# Combinatorial Properties of Self-Overlapping Curves and Interior Boundaries


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## Abstract

We study the interplay between the recently-defined concept of *minimum homotopy area* and the classical topic of *self-overlapping curves*. The latter are plane curves that are the image of the boundary of an immersed disk. Our first contribution is to prove new sufficient combinatorial conditions for a curve to be self-overlapping. We show that a curve $\gamma$ with Whitney index 1 and without any self-overlapping subcurves is self-overlapping. As a corollary, we obtain sufficient conditions for self-overlappingness solely in terms of the Whitney index of the curve and its subcurves. These results follow from our second contribution, which shows that any plane curve $\gamma$, modulo a basepoint condition, is transformed into an *interior boundary* by wrapping around $\gamma$ with Jordan curves. In fact, we show that $n + 1$ wraps suffice, where $\gamma$ has $n$ vertices. Our third contribution is to prove the equivalence of various definitions of self-overlapping curves and interior boundaries, often implicit in the literature. We also introduce and characterize *zero-obstinance curves*, a further generalization of interior boundaries defined by optimality in minimum homotopy area.



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**Related Version** This paper is based on the honors thesis of the first author [5]. A full version of this paper [7] is available at `https://arxiv.org/abs/2003.13595`.

**Supplementary Material** An accompanying computer program that can determine whether a plane curve is self-overlapping, compute its minimum homotopy area, and display the self-overlapping decomposition associated with a minimum homotopy is available for download [6], `http://www.cs.tulane.edu/~carola/research/code.html`. Figure 10 was created with this program.

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## 1 Introduction

Classically, a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is called **self-overlapping** if there is an orientation-preserving immersion $F : \mathbb{D}^2 \to \mathbb{R}^2$ of the unit disk $\mathbb{D}^2$, a map of full rank on the entire unit disk $\mathbb{D}^2$, such that $F|_{\partial\mathbb{D}^2} = \gamma$. One can think of such an immersion as distorting a unit disk

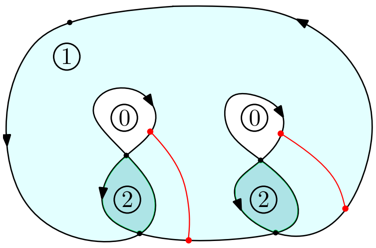


**Figure 1** A self-overlapping curve $\gamma$ with winding numbers for the faces circled. The Blank cuts, shown in red, slice $\gamma$ into a collection of simple positively oriented (counterclockwise) Jordan curves.

that lies flat in the plane and stretching and pulling it continuously without leaving the plane and without twisting or pinching it [15]. If the disk is painted blue on top and pink on the bottom, then one only sees blue. If we also imagine the disk being semi-transparent, then the blue will appear darker in the regions where it overlaps itself; see Figure 1. That means, any self-overlapping curve $\gamma$ must have non-negative winding numbers, $wn(x, \gamma) \geq 0$ for every $x \in \mathbb{R}^2$. We call this condition **positive consistent**. Another simple and intuitive view originates from Blank [1]: The curve is self-overlapping when we can cut it along simple curves into simple positively oriented Jordan curves, i.e., a collection of blue topological disks. **Interior boundaries** are generalizations of self-overlapping curves that are defined similarly, except that $F$ is an interior map which allows finitely many branch points [12]. Interior boundaries are composed of multiple self-overlapping curves (of the same orientation); see Figure 2 for an example. In this paper, all curves $\gamma : [0, 1] \to \mathbb{R}^2$ are assumed to be closed, immersed, and generic, i.e., with only finitely many intersection points, each of which are transverse double points. We also assume $\gamma'(t)$ exists and is nonzero for all $t \in [0, 1]$. We show new combinatorial properties of self-overlapping curves and interior boundaries by revealing new connections to the minimum homotopy area of curves.

## 1.1 Related Work

**Self-Overlapping Curves and Interior Boundaries.** Self-overlapping curves and interior boundaries have a rather rich history, and have been studied under the lenses of analysis, topology, geometry, combinatorics, and graph theory [1, 4, 10, 12–15, 17, 19]. In the 1960s, Titus [19] provided the first algorithm to test whether a curve is self-overlapping (or an interior boundary), by defining a set of cuts that must cut the curve into smaller subcurves that are self-overlapping (or interior boundaries). In a 1967 PhD thesis [1], Blank proved that a curve is self-overlapping iff there is a sequence of cuts (different from Titus cuts) that completely decompose the curve into simple pieces. He represents plane curves with words and showed that one can determine the existence of a cut decomposition by looking for algebraic decompositions of the word. In the 1970s, Marx [13] extended Blank's work to give an algorithm to test if a curve is an interior boundary. In the 1990s, Shor and Van Wyk [17] expedited Blank's algorithm to run in $O(N^3)$ time for a polygonal curve with $N$ line segments. Their dynamic programming algorithm is currently the fastest algorithm to test for self-overlappingness. It is not known whether this runtime bound is tight or whether a faster runtime might be achievable. In distantly related work, Eppstein and Mumford [4] showed that it is NP-complete to determine whether a fixed self-overlapping curve $\gamma$ is the 2D projection of an immersed surface in $\mathbb{R}^3$ defined over a compact two-manifold with boundary. Graver and Cargo [10] approached the problem from a graph-theoretical perspective using

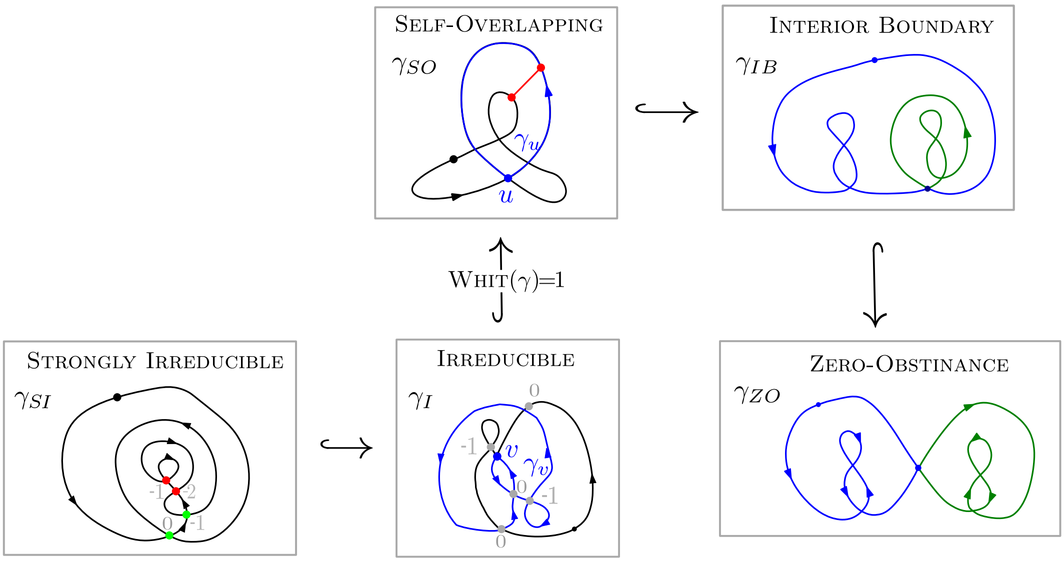


**Figure 2** Example curves of different curve classes and inclusion relationships between the classes. $\gamma_{SO}$ is self-overlapping as indicated by the Blank cuts in red. $\gamma_{IB}$ is an interior boundary consisting of two self-overlapping curves (of the same orientation), one in blue the other in green. The bottom row shows curve classes that are introduced in this paper: $\gamma_{SI}$ is strongly irreducible as can be seen from the non-positive Whitney indices (shown in gray) of its direct split subcurves. Similarly, $\gamma_I$ is irreducible; note that $\gamma_v$ has Whitney index 1 but is not self-overlapping. Also note that $\gamma_{SO}$ is not irreducible since $\gamma_u$ is self-overlapping. $\gamma_{ZO}$ also consists of two self-overlapping curves but of different orientation and is therefore not an interior boundary, but it has zero obstinance.

so-called covering graphs. All of these algorithms also compute the number of inequivalent immersions. Another fact we glean from Blank is that any self-overlapping curve $\gamma$ necessarily makes one full turn, i.e., it has Whitney index $\text{WHIT}(\gamma) = 1$. The necessity of Whitney index 1 and positive consistency to be self-overlapping are well-known and date back to [19].

