Decidability in Group Shifts and Group Cellular Automata

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Abstract

Many undecidable questions concerning cellular automata are known to be decidable when the cellular automaton has a suitable algebraic structure. Typical situations include linear cellular automata where the states come from a finite field or a finite commutative ring, and so-called additive cellular automata in the case the states come from a finite commutative group and the cellular automaton is a group homomorphism. In this paper we generalize the setup and consider so-called group cellular automata whose state set is any (possibly non-commutative) finite group and the cellular automaton is a group homomorphism. The configuration space may be any subshift that is a subgroup of the full shift and still many properties are decidable in any dimension of the cellular space. Decidable properties include injectivity, surjectivity, equicontinuity, sensitivity and nilpotency. Non-transitivity is semi-decidable. It also turns out that the the trace shift and the limit set can be effectively constructed, that injectivity always implies surjectivity, and that jointly periodic points are dense in the limit set. Our decidability proofs are based on developing algorithms to manipulate arbitrary group shifts, and viewing the set of space-time diagrams of group cellular automata as multidimensional group shifts.

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1 Introduction

Algebraic group shifts and group cellular automata operate on configurations that are colorings of the infinite grid $\mathbb{Z}^d$ by elements of a finite group $\mathbb{G}$, called the state set. The set $\mathbb{G}^{\mathbb{Z}^d}$ of all configurations, called the full shift, inherits the group structure as the infinite cartesian power of $\mathbb{G}$. A subshift (a set of configurations avoiding a fixed set of forbidden finite patterns) is a group shift if it is also a subgroup of $\mathbb{G}^{\mathbb{Z}^d}$. Group shifts are known to be of finite type, meaning that they can be defined by forbidding a finite number of patterns. A cellular automaton is a dynamical system on a subshift, defined by a uniform local update rule of states. A cellular automaton on a group shift is called a group cellular automaton if it is also a group homomorphism.

The purpose of this work is to demonstrate that group shifts and group cellular automata in arbitrarily high dimensions d are amenable to effective manipulations and algorithmic decision procedures. This is in stark contrast to general multidimensional subshifts of finite type and cellular automata that are plagued by undecidability. Our considerations generalize a long line of past results – see for example [2, 3] and citations therein – on algorithms for linear cellular automata (whose the state set is a finite commutative ring) and additive cellular

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automata (whose state set is a finite abelian group) to non-commutative group structures and to arbitrary dimensions, and from the full shift to arbitrary group shifts. Our methods are based on two classical results on group shifts: all group shifts – in any dimension – are of finite type, and they have dense sets of periodic points [13, 19]. By a standard argumentation these provide a decision procedure for the membership in the language of any group shift. Together with other considerations, this implies an effective procedure for constructing any lower dimensional projection of a given group shift (Corollary 10), as well as for constructing the image of a group shift under any group cellular automaton (Corollary 11).

To establish decidability results for d-dimensional group cellular automata we then view the set of valid space-time diagrams as a $(d+1)$-dimensional group shift. The local update rule of the cellular automaton provides a representation of this group shift. The one-dimensional projections in the temporal direction are the trace subshifts of the automaton that provide all possible temporal evolutions for a finite domain of cells, and the d-dimensional projection in the spatial dimensions is the limit set of the automaton. These can be effectively constructed. From the trace subshifts – which are one-dimensional group shifts themselves – one can analyze the dynamics of the cellular automaton and to decide, for example, whether it is periodic (Theorem 22), equicontinuous or sensitive to initial conditions (Theorem 24). There is a dichotomy between equicontinuity and sensitivivity (Lemma 23). We can semi-decide negative instances of mixing properties, i.e., non-transitive and non-mixing cellular automata (Theorem 25). The limit set reveals whether the automaton is nilpotent (Theorem 22), surjective or injective (Theorem 21). Note that all these considerations work for group cellular automata over arbitrary group shifts, not only over full shifts, and in all dimensions. We also note that in our setup injectivity implies surjectivity (Corollary 20), and that in all surjective cases jointly spatially and temporally periodic points are dense (Corollary 19).

The paper is organized as follows. We start by providing the necessary terminology and classical results about shift spaces and cellular automata; first in the general context of multidimensional symbolic dynamics and then in the algebraic setting in particular. In Section 3 we define projection operations on group shifts and exhibit effective algorithms to implement them. Then in Section 4 we apply the projections on space-time diagrams of cellular automata to effectively construct their traces and limit sets. These are then used to provide decision algorithms for a number of properties concerning group cellular automata. We finish with conclusions in Section 5.

2 Preliminaries

We first give definitions related to general subshifts and cellular automata, and then discuss concepts and properties particular to group shifts and group cellular automata.

Symbolic dynamics

A d-dimensional *configuration* over a finite alphabet A is an assignment of symbols of A on the infinite grid $\mathbb{Z}^d$. We call the elements of A the *states*. For any configuration $c \in A^{\mathbb{Z}^d}$ and any cell $\mathbf{u} \in \mathbb{Z}^d$, we denote by $c_{\mathbf{u}}$ the state $c(\mathbf{u})$ that c has in the cell $\mathbf{u}$. For any $a \in A$ we denote by $a^{\mathbb{Z}^d}$ the *uniform* configuration defined by $a^{\mathbb{Z}^d}_{\mathbf{u}} = a$ for all $\mathbf{u} \in \mathbb{Z}^d$.

For a vector $\mathbf{t} \in \mathbb{Z}^d$, the *translation* $\tau^{\mathbf{t}}$ shifts a configuration c so that the cell $\mathbf{t}$ is pulled to the cell $\mathbf{0}$, that is, $\tau^{\mathbf{t}}(c)_{\mathbf{u}} = c_{\mathbf{u}+\mathbf{t}}$ for all $\mathbf{u} \in \mathbb{Z}^d$. We say that c is *periodic* if $\tau^{\mathbf{t}}(c) = c$ for some non-zero $\mathbf{t} \in \mathbb{Z}^d$. In this case $\mathbf{t}$ is a *vector of periodicity* and c is also termed $\mathbf{t}$-*periodic*. If there are d linearly independent vectors of periodicity then c is called *totally* *periodic*. We denote by $\mathbf{e}_i = (0, \ldots, 0, 1, 0 \ldots, 0)$ the basic i'th unit coordinate vector, for $i = 1, \ldots, d$. A totally periodic $c \in A^{\mathbb{Z}^d}$ has automatically, for some $k > 0$, vectors of periodicity $k\mathbf{e}_1, k\mathbf{e}_2, \ldots, k\mathbf{e}_d$ in the d coordinate directions.

