illustrated in Figure 3. Cutting at the first and last maxima of the walk gives the factorization $\mathcal{W} = \mathcal{M}^+\mathcal{E}\mathcal{M}^-$, where the positive meander and the excursion are obtained after a 180^o rotation, and it is thus clear that the final altitude of this positive meander is the height of the initial walk. Similarly, cutting the walk at its first and last minima gives the factorization $\mathcal{W} = \mathcal{M}^-\mathcal{E}\mathcal{M}^+$. ◀

4 Lattice paths and symmetric functions

Building on the quantities introduced in the previous sections, we now show that three fundamental classes of symmetric polynomials evaluated at the small roots of the kernel have a natural combinatorial interpretation in terms of directed lattice paths. *En passant*, this also gives the generating function of generalized arches. For our main results see Table 2. We first recall the definitions of these symmetric polynomials (see e.g. [38] for more on these objects).

▶ **Definition 11.** *The complete homogeneous symmetric polynomials h_k of degree k in the d variables $x_1, \ldots, x_d$ are defined as*

$$h_k(x_1,\ldots,x_d) = \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant d} x_{i_1}\cdots x_{i_k}, \qquad \textit{thus} \sum_{k\geqslant 0} h_k(x_1,\ldots,x_d)u^k = \prod_{i=1}^{d}\frac{1}{1-ux_i}. \qquad (5)$$

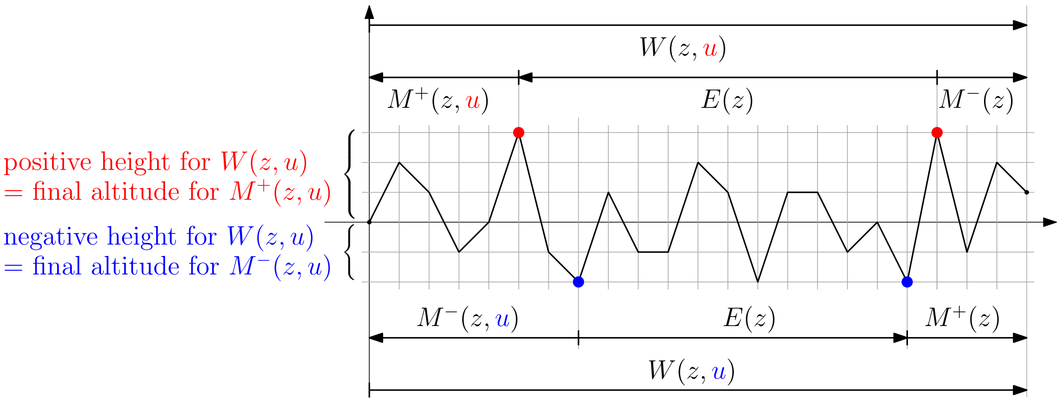


Figure 3 The Wiener–Hopf decomposition of a walk: $\mathcal{W} = \mathcal{M}^-\mathcal{E}\mathcal{M}^+$, a product of a negative meander, an excursion, and a positive meander. See e.g. [25] for the importance of this factorization for lattice path enumeration. It offers a link between two important parameters (height and final altitude): the proof uses a 180^o rotation of some of the factors (the ones indicated by a right to left arrow in the picture). The above picture crystallizes the key idea behind the theorems given by Feller in his nice introduction to the Wiener–Hopf factorization [19, Chapter XVIII.3 and XVIII.4]. It also explains why this decomposition holds for Lévy processes, which can be seen as the continuous time and space version of lattice paths, see [31].

The elementary homogeneous symmetric polynomials e_k of degree k in the d variables $x_1, \ldots, x_d$ are defined as

$$e_k(x_1, \ldots, x_d) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant d} x_{i_1} \cdots x_{i_k}, \qquad \textit{thus} \sum_{k=0}^{c} e_k(x_1, \ldots, x_d) u^k = \prod_{i=1}^{d} (1 + ux_i). \tag{6}$$

The power sum homogeneous symmetric polynomials p_k of degree k in the d variables $x_1, \ldots, x_d$ are defined as

$$p_k(x_1, \ldots, x_d) = \sum_{i=1}^{d} x_i^k, \qquad \textit{thus} \sum_{k \geqslant 0} p_k(x_1, \ldots, x_d) u^k = \sum_{i=1}^{d} \frac{1}{1 - ux_i}. \tag{7}$$

