The Stretch Factor of Hexagon-Delaunay Triangulations

Michael Dennis
Computer Science Division, University of California at Berkeley, Berkeley, CA, USA
michael_dennis@cs.berkeley.edu

Ljubomir Perković
School of Computing, DePaul University, Chicago, IL, USA
lperkovic@cs.depaul.edu

Duru Türkoğlu
School of Computing, DePaul University, Chicago, IL, USA
duru@cs.uchicago.edu

Abstract

The problem of computing the exact stretch factor (i.e., the tight bound on the worst case stretch factor) of a Delaunay triangulation is one of the longstanding open problems in computational geometry. Over the years, a series of upper and lower bounds on the exact stretch factor have been obtained but the gap between them is still large. An alternative approach to solving the problem is to develop techniques for computing the exact stretch factor of "easier" types of Delaunay triangulations, in particular those defined using regular-polygons instead of a circle. Tight bounds exist for Delaunay triangulations defined using an equilateral triangle and a square. In this paper, we determine the exact stretch factor of Delaunay triangulations defined using a regular hexagon: It is 2. We think that the main contribution of this paper are the two techniques we have developed to compute tight upper bounds for the stretch factor of Hexagon-Delaunay triangulations.

2012 ACM Subject Classification Theory of computation → Computational geometry; Theory of computation → Sparsification and spanners; Theory of computation → Shortest paths

Keywords and phrases Delaunay triangulation, geometric spanner, plane spanner, stretch factor, spanning ratio

Digital Object Identifier 10.4230/LIPIcs.SoCG.2020.34

Related Version A full version of the paper is available at [7], `https://arxiv.org/abs/1711.00068`.

1 Introduction

In this paper we consider the problem of computing a tight bound on the worst case stretch factor of a Delaunay triangulation. Given a set P of points on the plane, the Delaunay triangulation T on P is a plane graph such that for every pair $u, v \in P$, (u, v) is an edge of T if and only if there is a *circle* passing through u and v with no point of P in its interior. (This definition assumes that points in P are in general position which we discuss in Section 2.) In this paper, we refer to Delaunay triangulations defined using the *circle* as $\bigcirc$-Delaunay triangulations. The $\bigcirc$-Delaunay triangulation T of P is a plane subgraph of the complete, weighted Euclidean graph $\mathcal{E}^P$ on P in which the weight of an edge is the Euclidean distance between its endpoints. Graph T is also a *spanner*, defined as a subgraph of $\mathcal{E}^P$ with the property that the distance in the subgraph between any pair of points is no more than a constant multiplicative ratio of the distance in $\mathcal{E}^P$ between the points. The constant ratio is referred to as the *stretch factor* (or *spanning ratio*) of the spanner.

The problem of computing a tight bound on the worst case stretch factor of the $\bigcirc$-Delaunay triangulation has been open for more than three decades. In the 1980s, when $\bigcirc$-Delaunay triangulations were not known to be spanners, Chew considered related, "easier"

36th International Symposium on Computational Geometry (SoCG 2020).
Editors: Sergio Cabello and Danny Z. Chen; Article No. 34; pp. 34:1–34:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Table 1 Key stretch factor upper bounds (tight bounds are bold).

Paper	Graph	Upper Bound
[8]	○-Delaunay	$\pi(1+\sqrt{5})/2 \approx 5.08$
[9]	○-Delaunay	$4\pi/(3\sqrt{3}) \approx 2.41$
[10]	○-Delaunay	1.998
[6]	△-Delaunay	**2**
[5]	□-Delaunay	$\sqrt{10} \approx 3.16$
[2]	□-Delaunay	$\mathbf{\sqrt{4+2\sqrt{2}} \approx 2.61}$
[This paper]	⬡-Delaunay	**2**

structures. In 1986 [5], Chew proved that a □-Delaunay triangulation – defined using a *fixed-orientation square* instead of a circle – is a spanner with stretch factor at most $\sqrt{10}$. Following this, Chew proved that the △-Delaunay triangulation – defined using a *fixed-orientation equilateral triangle* – has a stretch factor of 2 [6]. Significantly, this bound is tight: one can construct △-Delaunay triangulations with stretch factor arbitrarily close to 2. Finally, Dobkin et al. [8] showed that the ○-Delaunay triangulation is a spanner as well. The bound on the stretch factor they obtained was subsequently improved by Keil and Gutwin [9] as shown in Table 1. The bound by Keil and Gutwin stood unchallenged for many years until Xia recently improved the bound to below 2 [10].

On the lower bound side, some progress has been made on bounding the worst case stretch factor of a ○-Delaunay triangulation. The trivial lower bound of $\pi/2 \approx 1.5707$ has been improved to 1.5846 [4] and then to 1.5932 [11].

After three decades of research, we know that the worst case stretch factor of ○-Delaunay triangulations is somewhere between 1.5932 and 1.998. Unfortunately, the techniques that have been developed so far seem inadequate for proving a tight stretch factor bound.

Rather than attempting to improve further the bounds on the stretch factor of ○-Delaunay triangulations, we follow an alternative approach. Just like Chew turned to △- and □-Delaunay triangulations to develop insights useful for showing that ○-Delaunay triangulations are spanners, we make use of Delaunay triangulations defined using regular polygons to develop techniques for computing tight stretch factor bounds. Delaunay triangulations based on regular polygons are known to be spanners (Bose et al. [3]). Tight bounds are known for △-Delaunay triangulations [6] and also for □-Delaunay triangulations (Bonichon et al. [2]) as shown in Table 1.

In this paper, we show that the worst case stretch factor of ⬡-Delaunay triangulations is 2. We present an overview of our proof in Section 3. The overview makes use of three lemmas whose detailed proofs are omitted; the proofs (briefly discussed in Sections 4, 5, and 6) are in the full version of the paper [7]. We think that our main contribution consists of two techniques that we use to compute tight upper bounds on the stretch factor of particular types of ⬡-Delaunay triangulations. In Section 7 we review the role of the techniques in the paper and explore their potential to be applied to other kinds of Delaunay triangulations.

2 Preliminaries

We consider a finite set P of points in the two-dimensional plane with an orthogonal coordinate system. The x- and y-coordinates of a point p will be denoted by $\mathtt{x}(p)$ and $\mathtt{y}(p)$, respectively. The Euclidean graph $\mathcal{E}^P$ of P is the complete weighted graph embedded in the plane whose nodes are identified with the points of P. For every pair of nodes p and q, the edge (p, q) represents the segment $[pq]$ and the weight of (p, q) is the Euclidean distance between p and q which is $d_2(p, q) = \sqrt{(\mathtt{x}(p) - \mathtt{x}(q))^2 + (\mathtt{y}(p) - \mathtt{y}(q))^2}$. Our arguments also use the x-coordinate distance between p and q which we denote as $d_x(p, q) = |\mathtt{x}(p) - \mathtt{x}(q)|$.

Let T be a subgraph of $\mathcal{E}^P$. The length of a path in T is the sum of the weights of the edges of the path and the distance $d_T(p, q)$ in T between two points p and q is the length of the shortest path in T between them. T is a t-spanner for some constant $t > 0$ if for every pair of points p, q of P, $d_T(p, q) \leq t \cdot d_2(p, q)$. The constant t is referred to as the *stretch factor* of T.

We define a *family of spanners* to be a set of graphs T^P, one for every finite set P of points in the plane, such that for some constant $t > 0$, every T^P is a t-spanner of $\mathcal{E}^P$. We say that the stretch factor t is exact (tight) for the family (or that the worst case stretch factor is t) if for every $\epsilon > 0$ there exists a set of points P such that T^P is *not* a $(t - \epsilon)$-spanner of $\mathcal{E}^P$.

The families of spanners we consider are various types of Delaunay triangulations on a set P of points in the plane. Given a set P of points on the plane, we say that a convex, closed, simple curve in the plane is empty if it contains no point of P in its interior. The ○-Delaunay triangulation T on P is defined as follows: For every pair $u, v \in P$, (u, v) is an edge of T if and only if there is an empty *circle* passing through u and v. (This definition assumes that points are in general position which in the case of ○-Delaunay triangulations means that no four points of P are co-circular.) If, in the definition, *circle* is replaced by *fixed-orientation square* (e.g., a square whose sides are axis-parallel) or by *fixed-orientation equilateral triangle* then different triangulations are obtained: the □- and the △-Delaunay triangulations.

