Visual Demo of Discrete Stratified Morse Theory

Youjia Zhou
University of Utah, Salt Lake City, UT, USA
zhou325@sci.utah.edu

Kevin Knudson
University of Florida, Gainesville, FL, USA
kknudson@ufl.edu

Bei Wang[1]
University of Utah, Salt Lake City, UT, USA
beiwang@sci.utah.edu

Abstract

Discrete stratified Morse theory, first introduced by Knudson and Wang, works toward a discrete analogue of Goresky and MacPherson's stratified Morse theory. It is inspired by the works of Forman on discrete Morse theory by generalizing stratified Morse theory to finite simplicial complexes. The class of discrete stratified Morse functions is much larger than that of discrete Morse functions. Any arbitrary real-valued function defined on a finite simplicial complex can be made into a discrete stratified Morse function with the proper stratification of the underlying complex. An algorithm is given by Knudson and Wang that constructs a discrete stratified Morse function on any finite simplicial complex equipped with an arbitrary real-valued function. Our media contribution is an open-sourced visualization tool that implements such an algorithm for 2-complexes embedded in the plane, and provides an interactive demo for users to explore the algorithmic process and to perform homotopy-preserving simplification of the resulting stratified complex.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Discrete Morse theory, stratified Morse theory, discrete stratified Morse theory, topological data analysis, data visualization

Digital Object Identifier 10.4230/LIPIcs.SoCG.2020.82

Category Media Exposition

Related Version A paper describing the theoretical foundation for the visualization is available at `https://arxiv.org/abs/1801.03183`.

Supplementary Material The visualization tool (our media contribution) is available on GitHub: `https://github.com/tdavislab/VIS-DSMT` with a link to a visual demo.

Funding NSF IIS-1513616, NSF DBI-1661375.

Acknowledgements We thank Yulong Liang who worked on the first prototype of the visualization.

1 Discrete stratified Morse theory

We begin with a brief review of discrete stratified Morse theory (DSMT) [4, 5]. Several key components differentiate DSMT from previous theories [1, 2, 3]. First, we work with *open simplices*, which is not surprising, since we are developing a discrete version of the strata in stratified spaces, which are typically open manifolds. Second, we define a *stratified simplicial complex*, which mimics the frontier condition of a Whitney stratification in a discrete setting. Third, we derive the notion of a *discrete stratified Morse function* from the

[1] Corresponding author

© Youjia Zhou, Kevin Knudson, and Bei Wang;
licensed under Creative Commons License CC-BY
36th International Symposium on Computational Geometry (SoCG 2020).
Editors: Sergio Cabello and Danny Z. Chen; Article No. 82; pp. 82:1–82:4
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

definition of a stratified Morse function. Finally, we establish the theoretical foundations of DSMT regarding its ability to capture the shape of data (homotopy type and homology) and its relation to DMT (generality). In particular, DSMT is applicable to *any* function f defined on a simplicial complex K since f can be made into a discrete stratified Morse function with respect to some stratification $\mathcal{S}$.

▶ **Definition 1** (Definition 3.2, [5]). *Let K be a simplicial complex. A* stratification *of K, $\mathcal{S} = \{S_i\}$, is a locally finite collection of disjoint locally closed subsets called* strata*, $S_i \subset K$, such that $K = \bigcup S_i$ and S_i satisfies the condition of the frontier: $S_i \cap \overline{S}_j \neq \emptyset$ if and only if $S_i \subset \overline{S_j}$. Each S_i is a union of (open) simplices; its connected components are called* strata pieces*. Let $\overline{S}_i$ denote its closure, and $\mathring{S}_i$ its interior.*

A stratification gives an *assignment* $s : K \to \mathcal{S}$ taking each open simplex σ in K to a particular stratum $s(\sigma)$. Let K be a simplicial complex equipped with a stratification s and a function $f : K \to \mathbb{R}$. For a p-simplex α, we define

$$U_s(\alpha) = \{\beta^{(p+1)} > \alpha \mid s(\beta) = s(\alpha) \text{ and } f(\beta) \leq f(\alpha)\},$$
$$L_s(\alpha) = \{\gamma^{(p-1)} < \alpha \mid s(\gamma) = s(\alpha) \text{ and } f(\gamma) \geq f(\alpha)\}.$$

Similarly, $U(\alpha) = \{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\}$, and $L(\alpha) = \{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\}$.