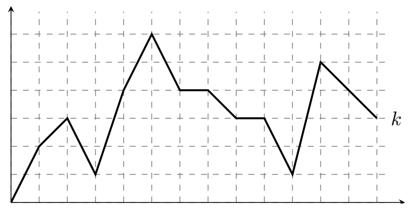
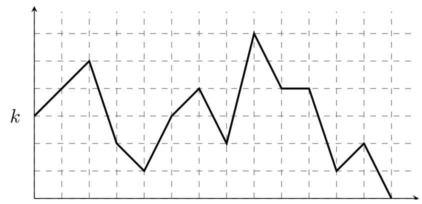
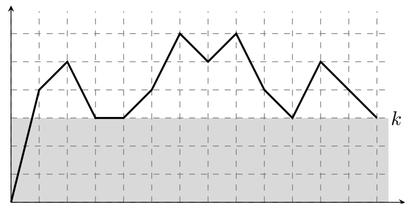
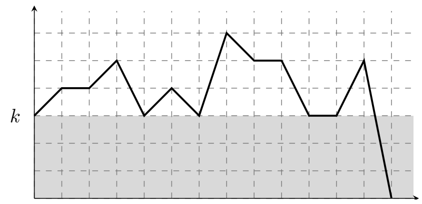
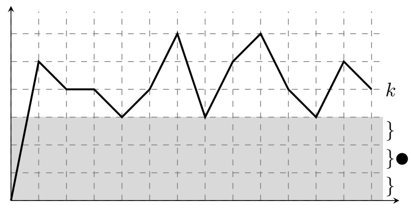
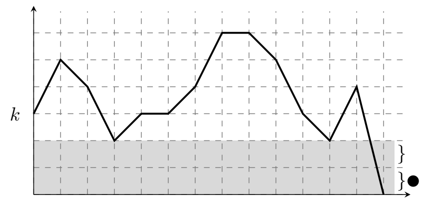
Many variants of directed lattice paths satisfy functional equations which are solvable by the kernel method and lead to formulas involving a quotient of Vandermonde-like determinants, see e.g. [4]. It is thus natural that Schur polynomials intervene, they e.g. play an important role for lattice paths in a strip, see [5,9]. It is nice that the other symmetric polynomials also have a combinatorial interpretation, as presented in the following table.

Let us now give a more formal definition of the corresponding objects and a proof of the formulas for the associated generating functions.

▶ **Definition 12.** *A positive meander is a path from $\ell \geqslant 0$ to $k \geqslant 0$ staying strictly above the x-axis (and possibly touching it at at most one of its end points). The generating function is denoted by $M^+_{\ell,k}(z)$. Negative meanders are defined similarly, with the condition to stay strictly below the x-axis.*

In Table 2, we focus on positive meanders from 0 to k and from k to 0. Note that it suffices to consider the paths from 0 to k as by time-reversion they are mapped to each other. In particular, let $u_i(z)$ and $v_j(z)$ be the small and large roots of the initial model. Then, after time-reversion the small roots are $\frac{1}{v_j(z)}$ and the large roots are $\frac{1}{u_i(z)}$. More details are given in the long version.

Table 2 In this article, we show that the fundamental symmetric polynomials (of the complete homogeneous, elementary, and power sum type) are counting families of positive meanders (walks touching the x-axis only at one of the end points and staying always above the x-axis). The functions $v_j(z)$ for $j = 1, \dots, d$ are the roots of the kernel equation $1 - zS(u) = 0$ with $\lim_{z=0} |v_j(z)| = +\infty$, whereas the functions $u_i(z)$ for $i = 1, \dots, c$ are the roots such that $\lim_{z=0} u_i(z) = 0$.

	from 0 to k	from k to 0
positive meander	 $M^{+}_{0,k}(z) = h_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right)$	 $M^{+}_{k,0}(z) = h_k\left(u_1(z), \dots, u_c(z)\right)$
positive meander avoiding $(0, k)$	 $M^{\geqslant}_{0,k}(z) = (-1)^{k-1} e_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right)$	 $M^{\geqslant}_{k,0}(z) = (-1)^{k-1} e_k\left(u_1(z), \dots, u_c(z)\right)$
positive meander marked below the minimum	 $M^{\bullet}_{0,k}(z) = p_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right)$	 $M^{\bullet}_{k,0}(z) = p_k\left(u_1(z), \dots, u_c(z)\right)$

▶ **Theorem 13** (Generating function of positive meanders).

$$M^{+}_{0,k}(z) = h_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right).$$

Proof. Observe that a meander ending at altitude k can be uniquely decomposed into an initial excursion followed by a positive meander from 0 to k. By [4, Theorem 2] their generating function is the coefficient of u^k in $\prod_{j=1}^{d} \frac{1}{1-u/v_j(z)}$. Consequently, by Equation (5) this is the generating function of the complete homogeneous symmetric polynomials $h_k(1/v_1(z), \dots, 1/v_d(z))$. ◀