If, in the definition of the ○-Delaunay triangulation, we change *circle* to *fixed-orientation regular hexagon*, then a ⬡-Delaunay triangulation is obtained. In this paper we focus on such triangulations. While any fixed orientation of the hexagon is possible, we choose w.l.o.g. the orientation that has two sides of the hexagon parallel to the y-axis as shown in Fig. 1-(a). In the remainder of the paper, *hexagon* will always refer to a regular hexagon with such an orientation. We find it useful to label the vertices of the hexagon N, E_N, E_S, S, W_S, and W_N, in clockwise order and starting with the top one. We also label the sides n_e, e, s_e, s_w, w, and n_w as shown in Fig. 1-(a); we will sometimes refer to the s_e and s_w sides as the s sides and to the n_e and n_w sides as the n sides.

The definition of the ⬡-Delaunay triangulation assumes that no four points lie on the boundary of an empty hexagon. Our arguments also assume that no two points lie on a line whose slope matches the slope of a side of the hexagon (i.e. slopes $\infty, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$). The *general position* assumption we therefore make in this paper consists of the above two restrictions. This assumption is made solely for the purpose of simplifying the presentation; the arguments in the paper could be extended so the results apply to all ⬡-Delaunay triangulations. Finally, we need to be aware that unlike the ○-Delaunay triangulation on P, the ⬡-Delaunay (and also the □- and △-Delaunay) triangulation on P may not contain all edges on the convex hull of P. To handle this and simplify our arguments, we add to P six additional points, very close to but not exactly (in order to satisfy the above assumptions) at coordinates $(0, \pm M)$ and $(\pm M \cos(\pi/6), \pm M \sin(\pi/6))$ where $M > 50 \max_{s,t \in P} d_2(s, t)$. The ⬡-Delaunay triangulation on this modified set of points P, consisting of the original triangulation plus additional edges between the new points and original points and also between the new points themselves, includes the edges on the convex hull of P. Also, any path in this triangulation between two points s and t from the original set P with length bounded by $2d_2(s, t)$ cannot possibly use the added points. Thus a proof of our main result for the modified triangulation will also be a proof for the original one and so we assume that the ⬡-Delaunay triangulation on P includes the edges on the convex hull of P.

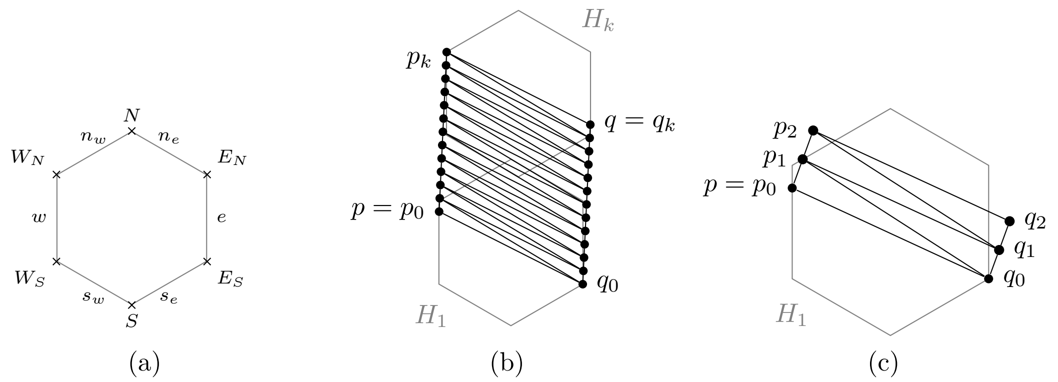

Figure 1 (a) The hexagon orientation and the side and vertex labels that we use (b) A ⬡-Delaunay triangulation with points p, q, p_k, and q_0 having coordinates $(0,0)$, $(1, \frac{1}{\sqrt{3}})$, $(\delta, \frac{2}{\sqrt{3}} - \sqrt{3}\delta)$, and $(1-\delta, -\frac{1}{\sqrt{3}} + \sqrt{3}\delta)$, respectively. For δ small enough, $d_T(p,q) \geq (2-\epsilon)d_2(p,q)$. (c) A closer look at the bottom faces of this triangulation.

We end this section with a lower bound, by Bonichon [1], on the worst case stretch factor of ⬡-Delaunay triangulations. The lower bound construction is illustrated in Fig. 1-(b) and Fig. 1-(c). The proof is omitted but appears in the full version of the paper [7].

▶ **Lemma 1.** *For every $\varepsilon > 0$, there exists a set P of points in the plane such that the ⬡-Delaunay triangulation on P has stretch factor at least $2 - \varepsilon$.*

3 Main result

In this section we state our main result and provide an overview of our proof. We start with a technical lemma that is used to prove the two key lemmas needed for the main result.

3.1 Technical lemma

Let T be the ⬡-Delaunay triangulation on a set of points P in the plane.

▶ **Definition 2.** *Let $T_1, T_2, \dots, T_n$ be a sequence of triangles of T that a line st of finite slope intersects. This sequence of triangles is said to be* linear *w.r.t. line st if for every $i = 1, \dots, n-1$:*

- *triangles T_i and T_{i+1} share an edge, and*
- *line st intersects the interior of that shared edge (not an endpoint).*

Our goal is to prove an upper bound on the length of the shortest path from the "leftmost" point of T_1 to the "rightmost" point of T_n, when certain conditions hold. We introduce some notation and definitions, illustrated in Fig. 2, to make this more precise.

We consider the $n-1$ shared triangle edges intersected by line st from left to right (where left and right are defined with respect to x-coordinates) and label the endpoints of the i-th edge u_i and l_i, with u_i being above line st and l_i below. We note that points typically get multiple labels and identify a point with its label(s). If line st goes through the vertex of T_1 other than u_1 and l_1, we assign that vertex both labels u_0 and l_0 (as shown in Fig. 4); otherwise, we assign labels u_0 and l_0 to the endpoints of the edge of T_1 intersected by line st other than (u_1, l_1), with u_0 being above line st (as shown in Fig. 2). Similarly, if line st goes through the vertex of T_n other than u_{n-1} and l_{n-1}, we assign it both labels u_n and l_n (as shown in Fig. 4); otherwise, we assign labels u_n and l_n to the endpoints of the other edge of T_n intersected by line st, with u_n being above line st (as shown in Fig. 2). Note that for $1 \le i \le n$:

- either $T_i = \triangle(u_i, l_i, l_{i-1})$, in which case we call l_{i-1} and l_i the *left* and *right* vertices of T_i
- or $T_i = \triangle(u_{i-1}, u_i, l_i)$, in which case we call u_{i-1} and u_i the *left* and *right* vertices of T_i.

Note that only one of the above holds, except for T_1 if $u_0 = l_0$ (in which case both hold) or for T_n if $u_n = l_n$ (in which case again both hold). For every $i = 1, \ldots, n$, when $u_i = u_{i-1}$ or $l_i = l_{i-1}$ we call the corresponding vertex of T_i the *base* vertex of T_i. Note that T_1 has no base vertex if $u_0 = l_0$ and T_n has no base vertex if $u_n = l_n$ (as is the case in Fig. 4). Let U and L be the sets of all point labels u_i and l_i, respectively, and let T_{1n} be the union of $T_1, T_2, \ldots, T_n$ which we will refer to as a linear sequence of triangles as well.

Let H_i, for $1 \le i \le n$, be the (empty) hexagon passing through the vertices of T_i; note that this hexagon is unique due to the general position assumption. A vertex of T_i is said to be a w, e, n, or s vertex of T_i if it lies on the w side, e side, one of the n sides, or one of the s sides, respectively, of H_i (see Fig. 2). A left vertex of T_i that is a w vertex of T_i is referred to as a *left induction vertex* of T_i; similarly, a right vertex that is a e vertex is referred to as a *right induction vertex* of T_i.

Note that a base vertex cannot be an induction vertex.

▶ **Definition 3.** *We call an edge* (u_i, l_j) gentle *if its slope is between* $-\frac{1}{\sqrt{3}}$ *and* $\frac{1}{\sqrt{3}}$.