▶ **Definition 2** (Definition 3.5, [5]). *A function $f : K \to \mathbb{R}$ (equipped with a stratification s) is a* discrete stratified Morse function *if for every p-dimensional simplex $\alpha^{(p)} \in K$, (i) $|U_s(\alpha)| \leq 1$, (ii) $|L_s(\alpha)| \leq 1$, and (iii) if one of these sets is nonempty, then the other must be empty.*

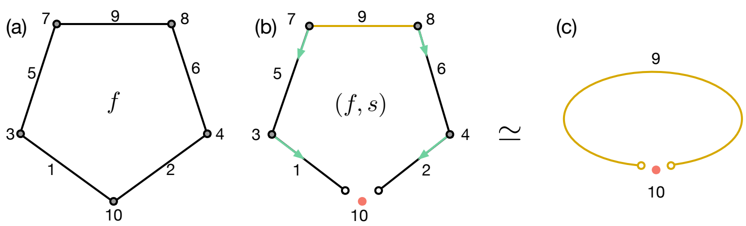

Figure 1 An example of a discrete stratified Morse function. Red are violators; yellow are critical simplicies. See **DSMT Exp. 1** in the demo.

▶ **Definition 3** (Definitions 3.6 and 3.7 [5]). *A simplex $\alpha^{(p)}$ is* globally critical *if $|U(\alpha)| = |L(\alpha)| = 0$. A simplex $\alpha^{(p)}$ is* locally critical *if it is not globally critical and if $|U_s(\alpha)| = |L_s(\alpha)| = 0$. A* critical value *of f is its value at a critical simplex. A simplex $\alpha^{(p)}$ is* globally noncritical *if $|U(\alpha)| + |L(\alpha)| = 1$. A simplex $\alpha^{(p)}$ is* locally noncritical *if it is not globally noncritical and exactly one of the following two conditions holds: (i) $|U_s(\alpha)| = 1$ and $|L_s(\alpha)| = 0$; or (ii) $|L_s(\alpha)| = 1$ and $|U_s(\alpha)| = 0$.*

▶ **Definition 4** (Definition 3.8, [5]). *A simplex $\alpha^{(p)}$ is a* violator *(of the conditions associated with a discrete Morse function) if it is neither critical nor noncritical.*

Violators are central to the algorithm in constructing a discrete stratified Morse function. See Figure 1 for an example: f in (a) is not a discrete stratified Morse function; however, it can be converted into one when it is equipped with an appropriate stratification s in (b).

Given a discrete stratified Morse function f equipped with a stratification s, f restricted to $S_i \in \mathcal{S}$, denoted as $f_i := f \mid_{S_i}$, is by definition a discrete Morse function. We may associate a discrete gradient vector field V_i to S_i as follows. Since any noncritical simplex $\alpha^{(p)} \in S_i$ has at most one of the sets $U_s(\alpha)$ and $L_s(\alpha)$ being nonempty, there is a unique face $\gamma^{(p-1)} < \alpha$ in S_i with $f(\gamma) \geq f(\alpha)$ or a unique coface $\beta^{(p+1)} > \alpha$ in S_i with $f(\beta) \leq f(\alpha)$. The pair $\{\alpha < \beta\}$ (or the pair $\{\gamma < \alpha\}$) is referred to as a *Morse pair*. Denote by V_i the collection of all such pairs. Such pairs are formed by (globally or locally) noncritical simplices. We visualize V_i by drawing an arrow from α to β for every Morse pair $\{\alpha < \beta\} \in V_i$. Such a discrete gradient provides a simplification (i.e., collapsing) order for the complex S_i, where Morse pairs can be removed to produce a reduced complex with the same homotopy type, see Figure 1(c). We have the following result [5, Theorem 3.1]:

▶ **Theorem 5** (Weak DSMT Theorem A, Theorem 3.3, [5]). *Given a discrete stratified Morse function (f, s), performing a collapse of either a global noncritical pair or a local noncritical pair is a stratum-preserving homotopy equivalence.*

2 Algorithm

Given a simplicial complex K, and any real-valued function $f : K \to \mathbb{R}$, we can construct a discrete stratified Morse function using the following algorithm described in [5, Section 3.5]:

1. Make a single pass of all simplices in K, and order the violators $\mathcal{V} = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ by increasing dimension and by increasing function value within each dimension.
2. Initialize $\mathcal{S} = \emptyset$, $i = 1$.
3. Remove σ_i from $\mathcal{V}$ and add σ_i to $\mathcal{S}$.
4. Consider $K_i = K \setminus \{\sigma_1, \ldots, \sigma_i\}$:
   - If the restriction of f to K_i, $f|_{K_i}$, is a discrete Morse function, then let J denote the set of indices $k \leq i$ such that $\sigma_k \in \overline{K_i}$ and add the following strata to $\mathcal{S}$ (which may contain more than two strata pieces): the frontier $\overline{K_i} \setminus (\mathring{K}_i \cup \{\sigma_j\}_{j \in J})$ and $\mathring{K}_i$.
   - Otherwise, if $f|_{K_i}$ is not a discrete Morse function, then at least one σ_j with $j > i$ remains a violator.
5. Remove simplices that are no longer violators from the list and repeat steps 2-4 above until no violators are left.