This theorem gives a shorter proof of [4, Corollary 3]:

▶ **Corollary 14.** *The generating function $M_k(z)$ of meanders ending at altitude k are given by*

$$M_k(z) = E(z) h_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right) = \frac{1}{s_d z} \sum_{\ell=1}^{d} \left(\prod_{j \neq \ell} \frac{1}{v_j(z) - v_\ell(z)}\right) \frac{1}{v_\ell(z)^{k+1}}.$$

Proof. As in the proof of Theorem 13, we use that positive meanders are classical meanders factored by excursions. Then a partial fraction decomposition of (5) yields the result. ◀

The last class we consider is the one of elementary symmetric polynomials. These are associated to a decorated class of paths.

▶ **Definition 15.** *A positive meander avoiding a strip of width k is a positive meander from 0 to k that always stays above any point of altitude $j < k$ except for its start point. The generating function is denoted by $M^{\geqslant}_{0,k}(z)$.*

▶ **Theorem 16** (Positive meanders avoiding the strip $[0, k]$)**.**

$$M^{\geqslant}_{0,k}(z) = (-1)^{k-1} e_k \left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)} \right).$$

Proof. We proceed by induction on k. The base case $k = 1$ holds due to $M^{\geqslant}_{0,1}(z) = M^{+}_{0,1}(z) = 1/v_1(z) + \cdots + 1/v_d(z)$. Next assume the claim holds for $M^{\geqslant}_{0,1}(z), \dots, M^{\geqslant}_{0,k-1}(z)$.

Take an arbitrary positive meander from 0 to k. Either it is a positive meander avoiding the strip of width k, or at least one of its lattice points has an altitude smaller than k.

Let $0 < i < k$ be the altitude of the last step below altitude k. Then the path can be uniquely decomposed into an initial part from altitude 0 to this altitude i and a part from this point to the end. Note that by the construction the initial part starts at altitude 0 and then always stays above the x-axis, whereas the last part avoids a strip of width $k - i$. In terms of generating functions this gives

$$M^{\geqslant}_{0,k}(z) = M^{+}_{0,k}(z) - \sum_{i=1}^{k-1} M^{+}_{0,i}(z) M^{\geqslant}_{0,k-i}(z).$$

Inserting the known expressions, we get

$$M^{\geqslant}_{0,k}(z) = \sum_{i=1}^{k} (-1)^{k-i} e_{k-i} \left(\frac{1}{v_1}, \dots, \frac{1}{v_d} \right) h_i \left(\frac{1}{v_1}, \dots, \frac{1}{v_d} \right) = (-1)^{k-1} e_k \left(\frac{1}{v_1}, \dots, \frac{1}{v_d} \right),$$

thanks to the fundamental involution relation [38, Equation (7.13)] between elementary symmetric polynomials and complete homogeneous symmetric polynomials. ◀

▶ **Corollary 17.** *The generating functions of generalized arches (as introduced in Definition 4) satisfy (for $k > 0$)*

$$A_k = \frac{(-1)^{k-c} s_{-c} z}{u_1(z) \cdots u_c(z)} \, e_k \left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)} \right),$$
$$A_{-k} = \frac{(-1)^{k-c} s_{-c} z}{u_1(z) \cdots u_c(z)} \, e_k \left(u_1(z), \dots, u_c(z) \right).$$

Proof. This follows from $A_k = M^{\geqslant}_{0,k}/E$ and $A_{-k} = M^{\geqslant}_{k,0}/E$. ◀

We end our discussion with a third class of positive meanders.

▶ **Definition 18.** *A positive meanders marked below the minimum is a positive meander with an additional marker in $\{1, \dots, m\}$ where m is its minimal positive altitude. The generating function for such paths from 0 to k is denoted by $M^{\bullet}_{0,k}(z)$.*

For example it is immediate that $M^{\bullet}_{0,1}(z) = M^{\geqslant}_{0,1}(z) = M^{+}_{0,1}(z)$ as the only restriction is to avoid the x-axis. Furthermore, $M^{\bullet}_{0,0}(z) = 0$ while $M^{\geqslant}_{0,0}(z) = M^{+}_{0,0}(z) = 1$.