Let $D \subseteq \mathbb{Z}^d$ be a finite set of cells, a *shape*. A *D-pattern* is an assignment $p \in A^D$ of symbols in the shape D. A *(finite) pattern* is a D-pattern for some shape D. We call D the *domain* of the pattern. We say that a finite pattern p of shape D *appears* in a configuration c if for some $\mathbf{t} \in \mathbb{Z}^d$ we have $\tau^{\mathbf{t}}(c)|_D = p$. We also say that c *contains* the pattern p. For a fixed D, the set of D-patterns that appear in a configuration c is denoted by $\mathcal{L}_D(c)$. We denote by $\mathcal{L}(c)$ the set of all finite patterns that appear in c, i.e., the union of $\mathcal{L}_D(c)$ over all finite $D \subseteq \mathbb{Z}^d$.

Let $p \in A^D$ be a finite pattern of a shape D. The set $[p] = \{c \in A^{\mathbb{Z}^d} \mid c|_D = p\}$ of configurations that have p in the domain D is called the *cylinder* determined by p. The collection of cylinders $[p]$ is a base of a compact topology on $A^{\mathbb{Z}^d}$, the *prodiscrete* topology. See, for example, the first few pages of [1] for details. The topology is equivalently defined by a metric on $A^{\mathbb{Z}^d}$ where two configurations are close to each other if they agree with each other on a large region around the cell $\mathbf{0}$. Cylinders are clopen in the topology: they are both open and closed.

A subset X of $A^{\mathbb{Z}^d}$ is called a *subshift* if it is closed in the topology and closed under translations. Note that – somewhat nonstandardly – we allow X to be the empty set. By a compactness argument one has that every configuration c that is not in X contains a finite pattern p that prevents it from being in X: no configuration that contains p is in X. We can then as well define subshifts using forbidden patterns: given a set P of finite patterns we define

$$\mathcal{X}_P = \{c \in A^{\mathbb{Z}^d} \mid \mathcal{L}(c) \cap P = \emptyset\},$$

the set of configurations that do not contain any of the patterns in P. The set $\mathcal{X}_P$ is a subshift, and every subshift is $\mathcal{X}_P$ for some P. If $X = \mathcal{X}_P$ for some finite P then X is a *subshift of finite type* (SFT). For a subshift $X \subseteq A^{\mathbb{Z}^d}$ we denote by $\mathcal{L}_D(X)$ and $\mathcal{L}(X)$ the sets of the D-patterns and all finite patterns that appear in elements of X, respectively. The set $\mathcal{L}(X)$ is called the *language* of the subshift.

A continuous function $F : X \longrightarrow Y$ between d-dimensional subshifts $X \subseteq A^{\mathbb{Z}^d}$ and $Y \subseteq B^{\mathbb{Z}^d}$ is a *shift homomorphism* if it is translation invariant, that is, $\tau_Y^{\mathbf{t}} \circ F = F \circ \tau_X^{\mathbf{t}}$ for every $\mathbf{t} \in \mathbb{Z}^d$, where we have denoted the translations $\tau^{\mathbf{t}}$ by a vector $\mathbf{t}$ with a subscript that indicates the space. A shift homomorphism from a subshift X to itself (i.e. a shift endomorphism) is called a *cellular automaton* on X. The Curtis-Hedlund-Lyndon-theorem [5] states that shift homomorphisms are precisely the functions $X \longrightarrow Y$ defined by a local rule as follows. Let $N \subseteq \mathbb{Z}^d$ be a finite *neighborhood* and let $f : \mathcal{L}_N(X) \longrightarrow B$ be a *local rule* that assigns a letter of B to every N-pattern that appears in X. Applying f at each cell yields a function $F_f : X \longrightarrow B^{\mathbb{Z}^d}$ that maps every c according to $F_f(c)_{\mathbf{u}} = f(\tau^{\mathbf{u}}(c)|_N)$ for all $\mathbf{u} \in \mathbb{Z}^d$. Shift homomorphisms $X \longrightarrow Y$ are precisely such functions F_f that also satisfy $F_f(X) \subseteq Y$.

The image $F(X)$ of a subshift under a shift homomorphism F is clearly also a subshift. Images of subshifts of finite type are called *sofic*. We refer to [14, 15] for more concepts and results on symbolic dynamics.

Group shifts and group cellular automata

Let $\mathbb{G}$ be a finite (not necessarily commutative) group. There is a natural group structure on the d-dimensional configuration space $\mathbb{G}^{\mathbb{Z}^d}$ where the group operation is applied cell-wise: $(ce)_{\mathbf{u}} = c_{\mathbf{u}} e_{\mathbf{u}}$ for all $c, e \in \mathbb{G}^{\mathbb{Z}^d}$ and $\mathbf{u} \in \mathbb{Z}^d$. A *group shift* is a subshift of $\mathbb{G}^{\mathbb{Z}^d}$ that is also a subgroup, i.e., closed under the group operations. A cellular automaton $F : \mathbb{X} \longrightarrow \mathbb{X}$ on a group shift $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$ is a *group cellular automaton* if it is a group homomorphism: $F(ce) = F(c)F(e)$ for all $c, e \in \mathbb{X}$. More generally, a shift homomorphism $F : \mathbb{X} \longrightarrow \mathbb{Y}$ that is also a group homomorphism between groups shifts $\mathbb{X}$ and $\mathbb{Y}$ is called a *group shift homomorphism.*

Group shifts have two important properties that are central in algorithmic decidability [12]: every group shift is of finite type, and totally periodic configurations are dense in all group shifts [13, 19].

▶ **Theorem 1** ([13]). *Every group shift is a subshift of finite type.*

It follows from this theorem that every group shift $\mathbb{X}$ has a finite representation using a finite collection P of forbidden finite patterns as $\mathbb{X} = \mathcal{X}_P$. This is the representation assumed in all algorithmic questions concerning given group shifts. Also when we say that we effectively construct a group shift $\mathbb{X}$ we mean that we produce a finite set P of finite patterns such that $\mathbb{X} = \mathcal{X}_P$.

▶ **Theorem 2** ([13]). *Totally periodic configurations are dense in group shifts, i.e., for every $p \in \mathcal{L}(\mathbb{X})$ there is a totally periodic $c \in \mathbb{X}$ such that $p \in \mathcal{L}(c)$.*

As an immediate corollary of these two fundamental properties we get that the language of a group shift is (uniformly) recursive.