In Fig. 2 no edge (u_i, l_j) is gentle while in Fig. 4 (u_0, l_1) and (u_8, l_8) are gentle.

▶ **Definition 4.** *The linear sequence of triangles* T_{1n} *is* regular *if* T_1 *has a left induction vertex,* T_n *has a right induction vertex, and if, for every* $i = 1, \ldots, n-1$:

- u_i *is* not *a* s *vertex of* T_i *and* T_{i+1},
- l_i *is* not *a* n *vertex of* T_i *and* T_{i+1}, *and*
- (u_i, l_i) *is not gentle.*

The linear sequence in Fig. 2 is regular while the one in Fig. 4 is not (because u_8 lies on the s_w side of H_9 – the red hexagon passing through the vertices of $T_9 = \triangle(u_8, u_9, l_9)$ – and also because edge (u_8, l_8) is gentle).

The proof of the following technical lemma is discussed in Section 5.

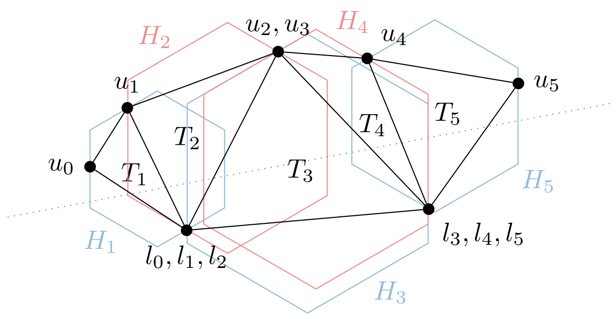

Figure 2 The dotted line (st) intersects the linear sequence of triangles $T_1, T_2, \ldots, T_5$. The vertices of each triangle T_i (u_{i-1}, u_i, l_{i-1}, l_i, two of which are equal) lie on the boundary of the hexagon H_i. Note that l_2 is the left, l_3 is the right, and $u_2 = u_3$ is the base vertex of T_3, for example. The linear sequence is regular since T_1 has a left induction vertex, T_5 has a right induction vertex, and, for $i = 1, \ldots, 4$, no u_i is a s vertex of T_i or T_{i+1}, no l_i is a n vertex of T_i or T_{i+1}, and no (u_i, l_i) is gentle.

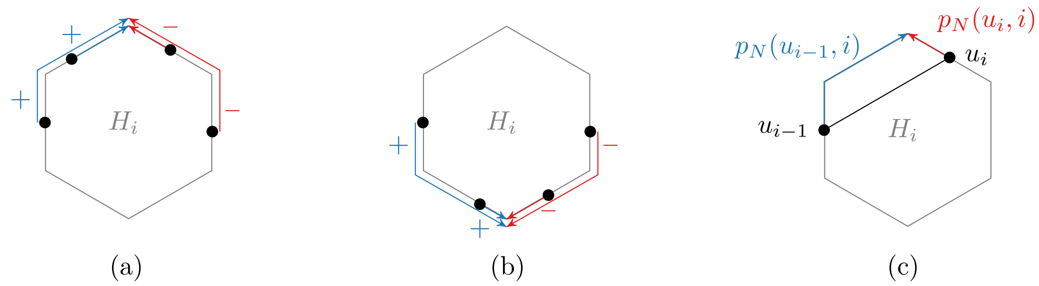

Figure 3 (a) The values of $p_N(o, i)$ are illustrated, for various points o lying on the boundary of H_i, as signed hexagon arc lengths. (b) The values of $p_S(o, i)$ are illustrated similarly.

▶ **Lemma 5** (The Technical Lemma). *If T_{1n} is a regular linear sequence of triangles then there is a path in T_{1n} from the left induction vertex p of T_1 to the right induction vertex q of T_n of length at most $\frac{4}{\sqrt{3}}d_x(p, q)$.*

Actually, what we show in Section 5 implies something stronger: If T_{1n} is regular then the lengths of the *upper* path $p, u_0, \ldots, u_n, q$ and of the *lower* path $p, l_0, \ldots, l_n, q$ add up to at most $\frac{8}{\sqrt{3}}d_x(p, q)$. It is useful to informally describe now the techniques we use to do this. For that purpose we introduce, for a point o on a side of H_i, functions $p_N(o, i)$ and $p_S(o, i)$ as the *signed* shortest distances around the perimeter of H_i from o to the N vertex and S vertex, respectively; the sign is positive if o lies on sides n_w, w, or s_w of H_i and negative otherwise (see Fig. 3-(a) and Fig. 3-(b)).

Note that the length of each edge (u_{i-1}, u_i) (assuming $u_{i-1} \neq u_i$) can be bounded by the distance from u_{i-1} to u_i when traveling clockwise along the sides of H_i. This distance is exactly $p_N(u_{i-1}, i) - p_N(u_i, i)$ as illustrated in Fig. 3-(c). This motivates the following *discrete* function, defined for $i = 0, 1, \ldots, n$ and, for convenience's sake, 1) assuming that $p = u_0$ and $q = u_n$ and 2) using an additional hexagon H_{n+1} of radius 0 centered at point q:

$$\bar{U}(i) = \sum_{j=1}^{i}(p_N(u_{j-1}, j) - p_N(u_j, j)) + p_N(u_i, i + 1).$$

Function $\bar{U}(i)$ can be used to bound the length of upper path fragments; in particular, $\bar{U}(n)$ bounds the length of the upper path from p to q. A function $\bar{L}(i)$ bounding the length of the lower path can be defined similarly. In Section 5, we will compute an upper bound for $\bar{U} + \bar{L}$ by 1) switching the analysis from a discrete one to a continuous one, with functions p_N and p_S defined not in term of index i but in terms of coordinate x for every x between $\mathtt{x}(p)$ and $\mathtt{x}(q)$ and 2) analyzing the growth rates, with respect to x, of the continuous functions p_N, p_S, and $\bar{U} + \bar{L}$. We will show that (the continuous versions of) p_N and p_S are piecewise linear functions with growth rates $\frac{2}{\sqrt{3}}$, $\frac{4}{\sqrt{3}}$, or $\frac{6}{\sqrt{3}}$, and that $\bar{U} + \bar{L}$ is also piecewise linear with growth rate equal to the growth rate of $p_N + p_S$ which can be $\frac{4}{\sqrt{3}}$, $\frac{6}{\sqrt{3}}$, or $\frac{8}{\sqrt{3}}$. Lemma 5 will follow from the last (largest) growth rate.

With the technical lemma in hand, we can now state the first of the two key lemmas that we need to prove our main result.

3.2 The amortization lemma

The first of our two key lemmas is a strengthening of the (Technical) Lemma 5 under two restrictions. The first restriction is that T_{1n} is defined with respect to a line st whose slope m_{st} is restricted to $0 < m_{st} < \frac{1}{\sqrt{3}}$. With that restriction we get the following properties:

▶ **Lemma 6.** *Let T_{1n} be a linear sequence with respect to line st with slope m_{st} such that $0 < m_{st} < \frac{1}{\sqrt{3}}$. For every i s.t. $1 \leq i \leq n$:*

- *If u_{i-1} lies on side s_w of H_i or l_i lies on side n_e of H_i then (u_{i-1}, l_i) is gentle.*
- *If l_{i-1} lies on side n_w of H_i or u_i lies on the s_e side of H_i then (l_{i-1}, u_i) is gentle.*
- *None of the following can occur: u_{i-1} lies on side s_e of H_i, l_i lies on side n_w of H_i, l_{i-1} lies on side n_e of H_i, and u_i lies on the s_w side of H_i.*

Note, for example, that u_8 lies on side s_w of hexagon H_9 in Fig. 4 and that edge (u_8, l_9) is gentle.

Proof. If u_{i-1} lies on side s_w of some hexagon H_i then, since $0 < m_{st} < \frac{1}{\sqrt{3}}$ and by general position assumptions, either $u_{i-1} = u_i$ and l_{i-1} and l_i must lie on sides s_e and e of H_i, respectively, or $l_{i-1} = l_i$ must lie on side s_e or e of H_i. Either way, the slope of the line going through u_{i-1} and l_i must be between $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$. Similar arguments can be used to handle the remaining three cases in the first two bullet points.

