On a Temporal Logic of Prefixes and Infixes

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Abstract

A classic result by Stockmeyer [16] gives a non-elementary lower bound to the emptiness problem for star-free generalized regular expressions. This result is intimately connected to the satisfiability problem for interval temporal logic, notably for formulas that make use of the so-called chop operator. Such an operator can indeed be interpreted as the inverse of the concatenation operation on regular languages, and this correspondence enables reductions between non-emptiness of star-free generalized regular expressions and satisfiability of formulas of the interval temporal logic of the chop operator under the homogeneity assumption [5]. In this paper, we study the complexity of the satisfiability problem for a suitable weakening of the chop interval temporal logic, that can be equivalently viewed as a fragment of Halpern and Shoham interval logic featuring the operators B, for "begins", corresponding to the prefix relation on pairs of intervals, and D, for "during", corresponding to the infix relation. The homogeneous models of the considered logic naturally correspond to languages defined by restricted forms of regular expressions, that use union, complementation, and the inverses of the prefix and infix relations.

2012 ACM Subject Classification Theory of computation → Logic and verification; Theory of computation

Keywords and phrases Interval Temporal Logic, Star-Free Regular Languages, Satisfiability, Complexity

Digital Object Identifier 10.4230/LIPIcs.MFCS.2020.21

1 Introduction

A classic result in formal languages proved by Stockmeyer states that the emptiness problem for star-free generalized regular expressions is non-elementarily decidable (tower-complete) for unbounded nesting of negation [15, 16] (it is in (k-1)-$EXPSPACE$-complete for expressions where the nesting of negation is at most $k \in \mathbb{N}^+$). Such a problem can be easily turned into the satisfiability problem for the logic C of the chop modality over finite domains [7, 13, 17], under the homogeneity assumption [14], and vice versa. C has one binary modality only, the so-called *chop* operator, that allows one to split the current interval in two parts and to state what is true over the first part and what over the second one. The homogeneity assumption forces a proposition letter to hold over an interval if and only if it holds over all of its points.

45th International Symposium on Mathematical Foundations of Computer Science (MFCS 2020).
Editors: Javier Esparza and Daniel Král'; Article No. 21; pp. 21:1–21:14
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

It can be easily shown that there is a $LOGSPACE$ reduction of the emptiness problem for star-free generalized regular expressions to the satisfiability problem for C with unbounded nesting of the chop operator, and vice versa.

The close relationships between formal languages and interval temporal logics have been already pointed out in [11, 12], where the interval temporal logic counterparts of regular languages, ω-regular languages, and extensions of them (ωB- and ωS-regular languages) have been provided. In this paper, we focus on some meaningful fragments of C (under the homogeneity assumption). Modalities for the prefix, the suffix, and the infix relations over (finite) intervals can be defined in C in a straightforward way. We have that a formula holds over a prefix of the current interval if and only if it is possible to split the interval in such a way that the formula holds over the first part and the second part contains at least two points. The case of suffixes is completely symmetric. As for infixes, a proper sub-interval of the current interval is a suffix of one of its prefixes or, equivalently, a prefix of one of its suffixes, that is, infixes can be defined in terms of prefixes and suffixes. The satisfiability problem for the logic D of the infix relation has been recently shown to be $PSPACE$-complete by a suitable contraction method [1]. Moreover, in [2], it has been proved that the problem for the logic BE of prefixes and suffixes is $EXPSPACE$-hard by a polynomial-time reduction from a domino-tiling problem for grids with rows of single exponential length. As for the upper bound, the only available one is given by the non-elementary decision procedure for full Halpern and Shoham's interval temporal logic HS [6] under the homogeneity assumption [10] (BE is a small fragment of it). Despite several attempts, no progress has been done in the reduction/closure of such a very large gap.[1]

Here, we study the satisfiability problem for the logic BD of prefixes and infixes that lies (strictly) in between D and BE. The addition of a modality for prefixes makes the satisfiability checking procedure for BD much more complex than the one for D, as the two relations/modalities may interact in a non-trivial way. In particular, while $PSPACE$ membership of the satisfiability problem for D was established by a reduction to the emptiness problem for a certain class of automata, we will prove $EXPSPACE$ membership of the problem for BD by means of a suitable small model theorem.

The rest of the paper is organised as follows. In Section 2, we introduce syntax and semantics of BD under the homogeneity assumption, and point out some interesting connections between BD formulas and restricted forms of star-free generalized regular expressions. Next, in Section 3, we introduce the notion of homogeneous compass structure, that provides a particularly useful representation for models of BD formulas. In Section 4, we give an EXPSPACE decision procedure for checking the satisfiability of BD formulas. Finally, in Section 5, we provide an assessment of the work done and outline future research directions.

2 The logic BD of prefixes and infixes

In this section, we introduce the logic BD of prefixes and infixes, we formally state the homogeneity assumption, and we define the satisfiability relation under such an assumption. We conclude the section with a short analysis of the relationships between BD and a suitable restriction of star-free generalized regular expressions.

[1] In fact, the only achieved result was a negative one showing that there is no hope in trying to tailor the proof techniques exploited for HS, which are based on the notion of BE-descriptor, to BE, as it is not possible to give an elementary upper bound on the size of BE-descriptors (in the case of BE) [3].

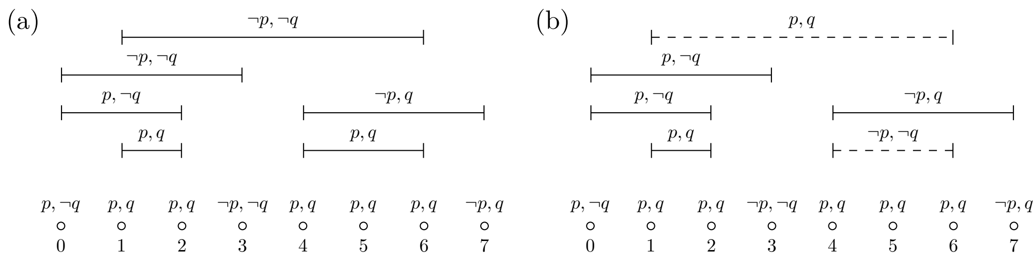

Figure 1 A homogeneous model (a - left) vs. a general one (b - right).

BD formulas are built up from a countable set Prop of proposition letters according to the following grammar: $\varphi ::= p \mid \neg\psi \mid \psi \vee \psi \mid \langle B\rangle\psi \mid \langle D\rangle\psi$, where $p \in \text{Prop}$ and $\langle B\rangle$ and $\langle D\rangle$ are the modalities for Allen's relations *Begins* and *During*, respectively.

Let $N \in \mathbb{N}$ be a natural number and let $\mathbb{I}_N = \{[x, y] : 0 \leq x \leq y \leq N\}$ be the set of all intervals over the prefix $0 \dots N$ of $\mathbb{N}$. A (finite) model for BD formulas is a pair $\mathbf{M} = (\mathbb{I}_N, \mathcal{V})$, where $\mathcal{V} : \mathbb{I}_N \to 2^{\text{Prop}}$ is a valuation that maps intervals in $\mathbb{I}_N$ to sets of proposition letters. Let $\mathbf{M}$ be a model and $[x, y]$ an interval. The semantics of a BD formula is defined as follows:

- $\mathbf{M}, [x, y] \models p$ iff $p \in \mathcal{V}([x, y])$;
- $\mathbf{M}, [x, y] \models \neg\psi$ iff $\mathbf{M}, [x, y] \not\models \psi$;
- $\mathbf{M}, [x, y] \models \psi_1 \vee \psi_2$ iff $\mathbf{M}, [x, y] \models \psi_1$ or $\mathbf{M}, [x, y] \models \psi_2$;
- $\mathbf{M}, [x, y] \models \langle B\rangle\psi$ iff there is y', with $x \leq y' < y$, such that $\mathbf{M}, [x, y'] \models \psi$;
- $\mathbf{M}, [x, y] \models \langle D\rangle\psi$ iff there are x' and y', with $x < x' \leq y' < y$, such that $\mathbf{M}, [x', y'] \models \psi$.

