The k-Cut Model in Conditioned Galton-Watson Trees

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Abstract

The k-cut number of rooted graphs was introduced by Cai et al. [7] as a generalization of the classical cutting model by Meir and Moon [16]. In this paper, we show that all moments of the k-cut number of conditioned Galton-Watson trees converge after proper rescaling, which implies convergence in distribution to the same limit law regardless of the offspring distribution of the trees. This extends the result of Janson [13].

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1 Introduction and main result

In order to measure the difficulty for the destruction of a resilient network Cai et al. [7] introduced a generalization of the cut model of Meir and Moon [16] where each vertex (or edge) needs to be cut $k \in \mathbb{N}$ times (instead of only once) before it is destroyed. More precisely, consider that the resilient network is a rooted tree $\mathbb{T}_n$, with $n \in \mathbb{N}$ vertices. We destroy it by removing its vertices as follows: **Step 1:** Choose a vertex uniformly at random from the component that contains the root and cut the selected vertex once. **Step 2:** If this vertex has been cut k times, remove the vertex together with the edges attached to it from the tree. **Step 3:** If the root has been removed, then stop. Otherwise, go to step **Step 1**. We let $\mathcal{K}_k(\mathbb{T}_n)$ denote the (random) total number of cuts needed to end this procedure the k-cut number, i.e., $\mathcal{K}_k(\mathbb{T}_n)$ models how much effort it takes to destroy the network. (For simplicity, we will omit the subscript k and write $\mathcal{K}(\mathbb{T}_n)$.) It should be plain that one can define analogously an edge deletion version of the previous algorithm, where one needs to cut an edge k times before removing it from the root component. Then, one would be interested in the number $\mathcal{K}_e(\mathbb{T}_n)$ of cuts needed to isolate the root of $\mathbb{T}_n$.

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The case $k = 1$ (i.e., the traditional cutting model of Meir and Moon [16]) has been well-studied by several authors in the past few decades. More precisely, Meir and Moon estimated the first and second moment of the 1-cut number in the cases when $\mathbb{T}_n$ is a Cayley tree [16] and a recursive tree [17]. Subsequently, several weak limit theorems for the 1-cut number have been obtained for Cayley trees (Panholzer [18, 19]), complete binary trees (Janson [12]), conditioned Galton-Watson trees (Janson [13] and Addario-Berry et al. [1]), recursive trees (Drmota et al. [8], Iksanov and Möhle [11]), binary search trees (Holmgren [9]) and split trees (Holmgren [10]). In the general case $k \geq 1$, the authors in [7] established first moment estimates of $\mathcal{K}(\mathbb{T}_n)$ for families of deterministic and random trees, such as paths, complete binary trees, split trees, random recursive trees and conditioned Galton-Watson trees. In particular, the authors in [7] have proven a weak limit theorem for $\mathcal{K}(\mathbb{T}_n)$ when $\mathbb{T}_n$ is a path consisting of n vertices. More recently, Cai and Holmgren [6] obtained also a weak limit theorem in the case when $\mathbb{T}_n$ is a complete binary tree.

In this work, we continue the investigation of this general cutting-down procedure in conditioned Galton-Watson trees and show that $\mathcal{K}(\mathbb{T}_n)$, after a proper rescaling, converges in distribution to a non-degenerate random variable. More precisely, let ξ be a non-negative integer-valued random variable such that

$$\mathbb{E}[\xi] = 1 \quad \text{and} \quad 0 < \sigma^2 := \operatorname{Var}(\xi) < \infty, \tag{1}$$

and consider a Galton-Watson process with (critical) offspring distribution ξ. Let $\mathbb{T}_n$ be the family tree conditioned on its number of vertices being $n \in \mathbb{N}$. The main result of this paper is the following. We write $\stackrel{d}{\to}$ to denote convergence in distribution. (In the rest of the paper CRT stands for Continuum Random Tree.)

▶ **Theorem 1.** *Let $k \in \mathbb{N}$. Let $\mathbb{T}_n$ be a Galton-Watson tree conditioned on its number of vertices being $n \in \mathbb{N}$ with offspring distribution ξ satisfying* (1)*. Then,*

$$\sigma^{-1/k} n^{-1+1/2k} \mathcal{K}(\mathbb{T}_n) \stackrel{d}{\to} Z_{\mathrm{CRT}}, \quad \text{as } n \to \infty, \tag{2}$$

where Z_{CRT} is a non-degenerate random variable whose law is determined entirely by its moments: $\mathbb{E}[Z^0_{\mathrm{CRT}}] = 1$, and for $q \in \mathbb{N}$, $\mathbb{E}[Z^q_{\mathrm{CRT}}] = \eta_{k,q}$ with

$$\eta_{k,q} := q! \int_0^\infty \cdots \int_0^\infty y_1(y_1 + y_2) \cdots (y_1 + \cdots + y_q) e^{-\frac{(y_1+\cdots+y_q)^2}{2}} F_q(\boldsymbol{y}_q)\, \mathrm{d}y_q \cdots \mathrm{d}y_1, \tag{3}$$

where $\boldsymbol{y}_q = (y_1, \ldots, y_q) \in \mathbb{R}^q_+$ and

$$F_q(\boldsymbol{y}_q) := \int_0^\infty \int_0^{x_1} \cdots \int_0^{x_{q-1}} \exp\left(- \frac{y_1 x_1^k + y_2 x_2^k + \cdots + y_q x_q^k}{k!} \right) \mathrm{d}x_q \cdots \mathrm{d}x_2\, \mathrm{d}x_1.$$

Furthermore, if $\mathbb{E}[\xi^p] < \infty$ for every $p \in \mathbb{Z}_{\geq 0}$, then for every $q \in \mathbb{Z}_{\geq 0}$,

$$\sigma^{-q/k} n^{-q+q/2k} \mathbb{E}[\mathcal{K}(\mathbb{T}_n)^q] \to \mathbb{E}[Z^q_{\mathrm{CRT}}]$$

as $n \to \infty$.

