

# Locally Solvable Tasks and the Limitations of Valency Arguments

Hagit Attiya 

Computer Science Department, Technion, Haifa, Israel  
hagit@cs.technion.ac.il

Armando Castañeda

Instituto de Matemáticas, UNAM, Mexico City, Mexico  
armando.castaneda@im.unam.mx

Sergio Rajsbaum 

Instituto de Matemáticas, UNAM, Mexico City, Mexico  
rajsbaum@matem.unam.mx

---

## Abstract

An elegant strategy for proving impossibility results in distributed computing was introduced in the celebrated FLP consensus impossibility proof. This strategy is *local* in nature as at each stage, one configuration of a hypothetical protocol for consensus is considered, together with future valencies of possible extensions. This proof strategy has been used in numerous situations related to consensus, leading one to wonder why it has not been used in impossibility results of two other well-known tasks: *set agreement* and *renaming*. This paper provides an explanation of why impossibility proofs of these tasks have been of a global nature. It shows that a protocol can always solve such tasks locally, in the following sense. Given a configuration and all its future valencies, if a single successor configuration is selected, then the protocol can reveal all decisions in this branch of executions, satisfying the task specification. This result is shown for both set agreement and renaming, implying that there are no local impossibility proofs for these tasks.

**2012 ACM Subject Classification** Theory of computation → Distributed computing models; Computing methodologies → Distributed algorithms; Computing methodologies → Concurrent algorithms; Theory of computation → Concurrent algorithms; Theory of computation → Distributed algorithms

**Keywords and phrases** Wait-freedom, Set agreement, Weak symmetry breaking, Impossibility proofs

**Digital Object Identifier** 10.4230/LIPIcs.OPODIS.2020.18

**Related Version** A full version of the paper is available at <https://arxiv.org/abs/2011.10436>.

**Funding** Hagit Attiya: Supported by ISF grant 380/18.

Armando Castañeda: Supported by UNAM-PAPIIT project IN108720.

Sergio Rajsbaum: Supported by UNAM-PAPIIT project IN106520.

**Acknowledgements** We thank Ulrich Schmid and the reviewers for helpful comments, and Dan Alistarh, James Aspnes, Faith Ellen, Rati Gelashvili and Leqi Zhu for helpful conversations.

## 1 Introduction

An elegant strategy for proving impossibility results in distributed computing was introduced in the celebrated FLP consensus impossibility proof [17]. This strategy is *local* in nature as at each stage, one configuration of a hypothetical protocol for consensus is considered, together with its future *valencies*, namely, the decisions the protocol may reach from this configuration. To apply it, one needs to consider only the interactions of pending transitions at the configuration, and analyze their commutativity properties. This local nature makes the strategy very powerful and flexible, and has therefore been used in numerous situations related to consensus (e.g., [1, 3, 6, 7, 8, 16, 21, 24, 25, 26, 27, 29]).



© Hagit Attiya, Armando Castañeda, and Sergio Rajsbaum;  
licensed under Creative Commons License CC-BY

24th International Conference on Principles of Distributed Systems (OPODIS 2020).

Editors: Quentin Bramas, Rotem Oshman, and Paolo Romano; Article No. 18; pp. 18:1–18:16

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

For this reason, it would be desirable to be able to use a local strategy, in the style of FLP, to prove impossibility results for two other important tasks: *k-set agreement* [15], an extension of consensus, where processes may decide on up to  $k$  different values, and *M-renaming* [4], where processes must pick distinct names from a given namespace of size  $M$ . Existing impossibility proofs for these tasks (e.g., [5, 9, 10, 12, 22, 23, 32]) are based on topological invariant properties of final configurations of a protocol, which are *global* in nature, namely, all final configurations are analyzed together to argue that there is no protocol for the task. For consensus, these configurations are connected, in the graph-theoretic sense. For set agreement and renaming, higher-dimensional connectivity properties are proved. Researchers have investigated why only global impossibility proofs have been used for these tasks [2].

This paper provides an explanation of why the impossibility proof strategies for set agreement and renaming have been of a global nature. It shows that one could not hope to prove that set agreement and renaming are unsolvable through a local argument, since *they are solvable in a local sense*. For a configuration  $C$  of the protocol, we denote by  $\chi(C)$  all its successor configurations. In a local FLP style of argument, one selects a configuration  $C' \in \chi(C)$ , based on the valencies of the configurations in  $\chi(C)$ . The observation is that valencies can be assigned to  $\chi(C)$ , such that for any chosen configuration  $C' \in \chi(C)$ , the protocol can reveal decisions in all final configurations extending  $C'$ , such that the decisions are consistent both with the valencies and with the task specification. Intuitively, a hypothetical protocol for set agreement or renaming can “hide” its errors, if one inspects it only locally.

Intuitively, the reason that a protocol can do this for set agreement and renaming, and not for consensus, is that the consensus specification is one-dimensional in nature, so one can “corner” the protocol to reveal a configuration violating agreement (assuming the protocol terminates). Formally, it is always possible to find a *bivalent* configuration for consensus, and it is impossible to locally solve consensus from such a configuration. For set agreement and renaming, the protocol can “move” its errors around, on a higher dimensional space, without being cornered, even if the protocol declares all its valencies.