▶ **Theorem 19** (Positive meanders marked below the minimum).

$$M^{\bullet}_{0,k}(z) = p_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right).$$

Proof (Sketch). Every path from 0 to k has to touch at least one of the altitudes $1, \dots, d$, as the largest possible up step is $+d$. We decompose any positive meander from 0 to k into two parts by cutting at the unique last positive minimum m. The first part is an arch avoiding the strip of width m, whereas the second part is a positive meander from m to k. Translating this decomposition into generating functions, we get

$$M^{\bullet}_{0,k}(z) = \sum_{m=1}^{d} m M^{\geqslant}_{0,m}(z) M^{+}_{0,k-m}(z),$$

where the factor m encodes the m possible ways to put a mark below the minimum, see Definition 18. Note that $M^{\geqslant}_{0,k}(z) = 0$ for $k > d$. Thus, by Theorems 13 and 16 we get

$$\sum_{k\geqslant 1} M^{\bullet}_{0,k}(z) u^k = \left(u \frac{\partial}{\partial u} \sum_{j\geqslant 0} M^{\geqslant}_{0,j}(z) u^j\right)\left(\sum_{i\geqslant 0} M^{+}_{0,i}(z) u^i\right) = \sum_{i=1}^{d} \frac{u/v_i(z)}{1 - u/v_i(z)}.$$

By Equation (7) this proves the claim. ◀

5 Asymptotics and limit laws

We end the discussion on the symmetric polynomial expressions by deriving their respective asymptotics: this allows us to revisit some limit laws in which the appearance of symmetric polynomials was so far unrecognized.

We only consider *aperiodic* step sets $\mathcal{S}$, which are defined by $\gcd\{|i-j| : i, j \in \mathcal{S}\} = 1$. For the treatment of periodic step sets see [6]. We only treat paths from k to 0, as the formulas are a bit simpler. The results for paths from 0 to k follow in an analogous fashion. The principal small branch $u_1(z)$ and the principal large branch $v_1(z)$ are defined by the property that they are real positive for near $0+$ and meet at $z = \rho$; see [4].

In the next theorem we give the asymptotics of our three classes of positive meanders.

▶ **Theorem 20.** *Consider an aperiodic step set $\mathcal{S}$. Let τ be the structural constant determined by $S'(\tau) = 0$, $\tau > 0$. For the different variants of positive meanders given in Table 2, the number of paths from k to 0 of size n has the following asymptotic expansions*

$$[z^n] M^{+}_{k,0}(z) = \alpha_1 \frac{S(\tau)^n}{2\sqrt{\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad \alpha_1 = \frac{\partial e_k}{\partial x_1}(\tau, u_2(\rho), \dots, u_c(\rho)).$$

The number of positive meanders avoiding $(0,k)$ from k to 0 of size n satisfies

$$[z^n] M^{\geqslant}_{k,0}(z) = \alpha_2 \frac{S(\tau)^n}{2\sqrt{\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad \alpha_2 = \frac{\partial h_k}{\partial x_1}(\tau, u_2(\rho), \dots, u_c(\rho)).$$

The number of positive meanders marked below the minimum from k to 0 of size n satisfies

$$[z^n] M^{\bullet}_{k,0}(z) = \alpha_3 \frac{S(\tau)^n}{2\sqrt{\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad \alpha_3 = \frac{\partial p_k}{\partial x_1}(\tau, u_2(\rho), \dots, u_c(\rho)).$$

Proof. Let $\mathcal{M}^{+}_{k,0}$, $\mathcal{M}^{\geqslant}_{k,0}$, and $\mathcal{M}^{\bullet}_{k,0}$ be the sets of positive meanders, positive meanders avoiding $(0,k)$, and positive meanders marked below the minimum, respectively; see Table 2. Let $\omega_k \in \mathcal{A}_k$ and $\omega_{-k} \in \mathcal{A}_{-k}$ be two generalized arches. Now, define the multiset $\mathcal{E}_k$ that consists of d copies of the set $\{w : \omega_k \cdot w \in \mathcal{E}\}$ of excursions factored by ω_k. Then, the following chain of inclusions holds:

$$\mathcal{E} \cdot \omega_{-k} \subseteq \mathcal{M}^{\geqslant}_{k,0} \subseteq \mathcal{M}^{+}_{k,0} \subseteq \mathcal{M}^{\bullet}_{k,0} \subseteq \mathcal{E}_k. \tag{8}$$

The first inclusion holds as every walk $e \cdot \omega_{-k}$ with $e \in \mathcal{E}$ is a positive meander avoiding $(0,k)$. The middle inclusions hold by definition (see Table 2). The last inclusion holds since, for every $m \in \mathcal{M}^{\bullet}_{k,0}$, we have $\omega_k \cdot m \in \mathcal{E}$ after removing the marker from m. Therefore, the exponential growth rates of the counting sequences of $\mathcal{E} \cdot \omega_{-k}$ and $\mathcal{E}_k$ are equal to the one of classical excursions $\mathcal{E}$, which has been explicitly computed in [4]. Hence, all 3 classes of meanders in (8) have the same asymptotic growth R^n.