In the case $k = 1$, Theorem 1 reduces to Z_{CRT} having a Rayleigh distribution with density $xe^{-x^2/2}$, for $x \in \mathbb{R}_+$. More precisely, one can verify that $\eta_{1,q} = 2^{q/2}\Gamma(1 + q/2)$, for $q \in \mathbb{Z}_{\geq 0}$, which are the moments of a random variable with the Rayleigh distribution; in this paper $\Gamma(\cdot)$ denotes the well-known gamma function. As we mentioned early, the case $k = 1$ has been shown in [13, Theorem 1.6] (or Addario-Berry et al. [1]). We henceforth assume throughout this paper that $k \geq 2$. It is also important to mention that we could not find a simpler expression (in general) for the moments $\eta_{k,q}$ except for some particular instances. For $q = 1$, we have

$$\eta_{k,1} = 2^{-\frac{1}{2k}} \frac{(k!)^{\frac{1}{k}}}{k} \Gamma\left(\frac{1}{k}\right) \Gamma\left(1 - \frac{1}{2k}\right).$$

Then Theorem 1 provides a proof of [7, Lemma 4.10], where an estimation for the first moment of $\mathcal{K}(\mathbb{T}_n)$ was first announced but whose proof was left to the reader, see Lemma 10. On the other hand, let $(U_1, \ldots, U_q)$ be q i.i.d. leaves of a Brownian CRT and define the vector $(L_0^{\text{CRT}}, L_1^{\text{CRT}}, \ldots, L_q^{\text{CRT}})$ where $L_0^{\text{CRT}} = 0$ and L_i^{CRT} is the total length of a Brownian CRT reduced to the leaves of $U_1, \ldots, U_i$; see [3, Lemma 21] from where one can deduce explicitly the distribution of $(L_0^{\text{CRT}}, L_1^{\text{CRT}}, \ldots L_q^{\text{CRT}})$. From the proof of Theorem 1, we obtain, for $q \in \mathbb{N}$, that

$$\eta_{k,q} = q! \int_0^\infty \int_0^{x_1} \cdots \int_0^{x_{q-1}} \mathbb{E}\left[\exp\left(-\frac{\sum_{i=1}^q (L_i^{\text{CRT}} - L_{i-1}^{\text{CRT}}) x_i^k}{k!}\right)\right] \mathrm{d}\bar{\mathbf{x}}_q,$$

where $\bar{\mathbf{x}}_q = (x_q, \ldots, x_1) \in \mathbb{R}_+^q$. This suggests that it ought to be possible to build the random variable Z_{CRT} by some construction that can be interpreted as the k-cut model on the Brownian CRT defined by Aldous [2, 3]. The appearance of the Brownian CRT in this framework should not come as a surprise since it is well-known that if we assign length $n^{-1/2}$ to each edge of the Galton-Watson tree $\mathbb{T}_n$, then the latter converges weakly to a Brownian CRT as $n \to \infty$.

The approach used in this work consists of implementing an extension of the idea of Janson [13], which was used in [7], in order to study the k-cut model on deterministic and random trees. The authors in [7] introduced an equivalent model that allows them to define $\mathcal{K}(\mathbb{T}_n)$ in terms of the number of records in $\mathbb{T}_n$ when vertices are assigned random labels. More precisely, let $(E_{i,v})_{i \geq 1, v \in \mathbb{T}_n}$ be a sequence of independent exponential random variables of parameter 1; Exp(1) for short. Let $G_{r,v} := \sum_{1 \leq i \leq r} E_{i,v}$, for $r \in \mathbb{N}$ and $v \in \mathbb{T}_n$. Clearly, $G_{r,v}$ has a gamma distribution with parameters $(r, 1)$, which we denote by Gamma(r). Imagine that each vertex $v \in \mathbb{T}_n$ has an alarm clock and v's clock fires at times $(G_{r,v})_{r \geq 1}$. If we cut a vertex when its alarm clock fires, then due to the memoryless property of exponential random variables, we are actually choosing a vertex uniformly at random to cut. However, this also means that we are cutting vertices that have already been removed from the tree. Thus, for a cut on vertex v at time $G_{r,v}$ (for some $r \in \{1, \ldots, k\}$) to be counted in $\mathcal{K}(\mathbb{T}_n)$, none of its strict ancestors can already have been cut k times, i.e.,

$$G_{r,v} < \min\{G_{k,u} : u \in \mathbb{T}_n \text{ and } u \text{ is a strict ancestor of } v\}.$$