In more detail, given a hypothetical full-information protocol for either set agreement or renaming, we introduce the notion of *valency task* for set agreement and for renaming. The inputs to such a task are the configurations  $\chi(C)$  of the protocol after  $\ell$  rounds,  $\ell \geq 1$  (one round after some configuration  $C$ ). For each configuration  $C' \in \chi(C)$ , there is a valency,  $val(C')$ , specifying the outputs of the protocol on executions starting in  $C'$ . The valency task is thus defined together by both  $\chi(C)$  and the valencies. A protocol *solves the valency task locally* in  $m \geq 1$  rounds, if starting on any  $C' \in \chi(C)$ , after  $m$  rounds it produces decisions that are consistent with the task specification (either set agreement or renaming), and additionally *complete*, that is, if a value  $v \in val(C')$ , then at least one process decides  $v$  in at least one execution starting in  $C'$ . This captures the notion that the values promised by valencies are indeed decided.

We present the notions of valency task and local solvability in Section 3, and define valency tasks for set agreement and renaming. We show in Section 4 that for both valency tasks, set agreement and renaming, for any  $\ell \geq 1$ , the task is locally solvable, in one round ( $m = 1$ ) in the wait-free model. This theorem implies our main result that there are no local proofs, in the style of FLP, for set agreement and renaming, as shown in Section 5, where we present a precise notion of *local impossibility proof*. The techniques are based on combinatorial topology arguments explaining how a protocol can “hide” the inevitable mistakes it must make in some final decisions.

The setting used is a round-based wait-free model, where  $n$  asynchronous processes communicate reading and writing shared variables. Since the model is wait-free, the impossibility results are related to  $k$ -set agreement,  $k = n - 1$ , and  $M$ -renaming,  $M = 2n - 1$ . Working in a round-based model facilitates identification of consistent layers of configurations, and talking about  $\ell$ -round configurations. Considering wait-free executions allows to assume the hypothetical protocol decides always after some number of rounds,  $R$ . The significance of these specific cases and the choice of the model is further discussed in Section 6, which also explains the relation of our results to the approach of Alistarh, Aspnes, Ellen, Gelashvili and Zhu [2], the first paper that has considered this question, which showed that *extension-based* techniques do not suffice for proving the impossibility of solving set agreement.

## 2 Model of Computation and Its Topological Interpretation

The model we consider is a standard shared-memory system with  $n \geq 2$  asynchronous wait-free processes,  $P_0, \dots, P_{n-1}$ , communicating by atomically reading and writing to shared variables.

**The IIS model.** A *protocol* specifies, for each process, the steps to perform in order to solve a task. We consider an *iterated immediate snapshot* (IIS) [31] model of computation in which the protocol proceeds in a sequence of asynchronous *rounds*. In each round  $r \geq 1$ , a process performs an *immediate snapshot* (IS) operation on a clean shared array  $M[r]$ . The execution of an IS operation on  $M[r]$  is described as a sequence of *concurrency classes*, i.e., non-empty sets of processes. Each concurrency class indicates that the processes in the class first write in  $M[r]$  (in some arbitrary order) and then read all entries of  $M[r]$  (in some arbitrary order). Each process appears in exactly one concurrency class for round  $r$ , namely, executes one IS, on each memory  $M[r]$ .

An *execution* starting in  $\sigma$  is defined by a sequence of IS executions, one for each  $M[r]$ : the sequence of concurrency classes on  $M[1]$ , followed by the sequence of concurrency classes on  $M[2]$ , and so on. Since processes access a clean memory  $M[r]$  in every round  $r$ , IIS executions can be equivalently defined as a sequence of concurrency classes with the property that, for each concurrency class  $C$ , the processes in it perform the same number of IS operations in the concurrency classes preceding  $C$ . This means that all of them are poised to perform an IS operation on the same  $M[r]$ .

A *configuration* of the protocol  $\sigma = \{(P_0, v_0), \dots, (P_{n-1}, v_{n-1})\}$  consists of the local state  $v_i$  for each process  $P_i$ , during an execution. Notice that the states of the processes define the values assigned to the entries of  $M[r]$ . In an *initial configuration*  $\sigma$ , each process of  $\sigma$  is in an initial state determined by its input value (and its id), and all shared variables hold their initial value. A *partial configuration* of a configuration  $\sigma$  is a subset of  $\sigma$ .

**Tasks.** A *task*  $T = (I, O, \Delta)$  is specified by a set of input assignments  $I$  to the processes participating in an execution, a set of possible output assignments  $O$  to the participating processes, and a mapping  $\Delta : I \mapsto 2^O$  specifying the allowable outputs for each input assignment. A protocol *solves a task*  $T$  if in every execution starting in any initial configuration  $\sigma \in I$ , every participating process of  $\sigma$  decides an output value, such that the output values of the processes respect  $\Delta$  for their input values. The *safety* property is that the decisions of the processes starting with inputs  $\sigma \in I$  define an output simplex  $\tau$ , such that  $\tau \in \Delta(\sigma)$ . The *liveness* property is that the protocol is *wait-free*, namely, a process does not take an infinite number of steps without deciding.