▶ **Corollary 3.** *There is an algorithm that determines, for any given group shift $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$ and any given finite pattern $p \in \mathbb{G}^D$ whether p is in the language $\mathcal{L}(\mathbb{X})$ of $\mathbb{X}$.*

Proof. This is a standard argumentation by Hao Wang [20]: There is a (non-deterministic) semi-algorithm for positive membership $p \in \mathcal{L}(\mathbb{X})$ that guesses a totally periodic configuration $c \in \mathbb{G}^{\mathbb{Z}^d}$, verifies that c contains the pattern p, and finally verifies that c does not contain any of the forbidden patterns in the given set P that defines $\mathbb{X} = \mathcal{X}_P$. Such a configuration c exists by Theorem 2 iff $p \in \mathcal{L}(\mathbb{X})$. Conversely, as for any SFT, there is a semi-algorithm for the negative cases $p \notin \mathcal{L}(\mathbb{X})$ that guesses a number n, makes sure that the domain D of $p \in \mathbb{G}^D$ is a subset of $E = \{-n, \ldots, n\}^d$, enumerates all finitely many patterns q with domain E that satisfy $q|_D = p$, and verifies that all such q contain a copy of a forbidden pattern in P that defines $\mathbb{X} = \mathcal{X}_P$. By compactness such a number n exists iff $p \notin \mathcal{L}(\mathbb{X})$. ◀

The representation of an SFT in terms of forbidden patterns is not unique. However, as soon as the language is recursive, we can effectively test if given representations define the same SFT.

▶ **Corollary 4.** *There are algorithms to determine*
(a) whether $\mathbb{X}_1 \subseteq \mathbb{X}_2$ holds for given group shifts $\mathbb{X}_1, \mathbb{X}_2 \subseteq \mathbb{G}^{\mathbb{Z}^d}$,
(b) whether $\mathbb{X}_1 = \mathbb{X}_2$ holds for given group shifts $\mathbb{X}_1, \mathbb{X}_2 \subseteq \mathbb{G}^{\mathbb{Z}^d}$,

Proof. To prove (a), let $P = \{p_1, \ldots, p_k\}$ be the given set of forbidden patterns that defines $\mathbb{X}_2 = \mathcal{X}_P$. We have $\mathbb{X}_1 \subseteq \mathbb{X}_2$ if and only if $p_1, \ldots, p_k \notin \mathcal{L}(\mathbb{X}_1)$, so (a) follows from Corollary 3. Now (b) follows trivially from (a) and the fact that $\mathbb{X}_1 = \mathbb{X}_2$ iff $\mathbb{X}_1 \subseteq \mathbb{X}_2$ and $\mathbb{X}_2 \subseteq \mathbb{X}_1$. ◀

Another important known property is that there are no infinite strictly decreasing chains $\mathbb{X}_1 \supsetneq \mathbb{X}_2 \supsetneq \mathbb{X}_3 \supsetneq \ldots$ of group shifts [13]. This is clear as the intersection $\mathbb{X}$ of such a chain is a group shift and hence, by Theorem 1, there is a finite set P such that $\mathbb{X} = \mathcal{X}_P$. If a pattern p is in the language of all $\mathbb{X}_k$ in the chain then p is also in the language of the intersection $\mathbb{X}$, proving that for large enough k the language of $\mathbb{X}_k$ does not contain any of the forbidden patterns in P. This implies that $\mathbb{X}_k = \mathbb{X}$ and the chain does not decrease any further. (Note, however, that while we presented here the decreasing chain property as a corollary to Theorem 1, in reality the proof is interweaved in the proof of Theorem 1, see [13].)

▶ **Theorem 5** ([13]). *There does not exist an infinite chain $\mathbb{X}_1 \supsetneq \mathbb{X}_2 \supsetneq \mathbb{X}_3 \supsetneq \dots$ of group shifts $\mathbb{X}_i \subseteq \mathbb{G}^{\mathbb{Z}^d}$.*

We also mention the obvious fact that pre-images of group shifts under group shift homomorphisms $F : \mathbb{X} \longrightarrow \mathbb{H}^{\mathbb{Z}^d}$ are group shifts and they can be effectively constructed. In particular, this applies to the kernel $\ker(F) = F^{-1}(\mathbf{1}_{\mathbb{H}}^{\mathbb{Z}^d})$ of F. (We denote the identity element of any group $\mathbb{G}$ by $\mathbf{1}_{\mathbb{G}}$, or simply by $\mathbf{1}$ if the group is clear from the context.)

▶ **Lemma 6.** *For any given d-dimensional group shifts $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$ and $\mathbb{Y} \subseteq \mathbb{H}^{\mathbb{Z}^d}$, and for a given group shift homomorphism $F : \mathbb{X} \longrightarrow \mathbb{H}^{\mathbb{Z}^d}$, the set $F^{-1}(\mathbb{Y})$ is a group shift that can be effectively constructed. In particular, the kernel $\ker(F)$ is a group shift that can be effectively constructed.*

Proof. The set $F^{-1}(\mathbb{Y})$ is clearly topologically closed, translation invariant, and a group, and therefore it is a group shift. Let P and Q be the given finite sets of forbidden patterns defining $\mathbb{X} = \mathcal{X}_P$ and $\mathbb{Y} = \mathcal{X}_Q$. Let $f : \mathcal{L}_N(\mathbb{X}) \longrightarrow \mathbb{H}$ be the given local rule with neighborhood $N \subseteq \mathbb{Z}^d$ that defines $F = F_f$. For each forbidden $q \in \mathbb{H}^D$ in Q we forbid all patterns $p \in \mathbb{G}^{D+N}$ that the local rule maps to q. We also forbid all patterns $p \in P$. The resulting subshift of finite type is $F^{-1}(\mathbb{Y})$. ◀

3 Algorithms for group shifts

To effectively manipulate group shifts we need algorithms to perform some basic operations. The main operations we consider are taking projections, either to lower the dimension of the space or to project into a subgroup of the state set but keeping the dimension. As a byproduct we obtain an algorithm to compute the image of a given group shift under a given group cellular automaton. We use derivatives of the symbol π for projections from $\mathbb{Z}^d$ to lower dimensional grids, and derivatives of the symbol ψ for projections that keep the dimension of $\mathbb{Z}^d$ but change the state set.

Notations for projections to lower dimensions

Let us first define the projection operators that cut from d-dimensional configurations $(d-1)$-dimensional slices of finite width in the first dimension. Let $d \geq 1$ be the dimension and $n \geq 1$ the width of the slice. For any d-dimensional configuration $c \in A^{\mathbb{Z}^d}$ over alphabet A the *n-slice* $\pi^{(n)}(c)$ is the $(d-1)$-dimensional configuration over alphabet A^n that has in any cell $\mathbf{u} \in \mathbb{Z}^{d-1}$ the n-tuple $(c(1,\mathbf{u}), \dots, c(n,\mathbf{u})) \in A^n$. The n-slice of a subshift $X \subseteq A^{\mathbb{Z}^d}$ is then the set $\pi^{(n)}(X)$ of the n-slices of all $c \in X$. Due to translation invariance of X, the fact that we cut slices at first coordinate positions $1, \dots, n$ is irrelevant: we could use any n consecutive first coordinate positions instead. Clearly $\pi^{(n)}(X)$ is a subshift, and if $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$ is a group shift then $\pi^{(n)}(\mathbb{X})$ is also a group shift over the group $\mathbb{G}^n = \mathbb{G} \times \dots \times \mathbb{G}$, the n-fold cartesian power of $\mathbb{G}$. Note that the projection $\pi^{(n)}(X)$ of a subshift of finite type is not necessarily of finite type – basically any effectively closed subshift can arise this way [6] – so group shifts are particularly well behaving as their projections are of finite type.