When the previous event happens, we say that $G_{r,v}$, or simply v, is an r-record and let

$$I_{r,v} := [\![G_{r,v} < \min\{G_{k,u} : u \in \mathbb{T}_n \text{ and } u \text{ is a strict ancestor of } v\}]\!], \tag{4}$$

where $[\![\cdot]\!]$ denotes the Iverson bracket, i.e., $[\![S]\!] = 1$ if the statement S is true and $[\![S]\!] = 0$ otherwise. Let $\mathcal{K}_r(\mathbb{T}_n)$ be the number of r-records, i.e., $\mathcal{K}_r(\mathbb{T}_n) := \sum_{v \in \mathbb{T}_n} I_{r,v}$. Then, it should be plain that

$$\mathcal{K}(\mathbb{T}_n) \stackrel{d}{=} \sum_{1 \leq r \leq k} \mathcal{K}_r(\mathbb{T}_n), \tag{5}$$

where $\stackrel{d}{=}$ denotes equal in distribution.

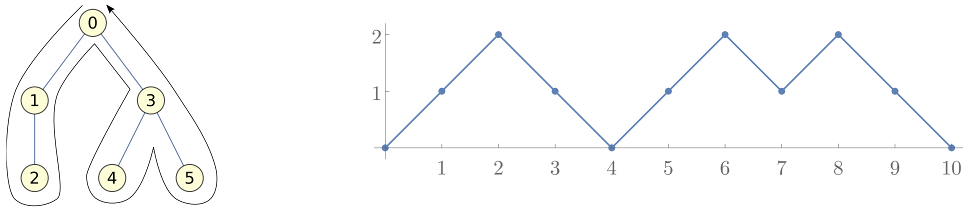

Figure 1 An example of a depth-first walk in a tree and the corresponding V_n.

Loosely speaking, we then consider the well-known *depth-first* walk $(V_n(t), t \in [0, 2(n-1)])$ of the tree $\mathbb{T}_n$ as depicted in Figure 1, that is, $V_n(t)$ is "the depth of the t-th vertex" visited in this walk; this will be made precise in the next section. As it is well-known (see Aldous [3, Theorem 23 with Remark 2] or [15, Theorem 1]), when T_n is a conditioned Galton-Watson with offspring distribution satisfying (1), we have that

$$(n^{-1/2}V_n(2(n-1)t), t \in [0,1]) \xrightarrow{d} 2\sigma^{-1}B^{\mathrm{ex}}, \qquad \text{as } n \to \infty.$$

in $C([0,1], \mathbb{R}_+)$, with its usual topology, and where $B^{\mathrm{ex}} = (B^{\mathrm{ex}}(t), t \geq 0)$ is a standard normalized Brownian excursion. It has been shown in [7, Lemma 1] that $\mathbb{E}[I_{r,v}] \sim C_{r,k} d_n(v)^{-r/k}$, for some (explicit) constant $C_{r,k} > 0$, where $d_n(v)$ is the depth of the vertex $v \in \mathbb{T}_n$. Let $\circ$ denote the root of $\mathbb{T}_n$. Thus, informally

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{K}_r(\mathbb{T}_n) \mid \mathbb{T}_n\right] \\
&\sim \sum_{v \in \mathbb{T}_n \setminus \{\circ\}} \frac{C_{r,k}}{d_n(v)^{r/k}} \sim \frac{C_{r,k}}{2} \int_0^{2(n-1)} \frac{\mathrm{d}t}{V_n(t)^{r/k}} \sim \frac{C_{r,k}}{n^{-1+\frac{r}{2k}}} \int_0^1 \left(\frac{V_n(2(n-1)t)}{\sqrt{n}}\right)^{-\frac{r}{k}} \mathrm{d}t \\
&\sim \frac{C_{r,k}}{n^{-1+\frac{r}{2k}}} \left(\frac{\sigma}{2}\right)^{\frac{r}{k}} \int_0^1 \frac{\mathrm{d}t}{B^{\mathrm{ex}}(t)^{r/k}},
\end{aligned}$$

as $n \to \infty$. By taking expectation, we deduce that

$$\sigma^{-r/k} n^{-1+\frac{r}{2k}} \mathbb{E}\left[\mathcal{K}_r(\mathbb{T}_n)\right] \sim C_{r,k} \mathbb{E}\left[\int_0^1 (2B^{\mathrm{ex}}(t))^{-r/k} \,\mathrm{d}t\right], \qquad \text{as } n \to \infty,$$

which coincides with the right-hand side of (3) when $r = q = 1$. Notice that this informal computation suggests that $\mathbb{E}\left[\mathcal{K}_r(\mathbb{T}_n)\right] = O(n^{1-\frac{r}{2k}})$, for $r \in \{1, \ldots, k\}$. As a consequence, the Markov's inequality implies $n^{-1+\frac{1}{2k}} \mathcal{K}_r(\mathbb{T}_n) \to 0$ in probability, as $n \to \infty$, for $r \in \{2, \ldots, k\}$. If so, by the identity in (5), it would be enough to prove Theorem 1 for $\mathcal{K}_1(\mathbb{T}_n)$ instead of $\mathcal{K}(\mathbb{T}_n)$.