A task is solvable in the IIS model if and only if it is solvable in the standard asynchronous read/write model [11, 18]. When one is interested only in computability (and not complexity), the protocol may be assumed to be *full-information*: a process remembers everything, and always writes all the information it has. Therefore, the protocol only needs to instruct a process when to *decide*, and on which output value.

The following tasks are defined over a domain of possible inputs  $V = \{0, 1, \dots, n-1\}$ . For proving impossibility results, it suffices to assume that a process  $P_i$  starts with input  $i$ .

► **Definition 1.** *In the  $k$ -set agreement task [15] processes decide on at most  $k$  different values, among the input values they have observed. The case where  $k = 1$  and  $V = \{0, 1\}$ , is the binary consensus task.*

► **Definition 2.** *In the  $M$ -renaming task [4] processes start with distinct values from a large domain, and decide on distinct values from a smaller domain  $\{0, \dots, M-1\}$ . In the weak symmetry breaking task [19] processes decide values in  $\{0, 1\}$ , such that not all of them decide the same value.*

If there is a protocol solving  $(2n-2)$ -renaming then there is a protocol solving weak symmetry breaking [19]. Due to its simpler structure and equivalence to  $(2n-2)$ -renaming, we study weak symmetry breaking instead of studying renaming.

**Topological Interpretation.** Since protocols preserve topological invariants of the model of computation, and these invariants, in turn, determine which tasks are solvable, it is convenient to describe protocols in the topological model of distributed computing [20].

In this model, the inputs of a task form an *input complex*  $I$ , which is a family of sets closed under containment. Each set in the family is called a *simplex*. An input simplex  $\sigma \in I$  has the form  $\sigma = \{(P_i, v_i)\}$ , for some subset of processes  $P_i$ , denoted  $ids(\sigma)$ . It indicates that process  $P_i \in ids(\sigma)$  starts with input  $v_i$ . The values  $v_i$  are taken from a universe  $V$  of possible input values. The *facets* of  $I$  are the simplexes of size  $n$ , defining the initial configurations of the system. (A *facet* is a simplex that is not contained in another simplex.) The output complex  $O$  is defined similarly.

For each input simplex  $\sigma \in I$ , a *task*  $T = (I, O, \Delta)$  specifies an output simplex  $\tau \in \Delta(\sigma)$ ,  $\tau = \{(P_i, v'_i)\}$ . This means that  $P_i$  may decide  $v'_i$ , in an execution starting with inputs defined by  $\sigma$ , where the processes observe steps by processes in  $ids(\sigma)$ .

Consider tuples of the form  $(P, view)$ , where  $P$  is in  $ids(\sigma)$  and  $view$  is the state of  $P$  after  $\ell$  rounds of communication. A configuration is a simplex, a set of such tuples, specifying the states of the processes after  $\ell$  rounds. The set of all configurations starting in  $\sigma$ , after some number of rounds  $\ell$  (including the partial configurations), defines the *protocol complex*  $\chi^\ell(\sigma)$ . The configurations of  $\chi^\ell(\sigma)$  are the simplexes of this complex. For a partial configuration  $\sigma' \subset \sigma$ ,  $\chi^\ell(\sigma')$  is the subset of  $\chi^\ell(\sigma)$  corresponding to executions where the processes of  $ids(\sigma')$  see only immediate snapshots by themselves.

In our model, the topological invariant preserved is that a full-information protocol subdivides the input complex.

The protocol complex is denoted  $\chi^\ell(\sigma)$ , since it turns out that it is the  $\ell$ -th *chromatic subdivision* of  $\sigma$ . For example, when  $n = 3$ , a configuration may be drawn as a triangle, as seen in Figure 1(left). The figure depicts the subdivision obtained after one round,  $\chi(\sigma)$ , for three processes ( $p = \text{black}$ ,  $q = \text{grey}$ ,  $r = \text{white}$ ), starting in one input simplex  $\sigma$ . It describes the sequences of concurrency classes that led to four of its simplexes. Notice that a partial configuration,  $\sigma' \subset \sigma$ ,  $|\sigma'| = 3$ , is depicted as a vertex (state of one process) or as an edge (state of two processes), contained in the triangle  $\sigma$ . The subdivision  $\chi^2(\sigma)$  is obtained by replacing each triangle  $\tau$  of  $\chi(\sigma)$ , by  $\chi(\tau)$ , and so forth.

A task  $T$  is *solvable* in  $\ell$  rounds if and only if there is a *simplicial map*  $\delta$  from the  $\ell$ -th chromatic subdivision  $\chi^\ell(I)$  to  $O$  that respects  $\Delta$ , i.e., for every  $\sigma \in I$ ,  $\delta(\chi^\ell(\sigma))$  is a subcomplex of  $\Delta(\sigma)$ . (A simplicial map sends vertices of one complex to vertices of another complex, preserving simplexes.)

If the input complex is finite (i.e., the universe  $V$  of possible input values is finite), it is well-known that there is an integer  $R$ , such that processes always decide at the end of the  $R$ -th round in a wait-free protocol. (This follows directly from König's Lemma.)

The dimension of the protocol complex, as well as the input complex, is  $n - 1$ . (The *dimension of a simplex*  $\sigma$  is  $|\sigma| - 1$ , and the *dimension of a complex* is the largest dimension of any of its simplexes.)