Patterns in $(d-1)$-dimensional slices of thickness n can be interpreted in a natural way as d-dimensional patterns having the width n in the first dimension. We introduce the notation $\hat{p}$ for such an interpretation of a pattern p. More precisely, for any $D \subseteq \mathbb{Z}^{d-1}$

and a $(d-1)$-dimensional pattern $p \in (G^n)^D$ over the alphabet $\mathbb{G}^n$ we denote by $\hat{p} \in \mathbb{G}^E$ the corresponding d-dimensional pattern over $\mathbb{G}$ whose domain is $E = \{1, \ldots, n\} \times D \subseteq \mathbb{Z}^d$ and $p(\mathbf{u}) = (\hat{p}(1, \mathbf{u}), \hat{p}(2, \mathbf{u}), \ldots, \hat{p}(n, \mathbf{u}))$ for every $\mathbf{u} \in D$. For a subshift X we then have that $p \in \mathcal{L}(\pi^{(n)}(X))$ if and only if $\hat{p} \in \mathcal{L}(X)$. In particular, using an algorithm for the membership of a pattern in $\mathcal{L}(X)$ we can also decide the membership of any given finite pattern in $\mathcal{L}(\pi^{(n)}(X))$. Based on Corollary 3 we then have immediately the following fact for groups shifts.

▶ **Lemma 7.** *One can effectively decide for any given d-dimensional group shift $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$, any given $n \geq 1$ and any given $(d-1)$-dimensional finite pattern $p \in (G^n)^D$ whether $p \in \mathcal{L}(\pi^{(n)}(\mathbb{X}))$.*

Projections $\pi^{(n)}(X)$ are elementary slicing operations that can be composed together, as well as with permutations of coordinates, to obtain more general projections of subshifts into lower dimensional grids. Very generally, for any subset $E \subseteq \mathbb{Z}^d$ we call the restriction $c|_E$ the *projection* of c on E, and the projection of a subshift X on E is $\pi_E(X) = \{c|_E \mid c \in X\}$. We mostly use operation π_E with sets of type $E = D \times \mathbb{Z}^k$ for some $k < d$ and a finite $D \subseteq \mathbb{Z}^{d-k}$, and we mostly apply π_E to group shifts $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$. The projection $\pi_E(\mathbb{X})$ is then viewed in the natural manner as the k-dimensional group shift over the finite group $\mathbb{G}^D$. One of the main results of this section is Corollary 10, stating that we can effectively construct $\pi_E(\mathbb{X})$ for a given $\mathbb{X}$ and $E = D \times \mathbb{Z}^k$.

Notations for projections that keep the dimension

Let $\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2$ be a cartesian product of two finite groups. For any $c \in \mathbb{G}^{\mathbb{Z}^d}$ we let $\psi^{(1)}(c) \in \mathbb{G}_1^{\mathbb{Z}^d}$ and $\psi^{(2)}(c) \in \mathbb{G}_2^{\mathbb{Z}^d}$ be the cell-wise projections to $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively, defined by $c_{\mathbf{u}} = (\psi^{(1)}(c)_{\mathbf{u}}, \psi^{(2)}(c)_{\mathbf{u}})$ for all $\mathbf{u} \in \mathbb{Z}^d$. By abuse of notation, for any $c^{(1)} \in \mathbb{G}_1^{\mathbb{Z}^d}$ and $c^{(2)} \in \mathbb{G}_2^{\mathbb{Z}^d}$ we denote by $(c^{(1)}, c^{(2)})$ the configuration $c \in (\mathbb{G}_1 \times \mathbb{G}_2)^{\mathbb{Z}^d}$ such that $\psi^{(i)}(c) = c^{(i)}$ for $i = 1, 2$. We also use the similar notation on finite patterns and implicitly use the obvious way to identify $\mathbb{G}_1^D \times \mathbb{G}_2^D$ and $(\mathbb{G}_1 \times \mathbb{G}_2)^D$.

Clearly, for any group shift $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$, the sets $\psi^{(1)}(\mathbb{X})$ and $\psi^{(2)}(\mathbb{X})$ are group shifts over $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. A pattern $p \in (\mathbb{G}_1)^D$ is in the language of $\psi^{(1)}(\mathbb{X})$ if and only if there is a pattern $q \in (\mathbb{G}_2)^D$ such that $(p, q) \in \mathcal{L}_D(\mathbb{X})$. Therefore we have the following counter part of Lemma 7.

▶ **Lemma 8.** *One can effectively decide for any given d-dimensional group shift $\mathbb{X} \subseteq (\mathbb{G}_1 \times \mathbb{G}_2)^{\mathbb{Z}^d}$, and any given d-dimensional finite pattern $p \in (\mathbb{G}_1)^D$ whether $p \in \mathcal{L}(\psi^{(1)}(\mathbb{X}))$.*

Let D, E be finite sets, $D \subseteq E$, and let $\mathbb{X} \subseteq (\mathbb{G}^E)^{\mathbb{Z}^d}$ be a group shift over the finite cartesian power $\mathbb{G}^E$ of the group $\mathbb{G}$. The group $\mathbb{G}^E$ is isomorphic to $\mathbb{G}^D \times \mathbb{G}^{E \setminus D}$, and $\psi^{(1)}$ projects then $\mathbb{X}$ into $(\mathbb{G}^D)^{\mathbb{Z}^d}$. We denote this projection by ψ_D. Notice that $\pi_{D \times \mathbb{Z}^k} = \psi_D \circ \pi_{E \times \mathbb{Z}^k}$ so that the projection into $D \times \mathbb{Z}^k$ can be obtained as a composition of projections $\pi^{(n)}$ into slices, permutations of coordinates, and a projection of the type $\psi^{(1)}$.

Effective constructions

Our main technical result is that projections of group shifts can be effectively constructed. We state this as a two-part lemma and give a short sketch of the proof ideas. (The detailed proof will be published elsewhere.) Corollaries 10 and 11 provide clean statements that we use in the next section.