Let the left and right intersection points of line *st* with hexagon H_i be h_{i-1} and h_i. Note that when traveling clockwise along the sides of H_i the points will be visited in this order: $h_{i-1}, u_{i-1}, u_i, h_i, l_i, l_{i-1}$. If u_{i-1} lies on side s_e of H_i then $i > 1$ and, because $0 < m_{st} < \frac{1}{\sqrt{3}}$, either u_i (if $u_{i-1} \neq u_i$) or l_i (if $l_{i-1} \neq l_i$) would have to lie on side s_e of H_i as well, which violates our general position assumption for the set of points P. The remaining three cases are handled similarly. ◀

By the above lemma, under the restriction $0 < m_{st} < \frac{1}{\sqrt{3}}$, if T_{1n} has no gentle edge then it is regular and (Technical) Lemma 5 applies. A narrower but much stronger version of (Technical) Lemma 5 applies as well if another restriction is made. To state the second restriction we need some additional terminology.

Let $l_i \in L$ and $u_j \in U$. If $i \leq j$ and $\mathtt{x}(l_i) < \mathtt{x}(u_j)$ then we say that l_i *occurs before* u_j, and if $j \leq i$ and $\mathtt{x}(u_j) < \mathtt{x}(l_i)$ then we say that u_j *occurs before* l_i.

▶ **Definition 7.** *Given points $l_i \in L$ and $u_j \in U$ such that one occurs before the other, a path between them is* gentle *if the length of the path is not greater than $\sqrt{3}d_x(u_j, l_i) - (\mathtt{y}(u_j) - \mathtt{y}(l_i))$.*

See Fig. 4 for an illustration of a gentle path. Note that a gentle edge is a gentle path (e.g., (u_0, l_1) and $(u_8, l_8 = l_9)$ in Fig. 4).

The following is the key to our proof of the main result of this paper:

▶ **Lemma 8** (The Amortization Lemma). *Let T_{1n} be a regular linear sequence with respect to line st with slope m_{st}. If $0 < m_{st} < \frac{1}{\sqrt{3}}$ and if T_{1n} contains no gentle path then there is a path in T_{1n} from the left induction vertex p of T_1 to the right induction vertex q of T_n of length at most $(\frac{5}{\sqrt{3}} - 1)d_x(p, q)$.*

We will discuss the proof of the Amortization Lemma in Section 6; the proof builds on the analysis done in Section 5 to prove (Technical) Lemma 5. Instead of using function $\bar{U}$, however, we consider the discrete function

$$U(i) = d_{T_{1i}}(p, u_i) + p_N(u_i, i + 1)$$

defined for $i = 0, 1, \ldots, n$ and, for convenience's sake, 1) assuming that $p = u_0$ and $q = u_n$ and 2) using additional hexagon H_{n+1} of radius 0 centered at point q. An equivalent discrete function $L(i)$ using points l_i instead of u_i can be defined. Note that $U(n) + L(n)$ is exactly twice the distance in T_{1n} from p to q. To bound $U + L$, we will switch the analysis to a continuous one just as we did for $\bar{U} + \bar{L}$. We will see that, except for a finite number of discontinuities, the continuous version of $U + L$ has the same growth rate as $\bar{U} + \bar{L}$, which

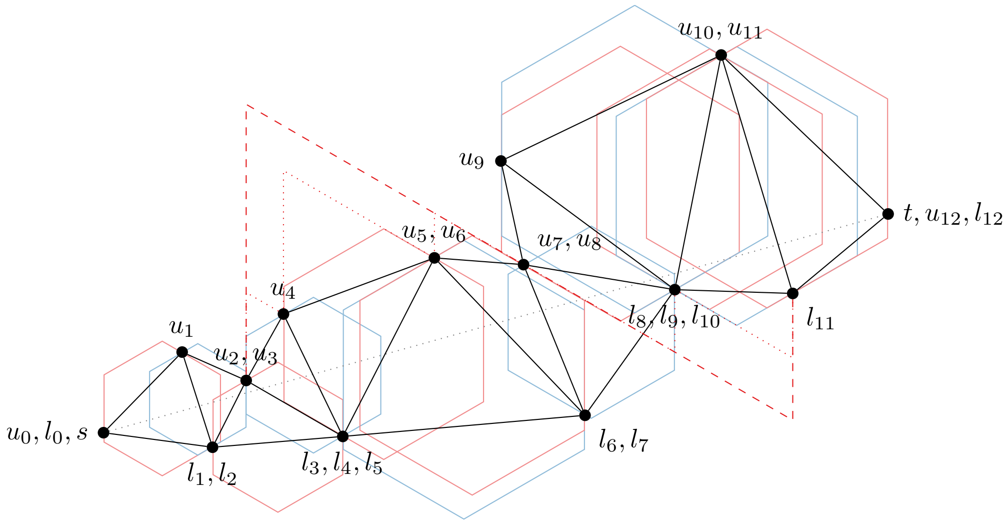

Figure 4 A gentle path from u_2 to l_{11} is one whose length is at most $\sqrt{3}d_x(u_2, l_{11}) - (\mathtt{y}(u_2) - \mathtt{y}(l_{11}))$, i.e. the length of the red dashed piecewise linear curve from u_2 to l_{11} (consisting of two vertical segments and a third with slope $-\frac{1}{\sqrt{3}}$). The path $u_2 = u_3, u_4, u_5 = u_6, u_7 = u_8, l_8 = l_9 = l_{10}, l_{11}$, easily seen to be bounded–in length–by the red dotted piecewise linear curve, is gentle. This path can be extended with edge (l_{11}, t) to a canonical gentle path from u_2 to t; the proof of (Main) Lemma 13, in this particular case, combines the bound on the length of this path together with the bound on the length of a path from s to u_2 obtained via induction.

is the growth rate of $p_N + p_S$. We will consider the intervals when the growth rate of (the continuous version of) $U + L$ is higher than $2(\frac{5}{\sqrt{3}} - 1)$ (i.e., when its growth rate is $\frac{8}{\sqrt{3}}$) and we will amortize the extra $2 - \frac{2}{\sqrt{3}}$ growth over intervals when its growth rate is smaller than $2(\frac{5}{\sqrt{3}} - 1)$ (i.e., when its growth rate is $\frac{4}{\sqrt{3}}$ or $\frac{6}{\sqrt{3}}$). The amortization can usually be done because when the growth rate of $U + L$ is large, the intervals must be relatively short compared to intervals when its growth is smaller, otherwise a gentle path can be shown to exist. To get our tight bound however, we will need to do more and show that at certain points (which are points of discontinuity) we need to use "cross-edges" (l_i, u_i). This is because when the amortization is not possible there is a long enough interval, say from hexagon H_i to hexagon H_j, when the growth rate of $U + L$ is $\frac{8}{\sqrt{3}}$ most of the time. It turns out that in that case one of U or L has growth rate bounded by $\frac{2}{\sqrt{3}}$ (say, U) and the other (L) by $\frac{6}{\sqrt{3}}$. This means that path $l_i, l_{i+1}, \ldots, l_j$ has relatively large length with respect to $\Delta(x)$ and that $u_i, u_{i+1}, \ldots, u_j$ is a relatively short path that can be used to replace the long subpath $l_i, l_{i+1}, \ldots, l_j$ with the shorter subpath $l_i, u_i, u_{i+1}, \ldots, u_j, l_j$ in a path from p to q. The $\frac{5}{\sqrt{3}} - 1$ stretch factor bound is the result of a min-max optimization between the two subpaths from l_i to l_j, and it is tight as we show in Section 7.

Next we turn to the case when the sequence of triangles T_{1n} contains a gentle path.

3.3 The gentle path lemma

Just as in the previous subsection, we consider a linear sequence of triangles T_{1n} defined with respect to a line st with slope m_{st} satisfying $0 < m_{st} < \frac{1}{\sqrt{3}}$. We now consider the case when T_{1n} contains a gentle path and state the other of our two key lemmas. We start with two definitions:

▶ **Definition 9.** *We say that linear sequence T_{1n} is* standard *if T_1 has a left induction vertex or $u_0 = l_0$, T_n has a right induction vertex or $u_n = l_n$, and neither the base vertex of T_1 (if any) nor the base vertex of T_n (if any) is the endpoint of a gentle path in T_{1n}.*

Note that if $u_0 = l_0$ and $u_n = l_n$ both hold (i.e., line st goes through those points) then T_{1n} is trivially standard because T_1 and T_n cannot have base vertices.