The logical constants $\top$ (true) and $\bot$ (false), the Boolean operators $\wedge, \rightarrow$, and $\leftrightarrow$, and the (universal) dual modalities $[B]$ and $[D]$ can be derived in the standard way. We say that a BD formula φ is *satisfiable* if and only if there exist a model $\mathbf{M}$ and an interval $[x, y]$ such that $\mathbf{M}, [x, y] \models \varphi$ (w.l.o.g., $[x, y]$ can be assumed to be the maximal interval $[0, N]$).

We say that a model $\mathbf{M} = (\mathbb{I}_N, \mathcal{V})$ is *homogeneous* if $\mathcal{V}$ satisfies the following property:

$$\forall p \in \text{Prop} \;\; \forall [x, y] \in \mathbb{I}_N \;\; \Big(p \in \mathcal{V}([x, y]) \;\Leftrightarrow\; \forall z \in [x, y] \; p \in \mathcal{V}([z, z])\Big).$$

In Fig. 1, we show a homogeneous model (a) and a non-homogeneous one (b). In homogeneous models, for any proposition letter, the labelling of point-intervals determines that of arbitrary intervals. This is not the case with arbitrary models (see, e.g., [4, 6]). One of the consequences of this fact is that, in homogeneous models, the labelling of the intersection of two intervals contains the labellings of the two intervals (this is the case with intervals $[1, 6]$ and $[4, 7]$ in Fig. 1 (a), whose intersection is the interval $[4, 6]$). Once again, this is not the case with arbitrary models (see the very same intervals in Fig. 1 (b)).

Satisfiability can be relativized to homogeneous models. We say that a BD formula φ is *satisfiable under homogeneity* if there is a homogeneous model $\mathbf{M}$ such that $\mathbf{M}, [0, N] \models \varphi$.

Satisfiability under homogeneity is clearly more restricted than plain satisfiability. We know from [8, 9] that dropping the homogeneity assumption makes D undecidable. This is not the case with the fragment B that, being extremely weak in terms of expressive power, remains decidable [4]. Hereafter, we always refer to BD under the homogeneity assumption.

We conclude the section with a short account of the relationships between BD and star-free generalized regular expressions. Let Σ be a finite alphabet. A *star-free generalized regular expression* (hereafter, simply *expression*) e over Σ is a term of the form: $e ::= \emptyset \mid a \mid \neg e \mid e + e \mid e \cdot e$, for any $a \in \Sigma$. We exclude the empty word ϵ from the syntax, as it makes the correspondence between finite words and finite models of BD formulas easier (such a simplification is quite common in the literature). An expression e defines a language $\mathrm{Lang}(e) \subseteq \Sigma^+$, which is inductively defined as follows:

- $\mathrm{Lang}(\emptyset) = \emptyset$;
- $\mathrm{Lang}(a) = \{a\}$, for every $a \in \Sigma$;
- $\mathrm{Lang}(\neg e) = \Sigma^+ \setminus \mathrm{Lang}(e)$;
- $\mathrm{Lang}(e_1 + e_2) = \mathrm{Lang}(e_1) \cup \mathrm{Lang}(e_2)$;
- $\mathrm{Lang}(e_1 \cdot e_2) = \{w_1 w_2 : w_1 \in \mathrm{Lang}(e_1), w_2 \in \mathrm{Lang}(e_2)\}$.

In [16], Stockmeyer proves that the problem of deciding the emptiness of $\mathrm{Lang}(e)$, for a given expression e, is non-elementary hard. Let us now consider the logic C of the chop operator (under the homogeneity assumption). C features one binary modality, the "chop" operator $\langle C\rangle$, plus the modal constant π. For any model $\mathbf{M}$ and any interval $[x, y]$, $\mathbf{M}, [x, y] \models \psi_1 \langle C\rangle \psi_2$ if and only if there exists $z \in [x, y]$ such that $\mathbf{M}, [x, z] \models \psi_1$ and $\mathbf{M}, [z, y] \models \psi_2$, and $\mathbf{M}, [x, y] \models \pi$ if and only if $x = y$. Modalities $\langle B\rangle$ and $\langle D\rangle$ of BD can be easily encoded in C as follows: $\langle B\rangle\psi = \psi\langle C\rangle\neg\pi$ and $\langle D\rangle\psi = \neg\pi\langle C\rangle(\psi\langle C\rangle\neg\pi)$.

It can be shown that, for any expression e over Σ, there exists a formula φ_e of C whose set of models is the language $\mathrm{Lang}(e)$, that is, $\mathrm{Lang}(e) = \{\mathcal{V}(0,0)\ldots\mathcal{V}(N,N) : (\mathbb{I}_N, \mathcal{V}) \models \psi_e\}$. Such a formula is the conjunction of two sub-formulas ψ_Σ and ψ_e, where ψ_Σ guarantees that each unitary interval of the model is labelled by exactly one proposition letter from Σ, and ψ_e constrains the valuation on the basis of the inductive structure of (the translation of) e. As an example, if $e = e_1 \cdot e_2$, then $\psi_e = \psi_{e_1}\langle C\rangle((\neg\pi \wedge \neg(\neg\pi\langle C\rangle\neg\pi))\langle C\rangle\psi_{e_2})$. Thanks to such a mapping of expressions in C formulas, we can conclude that the satisfiability problem for C is non-elementary hard (its non-elementary decidability follows from the opposite mapping).

A careful look at the expression-to-formula mapping reveals that the chop modality only comes into play in the translation of expressions featuring the operator of concatenation. In view of that, it is worth looking for subclasses of star-free generalized regular expressions where the concatenation operation is used in a very restricted manner, so as to avoid the need of the chop operator in the translation. Let us focus our attention on the following class of *restricted expressions*: $e ::= \emptyset \mid a \mid \neg e \mid e + e \mid \mathrm{Pre}(e) \mid \mathrm{Inf}(e)$, for any $a \in \Sigma$, where $\mathrm{Pre}(e)$ and $\mathrm{Inf}(e)$ are respectively a shorthand for $e \cdot (\neg\emptyset)$ (thus defining the language $\mathrm{Lang}(\mathrm{Pre}(e)) = \{wv : w \in \mathrm{Lang}(e), v \in \Sigma^+\}$), and $(\neg\emptyset) \cdot e \cdot (\neg\emptyset)$ (thus defining the language $\mathrm{Lang}(\mathrm{Inf}(e)) = \{uwv : u, v \in \Sigma^+, w \in \mathrm{Lang}(e)\}$). Every restricted expression e of the above form can be mapped into an equivalent formula φ_e of BD by applying the usual constructions for empty language, letters, negation, and union, plus the following two rules: (i) $\varphi_{\mathrm{Pre}(e)} = \langle B\rangle\psi_e$, and (ii) $\varphi_{\mathrm{Inf}(e)} = \langle D\rangle\psi_e$.

In the following, we show that the satisfiability problem for BD belongs to $EXPSPACE$. From the above mapping, it immediately follows that the emptiness problem for the considered subclass of expressions, that only uses prefixes and infixes, can be decided in exponential space (rather than in non-elementary time).

3 Homogeneous compass structures

In this section, we introduce a spatial representation of homogeneous models, called *homogeneous compass structures*, that will considerably ease the proofs.