The *carrier*,  $\text{carr}(\tau, \chi^\ell(\sigma))$ , is the smallest  $\sigma' \subseteq \sigma$ , such that  $\tau \in \chi^\ell(\sigma')$ . In the figure, for the two edges of  $\tau$ , we have  $\text{carr}(\tau', \chi^\ell(\sigma)) = \sigma'$ , and  $\text{carr}(\tau'', \chi^\ell(\sigma)) = \sigma$ .

A *carrier map*  $\Delta : I \mapsto 2^O$  sending each input simplex  $\sigma \in I$  to a subcomplex  $\Delta(\sigma)$  of  $O$ , such that  $\sigma \subseteq \sigma'$  implies  $\Delta(\sigma) \subseteq \Delta(\sigma')$ .

### 3 Valency Tasks and Local Solvability

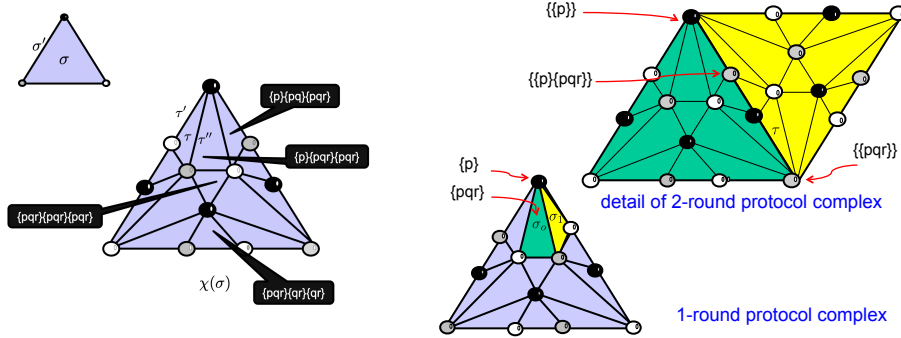
We introduce here the notions of valency task, and of locally solving such a task. Together, these notions provide the basic step in an impossibility proof in the FLP style, that will be formally defined in Section 5.

As discussed above, both for set agreement and weak symmetry breaking, one may consider, without loss of generality, a single input configuration,  $\sigma = \{(P_0, 0), \dots, (P_{n-1}, n-1)\}$ , meaning that the initial local states of the processes differ only in their ids. Thus, the input complex  $I$  consists of  $\sigma$  together with each subset of  $\sigma$ . For short, let  $\sigma = \{0, \dots, n-1\}$ , and we sometimes abuse notation and denote the input complex also by  $\sigma$ .

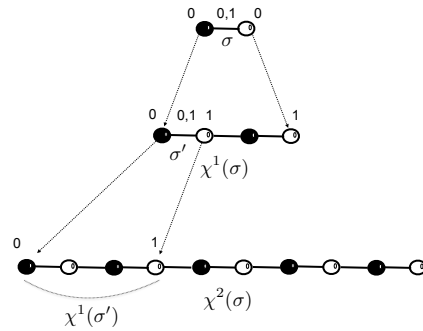
Now, assume by way of contradiction that there is a protocol  $\mathcal{P}$  solving an unsolvable task  $\mathcal{T}$  in  $R$  rounds, for some  $R \geq 1$ . Namely, the protocol complex is  $\chi^R(\sigma)$ , and each vertex  $v = (p, \text{view})$  of this complex corresponds to the state *view* of a process  $p$ , based on which,  $p$  produces an output, after executing an IS on  $M[R]$ . Solving the task means that the protocol determines a simplicial map  $\delta$ , a coloring of each vertex  $v$  of  $\chi^R(\sigma)$  with a decision value,  $\delta(v)$ , by the map  $\delta(v) = (p, \text{out})$ , in such a way that for any final configuration  $\tau \in \chi^R(\sigma)$ , the simplex of decision values  $\delta(\tau)$  belongs to  $\Delta(\sigma)$ . Since the task is unsolvable, there is no such  $\delta$ . Intuitively, a local proof demonstrates a contradiction by pinpointing a configuration  $\tau$  of the protocol complex where the decisions do not satisfy the task specification, through a local observation, as follows.

#### 3.1 Overview of the local solvability approach

Assume a protocol  $\mathcal{P}$  solving the task in  $R = \ell + m$  rounds, and consider all the configurations after  $\ell \geq 1$  rounds,  $\chi^\ell(\sigma)$ , and for each configuration  $\sigma' \in \chi^\ell(\sigma)$ , the *valencies*,  $\text{val}(\sigma')$  determined by  $\mathcal{P}$ . Namely, for each value  $v \in \text{val}(\sigma')$ , there is a final configuration  $\tau \in \chi^m(\sigma')$ , a successor of  $\sigma'$  after  $m$  rounds, such that at least one process decides on the value  $v$  in  $\tau$ . The successor configurations of  $\sigma'$  are all configurations after  $m$  additional rounds of computation by processes in  $\sigma'$ , namely, all simplexes in  $\chi^m(\sigma')$ . Figure 1 (right) depicts the case of  $\ell = m = 1$ . The successor  $\tau$  of  $\sigma'$  is reached from the initial configuration  $\sigma$  in  $\ell + m$  rounds. Given  $\chi^\ell(\sigma)$  and all the valencies of all these configurations, the impossibility argument consists of selecting one  $\sigma' \in \chi^\ell(\sigma)$ . If there are legal decisions  $\delta_{\sigma'}$  for all final configurations extending  $\sigma'$ , then the impossibility argument did not succeed in finding a contradiction, because  $\delta_{\sigma'}$  could be the map used by  $\mathcal{P}$ . This is precisely what we show for



■ Figure 1 Examples.