▶ **Lemma 9.** *Let $d \geq 1$ be a dimension, and let $\mathbb{G}$ and $\mathbb{G}_1, \mathbb{G}_2$ be finite groups.*

(a) *For any given d-dimensional group shift $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$ and any given $n \geq 1$ one can effectively construct the $d-1$ dimensional group shift $\pi^{(n)}(\mathbb{X}) \subseteq (\mathbb{G}^n)^{\mathbb{Z}^{d-1}}$.*

(b) *For any given d-dimensional group shift $\mathbb{X} \subseteq (\mathbb{G}_1 \times \mathbb{G}_2)^{\mathbb{Z}^d}$ one can effectively construct the d-dimensional group shift $\psi^{(1)}(\mathbb{X}) \subseteq \mathbb{G}_1^{\mathbb{Z}^d}$.*

Proof idea. The proof is by induction on the dimension d. In the inductive step we first prove (a) for a dimension d assuming that (b) holds in the dimension $d-1$, and then we prove (b) for a dimension d assuming (a) holds in the dimension d and that (b) holds for the dimension $d-1$. To start the induction we observe that (b) trivially holds for the dimension $d = 0$: In this case group shifts over $\mathbb{G}$ are precisely subgroups of $\mathbb{G}$.

Proving (a) for a dimension d assuming (b) holds for the dimension $d-1$: To construct the $(d-1)$-dimensional projection $\mathbb{Y} = \pi^{(n)}(\mathbb{X})$ we effectively enumerate patterns that are not in the language of $\mathbb{Y}$ using Lemma 7. We accumulate in a set Q patterns that are not in $\mathcal{L}(\mathbb{Y})$. At each stage, the subshift $\mathcal{X}_Q$ is an upper approximation of the desired projection, meaning that $\mathbb{Y} \subseteq \mathcal{X}_Q$, and we can also make sure that $\mathcal{X}_Q$ is a group shift. Since $\mathbb{Y}$ is a group shift and therefore of finite type, by systematically accumulating in Q patterns in the complement of $\mathcal{L}(\mathbb{Y})$ we eventually reach a set Q such that $\mathbb{Y} = \mathcal{X}_Q$. The problem is to identify when we have enumerated enough patterns and reached such a set Q. Fortunately this can be detected by checking that the left and the right slices of width $n-1$ of the upper approximation $\mathcal{X}_Q$ are identical with each other: this condition guarantees that any slice can be completed into a valid d-dimensional configuration. The slices are projections of type (b) of the $(d-1)$-dimensional group shift $\mathcal{X}_Q$, so by the inductive hypotheses they can be effectively constructed. By Corollary 4(b) we can then test whether the left and right slices are equal, and thus determine when to stop the enumeration of patterns.

Proving (b) for a dimension d assuming that (a) holds for the dimension d and (b) holds for the dimension $d-1$: To construct the d-dimensional projection $\mathbb{Y} = \psi^{(1)}(\mathbb{X})$ we – analogously to the proof of (a) above – use Lemma 8 to effectively enumerate patterns that are not in the language of $\mathbb{Y}$, thus obtaining upper approximations of $\mathbb{Y}$ by group shifts $\mathcal{X}_Q$. We eventually reach a set Q such that $\mathbb{Y} = \mathcal{X}_Q$, but the challenge is again to identify when we have reached such a set Q. We establish this by proving that we can effectively compute a number n such that $\mathbb{Y} = \mathcal{X}_Q$ if and only if $\pi^{(n)}(\mathcal{X}_Q) = \pi^{(n)}(\mathbb{Y})$. The projection $\pi^{(n)}(\mathcal{X}_Q)$ can be constructed by the inductive hypothesis that (a) holds in the dimension d. To construct the projection $\pi^{(n)}(\mathbb{Y})$ we observe that operators $\pi^{(n)}$ and $\psi^{(1)}$ commute so that we can first execute $\pi^{(n)}$ on $\mathbb{X}$ (using case (a) on the dimension d), and then $\psi^{(1)}$ on the result (using case (b) on the dimension $d-1$). ◀

The next corollary states that arbitrary projections can be effectively implemented on group shifts.

▶ **Corollary 10.** *Given a d-dimensional group shift $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$ and given $k < d$ and a finite $D \subseteq \mathbb{Z}^{d-k}$ we can effectively construct the k-dimensional group shift $\pi_{D\times\mathbb{Z}^k}(\mathbb{X}) \subseteq (\mathbb{G}^D)^{\mathbb{Z}^k}$.*

Proof. By shift invariance of $\mathbb{X}$ we arbitrarily translate D, so we may assume without loss of generality that D is a subset of $E = \{1, \dots, n\}^{d-k}$ for some n. By applying $d-k$ times Lemma 9(a), permuting the coordinates as needed, we can effectively construct $\mathbb{X}' = \pi_{E\times\mathbb{Z}^k}(\mathbb{X})$. Now $\pi_{D\times\mathbb{Z}^k}(\mathbb{X}) = \psi_D(\mathbb{X}')$, and by Lemma 9(b) the projection ψ_D from $\mathbb{G}^E$ to $\mathbb{G}^D$ can be effectively implemented. ◀

The second corollary tells that images of group shifts under group cellular automata can be also effectively constructed.

▶ **Corollary 11.** *Given a d-dimensional group shift* $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$ *and given a group shift homomorphism* $F : \mathbb{X} \longrightarrow \mathbb{H}^{\mathbb{Z}^d}$ *one can effectively construct the group shift* $F(\mathbb{X}) \subseteq \mathbb{H}^{\mathbb{Z}^d}$*.*

Proof. Let $\mathbb{X} = \mathcal{X}_P$ where P is the given finite set of forbidden patterns that defines $\mathbb{X}$, and let $F = F_f$ where $f : \mathcal{L}_N(\mathbb{X}) \longrightarrow \mathbb{H}$ is the given local rule of F with a neighborhood N. We can pad symbols to patterns to grow their domains, so we can assume without loss of generality that all patterns in P have the same domain D, that the neighborhood is the same set $N = D$, and that $\mathbf{0} \in D$.

We first effectively construct $\mathbb{X}' = \{(c, F(c)) \mid c \in \mathbb{X}\} \subseteq (\mathbb{G} \times \mathbb{H})^{\mathbb{Z}^d}$. This is a group shift over group $\mathbb{G} \times \mathbb{H}$ because F is a homomorphism. It is defined by forbidding all patterns $(p, q) \in (\mathbb{G} \times \mathbb{H})^D$ where $p \notin \mathcal{L}_D(\mathbb{X})$, or $p \in \mathcal{L}_D(\mathbb{X})$ but $q(\mathbf{0}) \neq f(p)$. So $\mathbb{X}'$ can indeed be effectively constructed. By Lemma 9(b) we can then effectively compute the second projection $F(\mathbb{X}) = \psi^{(2)}(\mathbb{X}')$. ◀

4 Algorithms for group cellular automata

In this part we apply the algorithms developed for group shifts to analyze group cellular automata. The basic idea is to view the set of space-time diagrams as a higher dimensional group shift and to effectively compute one-dimensional projections in the temporal direction. This way, trace subshifts are obtained. As these are one-dimensional group shifts, and hence of finite type, the long term dynamics can be analyzed. A projection in the spatial dimensions provides the limit set of the cellular automaton.