Let φ be a BD formula. We define the *closure* of φ, denoted by $\mathrm{Cl}(\varphi)$, as the set of all its subformulas and of their negations, plus formulas $\langle B\rangle\top$ and $[B]\bot$. For every BD formula φ, it holds that $\mathrm{Cl}(\varphi) \leq 2|\varphi| + 2$. A φ-*atom* (*atom* for short) is a maximal subset F of $\mathrm{Cl}(\varphi)$ that, for all $\psi \in \mathrm{Cl}(\varphi)$, satisfies the following two conditions: (i) $\psi \in F$ if and only if $\neg\psi \notin F$, and (ii) if $\psi = \psi_1 \vee \psi_2$, then $\psi \in F$ if and only if $\{\psi_1, \psi_2\} \cap F \neq \emptyset$. Let $\mathrm{At}(\varphi)$ be the set of all φ-atoms. We have that $|\mathrm{At}(\varphi)| \leq 2^{|\varphi|+1}$, where $|\varphi| = |\mathrm{Cl}(\varphi)|/2$. For all $R \in \{B, D\}$, we introduce the functions Req_R, Obs_R, and Box_R, that map each atom $F \in \mathrm{At}(\varphi)$ to the following subsets of $\mathrm{Cl}(\varphi)$:

- $\mathrm{Req}_R(F) = \{\psi \in \mathrm{Cl}(\varphi) : \langle R\rangle\psi \in F\}$;
- $\mathrm{Obs}_R(F) = \{\psi \in \mathrm{Cl}(\varphi) : \langle R\rangle\psi \in \mathrm{Cl}(\varphi), \psi \in F\}$;
- $\mathrm{Box}_R(F) = \{\psi \in \mathrm{Cl}(\varphi) : [R]\psi \in F\}$.

Note that, for each $F \in \mathrm{At}(\varphi)$ and each formula ψ, with $\psi \in \{\psi' : \langle B\rangle\psi' \in \mathrm{Cl}(\varphi)\}$, either $\psi \in \mathrm{Req}_B(F)$ or $\neg\psi \in \mathrm{Box}_B(F)$; the same for D (this means that, per se, $\mathrm{Box}_B(\cdot)$ and $\mathrm{Box}_D(\cdot)$ are not strictly necessary; we introduce them to simplify some proofs). By means of the above functions, we define two binary relations $\to_B$ and $\to_D$ over $\mathrm{At}(\varphi)$ as follows.

For all $F, G \in \mathrm{At}(\varphi)$ we let:

- $F \to_B G$ iff $\mathrm{Req}_B(F) = \mathrm{Req}_B(G) \cup \mathrm{Obs}_B(G)$;
- $F \to_D G$ iff $\mathrm{Req}_D(F) \supseteq \mathrm{Req}_D(G) \cup \mathrm{Obs}_D(G)$.

Notice that from the definition of $\to_B$ (resp., $\to_D$), it easily follows that $\mathrm{Box}_B(F) \subseteq G$ (resp., $\mathrm{Box}_D(F) \subseteq G$). Notice also that $\to_D$ is transitive (by definition of atom, from $\mathrm{Req}_R(F) \supseteq \mathrm{Req}_R(G)$, it immediately follows that $\mathrm{Box}_R(F) \subseteq \mathrm{Box}_R(G)$), while $\to_B$ is not.

▶ **Proposition 1.** *For each pair of atoms $F, G \in \mathrm{At}(\varphi)$, we have that $F = G$ iff $\mathrm{Req}_B(F) = \mathrm{Req}_B(G)$, $\mathrm{Req}_D(F) = \mathrm{Req}_D(G)$, and $F \cap \mathrm{Prop} = G \cap \mathrm{Prop}$.*

Given a formula φ, a φ-*compass structure* (compass structure, when φ is clear from the context) is a pair $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$, where $N \in \mathbb{N}$, $\mathbb{G}_N = \{(x, y) : 0 \leq x \leq y \leq N\}$, and $\mathcal{L} : \mathbb{G}_N \to \mathrm{At}(\varphi)$ is a labelling function that satisfies the following properties:

- (*initial formula*) $\varphi \in \mathcal{L}(0, N)$;
- (*B-consistency*) for all $0 \leq x \leq y < N$, $\mathcal{L}(x, y+1) \to_B \mathcal{L}(x, y)$, and for all $0 \leq x \leq N$, $\mathrm{Req}_B(\mathcal{L}(x, x)) = \emptyset$;
- (*D-consistency*) for all $0 \leq x < x' \leq y' < y \leq N$, $\mathcal{L}(x, y) \to_D \mathcal{L}(x', y')$;
- (*D-fulfilment*) for all $0 \leq x \leq y \leq N$ and all $\psi \in \mathrm{Req}_D(\mathcal{L}(x, y))$, there exist $x < x' \leq y' < y$ such that $\psi \in \mathcal{L}(x', y')$.

Observe that the definition of $\to_B$ and B-consistency guarantee that all the existential requests via the relation B (hereafter B-requests) are fulfilled in a compass structure.

We say that an atom $F \in \mathrm{At}(\varphi)$ is *B-reflexive* (resp., *D-reflexive*) if $F \to_B F$ (resp., $F \to_D F$). If F is not B-reflexive (resp., D-reflexive), it is *B-irreflexive* (resp., *D-irreflexive*).

Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure. We define the function $\mathcal{P} : \mathbb{G}_N \to 2^{\mathrm{Prop}}$ such that $\mathcal{P}(x, y) = \{p \in \mathrm{Prop} : p \in \mathcal{L}(x', x') \text{ for all } x \leq x' \leq y\}$. We say that a φ-compass structure $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ is *homogeneous* if for all $(x, y) \in \mathbb{G}_N$, $\mathcal{L}(x, y) \cap \mathrm{Prop} = \mathcal{P}(x, y)$. The proof of the following theorem is straightforward and thus omitted.

▶ **Theorem 2.** *A* BD *formula φ is satisfiable iff there is a homogeneous φ-compass structure.*

Hereafter, we will often write compass structure for homogeneous φ-compass structure.

In Figure 2, we depict the homogeneous model $\mathbf{M} = (\mathbb{I}_7, \mathcal{V})$ of Figure 1 (a) with the corresponding compass structure $\mathcal{G} = (\mathbb{G}_7, \mathcal{L})$, for a given formula φ. We assume that $\mathrm{Cl}(\varphi) \cap \mathrm{Prop} = \{p, q\}$, $\{\langle B\rangle\psi \in \mathrm{Cl}(\varphi)\} = \{\langle B\rangle\top, \langle B\rangle\neg p\}$, and $\{\langle D\rangle\psi \in \mathrm{Cl}(\varphi)\} = \{\langle D\rangle\neg q\}$. We know that, by the homogeneity assumption, the valuation of proposition letters at point-intervals determines that at non-point ones. As an example, if an interval $[x, y]$ contains time point 3, as, e.g., the interval $[1, 6]$, then $\{p, q\} \cap \mathcal{V}([x, y]) = \emptyset$. Similarly, if an interval $[x, y]$ contains time point 7 (resp., 0), then it must satisfy $\{p\} \cap \mathcal{V}([x, y]) = \emptyset$ (resp., $\{q\} \cap \mathcal{V}([x, y]) = \emptyset$). As for the compass structure $\mathcal{G}$, we first observe that each interval $[x, y]$ in $\mathbf{M}$ is mapped to a point in the second octant of the $\mathbb{N} \times \mathbb{N}$ grid (in Figure 2, we depict the