The result of the algorithm is shown in Figure 1(b). We have shown the correctness of the algorithm in [5, Theorem 3.4].

3 Visualization design

We provide an interactive visualization tool that demonstrates the algorithmic process of DSMT as described in Section 2; see Figure 2 for its user interface and the video for a demo.

The tool constructs a discrete stratified Morse function from any real-valued function defined on a 1- or 2-complex embedded in the plane (e.g., a planar triangulation). The tool provides 6 examples for DSMT, 4 of which are described in [5, Section 4], and for comparison, 3 examples for DMT (since a discrete Morse function is a discrete stratified Morse function for the trivial stratification). The tool enables the random perturbation of function values for each example. It also allows the import of user-specified examples in an appropriate format (see the user's guide for details). Using such a tool, we visualize a simplicial complex with function values attached to each simplex. We explore various stages of the algorithm by

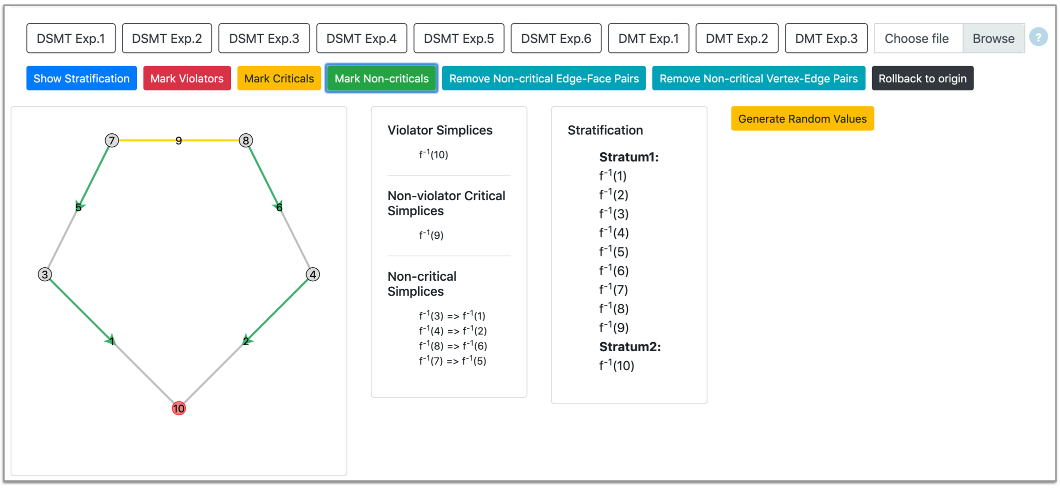

Figure 2 The user interface of our visualization tool.

marking violators, critical simplices, and noncritical simplices, which form Morse pairs that are visualized by green arrows. We also highlight the resulting stratification. We additionally perform homotopy-preserving simplification of the stratified simplicial complex by removing Morse pairs. Figure 3 illustrates such a process; see the visualization tool for more examples.

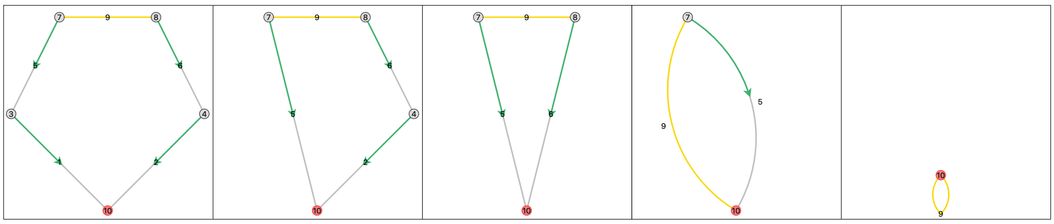

Figure 3 A homotopy-preserving simplification of a stratified simplicial complex.

References

1 Robin Forman. Morse theory for cell complexes. *Advances in Mathematics*, 134:90–145, 1998.

2 Robin Forman. A user's guide to discrete Morse theory. *Séminaire Lotharingien de Combinatoire*, 48, 2002.

3 Mark Goresky and Robert MacPherson. *Stratified Morse Theory*. Springer-Verlag, 1988.

4 Kevin Knudson and Bei Wang. Discrete stratified Morse theory: A user's guide. *International Symposium on Computational Geometry (SOCG)*, 2018.

5 Kevin Knudson and Bei Wang. Discrete stratified Morse theory: Algorithms and a user's guide, 2019. `arXiv:1801.03183`.