■ Figure 2 Consensus is not locally solvable.

set agreement and weak symmetry breaking: one can define valencies, such that for any such  $\sigma'$  there is a protocol  $\delta_{\sigma'}$  solving the task locally at  $\sigma'$ . The protocol  $\delta_{\sigma'}$  colors the vertices of  $\chi^m(\sigma')$  after executing  $m$  rounds starting in  $\sigma'$  and satisfies the task specification, and additionally, the a priori made commitments expressed by  $val(\sigma')$  (each  $val \in val(\sigma')$  is indeed decided, i.e., there is a vertex  $(p, view) \in \chi^m(\sigma')$ , with  $\delta_{\sigma'}(p, view) = (p, val)$ ). Thus, the protocol indeed preserves the valencies.

That is, an incorrect protocol can always hide the error locally. Given that the task is unsolvable, an error must exist somewhere. However, each particular configuration  $\sigma'$  inspected looks fine, and the error is moved elsewhere. We stress that this holds for every  $\ell \geq 1$  and  $m = 1$ , namely, even inspecting one round before the protocol terminates.

### 3.2 There is no Locally Solvable Valency Task for Consensus

For consensus, there is no way of defining a locally-solvable valency task. This is indeed what is expected, since there is a local impossibility proof for consensus. We show that there is no way to assign valencies, so that a protocol can hide its error. We present the case where the hypothetical protocol solves consensus in two rounds,  $\ell + m = 2$ , but the general case is analogous. (See Figure 2.)

Let  $\sigma = \{0, 1\}$  be the input edge, and the task specification  $\Delta(\{0\}) = \{(P_0, 0)\}$ ,  $\Delta(\{1\}) = \{(P_1, 1)\}$ ,  $\Delta(\{0, 1\}) = \{ \{(P_0, 0), (P_1, 0)\}, \{(P_0, 0), (P_1, 1)\} \}$ . In terms of valencies, for  $i \in \{0, 1\}$ , observe that  $val(\{i\}) = \{i\}$  (for any  $m$ ), because  $\chi^m(P_i, i)$  is the solo execution of  $P_i$  with input  $i$ , in which  $P_i$  must decide  $i$ . Thus,  $val(\sigma) = \{0, 1\}$  as  $\chi^m(P_i, i) \subset \chi^\ell(\sigma)$ .

Consider the complex  $\chi^1(\sigma)$ , which has the following edges:  $\{(P_0, \langle 0 \rangle), (P_1, \langle 0, 1 \rangle)\}$ , corresponding to the execution in which  $P_0$  goes first and then  $P_1$ ;  $\{(P_1, \langle 0, 1 \rangle), (P_0, \langle 0, 1 \rangle)\}$ , corresponding to the execution in which both processes run concurrently;  $\{(P_0, \langle 0, 1 \rangle), (P_1, \langle 1 \rangle)\}$ , corresponding to the execution in which  $P_1$  goes first and then  $P_0$ . As explained above,  $val(P_0, \langle 0 \rangle) = \{0\}$  and  $val(P_1, \langle 1 \rangle) = \{1\}$ , and the valency of any other vertex of  $\chi(\sigma)$ ,  $(P_0, \langle 0, 1 \rangle)$  and  $(P_1, \langle 0, 1 \rangle)$ , is either  $\{0\}$  or  $\{1\}$ . Thus, there must be an edge  $\sigma' = (u, v) \in \chi^1(\sigma)$  among the three edges with  $val(u) = \{0\}$  and  $val(v) = \{1\}$ , and hence  $val(\sigma') = \{0, 1\}$ . We pick such an edge  $\sigma'$  and observe that consensus is not locally solvable in  $\chi^1(\sigma')$ , i.e., the valency task with input  $\sigma'$  and outputs  $\chi^1(\sigma')$  with these valencies is not solvable. This is because any attempt to color the vertices of  $\chi^1(\sigma')$ , with one endpoint of the path colored 0 and the other colored 1, will produce an edge  $\tau$  whose vertices have different colors, violating the agreement requirement of consensus.

We have seen that for consensus (1-set agreement) it is impossible to define a valency task that is locally solvable. In Sections 3.3 and 3.4 we show how to specify valency tasks for set agreement and weak symmetry breaking that *are* locally solvable, and in Section 4 we describe protocols that solve them.

### 3.3 Valency Tasks and Local Solvability for Set Agreement

Consider now the unique input simplex  $\sigma = \{0, \dots, n-1\}$  for  $k$ -set agreement,  $k = n-1$ . Processes decide values from  $\sigma$  that they have seen, and such that at most  $n-1$  different values are decided in an execution.