Next, we observe that the corresponding generating functions have non-negative coefficients, and whence Pringsheim's Theorem [22, Theorem IV.6] guarantees the existence of a dominant singularity on the positive real axis $\mathbb{R}^+$. By [4] this is the only dominant singularity and we have $\rho = 1/R$. Furthermore, it was shown that on the radius of convergence $|z| = \rho$ only one root $u_1(z)$ is singular and has a square-root singularity, while the other ones are analytic. Then, we combine this result with the explicit shape of the symmetric polynomials from Definition 11. This gives the Puiseux expansion at $z = \rho$ on which we apply singularity analysis to derive the claimed formulas. ◀

Before we continue, let us comment on an often overlooked phenomenon concerning the analyticity of the small branches.

▶ Remark 21 (Singularities of the small roots). The small roots (and, in particular the principal small branch $u_1(z)$) can have a singularity inside the disk of convergence of $E(z)$. For example, for $S(u) = u + 13/u + 6/u^2$, one easily checks that the radius of convergence of $E(z)$ is $\rho = 8/61$ while $u_1(z)$ and $u_2(z)$ are singular at $z = -1/8$. However, their product $u_1 u_2$ is regular for $|z| < \rho$; more generally what is proven in [4] is that the product of the small roots is always regular for $0 < |z| < \rho$, while in general not each single small root is regular for $0 < |z| < \rho$.

Many theorems leading to a Gaussian distribution require that a key quantity (let us call it σ) is nonzero. In [22], this nonzero assumption is called "variability condition"; see therein Theorem IX.8 (Quasi-power theorem), Theorem IX.9 (Meromorphic schema), Theorem IX.10 (Positive rational systems). Now, many lattice path statistics have a variance with an expansion $\sigma n + o(n)$, where σ is defined as in the following lemma, and is therefore nonzero.

▶ **Lemma 22** (Universal positivity of the variability condition)**.** *For any Laurent series $S(u) = \sum_{i \geqslant -c} s_i u^i$, with $s_i \geqslant 0$ (at least two $s_i > 0$), one has $\sigma := S''(1)S(1) + S'(1)S(1) - S'(1)^2 > 0$.*

Proof. The trick is to introduce $\sigma(u) := uS''(u)S(u) + S'(u)S(u) - uS'(u)^2$. Then, all the monomials of $\sigma(u)$ have positive coefficients: this follows from $[s_i s_j]\sigma(u) = u^{i+j-1}(i-j)^2 \geqslant 0$, and thus $\sigma(u) > 0$ for $u > 0$. ◀

It is worth noting that an alternative version of this lemma is: « $uS(u)/S'(u) = n$ has no double root for $u > 0$ »; this plays a role in the tuning of Boltzmann random generation [17]. Such considerations are also related to Harald Cramér's trick of shifting the mean which transforms a problem with drift into a problem with zero drift, via the modification of the weights of the step set $\widetilde{S}(u) := S(\tau u)/S(\tau)$ (and choosing τ such that $S'(\tau) = 0$ indeed implies that $\widetilde{S}'(1) = 0$). Compare also with the proof of [21, Formula (2.37)].

As a consequence, Lemma 22 guarantees that we can apply the quasi-power theorem [22, Theorem IX.8], and obtain a Gaussian limit theorem. This explains why many statistics related to lattice paths are Gaussian. E.g., for paths with positive or zero drift, it furnishes a Gaussian limit theorem for the final altitude of meanders or for the height of walks. When the drift is negative, one gets some discrete limit laws of parameter given by our symmetric polynomial expressions:

▶ **Theorem 23** ([4, Theorem 6] and [41, Theorem 4.7]; negative drift cases)**.** *Assume a negative drift $\delta = S'(1) < 0$ and let $\rho = 1/P(\tau)$ and $\rho_1 = 1/P(1)$.*

1. *Let X_n be the random variable of the final altitude of a meander of length n. Then, the limit law is discrete and given by*

$$\lim_{n\to\infty} \Pr(X_n = k) = (1-\tau^{-1}) \frac{\sum_{i=0}^{k} \tau^{i-k} h_i(v_1(\rho)^{-1}, \dots, v_d(\rho)^{-1})}{\sum_{i\geqslant 0} h_i(v_1(\rho)^{-1}, \dots, v_d(\rho)^{-1})}.$$

2. *Let Y_n be the random variable of the height of a walk of length n. Then, the limit law is discrete and given by*

$$\lim_{n\to\infty} \Pr(Y_n = k) = \frac{h_k(v_1(\rho_1)^{-1}, \dots, v_d(\rho_1)^{-1})}{\sum_{i\geqslant 0} h_i(v_1(\rho_1)^{-1}, \dots, v_d(\rho_1)^{-1})}.$$

Proof (Sketch). Recall that for a path represented by a sequence of points $(\omega_0, \omega_1, \dots, \omega_n)$ the final altitude is ω_n and the height is $\max_i \omega_i$. In both cases the limit law follows from a rewriting of the closed form of the discrete probability generating function which basically consists of the generating function of h_k (alternatively, M^+) and proper rescaling. ◀

Note that the second case is an avatar of the Wiener–Hopf decomposition which links the height of walks with the final altitude of meanders; see Theorem 10 and [41].