**Minimum Homotopy Area.** The **minimum homotopy area $\sigma(\gamma)$** is the infimum of areas swept out by nullhomotopies of a closed plane curve $\gamma$. The key link between minimum area homotopies and self-overlapping curves arose in [8, 11], where the authors showed that any curve $\gamma$ has a minimum area homotopy realized by a sequence of nullhomotopies of self-overlapping subcurves (direct split subcurves; see Section 3.1 for the definition). The minimum homotopy area was introduced by Chambers and Wang [3] as a more robust metric for curve comparison than homotopy width (i.e., Fréchet distance or one of its variants) or homotopy height [2]. The minimum homotopy area can be computed in $O(N^2 \log N)$ time for consistent curves [3]. For general curves, Nie gave an algorithm to compute $\sigma(\gamma)$ based on an algebraic interpretation of the problem that runs in $O(N^6)$ time, while the self-overlapping decomposition result of [8] yields an exponential-time algorithm.

The **winding area $W(\gamma)$** is the integral over all winding numbers in the plane. A simple argument shows that $\sigma(\gamma) \geq W(\gamma)$; see [3]. Both self-overlapping curves and interior boundaries are characterized by positive consistency and optimality in minimum homotopy area, $\sigma(\gamma) = W(\gamma)$. A curve possessing both of these properties is self-overlapping when $\text{WHIT}(\gamma) = 1$ and an interior boundary when $\text{WHIT}(\gamma) \geq 1$.

## 1.2 New Results

We ask the following question: what are the sufficient combinatorial conditions for a plane curve to be self-overlapping? Such conditions provide novel mathematical foundations that could pave the way for speeding up algorithms for related problems, such as deciding self-overlappingness or computing the minimum homotopy area of a curve. The first contribution of this paper is to answer this question in the affirmative (Theorems 16 and 17 in Section 4): We show that a curve $\gamma$ with Whitney index 1 and without any self-overlapping subcurves is self-overlapping, and we obtain sufficient conditions for a curve to be self-overlapping solely in terms of the Whitney index of the curve and its subcurves. Here, we only consider **direct split** subcurves $\gamma_v$ that traverse $\gamma$ between the first and second appearance of vertex $v$ in the plane graph induced by $\gamma$. Our results apply to (strongly) irreducible curves; see Figure 2: We call $\gamma$ **irreducible**, if every (proper) direct split is not self-overlapping; if the Whitney index of each such direct split is non-positive, then we call $\gamma$ **strongly irreducible**.

These results follow from our second contribution (Theorems 13 and 14 in Section 4), which shows that any plane curve $\gamma$ is transformed into an interior boundary by wrapping around $\gamma$ with Jordan curves. Equivalently, this means that the minimum homotopy area of $\gamma$ is reduced to the minimal possible threshold, namely the winding area, through wrapping. See Figure 3 for an example of wrapping. Of course, we can make a curve positive consistent with repeated wrapping, since a single wrap increases the winding numbers of each face by one. However, our result shows a new and non-trivial connection between wrapping and the minimum homotopy area.

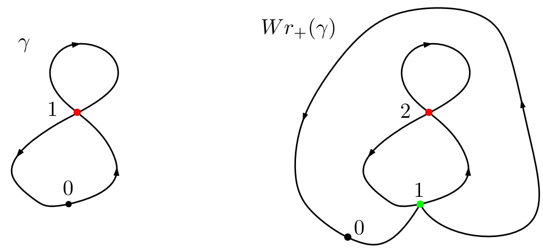


**Figure 3** The curve $\gamma$ is not self-overlapping, but its wrap $\mathrm{Wr}_+(\gamma)$ is self-overlapping.

The final contribution of this paper (in Section 3) is to unite the various definitions and perspectives on self-overlapping curves and interior boundaries. We prove the equivalence of five definitions of self-overlapping curves and four of interior boundaries (Theorems 10 and 9). To this end, we define the new concept of **obstinance** of a curve $\gamma$ as $\mathrm{obs}(\boldsymbol{\gamma}) = \sigma(\gamma) - W(\gamma) \geq 0$, and characterize **zero-obstinance** curves (Theorem 11), see Figure 2. Rephrasing our earlier characterization, self-overlapping curves and interior boundaries are positive-consistent curves with zero-obstinance and positive Whitney index.

# 2 Preliminaries

## 2.1 Regular and Generic Curves

We work with regular, generic, closed plane curves $\gamma : [0, 1] \to \mathbb{R}^2$ with basepoint $\gamma(0) = \gamma(1)$. Let $\mathscr{C}$ denote the set of such curves. A curve $\gamma$ is **regular** if $\gamma'(t)$ exists and is non-zero for all $t$; a curve is **generic** (or normal) if the embedding has only a finite number of intersection points, each of which are transverse crossings. Being generic is a weak restriction, as normal curves are dense in the space of regular curves [20]. Viewing a generic curve $\gamma$ by its image $[\gamma] \subseteq \mathbb{R}^2$, we can treat $\gamma$ as a directed plane multigraph $\boldsymbol{G(\gamma)} = (V(\gamma), E(\gamma))$. Here, $\boldsymbol{V(\gamma)} = \{p_0, p_1, \ldots, p_n\}$ is the set of ordered **vertices** (points) of $\gamma$, with basepoint $p_0 = \gamma(0)$ regarded as a vertex as well. An **edge** $(p_i, p_j)$ corresponds to a simple path along $\gamma$ between $p_i$ and $p_j$. The **faces** of $G(\gamma)$ are the path-connected components of $\mathbb{R}^2 \setminus [\gamma]$. Each $\gamma \in \mathscr{C}$ has exactly one unbounded face, the exterior face $\boldsymbol{F_{ext}}$. See Figure 4. Two curves are combinatorially equivalent when their planar multigraphs are isomorphic. We may therefore define a curve just by its image, orientation, and basepoint. A curve is **simple** if it has no intersection points. We notate $|\boldsymbol{\gamma}| = |V(\gamma) \setminus \{p_0\}|$ as the complexity of $\gamma$.

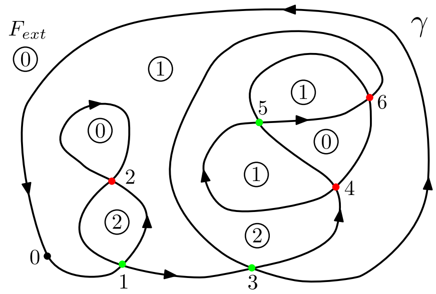


**Figure 4** A curve $\gamma$ that is self-overlapping. The winding numbers of each face are enclosed by circles. The signed intersection sequence of $\gamma$ is $0, 1_+, 2_-, 2_+, 1_-, 3_+, 4_-, 5+, 6_-, 4_+, 5_-, 6_+, 3_-, 0$; vertex labels are shown, and the sign of each vertex is indicated with green (positive) or red (negative). The combinatorial relations are: $p_2 \subset p_1$; $p_4, p_5, p_6 \subset p_3$; $p_1, p_2$ S $p_3, p_4, p_5, p_6$; $p_4, p_5$ L $p_6$; $p_4$ L $p_5$.

For any $x \in \mathbb{R}^2 \setminus [\gamma]$, the **winding number** $\boldsymbol{wn(x, \gamma)} = \sum_i a_i$ is defined using a simple path $P$ from $x$ to $F_{ext}$ that avoids the intersection points of $\gamma$. Here, $a_i = +1$ if $P$ crosses $\gamma$ from left to right at the $i$-th intersection of $P$ with $\gamma$, and $a_i = -1$ otherwise. Since this number is independent of the path chosen and is constant over each face $F$ of of $G(\gamma)$, we write $\boldsymbol{wn(F, \gamma)}$. If $wn(F, \gamma) \geq 0$ for every face $F$ on $G(\gamma)$, then we call $\gamma$ **positive consistent**. If $wn(F, \gamma) \leq 0$ for every face, then $\gamma$ is **negative consistent**. See Figure 4. The **winding area** of a curve $\gamma$ is given by $\boldsymbol{W(\gamma)} = \int_{\mathbb{R}^2} |wn(x, \gamma)|\, dx = \sum_F A(F)\, |wn(F, \gamma)|$, where $A(F)$ is the area of the face $F$ and $wn(x, \gamma) = 0$ for $x \in [\gamma]$. The **Whitney index WHIT($\boldsymbol{\gamma}$)** is the winding number of the derivative $\gamma'$ about the origin. A curve $\gamma$ is **positively oriented** if $\text{WHIT}(\gamma) > 0$ and **negatively oriented** if $\text{WHIT}(\gamma) < 0$.