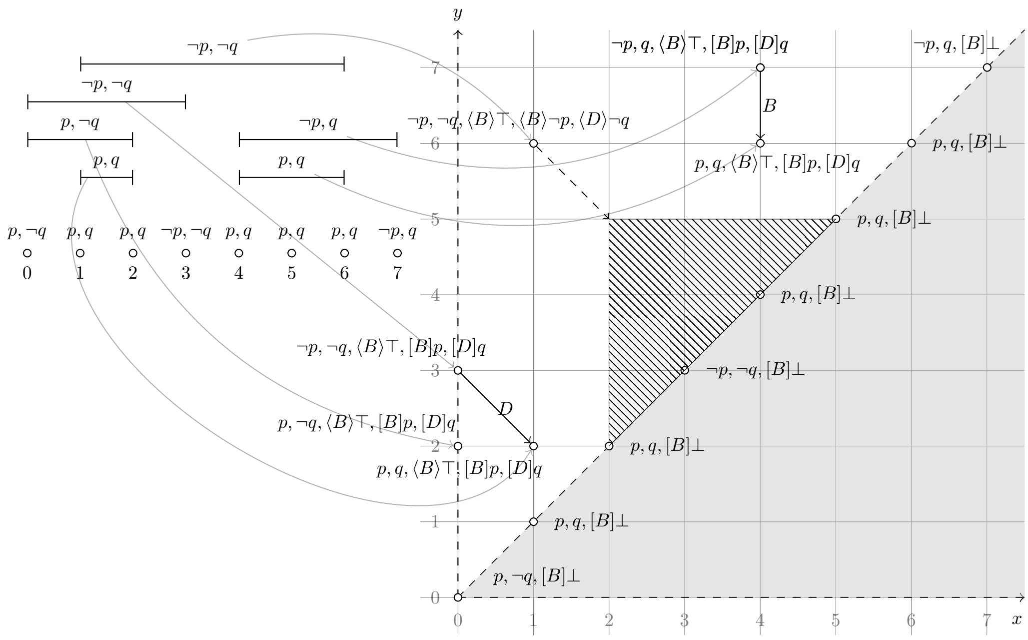

Figure 2 A homogeneous model and the corresponding compass structure.

first quadrant of such a grid, where the first octant is shaded). Thanks to such a mapping, interval relations are mapped into special relations between points (by a slight abuse of terminology, we borrow the names of the interval relations). As an example, in Figure 2 point $(0, 2)$ begins $(0, 3)$. Similarly, as enlightened by the hatched triangle, point $(1, 6)$ has points $(2, 2), (2, 3), (3, 3), (2, 4), (3, 4), (4, 4), (2, 5), (3, 5), (4, 5)$, and $(5, 5)$ as sub-intervals.

In general, all points (x, x) are labelled with irreflexive atoms containing $[B]\bot$, while all points (x, y), with $x < y$, are labelled with atoms containing $\langle B\rangle\top$. The variety of atoms is exemplified by the following cases. Atom $\mathcal{L}(0, 3)$ is both B-irreflexive and D-irreflexive, atom $\mathcal{L}(4, 6)$ is both B-reflexive and D-reflexive, atom $\mathcal{L}(4, 7)$ is B-irreflexive ($\mathrm{Box}_B(\mathcal{L}(4, 7)) = \{p\}$ and $\neg p \in \mathcal{L}(4, 7)$) and D-reflexive ($\mathrm{Box}_D(\mathcal{L}(4, 7)) = \{q\}$ and $q \in \mathcal{L}(4, 7)$), and atom $\mathcal{L}(0, 2)$ is B-reflexive ($\mathrm{Box}_B(\mathcal{L}(0, 2)) = \{p\}$ and $p \in \mathcal{L}(0, 2)$) and D-irreflexive ($\mathrm{Box}_D(\mathcal{L}(0, 2)) = \{q\}$ and $\neg q \in \mathcal{L}(0, 2)$). Finally, it holds that $\mathcal{L}(4, 7) \to_B \mathcal{L}(4, 6)$ ($\mathrm{Box}_B(\mathcal{L}(4, 7)) = \{p, q\}$ and $p, q \in \mathcal{L}(4, 6)$) and $\mathcal{L}(3, 0) \to_D \mathcal{L}(1, 2)$ ($\mathrm{Box}_D(\mathcal{L}(3, 0)) = \{q\}$ and $q \in \mathcal{L}(1, 2)$).

4 The satisfiability problem for BD is decidable in EXPSPACE

In this section, we show that the problem of checking whether a BD formula φ is satisfied by some homogeneous model can be decided in exponential space. We first prove that either φ is unsatisfiable or it is satisfied by a model of at most doubly-exponential size in $|\varphi|$; then, we show that this model of doubly-exponential size can be guessed in single exponential space.

▶ **Theorem 3.** *Let φ be a* BD *formula. The problem of deciding whether or not it is satisfiable belongs to EXPSPACE.*

The proof consists of the following four main steps, that will be detailed in the next sections (proofs will be given in an extended version of the paper).

1. We first show that for any compass structure and any of its X-axis coordinate x, the sequence $\mathcal{L}(x,0)\dots\mathcal{L}(x,N)$ is monotonic, i.e., for any triplet $0 \leq y_1 < y_2 < y_3 \leq N$, it cannot be the case that $\mathcal{L}(x,y_1) = \mathcal{L}(x,y_3)$ and $\mathcal{L}(x,y_1) \neq \mathcal{L}(x,y_2)$. Such a property allows us to represent relevant information associated with any column x in space (polynomially) bounded in $|\varphi|$.
2. Next, we define an equivalence relation over columns such that two columns are equivalent if they feature the same set of atoms. It is easy to verify that such an equivalence relation is of finite index and its index is exponentially bounded in $|\varphi|$. By exploiting the representation of step 1, we define a partial order over equivalent columns, and then we prove that, in a compass structure, such a relation totally orders equivalent columns.
3. By exploiting the total order of the elements of each equivalence class, we show a crucial property of the rows of a compass structure, which is the cornerstone of the proof. First, we associate with each point (x,y) on row y, with $0 \leq x \leq y$, a tuple consisting of: (i) $\mathcal{L}(x,y)$, (ii) the equivalence class $\sim_x$ of column x, and (iii) the set of pairs $(\mathcal{L}(x',y), \sim_{x'})$, for all $x < x' \leq y$, and then we prove that, for every pair of points $(x,y),(x',y)$ that feature the same tuple, $\mathcal{L}(x,y') = \mathcal{L}(x',y')$ for all $y' > y$, that is, columns x and x' behave the same way (i.e., exhibit the same labelling) from y to the upper end.
4. Thanks to the property proved at step 3, for every row y, there is a finite set of columns $C_y = \{x_1,\dots,x_n\}$ that behave pairwise differently for the portion of the compass structure above y. This means that each column $0 \leq x \leq y$, with $x \notin C_y$, behaves exactly as some $x_i \in C_y$ above y, that is, for all $y' > y$, $\mathcal{L}(x,y') = \mathcal{L}(x_i,y')$. We prove that n is bounded by $|\varphi|$, from which it immediately follows that, in any large enough model, there are two rows y and y', with $y < y'$, $C_y = \{x_1,\dots,x_n\}$, and $C_{y'} = \{x'_1,\dots,x'_n\}$, such that, for all $1 \leq i \leq n$, x_i and x'_i agree on conditions (i),(ii), and (iii) of step 3. Then, we can suitably contract the model into one whose Y-size is $y' - y$ shorter. By (possibly) repeatedly applying such a contraction, we obtain a model whose Y-size satisfies a doubly exponential bound. To complete the proof, it suffices to show that there exists a procedure that checks whether or not such a model exists in exponential space.