Following topology terminology, in the rest of the paper configurations are called simplexes. First, recall that for a simplex  $\tau \in \chi^\ell(\sigma)$ , the *carrier* of  $\tau$  in  $\chi^\ell(\sigma)$  is the smallest face  $\sigma' \subseteq \sigma$ , such that  $\tau \in \chi^\ell(\sigma')$ . From an operational perspective,  $carr(\tau, \chi^\ell(\sigma))$  identifies the set of processes seen in the  $\ell$ -round IIS execution that ends at configuration  $\tau$ .

The goal is to define, for each  $\ell$ , a *set agreement valency task*  $\mathcal{T} = \langle \chi^\ell(\sigma), \sigma, val \rangle$ . This is a task that respects the set agreement specification: a decided value should have been seen, namely, a process deciding  $v$  must have  $v$  in its view. Indeed, agreement tasks such as consensus and set agreement are specializations of a *validity* task [14], where this is the only requirement.

More formally, in a valency task for set agreement, for every simplex  $\tau \in \chi^\ell(\sigma)$ ,  $val(\tau) \subseteq carr(\tau, \chi^\ell(\sigma))$ . The set of inputs of  $\mathcal{T}$  are the configurations at round  $\ell$ , namely  $\chi^\ell(\sigma)$ . For each configuration  $\tau \in \chi^\ell(\sigma)$ , the set of possible decisions  $val(\tau)$  is a non-empty subset of  $\sigma$  (this is the standard hypothesis of Sperner's lemma). Notice that  $val$  can be formally defined as a carrier map.<sup>1</sup> The following is a particular set agreement valency task.

► **Definition 3** (Locally solvable set agreement valency task). *For every integer  $\ell \geq 1$ , let  $\mathcal{T} = \langle \chi^\ell(\sigma), \sigma, val \rangle$ , where  $val$  is the carrier map defined by*

1. *If  $|\tau| \leq n-2$ , then  $val(\tau) = ids(\tau)$ ,*
2. *else  $val(\tau) = carr(\tau, \chi^\ell(\sigma))$ .*

In the notion of *local solvability* of valency-tasks, we ask for a protocol that solves  $\mathcal{T}$  in  $m$  rounds, namely a decision map  $c_\tau : \chi^{\ell+m}(\sigma) \rightarrow \sigma$  that respects  $\mathcal{T}$ . Thus,  $c_\tau$  is a *global* solution to  $\mathcal{T}$ , but the  $k$ -set agreement task is solved only *locally* at  $\tau$ :  $c_\tau$  is determined by a specific input simplex  $\tau \in \chi^\ell(\sigma)$ , and  $c_\tau(\chi^m(\tau))$  does not have any simplex with  $k+1$

<sup>1</sup> Formally, the corresponding task specification  $\Delta$ , for  $\Delta(\tau)$ , consists of all output simplexes labeled by output values from  $val(\tau)$ .

decisions. Of course,  $c_\tau$  does not globally solve  $k$ -set agreement because indeed  $c_\tau(\chi^{\ell+m}(\sigma))$  is a Sperner's coloring and has at least one simplex colored with  $n$  different decisions, by Sperner's lemma [33]. Recall that a Sperner coloring  $c : \chi^{\ell+m}(\sigma) \rightarrow \sigma$  is a simplicial map such that  $c(v) \in \text{carr}(\tau, \chi^{\ell+m}(\sigma))$ , for every vertex  $v$  of  $\chi^{\ell+m}(\sigma)$ .

We have that if  $c_\tau$  solves  $\mathcal{T}$  in  $m$  rounds, for each  $\tau \in \chi^\ell(\sigma)$  and all configurations after  $m$  rounds,  $\chi^{\ell+m}(\tau)$ , it should hold that  $c_\tau$  is *consistent*, i.e.,  $c_\tau(\chi^{\ell+m}(\tau)) \subseteq \text{val}(\tau)$ . We require that  $c_\tau$  is additionally *complete*, meaning that every value committed by the valencies, is indeed decided, namely,  $c_\tau(\chi^{\ell+m}(\tau)) = \text{val}(\tau)$ .

► **Definition 4** (Local solvability of  $k$ -set agreement). *We say that a set agreement valency task  $\mathcal{T} = \langle \chi^\ell(\sigma), \sigma, \text{val} \rangle$  is  $k$ -locally solvable in  $m \geq 1$  rounds if for every input simplex  $\tau \in \chi^\ell(\sigma)$  there is a decision simplicial map  $c_\tau : \chi^{\ell+m}(\sigma) \rightarrow \sigma$  that is consistent and complete w.r.t.  $\mathcal{T}$  and  $c_\tau(\chi^m(\tau))$  does not have simplexes with more than  $k$  distinct decisions at its vertices.*

We stress that local solvability allows  $c_\tau$  (which depends on  $\tau$ ) to have simplexes not in  $\chi^m(\tau)$  with more than  $k$  distinct decisions, as it requires that  $c_\tau$  solves  $k$ -set agreement only in  $\chi^m(\tau)$ . Although it is unavoidable that there are simplexes with more than  $k$  distinct decisions *somewhere* (due to the  $k$ -set agreement impossibility), local solvability does not require that the task is globally unsolvable. Indeed, while we prove (Section 4.1) that the valency task for set agreement is  $(n-1)$ -locally solvable in a single round, we do not prove it is globally unsolvable.