▶ **Definition 10.** *Let T_{1n} be a standard linear sequence. A gentle path in T_{1n} from p to q, where p occurs before q, is* canonical *in T_{1n} (or simply* canonical *if T_{1n} is clear from the context) if p is a right induction vertex of T_i for some $i \geq 1$ or p is the left vertex of T_1 and if q is a left induction vertex of T_j for some $j \leq n$ or q is the right vertex of T_n.*

For example, the gentle path $u_2 = u_3, u_4, u_5 = u_6, u_7 = u_8, l_8 = l_9 = l_{10}, l_{11}, l_{12}$ in Fig. 4 is canonical.

The second key lemma, which we will use alongside (Amortization) Lemma 8 to prove our main result, is stated next; its proof is discussed in Section 4.

▶ **Lemma 11** (The Gentle Path Lemma). *Let T_{1n} be a linear sequence of triangles with respect to a line st with slope m_{st} such that $0 < m_{st} < \frac{1}{\sqrt{3}}$. If T_{1n} is standard and contains a gentle path then the path can be extended to a canonical gentle path in T_{1n}.*

The main idea behind the proof of this lemma is that a gentle path between $u_r \in U$ and $l_s \in L$ (where, say, $r \leq s$ and $\mathtt{x}(u_r) < \mathtt{x}(u_s)$) in T_{1n} can be extended using edge (u_{r-1}, u_r), unless $r = 0$ or u_r is a right induction vertex of T_r, or using edge (l_s, l_{s+1}), unless $s = n$ or l_s is a left induction vertex of T_{s+1}. In other words, a gentle path can be extended unless it is canonical.

We are now ready to state our main result and provide a proof that uses the two key lemmas.

3.4 The main result and the main lemma

▶ **Theorem 12.** *The stretch factor of a ⬡-Delaunay triangulation is at most* 2.

To prove Theorem 12 we need to show that between any two points s and t of a set of points P there is, in the ⬡-Delaunay triangulation T on P, a path from s to t of length at most $2d_2(s,t)$. Let m_{st} be the slope of the line st passing through s and t. Thanks to the hexagon's rotational and reflective symmetries as well as our general position assumptions, we can rotate the plane around s and possibly reflect the plane with respect to the x-axis to ensure that $0 < m_{st} < \frac{1}{\sqrt{3}}$. Given this assumption, our main theorem will follow from:

▶ **Lemma 13** (The Main Lemma). *For every pair of points $s, t \in P$ with $0 < m_{st} < \frac{1}{\sqrt{3}}$:*

$$d_T(s,t) \leq \max\left\{\frac{5}{\sqrt{3}} - 1, \sqrt{3} + m_{st}\right\} d_x(s,t). \tag{1}$$

Before we prove this lemma, we show that it implies the main theorem.

Proof of Theorem 12. W.l.o.g., we assume that s has coordinates $(0,0)$, t lies in the positive quadrant, $m_{st} < \frac{1}{\sqrt{3}}$, and $d_2(s,t) = 1$. With these assumptions it follows that $\frac{\sqrt{3}}{2} < \mathtt{x}(t) = d_x(s,t) < 1$ and we need to show that $d_T(s,t) \leq 2$.

By Lemma 13, either $d_T(s,t) \leq (\frac{5}{\sqrt{3}} - 1)d_x(s,t) \leq (\frac{5}{\sqrt{3}} - 1) < 2$ or

$$d_T(s,t) \leq (\sqrt{3} + m_{st})d_x(s,t) = \sqrt{3}d_x(s,t) + d_y(s,t) = \sqrt{3}d_x(s,t) + \sqrt{1 - d_x(s,t)^2}$$

which attains its maximum, over the interval $[\frac{\sqrt{3}}{2}, 1]$, at $d_x(s,t) = \frac{\sqrt{3}}{2}$ giving $d_T(s,t) \leq 2$. ◀

We now turn to the proof of (Main) Lemma 13. We start by noting that if there is a point p of P on the segment $[st]$ then (1) would follow if (1) holds for the pairs of points s, p and p, t; we can therefore assume that no point of P other than s and t lies on the segment $[st]$. We can also assume, as argued in Section 2, that segment $[st]$ does not intersect the outer face of the triangulation T. We assume w.l.o.g. that s has coordinates $(0, 0)$ and thus t lies in the positive quadrant.

Let $T_1, T_2, T_3, \ldots, T_n$ be the sequence of triangles of the triangulation T that line segment $[st]$ intersects when moving from s to t (refer to Fig. 4). (Recall that we assume that segment $[st]$ does not intersect the outer face of T.) Clearly, T_{1n} is a linear sequence of triangles and we assign labels u_i and l_i to the points and define sets U and L as described in Subsection 3.1. We note that all arguments in the rest of this paper use only points and edges of T_{1n}.

Note that, since st must intersect the interior of H_1, s can only lie on the n_w, w, or s_w sides of H_1; by Lemma 6, if s lies on the n_w side of H_1 then $(s, u_1) = (l_0, u_1)$ is gentle, and if s lies on the s_w side of H_1 then $(s, l_1) = (u_0, l_1)$ is gentle. Similarly, t can only lie on the n_e, e, or s_e sides of H_n; if t lies on the s_w side of H_n then $(t, l_{n-1}) = (u_n, l_{n-1})$ is gentle, and if t lies on the n_w side of H_n then $(t, u_{n-1}) = (l_n, u_{n-1})$ is gentle. Note that this means that if T_{1n} has no gentle edge then it is regular.

We now informally describe the approach we use to prove (Main) Lemma 13. We first note that (Amortization) Lemma 8 and (Gentle Path) Lemma 11 rely on (Technical) Lemma 5. We will prove (Main) Lemma 13 that bounds the length of the shortest path in T_{1n} from s to t as follows. If T_{1n} does not contain a gentle path then it is regular and the proof follows from (Amortization) Lemma 8. If T_{1n} contains a gentle path then by (Gentle Path) Lemma 11 it must contain a canonical gentle path $\mathcal{G}$ from, in general, a right induction vertex of T_i to a left induction vertex of T_j, where $0 \leq i < j \leq n$. We can assume, using (Gentle Path) Lemma 11, that $\mathcal{G}$ is maximal in the sense that it is not a subpath of any other gentle path in T_{1n}. The maximality of $\mathcal{G}$ will guarantee that neither T_{1i} nor T_{jn} contains a gentle path whose endpoint is the base vertex of T_i or the base vertex of T_j, respectively. Therefore T_{1i} and T_{jn} are standard and we then proceed by induction to prove a "more general" version of (Main) Lemma 13 for T_{1i} and T_{jn}. The obtained bounds on the lengths of shortest paths from s to the right induction vertex of T_i and from the left induction vertex of T_j to t are combined with the bound on the length of gentle path $\mathcal{G}$ to complete the proof of (Main) Lemma 13. Our reliance on induction means that we need to restate the Main Lemma so it is amenable to an inductive proof:

▶ **Lemma 14** (The Generalized Main Lemma). *Let $s, t \in P$ such that $0 < m_{st} < \frac{1}{\sqrt{3}}$ and let T_{1n} be the linear sequence of triangles that segment $[st]$ intersects. If T_{ij}, for some i, j such that $1 \leq i \leq j \leq n$, is standard, p is the left vertex of T_i, and q is the right vertex of T_j then*

$$d_{T_{ij}}(p, q) \leq \max\{\frac{5}{\sqrt{3}} - 1, \sqrt{3} + m_{st}\} d_x(p, q).$$

Note that (Main) Lemma 13 is a special case of this statement when $i = 1$ and $j = n$ since T_{1n} is (trivially) standard, s is the left vertex of T_1, and t is the right vertex of T_n.

Proof. We proceed by induction on $j - i$. If T_{ij} is standard and there is no gentle path in T_{ij} (the base case) then, by Lemma 6, the linear sequence of triangles in T_{ij} is regular and thus, by (Amortization) Lemma 8, we have $d_T(p, q) \leq (\frac{5}{\sqrt{3}} - 1) d_x(p, q)$.