A basepoint $p_0 = \gamma(0)$ is a **positive outer basepoint** if $p_0$ is incident to the two faces $F_{ext}$ and $F$, and $wn(F, \gamma) = 1$. If $wn(F, \gamma) = -1$, then $p_0$ is a negative outer basepoint. Several of our results require $\gamma$ to have a positive outer basepoint. A curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is **(positive) self-overlapping** when there is an orientation-preserving immersion $F : \mathbb{D}^2 \to \mathbb{R}^2$, a map of full rank, extending $\gamma$ to a map on the entire two-dimensional unit disk $\mathbb{D}^2$. If the reversal $\overline{\gamma}$ of a curve is self-overlapping, then we call $\gamma$ **negative self-overlapping**.

## 2.2 Combinatorial Relations and Intersection Sequences

Following Titus [18], we now describe how the intersection points of a curve $\gamma \in \mathscr{C}$ relate to each other; see Figure 4. Let $p_i, p_j \in V(\gamma)$ be two vertices such that $p_i = \gamma(t_i) = \gamma(t_i^*)$ and $p_j = \gamma(t_j) = \gamma(t_j^*)$ with $t_i < t_i^*$ and $t_j < t_j^*$. Then, one of the following must hold:

- $p_i$ **links** $p_j$, or $p_i$ **L** $p_j$, iff $t_i < t_j < t_i^* < t_j^*$ or $t_j < t_i < t_j^* < t_i^*$
- $p_i$ is **separate** from $p_j$, or $p_i$ **S** $p_j$, iff $t_i < t_i^* < t_j < t_j^*$ or $t_j < t_j^* < t_i < t_i^*$
- $p_i$ is **contained in** $p_j$, or $p_i \subset p_j$, iff $t_j \leq t_i < t_i^* \leq t_j^*$.

To define the intersection sequence of $\gamma$, the vertices are labeled in the order they appear on $\gamma$, starting with 0 for the basepoint $\gamma(0)$, and increasing by one each time an unlabeled vertex is encountered. The **signed intersection sequence** consists of the sequence of all vertex labels along $\gamma$ starting at the basepoint; the first time vertex $p_i$ is visited, the label is augmented with $\text{sgn}(p_i)$, and the second time with $-\text{sgn}(p_i)$. Here, $\text{sgn}(p_i) = \text{sgn}(p_i, \gamma)$ is the **sign** of vertex $p_i = \gamma(t_i) = \gamma(t_i^*)$, and is 1 if the vector $\gamma'$ rotates clockwise from $t_i$ to $t_i^*$, and $-1$ otherwise. Note that $\text{sgn}(p_i)$ depends on the basepoint of the curve. Interior boundaryness is invariant with respect to signed intersection sequences [19].

## 2.3 Minimum Homotopies

A **homotopy** between two generic curves $\gamma$ and $\gamma'$ is a continuous function $H : [0,1]^2 \to \mathbb{R}^2$ such that $H(0,\cdot) = \gamma$ and $H(1,\cdot) = \gamma'$. In $\mathbb{R}^2$, any curve is null-homotopic, i.e., homotopic to a constant map. Given a sequence of homotopies $(H_i)_{i=1}^k$, we notate the concatenation of these homotopies in order as $\boldsymbol{\sum_{i=1}^k H_i}$. We use the notation $\overline{H}$ for the reversal $\overline{H}(i,t) = H(1-i,t)$ of a homotopy. If $H(0,\cdot) = \gamma$ and $H(1,\cdot) = \gamma'$, we may write $\boldsymbol{\gamma \overset{H}{\rightsquigarrow} \gamma'}$.

**Homotopy moves** are basic local alterations to a curve defined by their action on $G(\gamma)$. These moves come in three pairs [8]; see Figure 5: The I-moves destroy/create an empty loop, II-moves destory/create a bigon, and III-moves flip a triangle. We denote the moves that remove vertices as $\mathbf{I}_{\boldsymbol{a}}$ and $\mathbf{II}_{\boldsymbol{a}}$, and moves that create vertices as $\mathbf{I}_{\boldsymbol{b}}$ and $\mathbf{II}_{\boldsymbol{b}}$. See Figure 5. It is well-known that any homotopy such that each intermediate curve is piecewise regular and generic, or almost generic, can be achieved by a sequence of homotopy moves. Thus, without loss of generality, we assume that each time the curve $H(i,\cdot)$ combinatorially changes is through a single homotopy move.

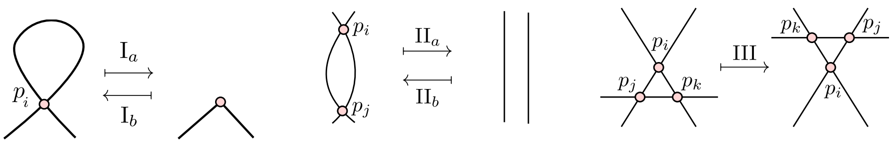


**Figure 5** All three homotopy moves and their reversals. Figure from [8].

Let $\gamma \in \mathscr{C}$ and $H$ be a nullhomotopty of $\gamma$. Define $E_H(x)$ as the number of connected components of $H^{-1}(x)$. Intuitively, this counts the number of times that $H$ sweeps over $x$. The **minimum homotopy area** of $\gamma$ is defined as $\boldsymbol{\sigma(\gamma)} = \inf_H\{\int_{\mathbb{R}^2} E_H(x)\,dx \mid H \text{ is a nullhomotopy of } \gamma\}$. The following was shown in [3,8]:

▶ **Lemma 1** (Homotopy Area $\geq$ Winding Area)**.** *Let $\gamma \in \mathscr{C}$. Then $\sigma(\gamma) \geq W(\gamma)$.*

We call a homotopy **left (right) sense-preserving** if $H(i+\epsilon, t)$ lies on or to the left (right) of the oriented curve $H(i,\cdot)$ for any $i, t \in [0,1]$ and any $\epsilon > 0$. The following two lemmas provide useful properties about sense-preserving homotopies; the first was proven in [3], the second in [8].

▶ **Lemma 2** (Monotonicity of Winding Numbers)**.** *Let $H$ be a homotopy. If $H$ is left (right) sense-preserving, then for any $x \in \mathbb{R}^2$ the function $a(i) = wn(x, H(i,\cdot))$ is monotonically decreasing (increasing).*

▶ **Lemma 3** (Sense-Preserving Homotopies are Optimal)**.** *Let $\gamma \in C$ be consistent. Then a nullhomotopy $H$ of $\gamma$ is optimal if and only if it is sense-preserving.*

# 3 Equivalences

In this section, we show the equivalence of different characterizations of interior boundaries (Theorem 9) and of self-overlapping curves (Theorem 10). Our analysis of curve classes hinges around the concept of obstinance. In Theorem 11 we classify zero obstinance curves, which are generalizations of interior boundaries and of self-overlapping curves.

## 3.1 Direct Splits

Let $\gamma \in \mathscr{C}$ and $p_i \in V(\gamma)$ with $p_i = \gamma(t_i) = \gamma(t_i^*)$ and $t_i < t_i^*$. Then, $\gamma$ can be split into two subcurves at $p_i$: The **direct split** is the curve with image $[\gamma|_{[t_i,t_i^*]}]$ with basepoint $p_i$, and the **indirect split** is the curve with image $[\gamma|_{[t_i^*,1]}] \cup [\gamma|_{[0,t_i]}]$ with basepoint $\gamma(0)$. We endow both of these curves with the same orientation as $\gamma$. Given a direct (or indirect) split $\widetilde{\gamma}$ on a curve $\gamma$, we write $\gamma \setminus \widetilde{\gamma}$ for the indirect (or direct) split complementary to $\widetilde{\gamma}$. We call a direct split **proper** if it is not the entire curve $\gamma$. See Figure 6. If $v = p_i \in V(\gamma)$, we may notate the direct split as $\gamma_i$ or $\gamma_v$. When removing multiple splits iteratively, we write $\gamma \setminus (\cup_{i=1}^{n} \gamma_i)$, where we require that $\gamma_i$ is a direct split of $\gamma \setminus \left(\cup_{j=1}^{i-1} \gamma_j\right)$. Being a direct split of a curve is a transitive property. I.e., if $\gamma_i$ is a direct split on $\gamma$, and $\gamma_j$ is a direct split on $\gamma_i$, then $\gamma_j$ is a direct split on $\gamma$. The parallel statement on indirect splits, however, is false.