6 Conclusion and perspectives

In this article we introduced the notion of prime walks, a class of walks which leads to natural decompositions of lattice paths and to concise proofs of several identities in probability theory that we are even able to further generalize by capturing some additional statistics. Moreover, these decompositions can keep track of some additional parameters (e.g. counting the number of occurrences of some given patterns, see [3]), which then gives access to many joint distribution studies, see e.g. [12].

Our work also offers new links with symmetric polynomials, adding to previous fundamental connections with algebraic combinatorics via Vandermonde determinants, the Jacobi–Trudi identity, and Schur functions (see [5,9]). In [6], we give an interpretation of Schur polynomials (for some appropriate index) in terms of meanders ending at a given altitude. Together with the results of the present work, this extends the table given in [38, Prop. 2.8.3]: therein, Stanley gives some nice combinatorial expressions for the bases of symmetric functions (Definition 11), when they are evaluated at specific values like $x_i = 1$ or $x_i = q^i$. This is what he calls the "principal specializations". Our work shows that what we could call the "kernel root specialization" of the symmetric function bases (i.e. evaluation at $x_i = u_i(z)$) is leading to the enumeration of fundamental lattice path classes, holding for any set of jumps.

En passant, we illustrate the old Schützenberger philosophy: most of the identities in the commutative world are images of structural identities in the non-commutative world. It is natural to ask how far we can extend the link between lattice paths and the non-commutative symmetric world; note that further non-commutative points of view are developed in [18,23,24].

It is striking that astonishingly powerful formulas can be obtained by astonishingly simple tools from symbolic combinatorics. Such formulas, e.g. the Spitzer formula for bridges, have some unexpected avatars. Indeed, bridges of length n can be seen as $[u^0]S(u)^n$ for some Laurent polynomial $S(u)$ and the same holds with multivariate polynomials; this leads to some interesting connections between the non-commutative world, the Laurent phenomenon (i.e. the fact that some expressions which by design are a priori rational functions are in fact some Laurent polynomial), and lattice paths (see [14, 29, 36]).

On the computer algebra side, the so-called "Platypus algorithm" from [4] is a way to get the algebraic equation satisfied by the generating function of excursions. Another nice consequence of our formulas is that they permit a generalization of this "Platypus algorithm": starting from the generating functions of the symmetric polynomials given in Definition 11, we show in the long version of this article how to get the algebraic equations of the different families of constrained meanders, bridges, etc. This offers an effective alternative to an approach by resultants or Gröbner bases, which are quickly time and memory consuming.

For Motzkin paths (that is, paths with step set $\mathcal{S} = \{-1, 0, +1\}$), the generating functions associated to starting/final altitude constraints can be expressed as continued fractions, and thus as quotients of orthogonal polynomials [20]. Our work, in one sense, gives the generalization of these formulas as soon as one has steps $> +1$ or < -1. Many combinatorial structures related to the Motzkin paths have some asymptotics in which the "algebra of orthogonal polynomials" plays a role (e.g. the height of binary trees, related to the Mandelbrot fractal equation involves Chebyshev polynomials, see e.g. [22]). It is thus natural to ask if there is a nice "algebra of symmetric polynomials" in which plugging the Puiseux expansions offered by the kernel method could lead to the limit laws of many parameters of lattice paths?

In conclusion, our work largely complements and extends [4], being part of a wider program illustrating how lattice path surgery (which we call *latticepathology*) leads directly to many neat enumerative, probabilistic, computational, and asymptotic formulas.

References

1 David Aldous, Grégory Miermont, and Jim Pitman. The exploration process of inhomogeneous continuum random trees, and an extension of Jeulin's local time identity. *Probab. Theory Related Fields*, 129(2):182–218, 2004. doi:10.1007/s00440-003-0334-7.

2 Erik Sparre Andersen. On the number of positive sums of random variables. *Scandinavian Actuarial Journal*, 1949(1):27–36, 1949. doi:10.1080/03461238.1949.10419756.