4.1 A finite characterisation of columns and of their relationships

In this section, we first show that, in every compass structure, the atoms that appear in a column x must respect a certain order, that is, they cannot be interleaved. Let F, G, and H be three pairwise distinct atoms. In Figure 3.(a), we give a graphical account of the property that we are going to prove, while, in Figure 3.(b), we show a violation of it (atom H appears before and after atom G moving upward along the column).

We preliminarily prove a fundamental property of B-irreflexive atoms.

▶ **Lemma 4.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure. For all $x \leq y < N$, if $\mathrm{Req}_B(\mathcal{L}(x,y)) \subset \mathrm{Req}_B(\mathcal{L}(x,y+1))$, then $\mathcal{L}(x,y)$ is B-irreflexive.*

Let us now provide a bound on the number of distinct atoms that can be placed above a given atom F in a column, that takes into account B-requests, D-requests, and negative literals in F. Formally, we define a function $\Delta_\uparrow : \mathrm{At}(\varphi) \to \mathbb{N}$ as follows:

$$\begin{aligned}\Delta_\uparrow(F) \quad = \quad & (2|\{\langle B\rangle\psi \in \mathrm{Cl}(\varphi)\}| - 2|\mathrm{Req}_B(F)| - |\mathrm{Obs}_B(F) \setminus \mathrm{Req}_B(F)|) + \\ & (|\{\langle D\rangle\psi \in \mathrm{Cl}(\varphi)\}| - |\mathrm{Req}_D(F)|) + \\ & (|\{\neg p : p \in \mathrm{Cl}(\varphi) \cap \mathrm{Prop}\}| - |\{\neg p : p \in \mathrm{Cl}(\varphi) \cap \mathrm{Prop} \wedge \neg p \in F\}|)\end{aligned}$$

To understand why a factor 2 comes into play in the case of B-requestes, notice that to move down from an atom including $\langle B\rangle\psi$ to an atom including $\neg\psi, [B]\neg\psi$ one must pass through an atom including $\psi, [B]\neg\psi$.

It can be easily checked that, for each $F \in \mathrm{At}(\varphi)$, $0 \le \Delta_\uparrow(F) \le 4|\varphi| + 1$. To explain how $\Delta_\uparrow$ works, we give a simple example. Let $\{\psi : \langle B\rangle\psi \in \mathrm{Cl}(\varphi)\} = \{\psi_1\}$ and let $F_3 \to_B F_2 \to_B F_1$, with $\mathrm{Req}_B(F_3) = \{\psi_1\}$ and $\mathrm{Req}_B(F_2) = \mathrm{Req}_B(F_1) = \emptyset$. For simplicity, let $\{\psi : \langle D\rangle\psi \in \mathrm{Cl}(\varphi)\} = \emptyset$, and thus $\mathrm{Req}_D(F_3) = \mathrm{Req}_D(F_2) = \mathrm{Req}_D(F_1) = \emptyset$, and $(F_3 \cap F_2 \cap F_1) \cap \mathrm{Prop} = \mathrm{Prop} = \{p\}$. It holds that $\Delta_\uparrow(F_1) = (2{\cdot}1-2{\cdot}0-0)+(0-0)+(1-0) = 3$, $\Delta_\uparrow(F_2) = (2{\cdot}1-2{\cdot}0-1)+(0-0)+(1-0) = 2$, and $\Delta_\uparrow(F_3) = (2{\cdot}1-2{\cdot}1-0)+(0-0)+(1-0) = 1$.

We say that an atom F is *initial* if and only if $\mathrm{Req}_B(F) = \emptyset$. A *$B$-sequence* is a sequence of atoms $\mathrm{Sh}_B = F_0 \dots F_n$ such that F_0 is initial and for all $0 < i \le n$ we have $F_i \to_B F_{i-1}$, $\mathrm{Req}_D(F_i) \supseteq \mathrm{Req}_D(F_{i-1})$, and $F_i \cap \mathrm{Prop} \subseteq F_{i-1} \cap \mathrm{Prop}$. It is worth pointing out that atoms in a B-sequence are monotonically non-increasing in $\Delta_\uparrow$, that is, $\Delta_\uparrow(F_0) \ge \dots \ge \Delta_\uparrow(F_n)$.

We say that a B-sequence $F_0 \dots F_n$ is *flat* if and only if it can be written as a sequence $\overline{F}_0^{k_0} \dots \overline{F}_m^{k_m}$, where $k_i > 0$, for all $1 \le i \le m$, and $\overline{F}_i \ne \overline{F}_j$, for all $1 \le i < j \le m$. Moreover, we say that a flat B-sequence $\overline{F}_0^{k_0} \dots \overline{F}_m^{k_m}$ is *decreasing* if and only if $\Delta_\uparrow(\overline{F}_0) > \dots > \Delta_\uparrow(\overline{F}_m)$. Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure for φ and $0 \le x \le N$. We define the *shading of x* in $\mathcal{G}$, written $\mathrm{Sh}(\mathcal{G})(x)$, as the sequence of atoms $\mathcal{L}(x,x) \dots \mathcal{L}(x,N)$. The next lemma easily follows from the definitions of B-sequence and shading (proof omitted).

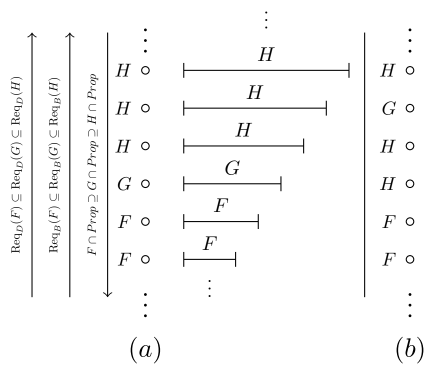

Figure 3 (a) Monotonicity of atoms along a column in a compass structure, together with a graphical account of the corresponding intervals and of how proposition letters and B/D requests must behave. (b) An example of a violation of monotonicity.

▶ **Lemma 5.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and $0 \le x \le N$. It holds that $\mathrm{Sh}(\mathcal{G})(x)$ is a B-sequence.*

The following lemma allows us to restrict our attention to decreasing flat B-sequences.

▶ **Lemma 6.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure (for a formula φ). For every $x \le y < N$, we have that $\mathcal{L}(x,y) = \mathcal{L}(x,y+1)$ iff $\mathcal{L}(x,y)$ is B-reflexive, $\mathcal{P}(x,y) = \mathcal{P}(x,y+1)$, and $\mathrm{Req}_D(x,y) = \mathrm{Req}_D(x,y+1)$.*

The next corollary immediately follows from Lemma 6. It states that the shading of each column x in $\mathcal{G}$ is a decreasing flat B-sequence, and gives a polynomial bound on the number of distinct atoms occurring in it.

▶ **Corollary 7.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure (for a formula φ). Then, for all $0 \le x \le N$, $\mathrm{Sh}(\mathcal{G})(x)$ is a decreasing flat B-sequence $\overline{F}_0^{k_0} \dots \overline{F}_m^{k_m}$, with $0 \le m \le 4|\varphi| + 1$.*

By exploiting the above (finite) characterisation of columns, we can define a natural equivalence relation of finite index over columns: we say that two columns x, x' are equivalent if and only if they feature the same set of atoms. Thanks to Corollary 7, if multiple copies of the same atom are present in a column, their occurrences are consecutive, and thus can be represented as blocks. Moreover, these blocks appear in the same order in equivalent columns because of the monotonicity of Req_B, Req_D, and *Prop*, the latter being forced by the homogeneity assumption (see Fig. 3.(a)).