We first define the central concepts of space-time diagrams, traces and limit sets, and show that they can be effectively constructed. Then we use this to prove properties and algorithms concerning several dynamical properties of group cellular automata. We refer to [9, 14] for more details and known results on the dynamical properties we consider.

Space-time diagrams

Let $\mathbb{X} \subseteq \mathbb{G}^{\mathbb{Z}^d}$ be a d-dimensional group shift and let $F : \mathbb{X} \longrightarrow \mathbb{X}$ be a group cellular automaton on $\mathbb{X}$. A bi-infinite *orbit* of F is a sequence $\ldots c^{(-1)}, c^{(0)}, c^{(1)}, \ldots$ of configurations $c^{(i)} \in \mathbb{X}$ such that $c^{(i+1)} = F(c^{(i)})$ for all $i \in \mathbb{Z}$. Such an orbit can be viewed as the $(d+1)$-dimensional configuration $c \in \mathbb{G}^{\mathbb{Z}^{d+1}}$ by concatenating the configurations c_i one after the other along the additional dimension, that is, $c_{\mathbf{u},i} = c^{(i)}_{\mathbf{u}}$ for all $i \in \mathbb{Z}$ and $\mathbf{u} \in \mathbb{Z}^d$. The first d dimensions are spatial dimensions while the $(d+1)$st dimension is the temporal dimension. The configuration c is a *space-time diagram* of the cellular automaton F. Note that the orbits and space-time-diagrams are temporally bi-infinite. The set of all space-time diagrams of F is denoted by $\mathrm{ST}(F)$. Because F is a group homomorphism we have the following.

▶ **Lemma 12.** $\mathrm{ST}(F) \subseteq \mathbb{G}^{\mathbb{Z}^{d+1}}$ *is a group shift.*

Given $\mathbb{X}$ and F we can effectively construct $\mathrm{ST}(F)$. Indeed, we just need to forbid in spatial slices all the forbidden patterns that define $\mathbb{X}$, and in temporally consecutive pairs of slices patterns where the local update rule of F is violated. More precisely, let P be the given finite set of forbidden patterns that defines $\mathbb{X} = \mathcal{X}_P$, and let $f : \mathcal{L}_N(\mathbb{X}) \longrightarrow \mathbb{G}$ be the given local update rule that defines F with the finite neighborhood $N \subseteq \mathbb{Z}^d$. For any $p \in P$ we forbid the $(d+1)$-dimensional pattern $\hat{p}$ over the domain $D \times \{0\}$ with $\hat{p}(\mathbf{u}, 0) = p(\mathbf{u})$ for all $\mathbf{u} \in D$, i.e., the spatial slices are forced to belong to $\mathbb{X}$, and for any neighborhood pattern $q \in \mathcal{L}_N(\mathbb{X})$ and for any $a \in \mathbb{G}$ such that $a \neq f(q)$ we forbid the pattern q'_a with the domain $N \times \{0\} \cup \{(\mathbf{0}, 1)\}$ where $q'_a(\mathbf{u}, 0) = q(\mathbf{u})$ for all $\mathbf{u} \in N$ and $q'_a(\mathbf{0}, 1) = a$, i.e. consecutive slices are prevented from having an update error according to the local rule f. Let P' be the set of all $\hat{p}$ and q'_a. Then clearly $\mathrm{ST}(F) = \mathcal{X}_{P'}$.

▶ **Lemma 13.** *Given* $\mathbb{X}$ *and* F *one can effectively construct* $\mathrm{ST}(F)$*.*

Traces

Let $D \subseteq \mathbb{Z}^d$ be finite. For any orbit $\dots, c^{(-1)}, c^{(0)}, c^{(1)}, \dots$ the sequence $\dots, c^{(-1)}|_D, c^{(0)}|_D, c^{(1)}|_D, \dots$ of consecutive views in the domain D is a D*-trace*. Each $c^{(i)}|_D$ is an element of the finite group $\mathbb{G}^D$, and hence the trace is a one-dimensional configuration over the group $\mathbb{G}^D$. Let us denote by $\mathrm{Tr}_D(F) \subseteq (\mathbb{G}^D)^{\mathbb{Z}}$ the set of all D-traces of F.

▶ **Lemma 14.** $\mathrm{Tr}_D(F)$ *is a one-dimensional group shift over* $\mathbb{G}^D$*. It is the projection of* $\mathrm{ST}(F)$ *on* $D \times \mathbb{Z}$*.*

We call the set $\mathrm{Tr}_D(F)$ the (D-)*trace subshift* of F. It can be effectively constructed: Given $\mathbb{X}$ and F we can use Lemma 13 to effectively construct the group shift $\mathrm{ST}(F)$ of space-time diagrams, and then by Corollary 10 we can effectively construct the projection $\mathrm{Tr}_D(F)$ of $\mathrm{ST}(F)$ on $D \times \mathbb{Z}$.

▶ **Lemma 15.** *Given* $\mathbb{X}$ *and* F *and any finite* $D \subseteq \mathbb{Z}^d$*, one can effectively construct* $\mathrm{Tr}_D(F)$*.*

Limit sets

The *limit set* Ω_F of a cellular automaton F consists of all configurations $c^{(0)} \in \mathbb{X}$ that are present in some bi-infinite orbit $\dots c^{(-1)}, c^{(0)}, c^{(1)}, \dots$ In other words, Ω_F is the set of the d-dimensional slices of thickness one of $\mathrm{ST}(F)$ in the d spatial dimensions. As a projection of the group shift $\mathrm{ST}(F)$, the set Ω_F is a group shift.

▶ **Lemma 16.** Ω_F *is a* d*-dimensional group shift over* $\mathbb{G}$*. It is the projection of* $\mathrm{ST}(F)$ *on* $\mathbb{Z}^d \times \{0\}$*.*

Using Corollary 10 we immediately get an algorithm to construct the limit set.

▶ **Lemma 17.** *Given* $\mathbb{X}$ *and* F*, one can effectively construct* Ω_F*.*

By definition it is clear that $F(\Omega_F) = \Omega_F$ so that F is surjective on its limit set. By a simple compactness argument we have that $\Omega_F = \bigcap_{n\in\mathbb{N}} F^n(\mathbb{X})$, stating that any configuration that has arbitrarily long sequences of pre-images has an infinite sequence of pre-images. Note that $\mathbb{X} \supseteq F(\mathbb{X}) \supseteq F^2(\mathbb{X}) \supseteq$ is a decreasing chain of group shifts. By Theorem 5 there are no infinite strictly decreasing chains of group shifts, so we have that $F^{k+1}(\mathbb{X}) = F^k(\mathbb{X})$ holds for some k. Then $F^j(\mathbb{X}) = F^k(\mathbb{X})$ for all $j > k$ so that $\Omega_F = F^k(\mathbb{X})$. So all group cellular automata reach their limit set after a finite time:

▶ **Lemma 18.** *Group cellular automata* $F : \mathbb{X} \longrightarrow \mathbb{X}$ *are stable in the sense that there exists* $k \in \mathbb{N}$ *such that* $F^k(\mathbb{X}) = \Omega_F$*.*

Basic properties

A well-known open problem due to Blanchard and Tisseur asks whether every surjective cellular automaton on a (one-dimensional) full shift has a dense set of temporally periodic points. This has been proved to be the case in a number of limited setups, including additive cellular automata on the one-dimensional full shift [2]. In fact, Theorem 2 implies the result for all group cellular automata, for any dimension and on any group shift, not just the full shift. Even jointly periodic configurations are dense: a configuration is called *jointly periodic* for a cellular automaton if it is temporally periodic and also totally periodic in space.