In the rest of the paper, we make the above argument precise and extend it to higher moments in order to apply the method of moments for proving Theorem 1. In a full version of this paper [4], we also apply the same idea to get all moments of the number of records in paths and several types of trees of logarithmic height, e.g., complete binary trees, split trees, uniform random recursive trees and scale-free trees. We omit the proofs of our more technical lemmas since they can be found in [4].

2 Preliminary results

The purpose of this section is to establish a general convergence result for the number of 1-records $\mathcal{K}_1(\mathbb{T}_n)$ of a deterministic rooted ordered tree $\mathbb{T}_n$. The results of this section can also be viewed as a generalization of those of Janson [13] and Cai, et al. [7]. Furthermore, these results will allow us to study the convergence of the cut number $\mathcal{K}(\mathbb{T}_n)$ not only for conditioned Galton-Watson trees in Section 3, but also for other classes of random trees in a full version of this paper [4].

We start by defining a probability measure through a continuous function in the same spirit as in [13, Theorem 1.9]. Let $I \subseteq \mathbb{R}_+$ be an interval. For a function $f : I \to \mathbb{R}_+$ and $t_1, \dots, t_q \in I$ with $q \in \mathbb{N}$, we define

$$L_f(t_1, \dots, t_q) \coloneqq \sum_{i=1}^{q} f(t_{(i)}) - \sum_{i=1}^{q-1} \inf_{t \in [t_{(i)}, t_{(i+1)}]} f(t), \tag{6}$$

where $t_{(1)}, \dots, t_{(q)}$ are $t_1, \dots, t_q$ arranged in nondecreasing order. Notice that $L_f(t_1, \dots, t_q)$ is symmetric in $t_1, \dots, t_q$ and that $L_f(t) = f(t)$ for $t \in I$. Define

$$D_f(t_1) \coloneqq L_f(t_1), \qquad D_f(t_1, \dots, t_q) \coloneqq L_f(t_1, \dots, t_q) - L_f(t_1, \dots, t_{q-1}), \quad \text{for } q \geq 2. \tag{7}$$

We also consider the functional

$$G_f(\mathbf{t}_q, \mathbf{x}_q) \coloneqq \exp\left(-\frac{D_f(t_1) x_1^k + \dots + D_f(t_1, \dots, t_q) x_q^k}{k!} \right), \tag{8}$$

for $\mathbf{x}_q = (x_1, \dots, x_q) \in \mathbb{R}_+^q$ and $\mathbf{t}_q = (t_1, \dots, t_q) \in I^q$. If $I = [0,1]$, we further define, for $q \in \mathbb{N}$, let $m_0(f) \coloneqq 1$ and

$$m_q(f) \coloneqq q! \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^\infty \int_0^{x_1} \cdots \int_0^{x_{q-1}} G_f(\mathbf{t}_q, \mathbf{x}_q) \, \mathrm{d}\overleftarrow{\mathbf{x}}_q \, \mathrm{d}\overleftarrow{\mathbf{t}}_q, \quad q \geq 2, \tag{9}$$

where $\overleftarrow{\mathbf{x}}_q = (x_q, \dots, x_1)$ and $\overleftarrow{\mathbf{t}}_q = (t_q, \dots, t_1)$.

▶ **Theorem 2.** *Let $k \in \mathbb{N}$. Suppose that $f \in C([0,1], \mathbb{R}_+)$ is such that $\int_0^1 f(t)^{-1/k} \mathrm{d}t < \infty$. Then there exists a unique probability measure ν_f on $[0, \infty)$ with finite moments given by*

$$\int_{[0,\infty)} x^q \nu_f(\mathrm{d}x) = m_q(f), \qquad \text{for } q \in \mathbb{Z}_{\geq 0}.$$

Consider a rooted ordered tree $\mathbb{T}_n$ with root $\circ$ and $n \in \mathbb{N}$ vertices. We now explain how $\mathbb{T}_n$ can be coded by a continuous function. We define the so-called *depth-first search function* [2, page 260], $\psi_n : \{0, 1, \dots, 2(n-1)\} \to \{\text{vertices of } \mathbb{T}_n\}$ such that $\psi_n(i)$ is the $(i+1)$-th vertex visited in a depth-first walk on the tree starting from the root $\circ$. Note that $\psi_n(i)$ and $\psi_n(i+1)$ always are neighbours, and thus, we extend ψ to $[0, 2(n-1)]$ by letting, for $1 \leq i < t < i+1 \leq 2(n-1)$, $\psi_n(t)$ to be the one of $\psi_n(i)$ and $\psi_n(i+1)$ that has largest depth (recall that the depth of a vertex $v \in \mathbb{T}_n$ is the distance, i.e., number of edges, between $\circ$ to v). Let $d_n(v)$ be the depth of a vertex $v \in \mathbb{T}_n$. We further define the *depth-first walk* V_n of $\mathbb{T}_n$ by

$$V_n(i) \coloneqq d_n(\psi(i)), \qquad 0 \leq i \leq 2(n-1),$$

and extend V_n to $[0, 2(n-1)]$ by linear interpolation. Thus $V_n \in C([0, 2(n-1)], \mathbb{R}_+)$. See Figure 1 for an example of V_n. Furthermore, we normalize the domain of V_n to $[0,1]$ by defining

$$\widetilde{V}_n(t) \coloneqq V_n(2(n-1)t) \quad \text{and} \quad \widehat{V}_n(t) \coloneqq \lceil V_n(2(n-1)t) \rceil, \tag{10}$$

for $t \in [0,1]$. Thus $\widetilde{V}_n, \widehat{V}_n \in C([0,1], \mathbb{R}_+)$. Note that $d_n(\psi(t)) = \lceil V_n(t) \rceil$, for $t \in [0, 2(n-1)]$. Moreover,

$$\max_{v \in \mathbb{T}_n} d_n(v) = \sup_{t \in [0, 2(n-1)]} V_n(t) = \sup_{t \in [0,1]} \widetilde{V}_n(t). \tag{11}$$