In the following, we prove that equivalent columns can be totally ordered according to a given partial order relation over their shadings. Formally, for any two equivalent columns x and x', $\mathrm{Sh}(\mathcal{G})(x) < \mathrm{Sh}(\mathcal{G})(x')$ if and only if for every row y the atom $\mathcal{L}(x', y)$ is equal to atom $\mathcal{L}(x, y')$ for some row y', with $0 \leq y' \leq y$. Intuitively, this means that moving upward column x' an atom cannot appear until it has appeared on column x. In Fig. 4.(a), we depict two equivalent columns that satisfy such a condition. In general, when moving upward, atoms on x' are often "delayed" with respect to atoms in x, the limit case being when atoms on the same row are equal. In Fig. 4.(b), a violation of the condition (boxed atoms) is shown. We are going to prove that this latter situation never occurs in a compass structure.

Let us now define an equivalence relation $\sim$ over decreasing flat B-sequences. Two decreasing flat B-sequences $\mathrm{Sh}_B = \overline{F}_0^{k_0} \ldots \overline{F}_m^{k_m}$ and $\mathrm{Sh}'_B = \overline{G}_0^{h_0} \ldots \overline{G}_{m'}^{h_{m'}}$ are *equivalent*, written $\mathrm{Sh}_B \sim \mathrm{Sh}'_B$, if and only if $m = m'$ and, for all $0 \leq i \leq m$, $\overline{F}_i = \overline{G}_i$. This amounts to say that two decreasing flat B-sequences are equivalent if and only if they feature exactly the same sequence of atoms regardless of their exponents. Then, we can represent equivalence classes as decreasing flat B-sequences where each exponent is equal to one, e.g., the B-sequence $\overline{F}_0^{k_0} \ldots \overline{F}_m^{k_m}$ belongs to the equivalence class $[\overline{F}_0 \ldots \overline{F}_m]_\sim$. Given an equivalence class $[\overline{F}_0 \ldots \overline{F}_m]_\sim$ and $0 \leq i \leq m$, we denote by $[\overline{F}_0 \ldots \overline{F}_m]^i_\sim$ the i^{th} atom in its sequence, i.e., $[\overline{F}_0 \ldots \overline{F}_m]^i_\sim = \overline{F}_i$ for all $0 \leq i \leq m$. We also define a function *next* that, given an equivalence class $[\overline{F}_0 \ldots \overline{F}_m]_\sim$ and one of its atom $\overline{F}_i$, returns the successor of $\overline{F}_i$ in the sequence $[\overline{F}_0 \ldots \overline{F}_m]_\sim$ (for $i = n$, it is undefined). It can be easily checked that $\sim$ is of finite index. From Corollary 7, it follows that its index is (roughly) bounded by $|\mathrm{At}(\varphi)|^{4|\varphi|+2} = 2^{(|\varphi|+1)(4|\varphi|+2)} = 2^{4|\varphi|^2+6|\varphi|+2}$ (remember that, for all atoms F, $\Delta_\uparrow(F)$ can take $4\,|\varphi| + 2$ distinct values).

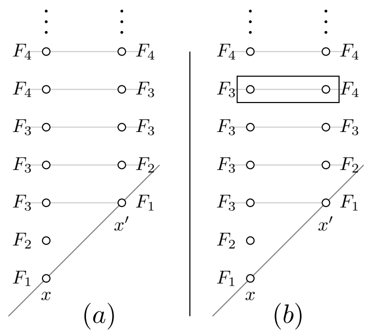

Figure 4 Two equivalent columns that respect the order (a) and two equivalent columns that violates it (b).

Let $\mathrm{Sh}_B = \overline{F}_0^{k_0} \ldots \overline{F}_m^{k_m}$ be a decreasing flat B-sequence. We define the length of Sh_B, written $|\mathrm{Sh}_B|$, as $\sum_{0 \leq i \leq m} k_m$. A partial order $<$ over the elements of each equivalence class $[\mathrm{Sh}_B]_\sim$ can be defined as follows. Let $\mathrm{Sh}_B = \overline{F}_0^{k_0} \ldots \overline{F}_m^{k_m}$ and $\mathrm{Sh}'_B = \overline{F}_0^{h_0} \ldots \overline{F}_m^{h_m}$ be two equivalent decreasing flat B-sequences. We say that Sh_B is dominated by Sh'_B, written $\mathrm{Sh}_B < \mathrm{Sh}'_B$, if and only if $|\mathrm{Sh}_B| > |\mathrm{Sh}'_B|$ and, for all $0 \le i \le m$, $\Sigma_{0\le j\le i} k_j \le (|\mathrm{Sh}_B| - |\mathrm{Sh}'_B|) + \Sigma_{0\le j\le i} h_j$. Finally, we introduce a notation for atom retrieval. Let $\mathrm{Sh}_B = \overline{F}_0^{k_0} \ldots \overline{F}_m^{k_m}$ be a decreasing flat B-sequence and $0 \le i \le |\mathrm{Sh}_B|$. We denote by $\mathrm{Sh}_B[i] = \overline{F}_j$, where j is such that $\sum_{0\le j'<j} k_{j'} < i \le \sum_{0\le j'\le j} k_{j'}$. The next lemma constrains the relationships between pairs of equivalent shadings (decreasing flat B-sequences) appearing in a compass structure.

▶ **Lemma 8.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure. For every pair of columns $0 \le x < x' \le N$ such that $\mathrm{Sh}(\mathcal{G})(x) \sim \mathrm{Sh}(\mathcal{G})(x')$, it holds that $\mathrm{Sh}(\mathcal{G})(x) < \mathrm{Sh}(\mathcal{G})(x')$.*

4.2 A spatial property of columns in homogeneous compass structures

In this section, we provide a very strong characterization of the rows of a compass structure by making use of a covering property, depicted in Fig. 5, stating that the sequences of atoms on two equivalent columns $x < x'$ must respect a certain order. To start with, we define the intersection of row y and column x, with $0 \le x \le y$, as the pair consisting of the equivalence class of x and the labelling of (x, y). We associate with each point (x, y) its intersection as well as the set $\mathbb{S}_\to(x, y)$ of intersections of row y with columns x', for all $x < x' \le y$. Let us denote by $fp(x, y)$ (fp stands for fingerprint) the triplet associated with point (x, y).

We prove that if a point (x, y) has $n + 1$ columns $(x <)\ x_0 < \ldots < x_n \le y$ on its right (with n large enough, but polynomially bounded by $|\varphi|$) such that, for all $0 \le i \le n$, $fp(x_i, y)$ is equal to $fp(x, y)$, then the sequence of atoms that goes from (x, y) to (x, N) is *exactly* the same as the sequence of atoms that goes from (x_0, y) to (x_0, N).

Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and let $0 \le x \le y$. We define $\mathbb{S}_\to(x, y)$ as the set $\{([\mathrm{Sh}(\mathcal{G})(x')]_\sim, \mathcal{L}(x', y)) : x' > x\}$. $\mathbb{S}_\to(x, y)$ collects the equivalence classes of $\sim$ which are witnessed to the right of x on row y plus a "pointer" to the "current atom", that is, the atom they are exposing on y. If $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ is homogeneous (as in our setting), for all $0 \le x \le y \le N$, the number of possible sets $\mathbb{S}_\to(x, y)$ is bounded by $2^{2^{4|\varphi|^2+6|\varphi|+2} \cdot 2^{|\varphi|+1}} = 2^{2^{4|\varphi|^2+7|\varphi|+3}}$, that is, it is doubly exponential in the size of $|\varphi|$.

The next lemma constrains the way in which two columns x, x', with $x < x'$ and $\mathrm{Sh}(\mathcal{G})(x) \sim \mathrm{Sh}(\mathcal{G})(x')$, evolve from a given row y on when $\mathbb{S}_\to(x, y) = \mathbb{S}_\to(x', y)$.