If T_{ij} is standard and there is a gentle path in T_{ij}, then, by Lemma 11, there exist points $u_{i'}$ and $l_{j'}$ in T_{ij} such that there is a canonical gentle path between $u_{i'}$ and $l_{j'}$ in T_{ij}. We also assume that the canonical path between $u_{i'}$ and $l_{j'}$ is maximal in the sense that it is not a proper subpath of a gentle path in T_{ij}. W.l.o.g., we assume that $u_{i'}$ occurs before $l_{j'}$, and so $i-1 \leq i' \leq j' \leq j$, $\mathtt{x}(u_{i'}) < \mathtt{x}(l_{j'})$, and $d_T(u_{i'}, l_{j'}) \leq \sqrt{3}d_x(u_{i'}, l_{j'}) - (\mathtt{y}(u_{i'}) - \mathtt{y}(l_{j'}))$. Since $u_{i'}$ is either s or above st and $l_{j'}$ is either t or below st, it follows that $-(\mathtt{y}(u_{i'}) - \mathtt{y}(l_{j'})) \leq m_{st}d_x(u_{i'}, l_{j'})$. Therefore, $d_T(u_{i'}, l_{j'}) \leq (\sqrt{3} + m_{st})d_x(u_{i'}, l_{j'})$.

Since the gentle path from $u_{i'}$ to $l_{j'}$ is canonical, either $u_{i'}$ is a right induction vertex of $T_{i'}$ and $i' \geq i$ or $u_{i'} = u_{i-1}$. In the first case, because $u_{i'}$ is on side e of $H_{i'}$ the base vertex $l_{i'-1} = l_{i'}$ of $T_{i'}$ must satisfy $\mathtt{x}(l_{i'}) < \mathtt{x}(u_{i'})$. Suppose that $l_{i'}$ is the endpoint of a gentle path in $T_{ii'}$ from, say, point $u_{i''}$ then we would have

$$\begin{aligned} d_{T_{ij}}(u_{i''}, l_{j'}) &\leq d_{T_{ij}}(u_{i''}, l_{i'}) + d_2(l_{i'}, u_{i'}) + d_{T_{ij}}(u_{i'}, l_{j'}) \\ &\leq \sqrt{3}d_x(u_{i''}, l_{i'}) - (\mathtt{y}(u_{i''}) - \mathtt{y}(l_{i'})) + \sqrt{3}d_x(l_{i'}, u_{i'}) - (\mathtt{y}(l_{i'}) - \mathtt{y}(u_{i'})) \\ &\quad + \sqrt{3}d_x(u_{i'}, l_{j'})) - (\mathtt{y}(u_{i'}) - \mathtt{y}(l_{j'})) \\ &\leq \sqrt{3}d_x(u_{i''}, l_{j'}) - (\mathtt{y}(u_{i''}) - \mathtt{y}(l_{j'}). \end{aligned}$$

This contradicts the maximality of the canonical gentle path from $u_{i'}$ to $l_{j'}$. This means that $l_{i'}$ is not the endpoint of a gentle path in $T_{ii'}$. Since $u_{i'}$ is a right induction vertex of $T_{i'}$, it follows that $T_{ii'}$ is standard, the inductive hypothesis applies, and $d_T(p, u_{i'}) \leq \max\{\frac{5}{\sqrt{3}} - 1, \sqrt{3} + m_{st}\}d_x(p, u_{i'})$. In the second case, because T_{ij} is standard, $u_{i'}$ cannot be the base vertex of T_i and so $u_{i'} = p$ and the same inequality holds trivially.

Similarly, we can show that $d_T(l_{j'}, q) \leq \max\{\frac{5}{\sqrt{3}} - 1, \sqrt{3} + m_{st}\}d_x(l_{j'}, q)$. Thus:

$$\begin{aligned} d_T(p, q) &\leq d_T(p, u_{i'}) + d_T(u_{i'}, l_{j'}) + d_T(l_{j'}, q) \\ &\leq \max\{\frac{5}{\sqrt{3}} - 1, \sqrt{3} + m_{st}\}(d_x(p, u_{i'}) + d_x(l_{j'}, q)) + (\sqrt{3} + m_{st})d_x(u_{i'}, l_{j'}) \\ &\leq \max\{\frac{5}{\sqrt{3}} - 1, \sqrt{3} + m_{st}\}d_x(p, q) \end{aligned}$$ ◀

4 Proof of (gentle path) lemma 11

The main idea behind the proof of the Gentle Path lemma is that a gentle path between $u_r \in U$ and $l_s \in L$ (where, say, $r \leq s$ and $\mathtt{x}(u_r) < \mathtt{x}(u_s)$) in T_{1n} can be extended using edge (u_{r-1}, u_r), unless $r = 0$ or u_r is a right induction vertex of T_r, or using edge (l_s, l_{s+1}), unless $s = n$ or l_s is a left induction vertex of T_{s+1}. In other words, a gentle path from r to s is either canonical or can be extended to a canonical path from $u_{r'}$ to $l_{s'}$ as illustrated in Fig. 5.

5 Proof of (technical) lemma 5

We prove this lemma via a framework that uses continuous versions of the discrete functions (p_N, p_S, etc.) informally introduced in Subsection 3.1. We start by defining functions $H(x)$, $T(x)$, $u(x)$, $\ell(x)$, $\mathtt{r}(x)$, $w(x)$, and $e(x)$ for $\mathtt{x}(p) \leq x \leq \mathtt{x}(q)$ as illustrated in Fig. 6. Let point c_i be the center of hexagon H_i, for $i = 1, \ldots, n$. For x such that $\mathtt{x}(c_i) \leq x < \mathtt{x}(c_{i+1})$, $H(x)$ is the hexagon whose center has abscissa x and that has points $u_i = u(x)$ and $l_i = \ell(x)$ on its boundary. Intuitively, function $H(x)$ from $x = \mathtt{x}(c_i)$ to $x = \mathtt{x}(c_{i+1})$ models the "pushing" of hexagon H_i through u_i and l_i up until it becomes H_{i+1}. Function $\mathtt{r}(x)$ is the minimum radius of $H(x)$ and $w(x) = x - \mathtt{r}(x)$ and $e(x) = x + \mathtt{r}(x)$ are the abscissas of the w and e sides, respectively, of $H(x)$. Finally, we define $T(x) = T_{1i}$ when $\mathtt{x}(c_i) \leq x < \mathtt{x}(c_{i+1})$.

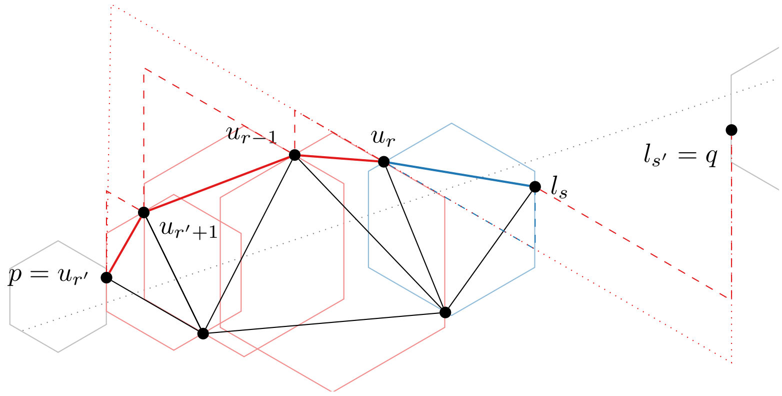

Figure 5 Illustration of the proof of Lemma 11 in the case when the gentle path from u_r to l_s is just a gentle edge. For every i such that $r' < i \le r$ and u_i is the right vertex of H_i, hexagon H_i and the edge (u_{i-1}, u_i) are shown in red. Each edge (u_{i-1}, u_i) has slope greater than $-\frac{1}{\sqrt{3}}$ and therefore has length bounded by $\sqrt{3}d_x(u_{i-1}, u_i) - (\mathtt{y}(u_{i-1}) - \mathtt{y}(u_i))$, a value equal to the total length of the two intersecting, red, dashed segments going north from u_{i-1} and north-west from u_i. The total length of the two dashed blue segments is an upper bound on the length of the edge (u_r, l_s) and the total length of the dotted red line segments represent the upper bound $\sqrt{3}d_x(p,q) - (\mathtt{y}(p) - \mathtt{y}(q))$ on the length of the path $p = u_{r'}, \ldots, u_r, l_s, l_{s+1}, \ldots, l_{s'} = q$.