▶ **Corollary 19.** *Let $F : \mathbb{X} \longrightarrow \mathbb{X}$ be a group cellular automaton on a d-dimensional group shift $\mathbb{X}$. Jointly periodic configurations are dense in Ω_F. In particular, if F is surjective then they are dense in $\mathbb{X} = \Omega_F$.*

Proof. By Lemma 12 the set $\mathrm{ST}(F)$ of space-time diagrams is a $(d+1)$-dimensional group shift, and by Theorem 2 totally periodic elements are dense in $\mathrm{ST}(F)$. The projection $\pi(c)$ of a totally periodic space-time diagram c on the domain $\mathbb{Z}^d \times \{0\}$ is a totally periodic element of Ω_F that is also temporally periodic. The density of totally periodic space-time diagrams c implies the density of their projections $\pi(c)$ in Ω_F. ◀

Another immediate implication of Theorem 2 is a *surjunctivity* property: every injective group cellular automaton $F : \mathbb{X} \longrightarrow \mathbb{X}$ is surjective.

▶ **Corollary 20.** *Let $F : \mathbb{X} \longrightarrow \mathbb{X}$ be a group cellular automaton on a d-dimensional group shift $\mathbb{X}$. If F is injective then it is surjective.*

Proof. If F is injective then it is injective among totally periodic configurations of $\mathbb{X}$. For any fixed $k > 0$ there are finitely many configurations in $\mathbb{X}$ that are $k\mathbf{e}_i$-periodic for all $i \in \{1, \ldots, d\}$. These are mapped by F injectively to each other. Any injective map on a finite set is also surjective, so we see that F is surjective among totally periodic configurations of $\mathbb{X}$. By Theorem 2 the totally periodic configurations are dense in $\mathbb{X}$ so that $F(\mathbb{X})$ is a dense subset of $\mathbb{X}$. By the continuity of F it is also closed which means that $F(\mathbb{X}) = \mathbb{X}$. ◀

We have that every injective group cellular automaton is bijective. Recall that a bijective cellular automaton F is automatically reversible, meaning that the inverse F^{-1} is also a cellular automaton. If F is a reversible group cellular automaton then clearly so is F^{-1}. Reversible cellular automata are of particular interest due to their relevance in modeling microscopic physics and in other application domains [10]. While it is decidable if a given one-dimensional cellular automaton is injective (=reversible) or surjective, the same questions are undecidable for general two-dimensional cellular automata [8]. As expected, the situation is different for group cellular automata.

▶ **Theorem 21.** *It is decidable if a given group cellular automaton $F : \mathbb{X} \longrightarrow \mathbb{X}$ over a given d-dimensional group shift $\mathbb{X}$ is injective (surjective).*

Proof. By Lemma 17 one can effectively construct the limit set Ω_F. The CA F is surjective if and only if $\Omega_F = \mathbb{X}$. As equality of given group shifts is decidable (Corollary 4(b)), it follows that surjectivity is decidable.

For injectivity, recall that a group homomorphism F is injective if and only if $\ker(F) = \{\mathbf{1}_\mathbb{X}\}$. Since $\ker(F)$ is a group shift that can be effectively constructed (Lemma 6), we can check injectivity by checking the equality of the two group shifts $\ker(F)$ and $\{\mathbf{1}_\mathbb{X}\}$. ◀

Nilpotency, equicontinuity and sensitivity

A cellular automaton is called *nilpotent* if there is only one configuration in the limit set Ω_F. (Clearly the limit set is never empty.) Nilpotency is undecidable even for cellular automata over one-dimensional full shifts [7, 18]. In the case of group cellular automata the identity configuration is a fixed point and hence automatically in the limit set. Nilpotency of group cellular automata can be easily tested by effectively constructing the limit set (Lemma 17) and testing equivalence with the singleton group shift $\{\mathbf{1}_\mathbb{X}\}$.

More generally, a cellular automaton F is *eventually periodic* if $F^{n+p} = F^n$ for some n and $p \geq 1$, and it is *periodic* if F^p is the identity map for some $p \geq 1$. Nilpotent cellular automata are clearly eventually periodic with $p = 1$. Note that eventually periodic cellular automata are periodic on the limit set and, conversely, if F is periodic on its limit set then it is eventually periodic on $\mathbb{X}$ because $\Omega_F = F^n(\mathbb{X})$ for some n by Lemma 18.

▶ **Theorem 22.** *It is decidable for a given group cellular automaton $G : \mathbb{X} \longrightarrow \mathbb{X}$ on a given d-dimensional group shift $\mathbb{X}$ whether F is nilpotent, periodic or eventually periodic.*

Proof. We have that F is
- nilpotent if and only if $\Omega_F = \{\mathbf{1}_{\mathbb{X}}\}$,
- eventually periodic if and only if $\mathrm{Tr}_{\{\mathbf{0}\}}(F)$ is finite,
- periodic if and only if it is injective and eventually periodic.

Group shifts Ω_F and $\mathrm{Tr}_{\{\mathbf{0}\}}(F)$ can be effectively constructed (Lemmata 15 and 17). Equivalence of Ω_F and $\{\mathbf{1}_{\mathbb{X}}\}$ can be tested (Corollary 4(b)) and finiteness of a given one-dimensional subshift of finite type is easily checked, so nilpotency and eventual periodicity are decidable. By Theorem 21 injectivity of F is decidable so also periodicity can be decided. ◀

A configuration $c \in \mathbb{X}$ is an *equicontinuity point* of $F : \mathbb{X} \longrightarrow \mathbb{X}$ if for every finite $D \subseteq \mathbb{Z}^d$ there exists a finite $E \subseteq \mathbb{Z}^d$ such that $e|_E = c|_E$ implies $F^n(e)|_D = F^n(c)|_D$ for all $n \geq 0$. Orbits of equicontinuity points can hence be reliably simulated even if the initial configuration is not precisely known. Let $\mathrm{Eq}(F) \subseteq \mathbb{X}$ be the set of equicontinuity points of F. We call F *equicontinuous* if $\mathrm{Eq}(F) = \mathbb{X}$.