We now state the central result of this section, that is, a general limit theorem in distribution for the number of 1-records $\mathcal{K}_1(\mathbb{T}_n)$ of a deterministic rooted tree $\mathbb{T}_n$ with n vertices. It is important to notice that $\mathcal{K}_1(\mathbb{T}_n)$ is a random variable since the 1-records are random.

▶ **Lemma 3.** *Let $k \in \mathbb{N}$. Suppose that $(\mathbb{T}_n)_{n\geq 1}$ is a sequence of (deterministic) ordered rooted trees, and denote the corresponding normalized depth-first walks by $\widetilde{V}_n$ and $\widehat{V}_n$. Suppose that there exists a sequence $(a_n)_{n\geq 1}$ of non-negative real numbers with $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} n a_n^{1/k} = \infty$ and a function $f \in C([0,1], \mathbb{R}_+)$ such that*

(a) *$a_n \widetilde{V}_n(t) \to f(t)$, in $C([0,1], \mathbb{R}_+)$, as $n \to \infty$.*

(b) *$\int_0^1 (a_n \widehat{V}_n(t))^{-1/k}\, \mathrm{d}t \to \int_0^1 f(t)^{-1/k}\, \mathrm{d}t < \infty$, as $n \to \infty$.*

Then, for each $q \in \mathbb{Z}_{\geq 0}$,

$$n^{-q} a_n^{-q/k} \mathbb{E}[\mathcal{K}_1(\mathbb{T}_n)^q] \to m_q(f),$$

as $n \to \infty$, where $m_q(f)$ is defined in (9). Moreover, $n^{-1} a_n^{-1/k} \mathcal{K}_1(\mathbb{T}_n) \xrightarrow{d} Z_f$, as $n \to \infty$, where Z_f is a random variable with distribution ν_f defined by Theorem 2.

We can apply similar ideas as in the proofs of Lemma 3 in order to estimate the mean of the number of r-records $\mathcal{K}_r(\mathbb{T}_n)$. It is important to mention that we have not tried to estimate higher moments of $\mathcal{K}_r(\mathbb{T}_n)$ in order to obtain a limit theorem in distribution for this quantity. We believe that our methods can be used but the computations will be more involved and we decided not to do it. Furthermore, the next results shows that $\mathcal{K}_r(\mathbb{T}_n)$ is of smaller order than $\mathcal{K}_1(\mathbb{T}_n)$ and hence it will not contribute (in the limit) to the distribution of the k-cut number $\mathcal{K}(\mathbb{T}_n)$.

▶ **Lemma 4.** *Let $k \in \mathbb{N}$. Let $\mathbb{T}_n$ be a (deterministic) rooted tree with $n \in \mathbb{N}$ vertices. Suppose that there exists a sequence $(a_n)_{n\geq 1}$ of non-negative real numbers with $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} n a_n = \infty$ and $\max_{v\in\mathbb{T}_n} d_n(v) = O(a_n^{-1})$. Then, for $r \in \{1, \ldots, k\}$, and uniformly over $\mathbb{T}_n$,*

$$n^{-1} a_n^{-r/k} \mathbb{E}[\mathcal{K}_r(\mathbb{T}_n)] = (1 + O(a_n^{\frac{1}{2k}})) \int_0^1 \int_0^\infty \frac{x^{r-1} e^{-a_n^{1/k} x}}{\Gamma(r)} e^{-\frac{a_n \widehat{V}_n(t) x^k}{k!}}\, \mathrm{d}x + o(1).$$

▶ **Lemma 5.** *Let $k \in \mathbb{N}$. Suppose that $(\mathbb{T}_n)_{n\geq 1}$ is a sequence of (deterministic) ordered rooted trees. Suppose that there exists a sequence $(a_n)_{n\geq 1}$ of non-negative real numbers with $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} n a_n = \infty$ and a function $f \in C([0,1], \mathbb{R}_+)$ such that $\widetilde{V}_n$ satisfies the condition (a) in Lemma 3 and that for $r \in \{1, \ldots, k\}$,*

$$\int_0^1 (a_n \widehat{V}_n(t))^{-r/k}\, \mathrm{d}t \to \int_0^1 f(t)^{-r/k}\, \mathrm{d}t < \infty, \quad \text{as } n \to \infty.$$

Then,

$$n^{-1} a_n^{-r/k} \mathbb{E}[\mathcal{K}_r(\mathbb{T}_n)] \to \frac{(k!)^{r/k}\Gamma(r/k)}{k\Gamma(r)} \int_0^1 f(t)^{-r/k}\, \mathrm{d}t, \quad \text{as } n \to \infty.$$

3 Proof of Theorem 1

Let $\mathbb{T}_n$ be a Galton-Watson tree conditioned on its number of vertices being $n \in \mathbb{N}$ with offspring distribution ξ satisfying (1). Notice that in this case both the r-records and the tree are random. Then we study $\mathcal{K}_r(\mathbb{T}_n)$ as random variable conditioned on $\mathbb{T}_n$. More precisely, we first choose a random tree $\mathbb{T}_n$. Then we keep it fixed and consider the number of r-records. This gives a random variable $\mathcal{K}_r(\mathbb{T}_n)$ with distribution that depends on $\mathbb{T}_n$. We have the following lemma that corresponds to [13, Lemma 4.8].