3 Andrei Asinowski, Axel Bacher, Cyril Banderier, and Bernhard Gittenberger. Analytic combinatorics of lattice paths with forbidden patterns: enumerative aspects. In *LATA'18*, volume 10792 of *Lecture Notes in Comput. Sci.*, pages 195–206. Springer, 2018. URL: https://lipn.univ-paris13.fr/~banderier/Papers/lata2018.pdf.

4 Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoretical Computer Science*, 281(1-2):37–80, 2002. doi:10.1016/S0304-3975(02)00007-5.

5 Cyril Banderier and Pierre Nicodème. Bounded discrete walks. *Discrete Math. Theor. Comput. Sci.*, AM:35–48, 2010. URL: https://dmtcs.episciences.org/2792/pdf.

6 Cyril Banderier and Michael Wallner. The kernel method for lattice paths below a rational slope. In *Lattice paths combinatorics and applications*, Developments in Mathematics Series, pages 119–154. Springer, 2019. doi:10.1007/978-3-030-11102-1_7.

7 Jean Bertoin and Jim Pitman. Path transformations connecting Brownian bridge, excursion and meander. *Bull. Sci. Math.*, 118(2):147–166, 1994. URL: https://digitalassets.lib.berkeley.edu/sdtr/ucb/text/350.pdf.

8 Olivier Bodini and Yann Ponty. Multi-dimensional Boltzmann sampling of languages. *Discrete Math. Theor. Comput. Sci. Proc.*, AM:49–63, 2010. URL: `https://hal.inria.fr/hal-00450763v4/document`.

9 Mireille Bousquet-Mélou. Discrete excursions. *Sém. Lothar. Combin.*, 57:23 pp., 2008. URL: `https://www.mat.univie.ac.at/~slc/wpapers/s57bousquet.pdf`.

10 Mireille Bousquet-Mélou and Marko Petkovšek. Linear recurrences with constant coefficients: the multivariate case. *Discrete Mathematics*, 225(1-3):51–75, 2000. `doi:10.1016/S0012-365X(00)00147-3`.

11 Mireille Bousquet-Mélou and Yann Ponty. Culminating paths. *Discrete Math. Theor. Comput. Sci.*, 10(2):125–152, 2008. URL: `https://hal.archives-ouvertes.fr/hal-00151979v2/document`.

12 Alan J. Bray, Satya N. Majumdar, and Grégory Schehr. Persistence and first-passage properties in nonequilibrium systems. *Advances in Physics*, 62(3):225–361, 2013. `doi:10.1080/00018732.2013.803819`.

13 Philippe Chassaing and Svante Janson. A Vervaat-like path transformation for the reflected Brownian bridge conditioned on its local time at 0. *Ann. Probab.*, 29(4):1755–1779, 2001. `doi:10.1214/aop/1015345771`.

14 Philippe Di Francesco and Rinat Kedem. Discrete non-commutative integrability: Proof of a conjecture by M. Kontsevich. *International Mathematics Research Notices*, 2010(21):4042–4063, 2010. `doi:10.1093/imrn/rnq024`.

15 Catherine Donati-Martin, Hiroyuki Matsumoto, and Marc Yor. On striking identities about the exponential functionals of the Brownian bridge and Brownian motion. *Period. Math. Hungar.*, 41(1-2):103–119, 2000. `doi:10.1023/A:1010308203346`.

16 Philippe Duchon. On the enumeration and generation of generalized Dyck words. *Discrete Mathematics*, 225(1-3):121–135, 2000. `doi:10.1016/S0012-365X(00)00150-3`.

17 Philippe Duchon, Philippe Flajolet, Guy Louchard, and Gilles Schaeffer. Boltzmann samplers for the random generation of combinatorial structures. *Combinatorics, Probability and Computing*, 13(4-5):577–625, 2004. `doi:10.1017/S0963548304006315`.

18 Kurusch Ebrahimi-Fard, Li Guo, and Dirk Kreimer. Spitzer's identity and the algebraic Birkhoff decomposition in pQFT. *J. Phys. A*, 37(45):11037–11052, 2004. `doi:10.1088/0305-4470/37/45/020`.

19 William Feller. *An Introduction to Probability Theory and its Applications. Vol. II.* Second edition. John Wiley & Sons, 1971.

20 Philippe Flajolet. Combinatorial aspects of continued fractions. *Discrete Mathematics*, 32(2):125–161, 1980. `doi:10.1016/0012-365X(80)90050-3`.

21 Philippe Flajolet and Andrew M. Odlyzko. Limit distributions for coefficients of iterates of polynomials with applications to combinatorial enumerations. *Math. Proc. Cambridge Philos. Soc.*, 96(2):237–253, 1984. `doi:10.1017/S0305004100062149`.

