Dots & Polygons

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──── **Abstract** ────

We present a new game, *Dots & Polygons*, played on a planar point set. We prove that its NP-hard and discuss strategies for the case when the point set is in convex position.

1 Introduction

Dots & Boxes [4] is a popular game, in which two players take turns in connecting nodes lying on the integer lattice, scoring when they surround unit squares. We introduce a more geometric variant of this game: *Dots & Polygons*.

The game is played on a planar point set P of size n. Two players, $\mathcal{B}$ (blue) and $\mathcal{R}$ (red), take turns, connecting two points $p, q \in P$ by a straight-line edge in a turn. The edge may not intersect other points or edges, and may not lie in a previously scored area. When a player closes a polygon, this player scores its area and makes another move. At the end, the player with the larger total area wins. We distinguish two variants. In *Dots & Polygons & Holes*, when a player closes a cycle, the player scores the enclosed area (excluding possibly
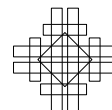
■ **Figure 1** A screenshot of the *Dots & Polygons* game. In the *Dots & Polygons & Holes* version, if $\mathcal{R}$ draws the dotted edge, $\mathcal{R}$ will score the interior minus the blue triangle. In *Dots & Simple Polygons*, $\mathcal{R}$ will not score in this way.

previously enclosed parts). In *Dots & Simple Polygons*, a player only scores, when they close a simple polygon with no points inside. Figure 1 illustrates the difference between the variants.

A similar game is *Monochromatic Complete Triangulation Game* [1], but in that game only triangles are scored, and the score is the number of triangles. There is another variant of *Dots & Boxes* also called Dots & Polygons [12] that is played on the integer lattice.

Dots & Polygons is implemented on top of the *Ruler of the Plane* framework [2]. Both variants of the game can be played online (see supplementary materials). The ruler of the plane framework can be used to demonstrate different interesting geometric concepts and their applications. For example, to show a dynamic representation of the trapezoidal decomposition the user can press the T key while a game is active. The framework is extended with an implementation of a Doubly-Connected Edge List (DCEL) [3], a trapezoidal decomposition [3] and the Graham Scan algorithm [11].

Contributions. In Section 2 we show that *Dots & Simple Polygons* is NP-hard. We do so by a reduction from vertex-disjoint cycle packing in cubic planar graphs, including a self-contained reduction from planar 3-Satisfiability to this cycle-packing problem, and from the cycle-packing problem to *Dots & Boxes*. In Section 3 we discuss a greedy strategy for the case that P is in convex position.

2 Hardness

We show that *Dots & Simple Polygons* is NP-hard by a reduction from the maximum cycle packing problem in planar cubic graphs. The reduction is similar to the proof of NP-hardness of *Dots & Boxes*. The book *Winning Ways for your Mathematical Plays* [5] mentions that a generalization of *Dots & Boxes* can be shown to be NP-hard by a reduction from the maximum vertex-disjoint cycle packing (VCP) problem. The VCP problem can be viewed as a generalization of the triangle packing problem [6], which is known to be NP-hard [10].
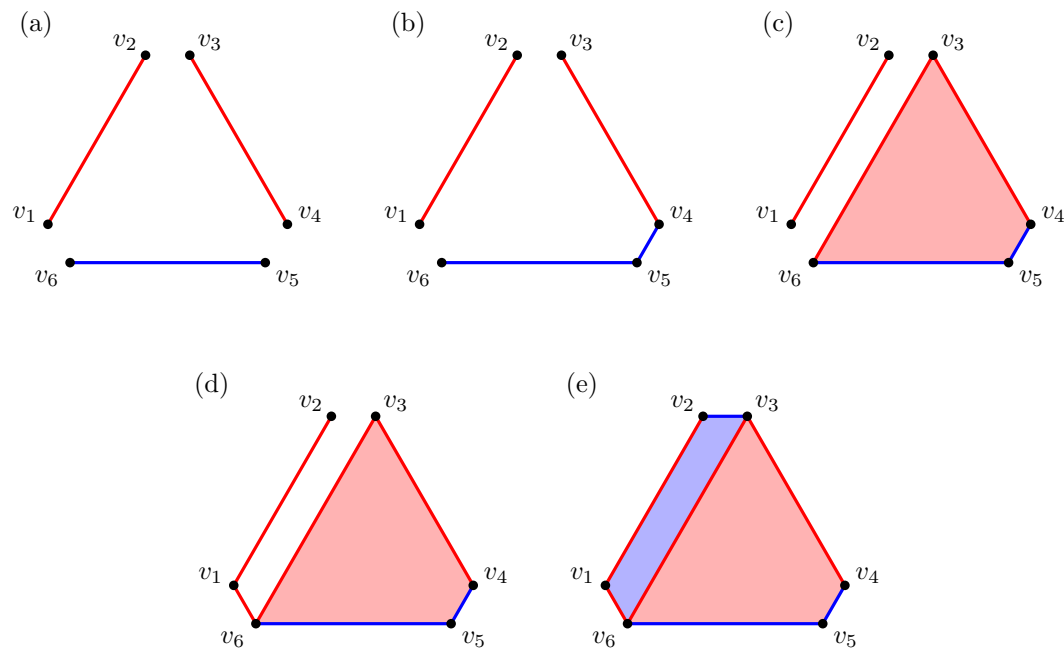
Figure 2 Player $\mathcal{R}$ could have won this game, but after reaching the state in (a) loses as shown in (b–e).