For a point o on a side of $H(x)$, we define functions $p_N(o,x)$ and $p_S(o,x)$ as the *signed* shortest distances around the perimeter of $H(x)$ to the N vertex and S vertex, respectively, with sign $\operatorname{sgn}(x - x(o))$. As Fig. 7-(a) and Fig. 7-(b) illustrate, these signs are positive for o on the n_w, w, or s_w sides of $H(x)$ and negative for o on n_e, e, or s_e sides. We omit o and use the shorthand notation $p_N(x)$ if $o = u(x)$ and $p_S(x)$ if $o = \ell(x)$.

Functions $U(x)$ and $L(x)$, used to bound the length of the shortest path from p to q and illustrated in Fig. 8-(a), are defined as follows for $\mathtt{x}(p) \le x \le \mathtt{x}(q)$:

$$U(x) = d_{T(x)}(p, u(x)) + p_N(x) \qquad\qquad L(x) = d_{T(x)}(p, \ell(x)) + p_S(x)$$

We note that $U(\mathtt{x}(q)) + L(\mathtt{x}(q))$ is exactly twice the distance in T_{1n} from p to q. We will compute an upper bound for function $U + L$ by bounding its growth rate.

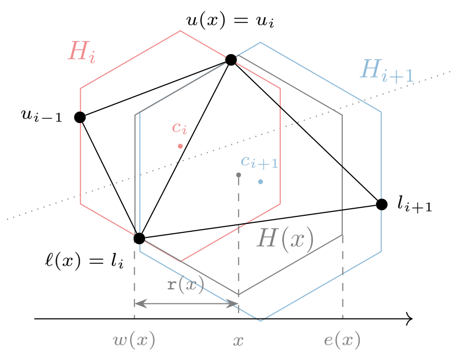

Figure 6 Intuitively, function $H(x)$ from $x = \mathtt{x}(c_i)$ to $x = \mathtt{x}(c_{i+1})$ models the “pushing” of hexagon H_i through u_i and l_i up until it becomes H_{i+1}.

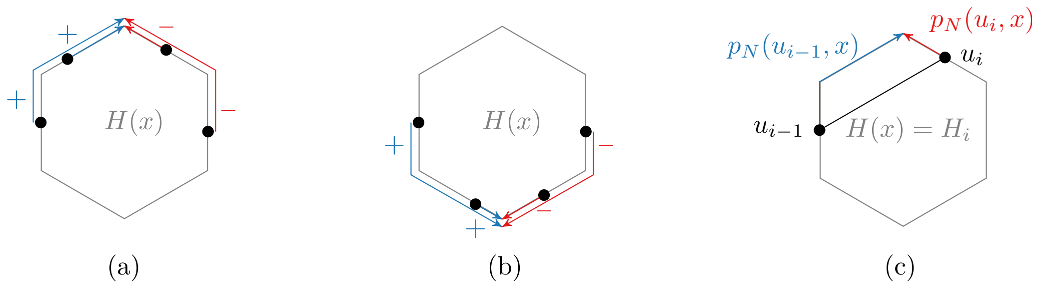

Figure 7 (a) The values of $p_N(o, x)$ are shown, for various points o lying on the boundary of $H(x)$, as signed hexagon arc lengths. (b) The values of $p_S(o, x)$ are shown similarly.

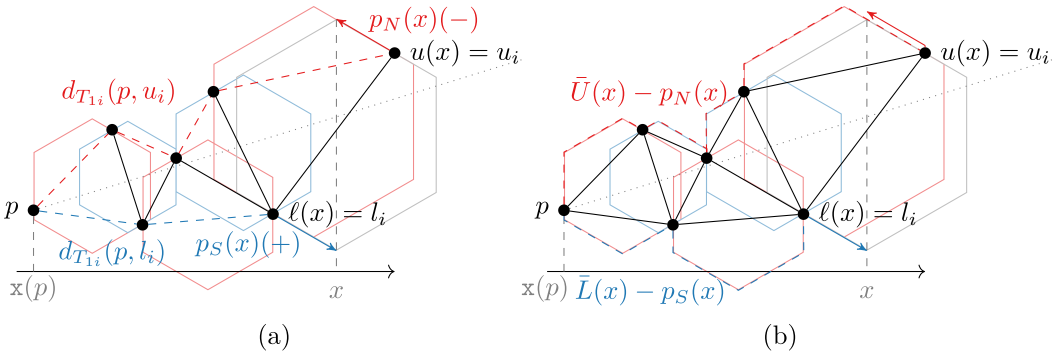

Figure 8 (a) Definition of $U(x)$ and $L(x)$. For example, $U(x)$ for $\mathtt{x}(c_i) \leq x < \mathtt{x}(c_{i+1})$ is the sum of the length of the shortest path from p to u_i in T_{1i} (illustrated as the red dashed path) and $p_N(x)$ (of negative value and represented as a red arrow). (b) Definition of $\bar{U}(x)$ and $\bar{L}(x)$. When $\mathtt{x}(c_i) \leq x < \mathtt{x}(c_{i+1})$ for example, $\bar{U}(x) - p_N(x)$ is an upper bound (equal to the length of the sequence of red dashed hexagon arcs going from p to u_i) on the length of the upper path $p, u_0, u_1, \ldots, u_{i-1}, u_i$.

As Fig. 7-(c) illustrates, the length of each edge (u_{i-1}, u_i), with u_{i-1}, u_i lying on the boundary of $H(x) = H_i$, is bounded by $p_N(u_{i-1}, x) - p_N(u_i, x)$. This and a similar insight about each (l_{i-1}, l_i) motivate functions $\bar{U}(x)$ and $\bar{L}(x)$ that bound the lengths of the upper and lower paths in T_{1n} and that are defined as follows for $\mathtt{x}(c_i) \leq x \leq \mathtt{x}(c_{i+1})$ (see Fig. 8-(b)):

$$\bar{U}(x) = \sum_{j=1}^{i} (p_N(u_{j-1}, \mathtt{x}(c_j)) - p_N(u_j, \mathtt{x}(c_j))) + p_N(x)$$

$$\bar{L}(x) = \sum_{j=1}^{i} (p_S(l_{j-1}), \mathtt{x}(c_j)) - p_S(l_j, \mathtt{x}(c_j))) + p_S(x)$$

When $\mathtt{x}(c_i) < x < \mathtt{x}(c_{i+1})$, functions $\bar{U}(x)$ and $\bar{L}(x)$ as well as $U(x)$ and $L(x)$ have rates of growth that depend solely on the last term ($p_N(x)$ or $p_S(x)$). We show that functions p_N and p_S are monotonically increasing piecewise linear and bound the rate of growth of p_N and p_S using elementary geometric arguments illustrated in Fig. 9. Figure 9-(c) illustrates a case when the growth rate of $p_N + p_S$, and therefore also of $\bar{U} + \bar{L}$ and of $U + L$, is $\frac{8}{\sqrt{3}}$.

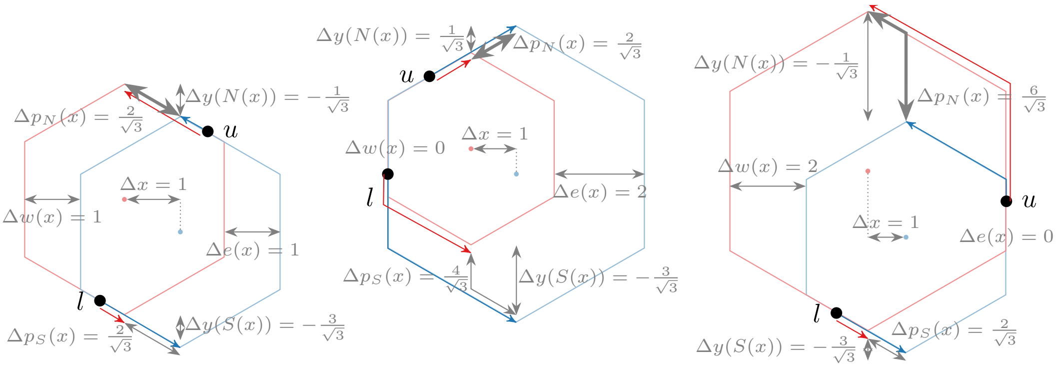

Figure 9 Constructions demonstrating growth rates, with respect to $\Delta x = 1$, of p_N, p_S and other functions for three different placements of $u(x) = u$ and $\ell(x) = l$ on the boundary of $H(x)$.