### 3.4 Valency Tasks and Local Solvability for Weak Symmetry Breaking

The weak symmetry breaking task with unique input  $(n-1)$ -simplex  $\sigma = \{0, \dots, n-1\}$  requires that the binary output coloring on the boundary of  $\chi^\ell(\sigma)$  has the next symmetry property (assuming the protocol terminates in  $\ell$  rounds) on the vertices  $V(\chi^\ell(\sigma))$ , e.g. [12, 13, 23]:

► **Definition 5** (Symmetric binary coloring). *A symmetric binary coloring of  $\chi^\ell(\sigma)$  is a simplicial map  $b : V(\chi^\ell(\sigma)) \rightarrow \{0, 1\}$  satisfying that, for any two distinct proper faces  $\sigma', \sigma''$  of  $\sigma$  of the same dimension,  $v \in V(\chi^\ell(\sigma'))$  and  $\phi(v) \in V(\chi^\ell(\sigma''))$  have the same binary color, i.e.  $b(v) = b(\phi(v))$ , where  $\phi$  is the simplicial bijection between  $V(\chi^\ell(\sigma'))$  and  $V(\chi^\ell(\sigma''))$  that maps vertices preserving order, namely, vertices with the smallest id in  $\text{ids}(\sigma')$  to vertices with the smallest id in  $\text{ids}(\sigma'')$ , vertices with the second smallest id in  $\text{ids}(\sigma')$  to vertices with the second smallest id in  $\text{ids}(\sigma'')$ , and so on.*

We remark that a weak symmetry breaking protocol can be transformed into a *comparison-based* protocol, in which processes only perform comparisons between inputs. Thus, actual input values are irrelevant, and only the relative order among them matters. In inputless weak symmetry breaking,  $i \in \sigma$  denotes the process with  $i$ -th input, in ascending order.

Output decisions in weak symmetry breaking are binary, hence in valency tasks for weak symmetry breaking the carrier map  $\text{val}$  goes from  $\chi^\ell(\sigma)$  to  $\{0, 1\}$ , the complex with a single edge, and its vertices. Since  $\text{val}$  models the valencies of a hypothetical protocol for weak symmetry breaking, the valencies must be symmetric on the boundary; this is the only requirement  $\text{val}$  must satisfy. The following is a particular weak symmetry breaking valency task, where it is not hard to check that  $\text{val}$  is indeed a carrier map.

► **Definition 6** (Locally solvable weak symmetry breaking valency task). *For every  $\ell \geq 1$ , let  $\mathcal{T} = \langle \chi^\ell(\sigma), \{0, 1\}, \text{val} \rangle$  where  $\text{val}$  is the carrier map defined by*

1. *If  $\dim(\tau) \leq n-3$ , then  $\text{val}(\tau) = \{1\}$ .*
2. *Otherwise,  $\text{val}(\tau) = \{0, 1\}$ .*



Analogous to set agreement, if a symmetric binary coloring  $b_\tau : V(\chi^{\ell+m}(\sigma)) \rightarrow \{0, 1\}$  solves  $\mathcal{T}$  in  $m$  rounds then it respects  $val$ , or is *consistent* with  $\mathcal{T}$ . This means that for every input simplex  $\tau \in \chi^\ell(\sigma)$ ,  $b_\tau(\chi^{\ell+m}(\tau)) \subseteq val(\tau)$ . We also require that it is *complete*, i.e.,  $b_\tau(\chi^{\ell+m}(\tau)) = val(\tau)$ .

It has been shown [9, 12] that if  $dim(\sigma) + 1$  is a prime power, then  $b(\chi^\ell(\sigma))$  has at least one *monochromatic* simplex (i.e. with all its vertices having the same binary color) of dimension  $dim(\sigma)$ , which implies the impossibility of weak symmetry breaking; those monochromatic simplexes are the errors that  $b$  makes, however,  $b$  is able to hide them locally: for the specified input simplex  $\tau \in \chi^\ell(\sigma)$ ,  $b(\chi^m(\tau))$  does not have monochromatic simplexes of dimension  $dim(\sigma)$ .

► **Definition 7** (Local solvability of weak symmetry breaking). *We say that a weak symmetry breaking valency task  $\mathcal{T} = \langle \chi^\ell(\sigma), \{0, 1\}, val \rangle$  is locally solvable in  $m \geq 1$  rounds if for every input simplex  $\tau \in \chi^\ell(\sigma)$  there is a symmetric binary decision map  $b_\tau : \chi^{\ell+m}(\sigma) \rightarrow \{0, 1\}$  (which is on function of  $\tau$ ) that is consistent and complete w.r.t.  $\mathcal{T}$  and  $b_\tau(\chi^m(\tau))$  does not have monochromatic simplexes of dimension  $dim(\sigma)$ .*

In the next section, we prove that the weak symmetry breaking valency task (Definition 6) is locally solvable in one round. This result is trivial when  $dim(\sigma) + 1$  is not a prime power because in those cases weak symmetry breaking is indeed solvable [13], and hence, there is a symmetric binary coloring with no monochromatic simplexes (i.e., without errors). The interesting case is when weak symmetry breaking is not solvable and unavoidable errors need to be hidden.