▶ **Lemma 6.** *Let $k \in \mathbb{N}$. Let $\mathbb{T}_n$ be a Galton-Watson tree conditioned on its number of vertices being $n \in \mathbb{N}$ with offspring distribution ξ satisfying (1). For $r \in \{1, \dots, k\}$. We have that $\mathbb{E}[\mathcal{K}_r(\mathbb{T}_n)] = O(n^{1-\frac{r}{2k}})$.*

Proof. By an application of Lemma 4 with $a_n = n^{-1/2}$, we see that

$$\begin{aligned}
\mathbb{E}[\mathcal{K}_r(\mathbb{T}_n)|\mathbb{T}_n] &\le (1 + O(a_n^{\frac{1}{2k}})) \sum_{v \in \mathbb{T}_n \setminus \{\circ\}} \int_0^\infty \frac{x^{r-1}}{\Gamma(r)} e^{-\frac{d_n(v) x^k}{k!}} \, \mathrm{d}x + o(n a_n^{r/k}) \\
&= (1 + O(a_n^{\frac{1}{2k}})) \sum_{v \in \mathbb{T}_n \setminus \{\circ\}} \frac{(k!)^{r/k} \Gamma(r/k)}{k \Gamma(r)} d_n(v)^{-r/k} + o(n a_n^{r/k}) \\
&= (1 + O(a_n^{\frac{1}{2k}})) \frac{(k!)^{r/k} \Gamma(r/k)}{k \Gamma(r)} \sum_{i=1}^{\infty} i^{-r/k} w_i(\mathbb{T}_n) + o(n a_n^{r/k}),
\end{aligned} \tag{12}$$

where $w_i(\mathbb{T}_n)$ denotes the number of vertices at depth $i \in \mathbb{N}$ in $\mathbb{T}_n$. Notice that

$$\sum_{i=1}^{\infty} i^{-r/k} w_i(\mathbb{T}_n) \le n^{1-\frac{r}{2k}} + \sum_{i=1}^{\lfloor n^{1/2} \rfloor} i^{-r/k} w_i(\mathbb{T}_n),$$

by the fact that $\sum_{i \ge 0} w_i(\mathbb{T}_n) = n$. Since $\mathbb{E}[\xi^2] < \infty$ by our assumption (1), [13, Theorem 1.13] implies that for all $n, i \in \mathbb{N}$, $\mathbb{E}[w_i(\mathbb{T}_n)] \le Ci$ for some constant $C > 0$ depending on ξ only. Therefore,

$$\sum_{i=1}^{\infty} i^{-r/k} \mathbb{E}[w_i(\mathbb{T}_n)] = n^{1-\frac{r}{2k}} + \sum_{i=1}^{\lfloor n^{1/2} \rfloor} \mathbb{E}[w_i(\mathbb{T}_n)] i^{-\frac{1}{k}} = O(n^{1-\frac{r}{2k}}). \tag{13}$$

By taking expectation in (12), our claim follows by (13). ◀

We continue by studying the moments of the number of 1-records $\mathcal{K}_1(\mathbb{T}_n)$. We denote by μ_n the (random) probability distribution of $\sigma^{-1/k} n^{-1+1/2k} \mathcal{K}_1(\mathbb{T}_n)$ given $\mathbb{T}_n$. Define the random variables

$$m_q(\mathbb{T}_n) \coloneqq \mathbb{E}[\mathcal{K}_1(\mathbb{T}_n)^q | \mathbb{T}_n], \qquad q \in \mathbb{Z}_{\ge 0}.$$

Notice that the moments of μ_n are given by $\sigma^{-q/k} n^{-q+q/2k} m_q(\mathbb{T}_n)$. We have the following lemma that corresponds to [13, Lemma 4.9].

▶ **Lemma 7.** *Let $k \in \mathbb{N}$. Let $\mathbb{T}_n$ be a Galton-Watson tree conditioned on its number of vertices being $n \in \mathbb{N}$ with offspring distribution ξ satisfying (1). Furthermore, suppose that for every fixed $q \in \mathbb{N}$ we have that $\mathbb{E}[\xi^{q+1}] < \infty$. Then $\mathbb{E}[m_q(\mathbb{T}_n)] = O(n^{q-\frac{q}{2k}})$.*

Let $\widetilde{V}_n$ and $\widehat{V}_n$ be the normalized depth-first walks associated with the conditioned Galton-Watson tree $\mathbb{T}_n$. Notice that in this case $\widetilde{V}_n$ and $\widehat{V}_n$ become random functions on $C([0,1], \mathbb{R}_+)$. Recall that a remarkable result due to Aldous [3, Theorem 23 with Remark 2] (see also [15, Theorem 1]) shows that

$$n^{-1/2} \widetilde{V}_n \xrightarrow{d} 2\sigma^{-1} B^{\text{ex}}, \qquad \text{as } \ n \to \infty, \tag{14}$$

in $C([0,1], \mathbb{R}_+)$, with its usual topology, and where $B^{\text{ex}} = (B^{\text{ex}}(t), t \ge 0)$ is a standard normalized Brownian excursion. Notice that B^{ex} is a random element in $C([0,1], \mathbb{R}_+)$; see for example [5] or [20].