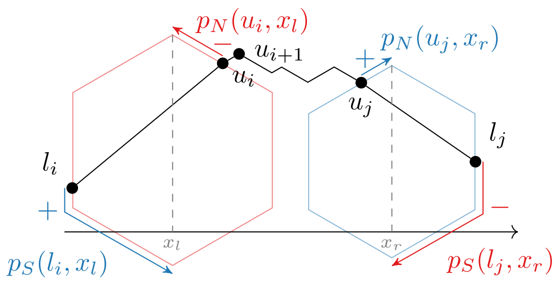

Figure 10 Illustrated is a situation in which the growth rate of $p_S(x)$ is $\frac{6}{\sqrt{3}}$ between $x = x_l$ and $x = x_r$. In that case the growth rate of $p_N(x)$ is $\frac{2}{\sqrt{3}}$. For large enough such intervals $[x_l, x_r]$, the path $l_i, u_i, u_{i+1}, \ldots, u_j, l_j$ is a shortcut for $l_i, l_{i+1}, \ldots, l_j$ and therefore $L(x_r)$ is smaller than what the growth rate of $p_S(x)$ would indicate. The stretch factor bound we obtain is the result of a min-max optimization between the two subpaths from l_i to l_j, and it is tight as we show in Fig. 11.

6 Proof of (amortization) lemma 8

The proof of the lemma builds on the framework discussed in the previous section and on a careful analysis of the growth rates of p_N and p_S when T_{1n} contains no gentle path. We show that in that case the average growth rate of $U + L$ is at most $2\left(\frac{5}{\sqrt{3}} - 1\right)$.

Our main approach is to spread (i.e., amortize) the "extra" $\frac{2}{\sqrt{3}}$ of the $\frac{8}{\sqrt{3}}$ growth rate over wider intervals of time that, as we show, include time intervals during which the growth rate is smaller. To achieve our tight bound of $2\left(\frac{5}{\sqrt{3}} - 1\right)$, however, we need to do more and also include "cross-edges" (l_i, u_i) as illustrated in Fig. 10.

7 Conclusion

The approach we use to bound the length of the shortest path in a Delaunay triangulation T between points s and t is to consider the linear sequence T_{1n} of triangles of T that segment $[st]$ intersects. We show that, in general, T_{1n} can be split into 1) disjoint linear sequences of triangles $T_{i_1 j_1}, T_{i_2 j_2}, \ldots, T_{i_k, j_k}$ that contain no gentle path and 2) $k-1$ gentle paths with a gentle path connecting the right vertex of T_{j_l} with the left vertex of $T_{i_{l+1}}$ for $l = 1, \ldots, k-1$.

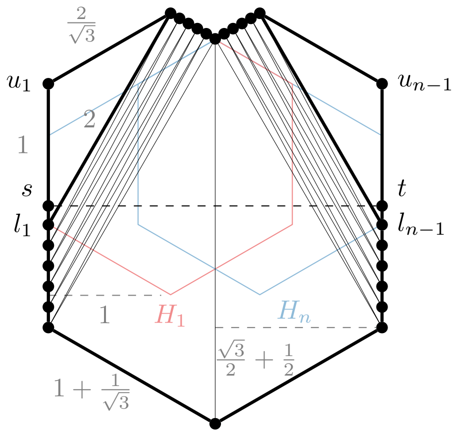

Figure 11 The Mickey Mouse ⬡-Delaunay triangulation. The inradii of H_1 and H_n are both set to 1. Edges that belong to a shortest path from s to t are in bold.

The worst case stretch factor for the Delaunay triangulation is then the maximum between the worst case stretch factors for 1) a path connecting the leftmost and rightmost points in a linear sequence T_{ij} that contains no gentle path and 2) a gentle path.

(Main) Lemma 13 and Lemma 1 show that the worst case stretch factor for ⬡-Delaunay triangulations comes from gentle path constructions. It turns out that similar conclusions can also be made regarding △- and □-Delaunay triangulations.

For ○-Delaunay triangulations, the situation seems to be different. The lower bound construction by Bose et al. [4] corresponds to a gentle path construction and has stretch factor 1.5846. The lower bound construction by Xia and Zhang [11] corresponds to a linear sequence that contains no gentle path and has stretch factor 1.5932. We think that the worst case stretch factor for ○-Delaunay triangulations will come from a construction similar to the one by Xia and Zhang [11]. Therefore, to get a tight bound on the stretch factor of a ○-Delaunay triangulation one needs to develop techniques that give tight bounds on the stretch factor of a linear sequence that contains no gentle path.

We have done so for ⬡-Delaunay triangulations. Our (Amortization) Lemma 8 implies that for ⬡-Delaunay triangulations the worst case stretch factor for a linear sequence T_{ij} with no gentle paths is $(\frac{5}{\sqrt{3}} - 1)$. It turns out that our analysis is tight: Figure 11 shows a construction–which we name the Mickey Mouse ⬡-Delaunay triangulation–that, for any $\epsilon > 0$, can be extended to a ⬡-Delaunay triangulation whose shortest path between s and t is at least $(\frac{5}{\sqrt{3}} - 1)d_x(s,t) - \epsilon$. Unsurprisingly, the construction corresponds to the lower bound construction by Xia and Xhang [11] for ○-Delaunay triangulations.

Based on this we think that the techniques we developed for obtaining the tight bound in Lemma 8 will be useful in obtaining better upper bounds for the stretch factor of other kinds of Delaunay triangulations.

References

1 Nicolas Bonichon, 2011. Personal communication.

2 Nicolas Bonichon, Cyril Gavoille, Nicolas Hanusse, and Ljubomir Perković. Tight stretch factors for L_1- and L_∞-Delaunay triangulations. *Computational Geometry*, 48(3):237–250, 2015. doi:10.1016/j.comgeo.2014.10.005.

3 Prosenjit Bose, Paz Carmi, Sebastien Collette, and Michiel Smid. On the stretch factor of convex Delaunay graphs. *Journal of Computational Geometry*, 1(1):41–56, 2010. doi:10.20382/jocg.v1i1a4.

4 Prosenjit Bose, Luc Devroye, Maarten Löffler, Jack Snoeyink, and Vishal Verma. Almost all Delaunay triangulations have stretch factor greater than $\pi/2$. *Computational Geometry*, 44(2):121–127, 2011. doi:10.1016/j.comgeo.2010.09.009.

5 L. Paul Chew. There is a planar graph almost as good as the complete graph. In *Proceedings of the 2nd Annual ACM Symposium on Computational Geometry (SoCG)*, pages 169–177, 1986. doi:10.1145/10515.10534.

6 L. Paul Chew. There are planar graphs almost as good as the complete graph. *Journal of Computer and System Sciences*, 39(2):205–219, 1989. doi:10.1016/0022-0000(89)90044-5.

7 Michael Dennis, Ljubomir Perković, and Duru Türkoğlu. The stretch factor of hexagon-delaunay triangulations, 2017. arXiv:1711.00068.

8 David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete & Computational Geometry*, 5(4):399–407, 1990. doi:10.1007/BF02187801.

9 J. Mark Keil and Carl A. Gutwin. Classes of graphs which approximate the complete Euclidean graph. *Discrete & Computational Geometry*, 7(1):13–28, 1992. doi:10.1007/BF02187821.

10 Ge Xia. The stretch factor of the Delaunay triangulation is less than 1.998. *SIAM Journal on Computing*, 42(4):1620–1659, 2013. doi:10.1137/110832458.

11 Ge Xia and Liang Zhang. Toward the tight bound of the stretch factor of Delaunay triangulations. In *Proceedings of the 23rd Annual Canadian Conference on Computational Geometry (CCCG)*, 2011. URL: http://www.cccg.ca/proceedings/2011/papers/paper57.pdf.