22 Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics.* Cambridge University Press, 2009. URL: `http://algo.inria.fr/flajolet/Publications/book.pdf`.

23 Sergey Fomin and Curtis Greene. Noncommutative Schur functions and their applications. *Discrete Math.*, 193(1-3):179–200, 1998. `doi:10.1016/S0012-365X(98)00140-X`.

24 Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, and Jean-Yves Thibon. Noncommutative symmetric functions. *Adv. Math.*, 112(2):218–348, 1995. `doi:10.1006/aima.1995.1032`.

25 Ira M. Gessel. A factorization for formal Laurent series and lattice path enumeration. *Journal of Combinatorial Theory, Series A*, 28(3):321–337, 1980. `doi:10.1016/0097-3165(80)90074-6`.

26 Priscilla Greenwood. Wiener–Hopf methods, decompositions, and factorisation identities for maxima and minima of homogeneous random processes. *Advances in Appl. Probability*, 7(4):767–785, 1975. `doi:10.2307/1426398`.

27 John E. Hopcroft, Raghavan Motwani, and Jeffrey D. Ullman. *Introduction to Automata Theory, Languages, and Computation.* Addison-Wesley, 3rd edition, 2006.

28 Thierry Jeulin and Marc Yor. Grossissements de filtrations: exemples et applications. In *Séminaire de Calcul Stochastique 1982/83, Université Paris VI*, volume 1118 of *Lecture Notes in Math.* Springer, 1985. doi:10.1007/BFb0075765.

29 Maxim Kontsevich. Noncommutative identities. *arXiv*, pages 1–10, 2011. arXiv:1109.2469v1.

30 Christian Krattenthaler. *Lattice Path Enumeration.* CRC Press, 2015. In: Handbook of Enumerative Combinatorics, M. Bóna (ed.), Discrete Math. and Its Appl. arXiv:1503.05930.

31 Alexey Kuznetsov, Andreas E. Kyprianou, and Juan C. Pardo. Meromorphic Lévy processes and their fluctuation identities. *Ann. Appl. Probab.*, 22(3):1101–1135, 2012. doi:10.1214/11-AAP787.

32 Jacques Labelle and Yeong N. Yeh. Generalized Dyck paths. *Discrete Mathematics*, 82(1):1–6, 1990. doi:10.1016/0012-365X(90)90039-K.

33 Titus Lupu, Jim Pitman, and Wenpin Tang. The Vervaat transform of Brownian bridges and Brownian motion. *Electron. J. Probab.*, 20:no. 51, 31, 2015. doi:10.1214/EJP.v20-3744.

34 Philippe Marchal. On a new Wiener–Hopf factorization by Alili and Doney. In *Sém. de Probabilités, XXXV*, Lecture Notes in Math. #1755, pages 416–420. Springer, 2001. doi:10.1007/978-3-540-44671-2_28.

35 Donatella Merlini, Douglas G. Rogers, Renzo Sprugnoli, and Cecilia Verri. Underdiagonal lattice paths with unrestricted steps. *Discrete Appl. Math.*, 91(1-3):197–213, 1999. doi:10.1016/S0166-218X(98)00126-7.

36 Christophe Reutenauer and Marco Robado. On an algebraicity theorem of Kontsevich. *Discrete Math. Theor. Comput. Sci.*, AR:239–246, 2012. 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012). URL: https://hal.archives-ouvertes.fr/hal-01283155/document.

37 Frank Spitzer. A combinatorial lemma and its application to probability theory. *Trans. Amer. Math. Soc.*, 82:323–339, 1956. doi:10.2307/1993051.

38 Richard P. Stanley. *Enumerative Combinatorics. Vol. 2*, volume 62. Cambridge University Press, 1999. doi:10.1017/CBO9780511609589.

39 J. Michael Steele. The Bohnenblust–Spitzer algorithm and its applications. *J. Comput. Appl. Math.*, 142(1):235–249, 2002. doi:10.1016/S0377-0427(01)00472-1.

40 Wim Vervaat. A relation between Brownian bridge and Brownian excursion. *Ann. Probab.*, 7(1):143–149, 1979. URL: https://www.jstor.org/stable/2242845.

41 Michael Wallner. A half-normal distribution scheme for generating functions. *European J. Combin.*, 87(103138):1–21, 2020. arXiv:1610.00541, doi:10.1016/j.ejc.2020.103138.

42 James G. Wendel. Spitzer's formula: a short proof. *Proc. Amer. Math. Soc.*, 9:905–908, 1958. doi:10.2307/2033326.