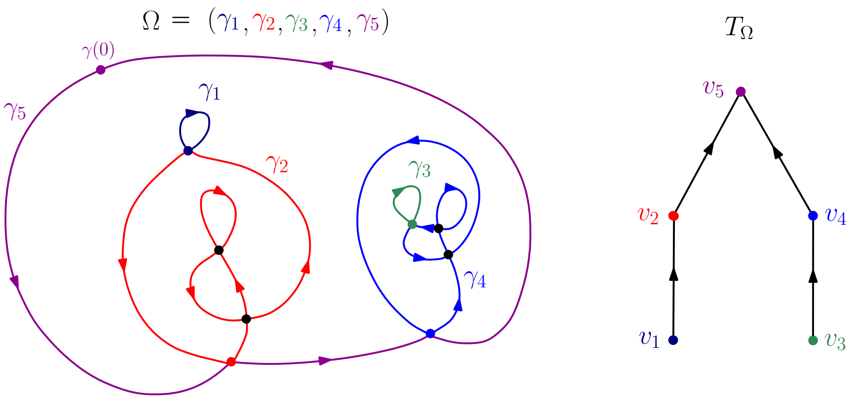


**Figure 6** A self-overlapping decomposition of a self-overlapping curve $\gamma$. Here, $\gamma_1$ and $\gamma_3$ are (proper) direct splits of $\gamma$, while $\gamma_2, \gamma_4$, and $\gamma_5$ are neither direct nor indirect splits of $\gamma$.

## 3.2 Decompositions and Loops

A curve $\gamma \in \mathscr{C}$ can be entirely decomposed by iteratively removing direct splits. For a sequence of subcurves $\Omega = (\gamma_i)_{i=1}^{k}$, define $C_0 = \gamma$ and inductively $C_i = C_{i-1} \setminus \gamma_i$ for $i \geq 1$; the basepoint of $\gamma_i$ is $v_i = C_i \cap \gamma_i$. We call $\Omega$ a **direct split decomposition** if $\gamma_i$ is a direct split of $C_{i-1}$, for all $i \in \{1, 2, \ldots, k\}$, and $\gamma_k = C_{k-1}$. Given a direct split decomposition $\Omega = (\gamma_i)_{i=1}^{k}$, we write $\boldsymbol{V(\Omega)}$ for the set of basepoints of all $\gamma_i \in \Omega$. See Figure 6. Observe that no two vertices $v_i, v_j \in V(\Omega)$ may be linked. Hence, we obtain a partial order $\prec$ on $V(\Omega)$ by declaring $v_i \prec v_j$ whenever $v_i \subset v_j$. We define $T_\Omega$ to be the tree with vertex set $V(T_\Omega) = V(\Omega)$ and edges $e = (v_i, v_j)$ whenever $v_i \subset v_j$ and there is no other vertex $v_k \neq v_i, v_j$ such that $v_i \subset v_k \subset v_j$. We consider two subcurve decompositions $\Omega, \Gamma$ equivalent, $\Omega \sim \Gamma$, when $T_\Omega = T_\Gamma$. This means that $\Omega$ and $\Gamma$ contain the same set of subcurves, just in a different order. If every $\gamma_i$ is self-overlapping, we call $\Omega$ a **self-overlapping decomposition**; it may contain self-overlapping subcurves of positive and negative orientations. We now observe that the vertex set of a decomposition already determines the subcurves in the decomposition:

▶ **Observation 4.** *Given a curve $\gamma \in \mathscr{C}$ and a subset $S \subset V(\gamma)$ such that $p_0 \in S$ and no two vertices in $S$ are linked, there is a unique equivalence class $\mathscr{E}$ of direct split decompositions with $V(\Omega) = S$ for all $\Omega \in \mathscr{E}$.*

The observation below follows directly from the definition of winding numbers.

▶ **Observation 5.** *Let $\Omega$ be a direct split decomposition of a curve $\gamma \in \mathscr{C}$. Then for any face $F$ in the plane multigraph $G(\gamma)$, $wn(F, \gamma) = \sum_{\gamma_i \in \Omega} wn(F, \gamma_i)$.*

We define a **loop** as a simple direct split $\gamma_v$ of a curve $\gamma \in \mathscr{C}$. Intersection points of $\gamma$ may lie on $\gamma_v$, but none occur as intersections of $\gamma_v$ with itself. Every non-simple plane curve has a loop; e.g., the direct split $\gamma_w$, where $w$ is the highest index vertex on $\gamma$ in the signed intersection sequence. A loop $\gamma_v$ is **empty** if $v$ links no vertex $w \in V(\gamma)$. Let $\text{int}(\gamma_v)$ denote its interior. We call $\gamma_v$ an **outwards loop** if the edges $e_1, e_4$, that are incident on $v$ and lie on $\gamma \setminus \gamma_v$, both lie outside $\text{int}(\gamma_v)$. Otherwise $\gamma_v$ is an **inwards loop**. See Figure 7.

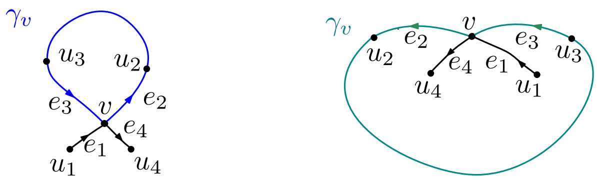


**Figure 7** An outwards loop (left) and an inwards loop (right).

The lemma below follows from [9, 16] and is proven in [7].

▶ **Lemma 6** (Whitney Index Through Decompositions). *Let $\gamma \in \mathscr{C}$ and $\Omega$ be a direct split decomposition of $\gamma$. Then $WHIT(\gamma) = \sum_{C \in \Omega} WHIT(C)$.*

A consequence of Lemma 6 is that iteratively removing loops and summing $\pm 1$ for their signs allows one to quickly compute Whitney indices. Assuming $\gamma$ is given as a directed plane multigraph, one can adapt a depth-first traversal to compute such a loop decomposition of $\gamma$ in $O(|\gamma|)$ time, which yields the following corollary:

▶ **Corollary 7** (Compute Whitney Index). *Let $\gamma \in \mathscr{C}$ be of complexity $n = |\gamma| = |V(\gamma)|$. One can compute a loop decomposition of $\gamma$, and $WHIT(\gamma)$, in $O(n)$ time.*

Now, let $H$ be a nullhomotopy of a curve $\gamma$, and consider all the points $A = \{v_i\}_{i=1}^k$ of $\mathbb{R}^2$ such that $H$ performs a $\text{I}_a$ move to contract a loop to that point. All such points are called **anchor points** of the homotopy $H$. Following [8] we call a homotopy $H$ **well-behaved** when the anchor points $A$ of $H$ satisfy $A \subseteq V(\gamma)$, i.e., $H$ only contracts loops to vertices of the original curve. The theorem below from [8] shows that computing minimum homotopy area is reduced to finding an optimal self-overlapping decomposition. The homotopy $H$ guaranteed in the following theorem is well-behaved.

▶ **Theorem 8** (Minimum Homotopy Decompositions). *Let $\gamma \in \mathscr{C}$. Then there is a self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^k$ of $\gamma$ as well as an associated minimum homotopy $H_\Omega$ of $\gamma$ such that $H_\Omega = \sum_{i=1}^k H_i$ and each $H_i$ is a nullhomotopy of $\gamma_i$. In particular, $\sigma(\gamma) = \min_{\Omega \in \mathscr{D}(\gamma)} \sum_{C \in \Omega} W(C)$, where $\mathscr{D}(\gamma)$ is the set of all self-overlapping decompositions of $\gamma$.*

## 3.3 Equivalence of Interior Boundaries

In this section, we unify different definitions and characterizations of interior boundaries by showing their equivalence. We call a curve $\gamma$ a **k-interior boundary** when (1) $\text{obs}(\gamma) = 0$, (2) $\text{WHIT}(\gamma) = k > 0$, and (3) $\gamma$ is positive consistent. We call $\gamma$ a **$(-$k$)$-interior boundary** when its reversal $\overline{\gamma}$ is a $k$-interior boundary. In accordance with Titus [19], we call a curve $\zeta : [0,1] \to \mathbb{R}^2$ a **Titus interior boundary** if there exists a map $F : \mathbb{D}^2 \to \mathbb{R}^2$ such that $F$ is continuous, **light** (defined as: pre-images are totally disconnected), open, orientation-preserving, and $F|_{\partial\mathbb{D}^2} = \zeta$. The map $F$ is called **properly interior**.