Eppstein notes that the NP-hardness, mentioned in [5], should apply to the classic *Dots & Boxes* by a reduction from the VCP problem in planar cubic graphs [9]. However, he does not cite a source of the hardness proof for this VCP variant. Furthermore, triangle packing is polynomial-time solvable in planar graphs with maximum degree three [8], and thus can no longer be used to justify the hardness of the VCP in planar cubic graphs. Thus, for the sake of completeness, we also show Theorem 1 and Theorem 2, which are used to prove Theorem 3. The full proofs for these theorems are given in [7].

▶ **Theorem 1.** *Maximal vertex-disjoint cycle packing in planar cubic graphs is NP-complete.*

▶ **Theorem 2.** *Given a state of* Dots & Boxes*, it is NP-hard to decide whether* $\mathcal{B}$ *can win.*

▶ **Theorem 3.** *Given a state of* Dots & Simple Polygons*, it is NP-hard to decide whether* $\mathcal{B}$ *can win.*

3 Strategy

What follows is a discussion of greedy strategies for *Dots & Polygons* played on a set of points P in convex position. Trivially, when the points are places in convex position, there exists no distinction between *Dots & Polygons & Holes* and *Dots & Simple Polygons*. In the related *Monochromatic Complete Triangulation Game* a greedy strategy is optimal for such points [1].

We first observe that in this setting the number of turns is exactly $n = |P|$: Consider connected components of the edges drawn by the players. If a player connects two points in the same component, this closes a polygon, and therefore the turn continues. If, however, the two points are in different components, the turn ends and the number of connected components decreases. Thus, the number of turns equals to the number of initial components.

Consider a game state in which the current player cannot close a polygon. Let E be the set of all edges that can still be drawn. Define the weight $w(e)$ for $e \in E$ to be the area the opponent can claim on their next turn if the current player draws e. For example an edge e between two isolated points has weight $w(e) = 0$. A *simple greedy strategy* is the following: if there is an edge that can close some area, immediately draw that edge. Otherwise, draw the edge $e_{min} = \min\limits_{\forall e \in E} w(e)$. This strategy is not optimal, as shown in Figure 2.

The edges drawn partition the remaining area into *subproblems*. For an edge $e \in E$, $w(e)$ can only change if an edge in the same subproblem is drawn. Let $E' \subset E$ be the set of edges within a subproblem. We call a subproblem *easy*, if only two of the edges $e, e' \in E'$ lie on the convex hull of P. In such a subproblem, all edges have the same weight, namely the area of the subproblem. We call a game state in which all subproblems are easy, an *easy endgame*.

In the following we assume that points are placed in such a way that a draw is not possible. Consider the player that will go last (i.e., $\mathcal{B}$ for odd n, $\mathcal{R}$ for even n). If this player plays the *simple greedy strategy* in such a way that they reach an easy endgame, then this player wins. The reason is that from that point onward, anytime the opponent scores an area, this player will score an area that is at least as large in their next turn. For $n \leq 5$, an easy endgame is always reached. For $n = 6$ and $n = 7$, the player that will go last can enforce an easy endgame by playing a diagonal in their first move, preventing the situation of Figure 2. In this way, $\mathcal{B}$ can always win for $n = 3, 5, 7$, and $\mathcal{R}$ for $n = 4, 6$. We leave the problem for $n > 7$ open.

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