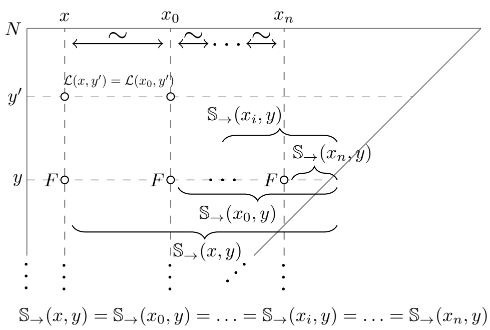

Figure 5 A graphical account of the behaviour of covered points. We have that x is covered by $x_0 < \ldots < x_n$ on row y and thus the labelling of points on column x above (x, y) is exactly the same of the correspondent points on column x_0 above (x_0, y), that is, $\mathcal{L}(x, y') = \mathcal{L}(x_0, y')$, for all $y \le y' \le N$.

▶ **Lemma 9.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and let $0 \le x < x' \le y \le N$. If $fp(x, y) = fp(x', y)$ and y' is the smallest point greater than y such that $\mathcal{L}(x, y') \neq \mathcal{L}(x, y)$, if any, and N otherwise, then, for all $y \le y'' \le y'$, $\mathcal{L}(x, y'') = \mathcal{L}(x', y'')$.*

From Lemma 9, the next corollary follows.

▶ **Corollary 10.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and let $0 \leq x < x' \leq y \leq N$. If $fp(x, y) = fp(x', y)$ and y' is the smallest point greater than y such that $\mathcal{L}(x, y') \neq \mathcal{L}(x, y)$, if any, and N otherwise, then, for every pair of points $\overline{x}, \overline{x}'$, with $x < \overline{x} < x' < \overline{x}'$, with $\mathcal{L}(\overline{x}, y) = \mathcal{L}(\overline{x}', y)$ and $\mathrm{Sh}(\mathcal{G})(\overline{x}) \sim \mathrm{Sh}(\mathcal{G})(\overline{x}') \not\sim \mathrm{Sh}(\mathcal{G})(x)$, it holds that $\mathcal{L}(\overline{x}, y'') = \mathcal{L}(\overline{x}', y'')$, for all $y \leq y'' \leq y'$.*

The above results lead us to the identification of those points (x, y) whose behaviour perfectly reproduces that of a number of points (x', y) on their right with $fp(x, y) = fp(x', y)$. These points (x, y), like all points "above" them, are useless with respect to fulfilment in a compass structure. We call them *covered points*.

▶ **Definition 11.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and $0 \leq x \leq y \leq N$. We say that (x, y) is* covered *iff there exist $n + 1 = \Delta_\uparrow(\mathcal{L}(x, y))$ distinct points $x_0 < \ldots < x_n \leq y$, with $x < x_0$, such that for all $0 \leq i \leq n$, $fp(x, y) = fp(x_i, y)$. In such a case, we say that x is covered by $x_0 < \ldots < x_n$ on y.*

▶ **Lemma 12.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and let x, y, with $0 \leq x \leq y \leq N$, be two points such that x is covered by points $x_0 < \ldots < x_n$ on y. Then, for all $y \leq y' \leq N$, it holds that $\mathrm{Sh}(\mathcal{G})(x)[y'] = \mathrm{Sh}(\mathcal{G})(x_0)[y']$.*

In Figure 6, we give an intuitive account of the notion of covered point and of the statement of Lemma 12. First of all, we observe that, since $\mathbb{S}_\rightarrow(x, y) = \mathbb{S}_\rightarrow(x_0, y) = \ldots = \mathbb{S}_\rightarrow(x_n, y)$ and, for all $0 \leq j, j' \leq n$, it holds that $(\mathrm{Sh}(\mathcal{G})(x_j), \mathcal{L}(x_j, y)) = (\mathrm{Sh}(\mathcal{G})(x_{j'}), \mathcal{L}(x_{j'}, y))$, there exists $x_n < \hat{x} \leq y$ such that $(\mathrm{Sh}(\mathcal{G})(x_n), \mathcal{L}(x_n, y)) = (\mathrm{Sh}(\mathcal{G})(\hat{x}), \mathcal{L}(\hat{x}, y))$, and $\hat{x}$ is the smallest point greater than x_n that satisfies such a condition. Now, it may happen that $\mathbb{S}_\rightarrow(x_n, y) \supset \mathbb{S}_\rightarrow(\hat{x}, y)$, and all points $\overline{x}' > x_n$ with $(\mathrm{Sh}(\mathcal{G})(\overline{x}'), \mathcal{L}(\overline{x}', y)) = (\mathrm{Sh}(\mathcal{G})(\overline{x}), \mathcal{L}(\overline{x}, y))$, for some $x < \overline{x} < x_n$, are such that $x_n < \overline{x}' < \hat{x}$. Then, it can be the case that, for all $0 \leq i \leq n$, $\mathcal{L}(x_i, y') = F_{i+1}$, as all points (x_i, y') satisfy some D-request ψ that only belongs to $\mathcal{L}(\overline{x}', y' - 1)$. In such a case, as shown in Figure 6, $\mathcal{L}(\hat{x}, y') = F_i$, because for all points $(\hat{x}', \hat{y}')$, with $\hat{x} < \hat{x}' \leq \hat{y}' < y'$, $\psi \notin \mathcal{L}(\hat{x}', \hat{y}')$. Hence, $(\mathrm{Sh}(\mathcal{G})(x_n), F_{i+1}) \in \mathbb{S}_\rightarrow(x_j, y')$ for all $0 \leq j < n$, but $(\mathrm{Sh}(\mathcal{G})(x_n), F_{i+1}) \notin \mathbb{S}_\rightarrow(x_n, y')$. Then, by applying Corollary 10, we have that $\mathbb{S}_\rightarrow(x_0, y') = \mathbb{S}_\rightarrow(x_{n-1}, y')$. Since $\Delta_\uparrow(F_{i+1}) < \Delta_\uparrow(F_i)(= n)$, it holds that $\Delta_\uparrow(F_{i+1}) \leq n - 1$ The same argument can then be applied to $x, x_0, \ldots, x_{n-1}$ on y', and so on.

4.3 A contraction method for homogeneous compass structures

In this section, we complete the proof of Theorem 3 by providing a small model theorem for compass structures. By exploiting Lemma 12, we can show that, for each row y, the cardinality of the set of columns $x_1, \ldots, x_m$ which are not covered on y is exponential in $|\varphi|$. Then, the sequence of triplets for non-covered points that appear on y is bounded by an exponential value on $|\varphi|$. It follows that, in a compass structure of size more than doubly exponential in $|\varphi|$, there exist two rows y, y', with $y < y'$, such that the sequences of the triplets for non-covered points that appear on y and y' are exactly the same. This allows us to apply a "contraction" between y and y' on the compass structure.

An example of how contraction works is given in Figure 7.

First of all, notice that rows 7 and 11 feature the same sequences for triplets of non-covered points, and that, on any row, each covered point is connected by an edge to the non-covered point that "behaves" in the same way. More precisely, we have that column 2 behaves as column 4 between $y = 7$ and $y' = 15$, columns $3, 5$, and 7 behave as column 8 between $y = 11$ and $y' = 15$, and column 4 behaves as column 6 between $y = 11$ and $y' = 15$.