▶ **Lemma 8.** *Let $k \in \mathbb{N}$. For $r \in \{1, \dots, k\}$, we have that $\int_0^1 B^{\mathrm{ex}}(t)^{-r/k}\,\mathrm{d}t < \infty$ almost surely.*

Proof. One only needs to show that $\mathbb{E}[\int_0^1 B^{\mathrm{ex}}(t)^{-r/k}\,\mathrm{d}t] < \infty$. This follows by computing $\mathbb{E}[B^{\mathrm{ex}}(t)^{-r/k}]$, for every $t \in [0,1]$, from the well-known density function of $B^{\mathrm{ex}}(t)$; see [5, Chapter II, Equation (1.4)]. ◀

Therefore, Theorem 2 and Lemma 8 imply that there exists almost surely a (unique) measure $\nu_{2B^{\mathrm{ex}}}$ with moments given by $m_q(2B^{\mathrm{ex}})$. The next result provides a generalization of [13, Theorem 1.10] and it will be used in the proof of Theorem 1.

▶ **Theorem 9.** *Let $k \in \mathbb{N}$. Let $\mathbb{T}_n$ be a Galton-Watson tree conditioned on its number of vertices being $n \in \mathbb{N}$ with offspring distribution ξ satisfying (1). Then*

$$\mu_n \xrightarrow{d} \nu_{2B^{\mathrm{ex}}}, \qquad \text{as } n \to \infty, \tag{15}$$

in the space of probability measures on $\mathbb{R}$. Moreover, we have that for every $q \in \mathbb{N}$,

$$\sigma^{-q/k} n^{-q+q/2k} m_q(\mathbb{T}_n) \xrightarrow{d} m_q(2B^{\mathrm{ex}}), \qquad \text{as } n \to \infty. \tag{16}$$

The convergences in (14), (15) and (16), for all $q \in \mathbb{N}$, hold jointly. In particular, if $\mathbb{E}[\xi^p] < \infty$ for all $p \in \mathbb{N}$, then for all $q \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$\sigma^{-lq/k} n^{-lq/k+lq/2k} \mathbb{E}[m_q(\mathbb{T}_n)^l] \to \mathbb{E}[m_q(2B^{\mathrm{ex}})^l], \qquad \text{as } n \to \infty. \tag{17}$$

Proof. A simple adaptation of the proof for [13, Lemma 4.7] easily shows that

$$\left(\widetilde{V}_n, \int_0^1 \widehat{V}_n(t)^{-1/k}\,\mathrm{d}t\right) \xrightarrow{d} \left(2\sigma^{-1}B^{\mathrm{ex}}, 2^{-1/k}\sigma^{1/k}\int_0^1 B^{\mathrm{ex}}(t)^{-1/k}\,\mathrm{d}t\right), \quad \text{in } C([0,1],\mathbb{R}_+), \tag{18}$$

as $n \to \infty$. By the Skorohod coupling theorem (see e.g. [14, Theorem 4.30]), we can assume that the trees $(\mathbb{T}_n)_{n\geq 1}$ are defined on a common probability space such that the convergence in (18) holds almost surely. Therefore, the convergences (15) and (16) follow immediately from Lemma 3. It only remains to prove (17). Recall that we assume that $\mathbb{E}[\xi^p] < \infty$ for every $p \in \mathbb{N}$. By Jensen's inequality, we notice that $m_q(\mathbb{T}_n)^l \leq m_{lq}(\mathbb{T}_n)$ for $l, q \in \mathbb{N}$. Hence Lemma 7 implies that $\mathbb{E}[m_q(\mathbb{T}_n)^l] = O(n^{lq-\frac{lq}{2k}})$. This shows that every moment of the right-hand side of (16) stays bounded as $n \to \infty$ which implies (17). ◀

Proof of Theorem 1. Lemma 6 establishes that $\mathbf{E}[\mathcal{K}_r(\mathbb{T}_n)] = O(n^{1-\frac{r}{2k}})$ for $r \in \{1, \dots, k\}$. As a consequence, the Markov's inequality implies $n^{-1+\frac{1}{2k}}\mathcal{K}_r(\mathbb{T}_n) \to 0$ in probability, as $n \to \infty$, for $r \in \{2, \dots, k\}$. Then, by the identity in (5), it is enough to prove Theorem 1 for $\mathcal{K}_1(\mathbb{T}_n)$ instead of $\mathcal{K}(\mathbb{T}_n)$. By the definition of μ_n and Theorem 9, for any bounded continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathbb{E}[g(\sigma^{-1/k} n^{-1+1/2k}\mathcal{K}_1(\mathbb{T}_n))|\mathbb{T}_n] = \int g\,\mathrm{d}\mu_n \xrightarrow{d} \int g\,\mathrm{d}\nu_{2B^{\mathrm{ex}}}, \qquad \text{as } n \to \infty.$$