Cellular automaton $F : \mathbb{X} \longrightarrow \mathbb{X}$ is *sensitive to initial conditions*, or just *sensitive*, if there exists a finite observation window $D \subseteq \mathbb{Z}^d$ such that for every configuration $c \in \mathbb{X}$ and every finite $E \subseteq \mathbb{Z}^d$ there is $e \in \mathbb{X}$ with $e|_E = c|_E$ but $F^n(e)|_D \neq F^n(c)|_D$ for some $n \geq 0$. Clearly c cannot be an equicontinuity point so for all sensitive F we have $\mathrm{Eq}(F) = \emptyset$. For group cellular automata also the converse holds.

▶ **Lemma 23.** *Let $F : \mathbb{X} \longrightarrow \mathbb{X}$ be a group cellular automaton over a d-dimensional group shift $\mathbb{X} \neq \emptyset$. Then exactly one of the following two possibilities holds:*
- $\mathrm{Eq}(F) = \mathbb{X}$ *and F is equicontinuous, or*
- $\mathrm{Eq}(F) = \emptyset$ *and F is sensitive.*

Proof. Assume that some $c \notin \mathrm{Eq}(F)$ exists, which means that there exists a finite $D \subseteq \mathbb{Z}^d$ such that for all finite $E \subseteq \mathbb{Z}^d$ there is $e \in \mathbb{X}$ and $n \geq 1$ with $e|_E = c|_E$ but $F^n(e)|_D \neq F^n(c)|_D$. Consider an arbitrary $c' \in \mathbb{X}$. For $c'' = c'ec^{-1} \in \mathbb{X}$ we then have that $c''|_E = c'|_E$ but $F^n(c'')|_D \neq F^n(c')|_D$. This proves that $c' \notin \mathrm{Eq}(F)$.

We can conclude that for group cellular automata either $\mathrm{Eq}(F) = \mathbb{X}$ or $\mathrm{Eq}(F) = \emptyset$. By definition, $\mathrm{Eq}(F) = \mathbb{X}$ is equivalent to equicontinuity of F.

If F is sensitive then $\mathrm{Eq}(F) = \emptyset$ holds. Conversely, if F is not sensitive then, by definition, for all finite $D \subseteq \mathbb{Z}^d$ there exists $c \in \mathbb{X}$ and a finite $E \subseteq \mathbb{Z}^d$ such that $e|_E = c|_E$ implies that $F^n(e)|_D = F^n(c)|_D$ for all $n \geq 0$. As above, we can replace c by any other configuration c', which implies that all configurations are equicontinuity points, i.e., $\mathrm{Eq}(F) \neq \emptyset$. ◀

We can decide equicontinuity and sensitivity.

▶ **Theorem 24.** *It is decidable for a given group cellular automaton $G : \mathbb{X} \longrightarrow \mathbb{X}$ on a given d-dimensional group shift $\mathbb{X}$ whether F is equicontinuous or sensitive to initial conditions.*

Proof. By the dichotomy in Lemma 23 it is enough to decide equicontinuity. Let us show that F is equicontinuous if and only if it is eventually periodic, after which the decidability follows from Theorem 22.

If F is eventually periodic then it is trivially equicontinuous since there are only finitely many different functions F^k, $k \geq 0$, and all these functions are continuous. Conversely, if F is equicontinuous then one easily sees that there are only finitely many different traces in $\mathrm{Tr}_{\{\mathbf{0}\}}(F)$. Indeed, equicontinuity at configuration c implies that there is a finite set $E \subseteq \mathbb{Z}^d$ such that $e|_E = c|_E$ implies that $F^n(e)_{\mathbf{0}} = F^n(c)_{\mathbf{0}}$ for all $n \geq 0$. As in the proof of Lemma 23 we see that the same set E works for all configurations c. But then $|\mathcal{L}_E(\mathbb{X})|$ is an upper bound on the number of different traces in $\mathrm{Tr}_{\{\mathbf{0}\}}(F)$ because $c|_E$ uniquely identifies the positive trace of c (and by the translation invariance of the trace subshift any k different traces can be shifted to provide k different positive traces.)

Finiteness of $\mathrm{Tr}_{\{\mathbf{0}\}}(F)$ implies that all traces are periodic with a common period, so that cellular automaton F is periodic on its limit set. Hence F is eventually periodic. ◀

Mixing properties

A cellular automaton $F : \mathbb{X} \longrightarrow \mathbb{X}$ is *transitive* if there is an orbit from every non-empty open set to every non-empty open set, that is, if for any finite $D \subseteq \mathbb{Z}^d$ and all $p, q \in \mathcal{L}_D(\mathbb{X})$ there exists $c \in \mathbb{X}$ and $n \geq 0$ such that $c|_D = p$ and $G^n(c)|_D = q$. It is *mixing* if there exists such c for every sufficiently large n, that is, if for all D, p and q as above there is m such that for every $n \geq m$ there exists $c \in \mathbb{X}$ such that $c|_D = p$ and $G^n(c)|_D = q$.

For these properties we obtain only semi-algorithms for the negative instances. Decidability remains open.

▶ **Theorem 25.** *It is semi-decidable for a given group cellular automaton $G : \mathbb{X} \longrightarrow \mathbb{X}$ on a given d-dimensional group shift $\mathbb{X}$ whether F is non-transitive or non-mixing.*

Proof. A non-deterministic semi-algorithm guesses a finite $D \subseteq \mathbb{Z}^d$, forms the trace subshift $\mathrm{Tr}_D(F)$, and verifies that the trace subshift is not transitive (not mixing, respectively). Clearly F is not transitive (not mixing, respectively) if and only if such a choice of D exists. For one-dimensional subshifts of finite type, such as $\mathrm{Tr}_D(F)$, it is easy to decide transitivity and the mixing property [15]. ◀

5 Conclusions

We have demonstrated how the "swamp of undecidability" [16] of multidimensional SFTs and cellular automata is mostly absent in the group setting. For general cellular automata nilpotency [7, 18], as well as eventual periodicity, equicontinuity and sensitivity [4] are undecidable on one-dimensional full shifts, and periodicity [11], as well as sensitivity, mixingness and transitivity [17] are undecidable even among reversible one-dimensional cellular automata on the full shift; injectivity and surjectivity are undecidable for two-dimensional cellular automata on the full shift [8]. Algorithms and characterizations have been known for linear and additive cellular automata (on full shifts, sometimes depending on the dimension [2, 3]). Our results improve these to the greater generality of non-commutative groups and cellular automata on higher dimensional subshifts. However, it should be noted that the existing characterizations in the literature typically provide easy to check conditions on the local rule of the cellular automaton for the considered properties, while algorithms extracted from our proofs are impractical and only serve the purpose of proving decidability.

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