▶ **Theorem 9** (Equivalence of Interior Boundaries). *Let $\gamma \in \mathscr{C}$ and suppose $WHIT(\gamma) = k > 0$. Then, the following are equivalent:*

1. *$\gamma$ is an interior boundary.*
2. *$\gamma$ is a Titus interior boundary.*
3. *$\gamma$ admits a self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^k$, where each $\gamma_i$ is positive self-overlapping.*
4. *$\gamma$ admits a well-behaved left sense-preserving nullhomotopy $H$ with exactly $k$ $I_a$-moves.*

**Proof.**

**1 ⇒ 3:** Let $\gamma$ be an interior boundary. By Theorem 8, we have an optimal self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^j$ of $\gamma$. Suppose, by contradiction, that there exists an $l \leq j$ such that $\gamma_l$ is negative self-overlapping. Let $F$ be any face contained in the interior $\text{int}(\gamma_l)$. We know by Observation (5) that $wn(F,\gamma) = \sum_{i=1}^j wn(F,\gamma_i)$, and since $\gamma$ is positive consistent $wn(F,\gamma) \geq 0$. Thus there must exist a positive self-overlapping curve $\gamma_i \in \Omega$ with $F \subseteq \text{int}(\gamma_i)$. Consider the nullhomotopies $H_l$ and $H_i$ that are part of the canonical optimal homotopy $H_\Omega$. Then $H_l$ contracts $\gamma_l$ and is right sense-preserving, while $H_i$ contracts $H_i$ and is left sense-preserving. Thus by Lemma 2, $H_l$ increases the winding number on $F$ and $H_i$ decreases the winding number, which means $F$ is swept more than $W(F)$ times, a contradiction. Thus, no negative self-overlapping subcurve $\gamma_l$ may exist in $\Omega$. Since $\text{WHIT}(C) = 1$ for any positive self-overlapping subcurve and $\text{WHIT}(\gamma) = \sum_{i=1}^k \text{WHIT}(\gamma_i)$ by Lemma 6, we we must have $k = j$.

**1 ⇔ 4:** If $\gamma$ has a well-behaved left sense-preserving nullhomotopy $H$ with exactly $k$ $\text{I}_a$-moves, then $H$ comes naturally with an associated self-overlapping decomposition $\Omega$ of $\gamma$ with $|\Omega| = k$, and $\text{WHIT}(\gamma) = k > 0$ by Lemma 6. We now show that $\sigma(\gamma) = W(\gamma)$. Consider the reversal $\overline{H}$ from the constant curve $\gamma_{p_0}(t) = p_0$ to $\gamma$. Then, $\overline{H}$ is right sense-preserving and by Lemma 2 the function $a(i) = wn(x, \overline{H}(i,\cdot))$ is monotonically increasing for any $x \in \mathbb{R}^2$. Since $wn(x,\gamma_{p_0}) = 0$ for all $x \in \mathbb{R}^2$, we have that $wn(x,\gamma) \geq 0$ for all $x \in \mathbb{R}^2$. Thus, $\gamma$ is an interior boundary. Conversely, if $\gamma$ is a positive interior boundary, then $\text{obs}(\gamma) = 0$ and by Lemma 3, and since $\gamma$ is positive, $H$ is left sense-preserving. Again, by Lemma 6, $\text{WHIT}(\gamma) = j$, where $j$ is the number of $\text{I}_a$-moves in any well-behaved nullhomotopy $H$ of $\gamma$. Hence, we must have $j = k$. The remaining cases are proved in [7]. ◀

## 3.4 Equivalences of Self-Overlapping Curves

In this section, we study different characterizations of self-overlapping curves and show their equivalence. First we describe a geometric formulation of self-overlappingness, inspired by the work of Blank and Marx [1, 13]. Let $\gamma \in \mathscr{C}$ be self-overlapping. Let $P : [0,1] \to \mathbb{R}^2$ be a simple path so that $P(0) = q = \gamma(t_q)$ and $P(1) = p = \gamma(t_p)$ lie on $[\gamma]$ but are not vertices of $\gamma$. Without loss of generality, assume $t_q < t_p$. Let $\widetilde{P} = \gamma|_{[t_q,t_p]}$, and suppose that (1) $P \cap \widetilde{P} = \{p,q\}$, (2) $C = \widetilde{P} * \overline{P}$ is a simple closed curve, and (3) $C$ is positively oriented; see Figure 8. Then we call $P$ a **Blank cut** of $\gamma$. By cutting along $P$, $\gamma$ is split into two curves of strictly smaller complexity, $\gamma_1$ and $C$. We call a sequence $(P_i)_{i=1}^k$ of Blank cuts a **Blank cut decomposition** if the final curve is a simple positively oriented curve.

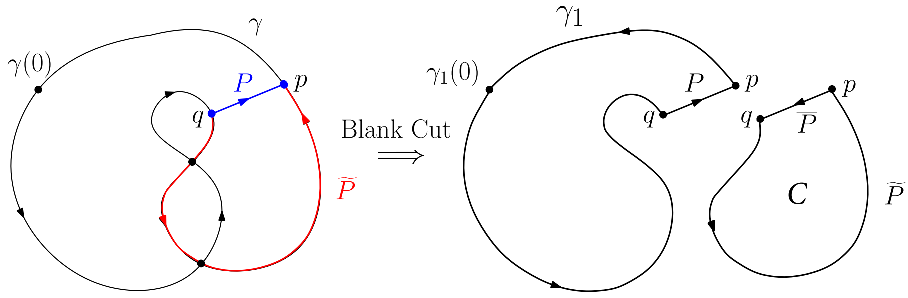


**Figure 8** A Blank cut on a small self-overlapping curve.

▶ **Theorem 10** (Equivalent Characterizations of Self-Overlapping Curves). *Let $\gamma \in \mathscr{C}$. Then the following are equivalent:*

1. *(Analysis) There is an immersion $F : D^2 \to \mathbb{R}^2$ so that $F|_{\partial D^2} = \gamma$.*
2. *(Geometry) $\gamma$ admits a Blank cut decomposition.*
3. *(Geometry/Topology) $\gamma$ is a $(+1)$-interior boundary, i.e., self-overlapping.*
4. *(Topology) $\gamma$ admits a left-sense preserving nullhomotopy $H$ with exactly one $\mathrm{I}_a$-move.*
5. *(Analysis) $\gamma$ is a Titus interior boundary with $WHIT(\gamma) = 1$.*

**Proof.** By property 3 in Theorem 9, self-overlapping curves are 1-interior boundaries, since any self-overlapping curve $\gamma$ has the trivial self-overlapping decomposition $\Omega = (\gamma)$. Thus, we have already established $1 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$ in Theorem 9. We now prove $2 \Leftrightarrow 4$. Any Blank cut $P$ can be performed by a left sense-preserving homotopy that deforms $\widetilde{P}$ to $P$. Hence the Blank cut decomposition corresponds to a left sense-preserving homotopy to a simple positively oriented curve. Finally we perform a single $\mathrm{I}_a$-move to complete a left sense-preserving nullhomotopy of $\gamma$. Conversely, let $\gamma$ have a left sense-preserving nullhomotopy $H$. From $3 \Leftrightarrow 4$ we know that every intermediary curve $\gamma_i = H(i, \cdot)$ is self-overlapping since the subhomotopy $H_i = H|_{[i,1]\times[0,1]}$ is a left sense-preserving nullhomotopy of $\gamma_i$ with one $\mathrm{I}_a$-move. As $H$ ends with a $\mathrm{I}_a$-move, we may select a subhomotopy $H'$ such that $\gamma \overset{H'}{\rightsquigarrow} C$, where $C$ is a simple self-overlapping curve. Moreover, we see that $H = H' + H''$, where the unique $\mathrm{I}_a$-move of $H$ occurs during $H''$. Thus, $H'$ is regular, i.e., consists of a sequence of homotopy moves only of types $\mathrm{II}_a$, $\mathrm{II}_b$, or III, which deform $\gamma$ to $C$. Each of these homotopy moves can be performed by a Blank cut, as shown in Figure 9. Since all of the intermediary curves are self-overlapping, this induces a Blank cut decomposition. ◀