Taking expectations, the dominated convergence theorem implies that

$$\sigma^{-1/k} n^{-1+1/2k}\mathcal{K}_1(\mathbb{T}_n) \xrightarrow{d} Z_{\mathrm{CRT}},$$

as $n \to \infty$, where Z_{CRT} has distribution $\nu(\cdot) = \mathbb{E}[\nu_{2B^{\mathrm{ex}}}(\cdot)]$. Suppose that $\mathbb{E}[\xi^p] < \infty$ for every $p \in \mathbb{N}$. Lemma 7 implies that every moment of $n^{-1+1/2k}\mathcal{K}_1(\mathbb{T}_n)$ stays bounded as $n \to \infty$ which implies the moment convergence in Theorem 1. It remains to identify the moments of Z_{CRT} (or equivalently ν). Notice that

$$\mathbb{E}[Z_{\mathrm{CRT}}^q] = \int x^q \,\mathrm{d}\nu = \mathbb{E}\left[\int x^q \,\mathrm{d}\nu_{2B^{\mathrm{ex}}}\right] = \mathbb{E}[m_q(2B^{\mathrm{ex}})], \qquad \text{for } q \in \mathbb{N}.$$

For $q \in \mathbb{N}$, let $U_1, \dots, U_q$ be independent random variables with the uniform distribution on $[0,1]$. Let $Y_1, \dots, Y_q$ be the first q points in a Poisson process on $(0,\infty)$ with intensity $x\,\mathrm{d}x$, i.e., $Y_1, \dots, Y_q$ have joint density function $y_1 \cdots y_q e^{-y_q^2/2}$ on $0 < y_1 < \cdots < y_q < \infty$. It is well-known that $L_{2B^{\mathrm{ex}}}(U_1, \dots, U_q) \stackrel{d}{=} Y_q$, see, e.g., [13, Proof of Lemma 5.1]. Defining the function

$$H_{f,q}(\mathbf{t}_q) \coloneqq \int_0^\infty \int_0^{x_1} \cdots \int_0^{x_{q-1}} G_f(\mathbf{t}_q, \mathbf{x}_q)\,\mathrm{d}\bar{\mathbf{x}}_q, \tag{19}$$

we see that

$$\mathbb{E}[m_q(2B^{\mathrm{ex}})] = q!\mathbb{E}[H_{2B^{\mathrm{ex}},q}(\mathbf{U}_q)] = q! \int_0^{y_1} \cdots \int_0^{y_{q-1}} \int_0^\infty y_1 \cdots y_q e^{-y_q^2/2} \widetilde{F}_q(\mathbf{y}_q)\,\mathrm{d}\mathbf{y}_q, \tag{20}$$

where $\mathbf{U}_q = (U_1, \dots, U_q)$, $\mathbf{y}_q = (y_1, \dots, y_q) \in \mathbb{R}_+^q$ and

$$\widetilde{F}_q(\mathbf{y}_q) \coloneqq \int_0^\infty \int_0^{x_1} \cdots \int_0^{x_{q-1}} \exp\left(-\frac{y_1 x_1^k + (y_2 - y_1)x_1^k + \cdots + (y_q - y_{q-1})x_q^k}{k!}\right)\mathrm{d}\bar{\mathbf{x}}_q.$$

Finally, the expression for the moments in Theorem 1 follows by first changing the order of integration in (20) and then by making the change of variables $w_i = y_i - y_{i-1}$ for $2 \le i \le q$. ◀

Following the idea of the proof of Theorem 1, we obtain the following convergence of the first moment of the number of r-records $\mathcal{K}_r(\mathbb{T}_n)$. This provides a proof of [7, Lemma 4.10].

▶ **Lemma 10.** *Let $k \in \mathbb{N}$. Let $\mathbb{T}_n$ be a Galton-Watson tree conditioned on its number of vertices being $n \in \mathbb{N}$ with offspring distribution ξ satisfying (1). For $r \in \{1, \dots k\}$, we have that*

$$n^{-1+\frac{r}{2k}}\mathbb{E}[\mathcal{K}_r(\mathbb{T}_n)] \to \frac{(k!)^{\frac{r}{k}}}{k} \frac{\Gamma(\frac{r}{k})\Gamma\left(1 - \frac{r}{2k}\right)}{\Gamma(r)} \left(\frac{\sigma}{\sqrt{2}}\right)^{\frac{r}{k}}, \quad \text{as } n \to \infty.$$

Proof. The proof follows by a simple adaptation of the argument used in the proof of Theorem 1 by using Lemma 5 (with $a_n = n^{-1/2}$), Lemma 6 and Lemma 8. One only needs to notice that

$$\mathbb{E}\left[\int_0^1 B^{\mathrm{ex}}(t)^{-r/k}\,\mathrm{d}t\right] = 2^{\frac{r}{2k}}\Gamma\left(1 - \frac{r}{2k}\right)$$

which follows from the well-known density function of $B^{\mathrm{ex}}(t)$; see [5, II.1.4]. ◀

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