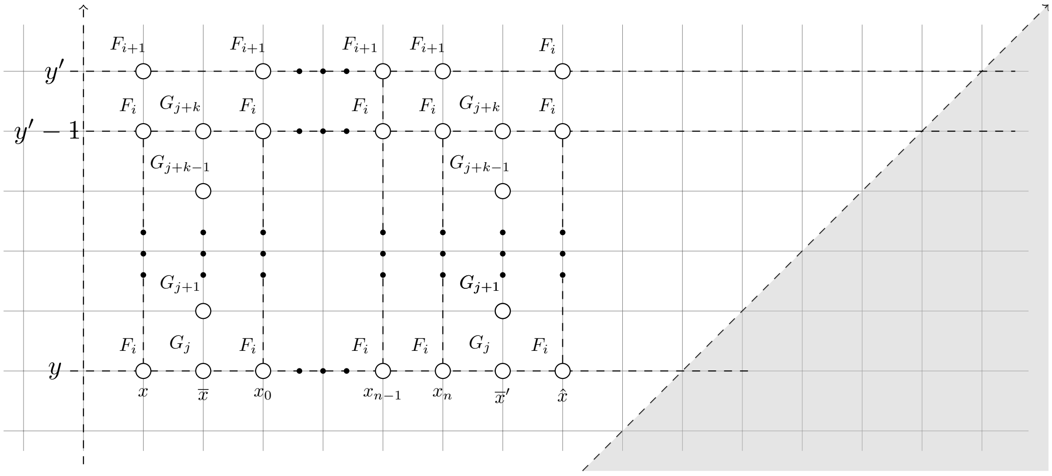

Figure 6 An intuitive account of the statement of Lemma 12.

The compass structure in Figure 7.(a) can thus be shrinked into the compass structure in Figure 7.(b), where each column of non-covered points x on y' is copied above the corresponding non-covered point x' on y. Moreover, the column of a non-covered point x on y' is copied over all the points which are covered by the non-covered point x' corresponding to x on y. This is the case with point 2 in Figure 7.(b) which takes the new column of its "covering" point 4. The resulting compass structure is $y' - y$ shorter than the original one, and we can repeatedly apply such a contraction until we achieve the desired bound.

The next corollary, which easily follows from Lemma 12 , turns out to be crucial for the proof of the EXPSPACE membership of the satisfiability problem for BD. Roughly speaking, it states that the property of "being covered" propagates upward.

▶ **Corollary 13.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure. Then, for every covered point (x, y), it holds that, for all $y \leq y' \leq N$, point (x, y') is covered as well.*

From Corollary 13, it immediately follows that, for every covered point (x, y) and every $y \leq y' \leq N$, there exists $x' > x$ such that $\mathcal{L}(x', y') = \mathcal{L}(x, y')$. Hence, for all $\overline{x}, \overline{y}$, with $\overline{x} < x \leq y' < \overline{y}$, and any D-request $\psi \in \mathrm{Req}_D(\mathcal{L}(\overline{x}, \overline{y})) \cap \mathrm{Obs}_D(\mathcal{L}(x, y))$, we have that $\psi \in \mathcal{L}(x', y)$, with $x' > x$. This allows us to conclude that if (x, y) is covered, then all points (x, y'), with $y' \geq y$, are "useless" from the point of view of D-requests.

Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and $0 \leq y \leq N$. We define the set of *witnesses* of y as the set $\mathrm{Wit}_{\mathcal{G}}(y) = \{x : (x, y) \text{ is not covered}\}$. Corollary 13 guarantees that, for any row y, the shading $\mathrm{Sh}(\mathcal{G})(x)$ and the labelling $\mathcal{L}(x, y)$ of witnesses $x \in \mathrm{Wit}_{\mathcal{G}}(y)$ are sufficient, bounded, and unambiguous pieces of information that one needs to maintain about y.

Given a compass structure $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ and $0 \leq y \leq N$, we define the *row blueprint* of y in $\mathcal{G}$, written $\mathrm{Row}_{\mathcal{G}}(y)$, as the sequence $\mathrm{Row}_{\mathcal{G}}(y) = ([\mathrm{Sh}_B^0]_\sim, F_0) \ldots ([\mathrm{Sh}_B^m]_\sim, F_m)$ such that $m + 1 = |\mathrm{Wit}_{\mathcal{G}}(y)|$ and there exists a bijection $b : \mathrm{Wit}_{\mathcal{G}}(y) \to \{0, \ldots, m\}$ such that, for every $x \in \mathrm{Wit}_{\mathcal{G}}(y)$, it holds that $\mathrm{Sh}(\mathcal{G})(x) \in [\mathrm{Sh}_B^{b(x)}]_\sim$ and $F_{b(x)} = \mathcal{L}(x, y)$, and for every x, x' in $\mathrm{Wit}_{\mathcal{G}}(y)$, $b(x) < b(x') \leftrightarrow x < x'$.

Now, we are ready to prove the following *small model* theorem.

▶ **Theorem 14.** *Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure. If there exist two points y, y', with $0 \leq y < y' \leq N$, such that $\mathrm{Row}_{\mathcal{G}}(y) = \mathrm{Row}_{\mathcal{G}}(y')$, then there exists a compass structure $\mathcal{G}' = (\mathbb{G}_{N'}, \mathcal{L}')$ with $N' = N - (y' - y)$.*

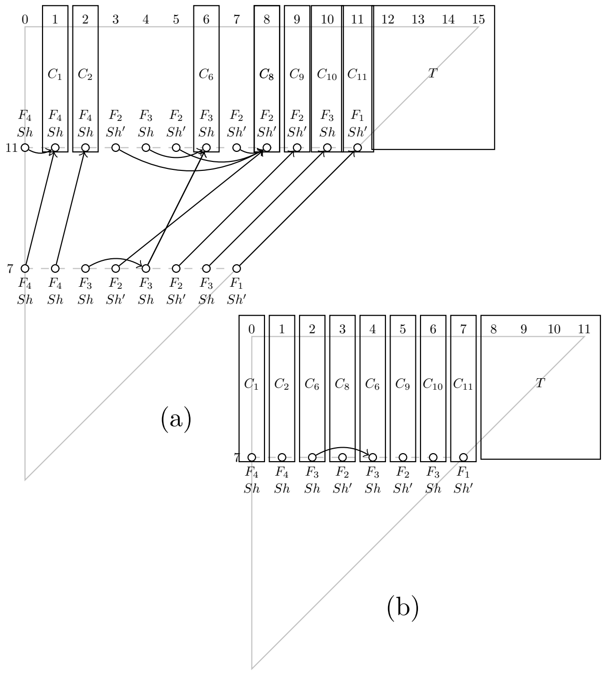

Figure 7 An example of contraction, where compass structure (a) is contracted into compass structure (b).

To conclude the proof of Theorem 3, it suffices to show that if a BD formula is satisfiable, then it is satisfied by a doubly exponential compass structure, whose existence can be checked in exponential space.

▶ **Theorem 15.** *Let φ be a* BD *formula. It holds that φ is satisfiable iff there is a compass structure $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ for it such that $N \leq 2^{2(|\varphi|+1)(4|\varphi|^2+7|\varphi|+3)2^{8|\varphi|^2+14|\varphi|+6}}$, whose existence can be checked in $EXPSPACE$.*

5 Conclusions

In this paper, we proved that the satisfiability problem for BD over homogeneous compass structures is in EXPSPACE. This result improves the previously-known non-elementary upper bound [10]. The problem of determining the exact complexity of the fragment BE, which is known to be EXPSPACE-hard and subsumes BD, [2], remains open.

As already pointed out, as a by-product, we obtained a better complexity bound to the problem of checking the emptiness of the fragment of star-free generalized regular expressions that replaces concatenation by prefix and infix relations.

In a similar way, a precise characterization of the complexity of the satisfiability problem for BE would be immediately transferable to the emptiness problem for languages in the fragment of star-free generalized regular expressions that replaces concatenation by prefix and suffix relations.

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