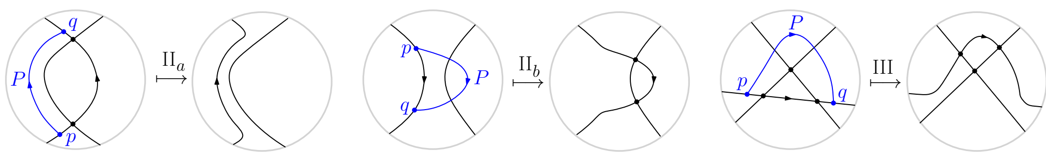


**Figure 9** Homotopy moves $\mathrm{II}_a$, $\mathrm{II}_b$, and III each correspond to a Blank cut (shown in blue).

## 3.5 Zero Obstinance Curves

In this section, we classify curves $\gamma \in \mathscr{C}$ with **zero obstinance**, $\mathrm{obs}(\gamma) = \sigma(\gamma) - W(\gamma) = 0$. See Figures 2 and 10 for examples of zero-obstinance curves. We show that just as interior boundaries can be decomposed into self-overlapping curves, so too can zero-obstinance curves be decomposed into interior boundaries.

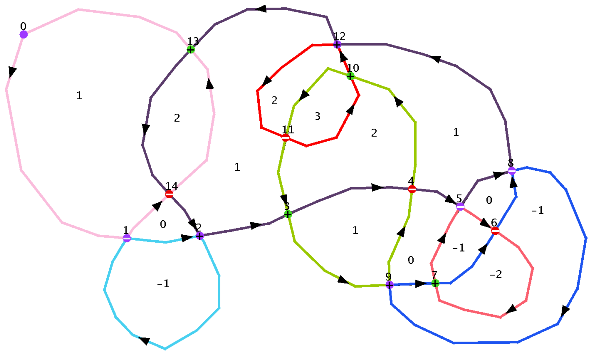

**Figure 10** A zero obstinance curve, with its minimum homotopy decomposition, and winding numbers shown. Each curve in the decomposition is self-overlapping and shown in a different color. The vertices with labels $1, 2, 5, 8, 9$ are sign-changing.

If a curve $\gamma$ has zero obstinance, then there is a nullhomotopy $H$ which sweeps each face $F$ on $G(\gamma)$ exactly $wn(F, \gamma)$ times. Note that such a homotopy $H$ is necessarily minimal by Lemma 1. Intuitively, this implies that the homotopy $H$ should be locally sense-preserving. We expect it to sweep leftwards on positive consistent regions and rightwards on negative consistent regions. Hence, we might expect regions of the curve where the winding numbers change from positive to negative to be especially problematic. Indeed, let $v \in V(\gamma)$ be incident to the faces $\{F_1, F_2, F_3, F_4\}$. We call $v$ **sign-changing** when, as a multiset, $\{wn(\gamma, F_1), wn(\gamma, F_2), wn(\gamma, F_3), wn(\gamma, F_4)\} = \{-1, 0, 0, 1\}$; see Figures 10 and 11.

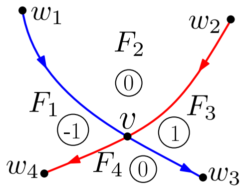


**Figure 11** A sign-changing vertex $v$. The winding numbers of the incident faces are $-1$, $0$, $1$, $0$.

▶ **Theorem 11** (Zero Obstinance Characterization)**.** *Let $\gamma \in \mathscr{C}$ and let $\mathscr{S}$ be the sign-changing vertices of $\gamma$. Then $obs(\gamma) = 0$ iff no two vertices in $\mathscr{S}$ are linked and any direct split subcurve decomposition $\Omega$ with vertex set $V(\Omega) = \mathscr{S} \cup \{p_0\}$ contains only interior boundaries.*

The proof is available in [7].

## 4 Wraps and Irreducability

In this section, we show (Theorems 13 and 14) that wrapping around a curve $\gamma$ until its obstinance is reduced to zero results in an interior boundary. This key result is used to prove sufficient combinatorial conditions for a curve to be self-overlapping based on the Whitney index of the curve and its direct splits (Theorems 16 and 17).

### 4.1 Wraps

Let $\gamma \in \mathscr{C}$, and let $I$ be its signed intersection sequence. Form $I'$ by incrementing each label by one and removing the occurrences of 0 corresponding to the basepoint. If $\gamma$ has a positive outer basepoint $\gamma(0)$, then its **(positive) wrap $\boldsymbol{Wr_+(\gamma)}$** is the unique (class of) curve with signed intersection sequence $0, 1_+, I', 1_-, 0$. This corresponds to gluing a simple positively oriented curve $\alpha$ to $\gamma$ at $\gamma(0)$, where the interior $\text{int}(\alpha) \supseteq [\gamma]$; the new basepoint is on $\alpha$. See Figure 12. The **negative wrap $Wr_-(\gamma)$** is defined analogously if $\gamma$ has a negative outer basepoint. We write $Wr_+^k(\gamma)$ for the curve achieved from $\gamma$ by wrapping $k$ times.

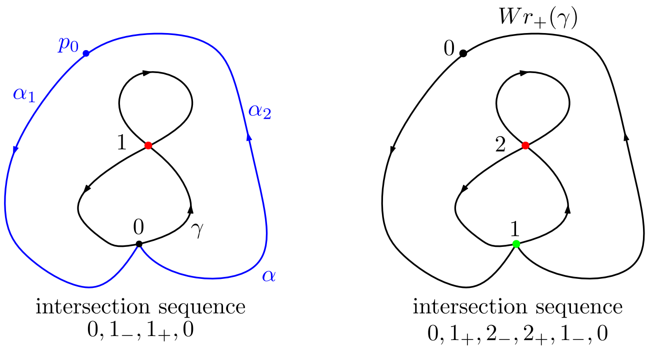


**Figure 12** A curve $\gamma$ with positive outer basepoint and its *positive* wrap $Wr_+(\gamma)$.

To wrap a curve in the direction opposed to the sign of the basepoint, we must be more careful. Without loss of generality, we describe the construction of $Wr_-(\gamma)$ when $\gamma$ has a positive outer basepoint. Perform a $\text{I}_b$-move to add a simple loop $\tilde{\gamma}$ of the opposite orientation tangent to the basepoint $\gamma(0)$. Let $\gamma'$ be the curve after the $\text{I}_b$-move, with a basepoint chosen to lie on $\tilde{\gamma}$. We then define $Wr_-(\gamma) = Wr_-(\gamma')$. See Figure 13.

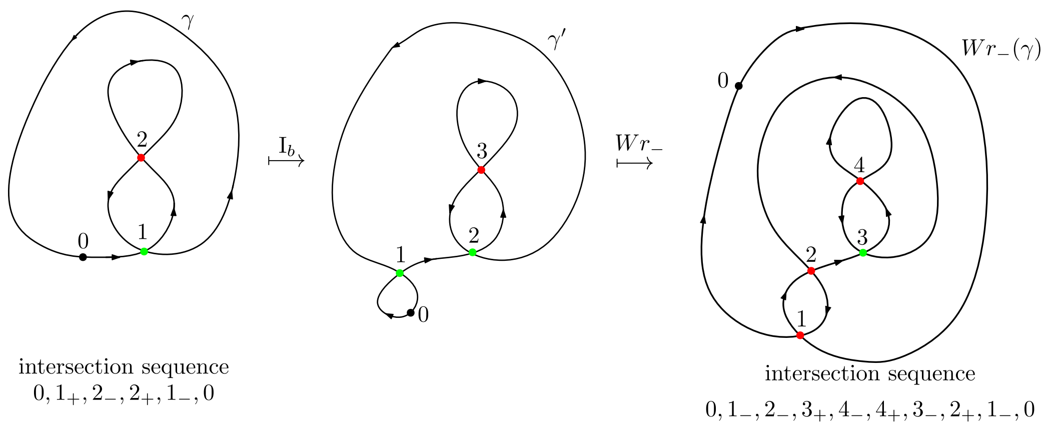


**Figure 13** A curve $\gamma$ with positive outer basepoint and its transformation into its *negative* wrap $Wr_-(\gamma)$. First, we perform a $\text{I}_b$-move and then wrap normally on $\gamma'$.

## 4.2 Wrapping Resolves Obstinance

First, we prove a simple lemma:

▶ **Lemma 12** (Existence of an Outwards Loop). *Let $\gamma \in \mathscr{C}$ have an outer basepoint. Then, if $\gamma$ is non-simple, it has an outwards loop.*

**Proof.** Let $v$ be the first self-intersection of $\gamma$. Then $\gamma_v$ is a loop. Write $\gamma^{-1}(v) = \{t, t^*\}$, where $t < t_*$. Since $\gamma(0)$ lies outside of $\text{int}(\gamma_v)$, as an outer basepoint, we note that if $\gamma_v$ were inwards, the path $P = \gamma_{[0,t_1]}$ would cross $[\gamma_v]$ to get from outside the simple curve to inside it. This is then a contradiction, for if the crossing occurred at a point $q$ on $[\gamma_v]$, then $q$ would be the first self-intersection of $\gamma$. Indeed, we would reach $q$ a second time before we reach $v$ a second time. Thus, $\gamma_v$ is outwards. ◀

Now, let $t_1 < t_1^* \in [0,1]$ be the smallest value so that $\gamma_1 = \gamma_{[t_1,t_1^*]}$ is a loop. We call this the **first loop** of $\gamma$. Since $t_1^*$ is the first time that $\gamma$ self-intersects, we know that such a loop is outwards by the argument in Lemma 12.

▶ **Theorem 13** (Wrapping Resolves Obstinance). *Let $\gamma \in \mathscr{C}$ have positive outer basepoint. Then there is a positive integer $k$ so that obs$(Wr_+^k(\gamma)) = 0$. Moreover, $Wr_+^k(\gamma)$ is a positive interior boundary.*

**Proof.** Let $l$ be the number of negative vertices in $V(\gamma)$. Set $k = l+1$. We claim that $\mathrm{Wr}_+^k(\gamma)$ is an interior boundary. We will show this by iteratively constructing a left sense-preserving nullhomotopy $H$ for $\gamma$. By property 4 of Theorem 9 it then follows that $\gamma$ is a positive interior boundary and $\mathrm{obs}(\gamma) = 0$.

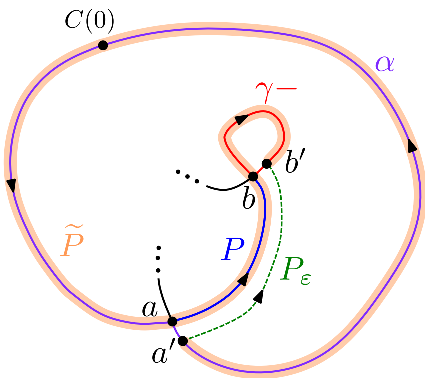


**Figure 14** The combinatorial structure necessary to apply balanced loop deletion: a wrapped curve, with outer wrap $\alpha$ and a negatively oriented loop $\gamma_-$ as first loop of $C$.

We first introduce a trick that we call **balanced loop deletion**. See Figure 14, where all of the following objects are shown. Suppose that $C \in \mathscr{C}$ is a curve that is positively wrapped, $C = Wr_+(C')$ for some curve $C' \in \mathscr{C}$, and suppose that the first loop $\gamma_-$ of $C$, shown in red, is negatively oriented. Let $b = C(t_b) = C(t_b^*)$, with $t_b < t_b^*$, be the basepoint of $\gamma_-$. Balanced loop deletion performs a left sense-preserving homotopy $H$ so that $C \overset{H}{\rightsquigarrow} C \setminus (\alpha \cup \gamma_-)$, where $\alpha$, shown in purple, is the positive wrap on $C$. Let $P$, shown in blue, be the simple subpath of $C$ from $a = C(t_a) = C(t_a^*)$ to $b$, where $a$ is the unique outer intersection point on $[C]$, i.e., the basepoint of the wrap $\alpha$, and $t_a < t_a^*$. For $\varepsilon > 0$ sufficiently small, let $a' = C(t_a^* + \varepsilon)$ and $b' = C(t_b^* - \varepsilon)$ and let $P_\varepsilon$, shown in dashed green, be a simple path between $a'$ and $b'$ that is $\varepsilon$-close to $P$ in Hausdorff distance. Let $\widetilde{P} = C|_{[t_a^*+\varepsilon, t_b^*-\varepsilon]}$ (shown in thick beige) be the simple subpath of $C$ from $a'$ to $b'$. Then, $\widetilde{P}$ is the concatenation of (i) the path from $a'$ to $a$ along $\alpha$, (ii) the path $P$ from $a$ to $b$, and (iii) the path from $b$ to $b'$ along $\gamma_-$. The path $\widetilde{P}$ is simple, because each of these subpaths are simple and none of them intersect each other since $b$ is the first self-intersection point of the curve. Observe that $\widetilde{P} * P_\varepsilon$ is a simple, positively oriented, closed curve. It follows that we can perform a Blank cut along $P_\varepsilon$ that replaces $\widetilde{P}$ on $C$ with the path $P_\epsilon$. The effect of this cut on $C$ is that both the outer wrap $\alpha$ and the negatively oriented loop $\gamma_-$ are deleted, and the path $P$ is replaced by $P_\varepsilon$. This Blank cut can be performed by a left sense-preserving homotopy, so we have established the existence of left sense-preserving balanced loop deletion.

Now we construct a left sense-preserving nullhomotopy $H$ of $\mathrm{Wr}_+^k(\gamma) = \gamma_1$ by iteratively concatenating several left sense-preserving subhomotopies, so $H = \sum_i H_i$. We proceed inductively as follows. Suppose $H_1, \ldots, H_{i-1}$ have been defined and $\gamma_i$ is the current curve. Consider the first loop $C_i$ of $\gamma_i$. If $C_i$ is positively oriented we let $H_i$ be the left sense-preserving nullhomotopy that contracts this loop. Otherwise $C_i$ is negatively oriented and we let $H_i$ be the homotopy performing balanced loop deletion.

We claim that there is a wrap available to perform this balanced loop deletion. Each homotopy $H_j$ for $j = 1 \ldots i-1$ deletes one direct split and at most one indirect split of $\gamma_j$. Therefore the signs of the remaining intersection points are not affected. Observe that if a vertex $v$ is the basepoint of an outwards loop $\gamma_v$, then $\mathrm{sgn}(v) = 1$ iff $\gamma_v$ is positively oriented, and $\mathrm{sgn}(v) = -1$ iff $\gamma_v$ is negatively oriented; see Figure 15. By definition of $l$, we have

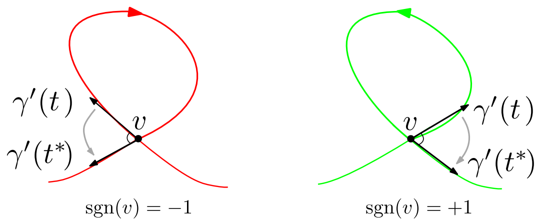


**Figure 15** Two outwards loops; negatively oriented (left) and positively oriented (right).

$n_i \leq l$, where $n_i$ is the number of negative vertices on $\gamma_i$. Therefore there can be at most $l$ distinct integers $i_1, \ldots, i_l$ such that the first loop on $\gamma_{i_\nu}$ is negatively oriented, by Lemma 12 the first loop is always outwards. Since $k = l + 1$, there is a wrap available on $Wr_+^k(\gamma)$.

The process of constructing homotopies $H_i$ never gets stuck, and $|\gamma_{i+1}| < |\gamma_i|$. Therefore we must eventually reach a point when the current curve $\gamma_m$ has $|\gamma_m| = 0$. We now show that this final curve $\gamma_m$ is positively oriented. Note that if $\gamma$ is simple then $\mathrm{Wr}_+^k(\gamma)$ is trivially a positive $k$-interior boundary. So, assume $\gamma$ is not simple.

By definition, the intersection sequence of $\mathrm{Wr}_+^k(\gamma)$ has the form $0, 1+, 2+, \ldots, k+, I'$, $k-, \ldots, 1-, 0$, where $I'$ is obtained from the signed intersection sequence of $\gamma$ by incrementing each label by $k$ and removing occurrences of 0. The basepoint of the first loop on $\gamma_1 = \mathrm{Wr}_+^k(\gamma)$ must be a vertex from $\gamma$. Then $H_1$ modifies the intersection sequence by removing two labels from $I'$ corresponding to this loop. If the loop is negatively oriented, then balanced loop deletion also removes a pair $a+ \ldots a-$ for the wrap. The same modification happens for each homotopy $H_i$ until $I'$ is empty, and the homotopies after that contract wraps $a+ \ldots a-$ which are all positively oriented loops. We know that $\gamma$ has at most $l = k-1$ negative vertices, hence there can only be $k-1$ balanced loop deletions, but there are $k$ wraps. Thus, $\gamma_m$ must be a wrap that $Wr_+^k(\gamma)$ added to $\gamma$, so $\gamma_m$ is a positively oriented loop which can be contracted to its basepoint using a final left sense-preserving homotopy $H_m$. This shows that $H = \sum_{i=1}^m H_i$ is a left sense-preserving nullhomotopy of $\mathrm{Wr}_+^k(\gamma)$ as desired. ◀

The example in Figure 16 shows that the number of wraps used in Theorem 13 is nearly tight. We now show that wrapping resolves obstinance in either direction of wrapping. The proof is straight-forward and given in [7].

▶ **Theorem 14** (Wrapping Resolves Obstinance (General)). *Let $\gamma \in \mathscr{C}$ with outer basepoint and set $n = |\gamma|$. Then there are constants $k_-, k_+ \leq n+2$ so that $obs(Wr_-^{k_-}(\gamma)) = obs(Wr_+^{k_+}(\gamma)) = 0$, $Wr_-^{k_-}(\gamma)$ is a negative interior boundary, and $Wr_+^{k_+}(\gamma)$ is a positive interior boundary.*

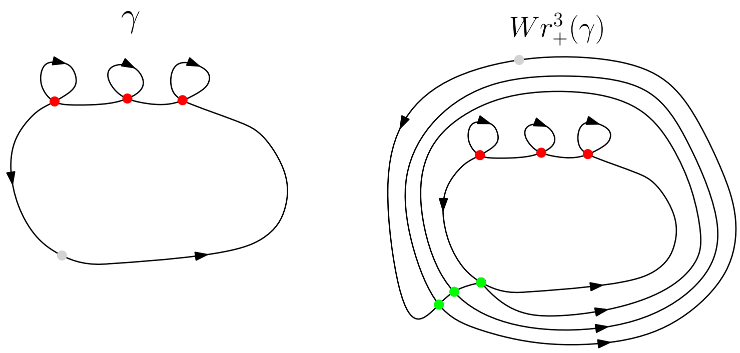


**Figure 16** An example of a family of curves that require $k = l$ wraps to resolve obstinance, where $l$ is the number of negative vertices in $V(\gamma)$. Here, $k = l = 3$.

Let us make a simple observation: once $Wr_\pm^k(\gamma)$ is an interior boundary, so too is $Wr_\pm^j(\gamma)$ for any integer $j \geq k$. This holds because we can simply add the extra $j - k$ wraps to the self-overlapping decomposition $\Omega$ of $Wr_+^k(\gamma)$.

## 4.3 Irreducible and Strongly Irreducible Curves

We are now ready to apply Theorem 14 to prove sufficient combinatorial conditions for a curve $\gamma$ to be self-overlapping based on $\text{WHIT}(\gamma)$ and properties of its direct splits. If $\gamma \in \mathscr{C}$ has no proper positive self-overlapping direct splits, we call $\gamma$ **irreducible**. A special case of irreducibility is of particular interest to us: If $\text{WHIT}(\gamma_v) \leq 0$ for all proper direct splits, we call $\gamma$ **strongly irreducible**. See Figures 4 and 6 for examples of strongly irreducible curves. Note that a strongly irreducible curve is irreducible since any positive self-overlapping curve $\gamma$ has $\text{WHIT}(\gamma) = 1$.

▶ **Lemma 15** (Existence of a Direct Split). *Let $\gamma \in \mathscr{C}$ and $\Omega$ be a direct split decomposition of $\gamma$, with $|\Omega| \geq 2$. Then $\Omega$ contains a proper direct split.*

**Proof.** A leaf $v_i$ in the tree $T_\Omega$ necessarily corresponds to the basepoint of a direct split $\gamma_i$ in the decomposition $\Omega$. Since $|\Omega| \geq 2$, this direct split $\gamma_i$ must be proper. ◀

▶ **Theorem 16** (Irreducible Curves are Self-Overlapping). *Assume $\gamma$ has $\text{WHIT}(\gamma) = 1$ and positive outer basepoint. If $\gamma$ is irreducible, then it is self-overlapping.*

**Proof.** Apply Theorem 13 to find a $k \in \mathbb{Z}$ such that $Wr_+^k(\gamma)$ is a positive interior boundary. We know from property 3 of Theorem 9 that there is a self-overlapping decomposition $\Omega$ of $Wr_+^k(\gamma)$ into positive self-overlapping subcurves. By Lemma 15 we know that $\Omega$ must have a self-overlapping direct split of $Wr_+^k(\gamma)$, and we will show that $\gamma$ is the only direct split of $Wr_+^k(\gamma)$ that can be self-overlapping.

Let $w_i$ be the vertex created by the $i^{th}$ wrap. The intersection sequence of $Wr_+^k(\gamma)$ therefore has the prefix $w_k, w_{k-1}, \ldots, w_1$. Then the direct split $Wr_+^k(\gamma)_{w_i}$ at $w_i$ on $Wr_+^k(\gamma)$ has $\text{WHIT}(Wr_+^k(\gamma)_{w_i}) = 1 + (i - 1) = i$ by Lemma 6, and is therefore not self-overlapping for $i \geq 2$. And any direct split $Wr_+^k(\gamma)$ at a vertex of $\gamma$ which is also a proper direct split on $\gamma$ cannot be self-overlapping since $\gamma$ is irreducible. Note that by our notation $w_1$ is the vertex corresponding to the original basepoint $\gamma(0)$. And this is the only vertex at which the direct split $Wr_+^k(\gamma)_{w_1} = \gamma$ could potentially be self-overlapping. Thus, it follows with Lemma 15 that $\gamma$ is self-overlapping. ◀

We now have a nice corollary: conditions on the Whitney indices of a curve and its subcurves alone can be sufficient for self-overlappingness.

▶ **Theorem 17** (Strongly Irreducible Curves are Self-Overlapping). *Assume $\gamma$ has $WHIT(\gamma) = 1$ and positive outer basepoint. If $\gamma$ is strongly irreducible, then it is self-overlapping.*

Note that strongly irreducible curves are a proper subset of irreducible curves; see $\gamma_I$ in Figure 2. And Theorem 17 is false without the basepoint assumption; see Figure 17.

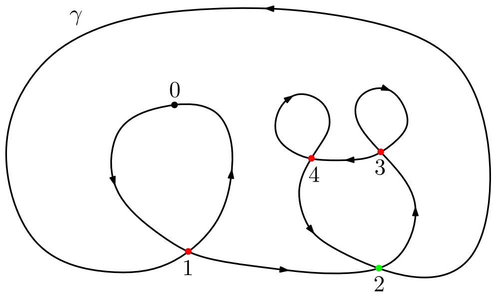


**Figure 17** This curve $\gamma$ does not have an outer basepoint. It is not self-overlapping, yet $\gamma$ is strongly irreducible due to the empty positively oriented loop on the indirect split $\gamma_{1^*}$.

One can decide whether a piecewise linear curve $\gamma$ is (strongly) irreducible by checking the required condition for each direct split. Let $N$ be the number of line segments of $\gamma$ and $n = |\gamma| = |V(\gamma)| \in O(N^2)$. Then irreducibility can be tested in $O(nN^3)$ time, using Shor and Van Wyk's algorithm to test for self-overlappingness in $O(N^3)$ time [17]. Strong irreducibility can be decided in $O(n^2)$ time by applying Corollary 7 to each direct split of $\gamma$.

## 5 Discussion

We introduced new curve classes (zero-obstinance, irreducable, and strongly irreducable curves; see Figure 2), which help us understand self-overlapping curves and interior boundaries. We proved combinatorial results and showed that wrapping a curve resolves obstinance. These new mathematical foundations for self-overlapping curves and interior boundaries could pave the way for related algorithmic questions. For example, is it possible to decide whether a curve is self-overlapping in $o(N^3)$ time? How fast can one decide self-overlappingness of a curve on the sphere? Can one decide irreducability in $o(n^2)$ time, even in the presence of a large number of linked subcurves?

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