## 4 Solving Valency Tasks

This section contains the proof of Theorems 8 and 10, stating that the set agreement and weak symmetry breaking valency tasks defined in the previous section, Definitions 3 and 6, are locally solvable in one round.

### 4.1 Set Agreement

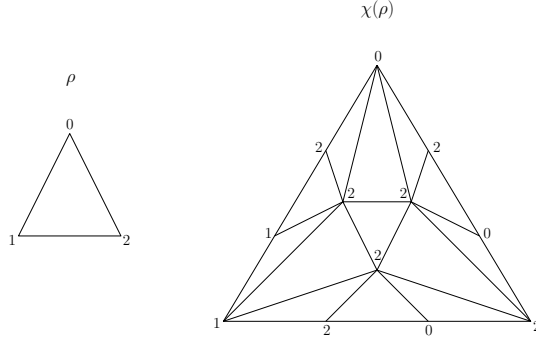
The following theorem shows that the valency tasks for set agreement defined in the previous section are locally solvable.

► **Theorem 8.** *For any  $n \geq 3$  and  $\ell \geq 1$ , the set agreement valency task  $\mathcal{T} = \langle \chi^\ell(\sigma), \sigma, val \rangle$  in Definition 3 is  $(n - 1)$ -locally solvable in one round.*

The proof of Theorem 8 relies on the following lemma, regarding vertex colorings of the first standard chromatic subdivision. Roughly speaking, the lemma identifies colorings that, to some extent, satisfy the properties of a Sperner coloring, but without simplexes with  $n$  different decisions. Figure 3 presents an example of these colorings.

► **Lemma 9.** *Consider the  $(n - 1)$ -dimensional simplex  $\rho = \{0, \dots, n - 1\}$  with  $n \geq 3$ . There is a coloring (simplicial map)  $c : \chi(\rho) \rightarrow \rho$  such that:*

1. *for every  $\rho' \subset \rho$  with  $dim(\rho') \leq n - 3$ ,  $c(\chi(\rho')) = \rho'$ ,*
2. *one of the following holds:*
  - a. *for every  $(n - 2)$ -dimensional face  $\rho' \subseteq \rho$ ,  $c(\chi(\rho')) = \rho$ ,*
  - b. *for a chosen  $(n - 2)$ -dimensional  $\rho'' \subset \rho$ ,  $c(\chi(\rho'')) = \rho''$ , and for every other  $(n - 2)$ -dimensional face  $\rho' \subseteq \rho$ ,  $c(\chi(\rho')) = \rho$ ,*
3.  *$c(\chi(\rho)) = \rho$  and there is no fully colored  $(n - 1)$ -simplex in  $c(\chi(\rho))$ .*



■ **Figure 3** Example for Lemma 9. Since every vertex  $v$  of  $\rho$  has dimension  $n - 3$ , the only vertex in  $\chi(v)$  has color  $v$ , implying (1). The coloring satisfies requirement (2.b) of the lemma: for the  $(n - 2)$ -face  $\{0, 2\}$  of  $\rho$ , all vertices in  $\chi(\{0, 2\})$  have colors in  $\{0, 2\}$ , while for any other  $(n - 2)$ -face  $\rho'$  of  $\rho$ , every vertex in  $\chi(\rho')$  has a color in  $\rho$ . Finally,  $\chi(\rho)$  has no  $(n - 1)$ -simplex with the three colors at its vertices, implying (3).

For every simplex  $\tau \in \chi^\ell(\sigma)$ , let  $ids(\tau)$  be the simplex containing the first entries of the vertices in  $\tau$  (recall that each vertex of  $\chi^\ell(\sigma)$  is a pair  $(v, view)$  where  $v$  is the id of a process and  $view$  is its view after  $\ell$  rounds); note that  $ids(\tau)$  is a  $dim(\tau)$ -face of  $\sigma$ , and  $ids(\tau) \subseteq carr(\tau, \sigma, \chi^\ell)$ , since  $\chi^\ell(\sigma)$  is a chromatic subdivision of  $\sigma$ .

**Proof of Theorem 8.** We prove now that set agreement valency task  $\mathcal{T} = \langle \chi^\ell(\sigma), \sigma, val \rangle$  is  $(n - 1)$ -locally solvable in one round. To do so, we define a Sperner coloring  $c_\tau$  of  $\chi^{\ell+1}(\sigma)$  that is consistent and complete w.r.t. the task and has no fully colored  $(n - 1)$ -simplexes in  $c_\tau(\chi(\tau))$ , for any input simplex  $\tau \in \chi^\ell(\sigma)$ . We focus on the case when  $|\tau| = n$  because for any simplex  $\tau'$  of a smaller dimension, we can just pick any  $\tau$  containing  $\tau'$ , and set  $c_{\tau'}$  to  $c_\tau$  restricted to  $\chi(\tau')$ , i.e.  $c_\tau|_{\chi(\tau')}$ .

Thus, for the rest of the proof fix an  $(n - 1)$ -dimensional simplex of  $\tau \in \chi^\ell(\sigma)$ . We define a Sperner coloring  $c_\tau$  that is consistent with  $val$  and has no fully colored  $(n - 1)$ -simplexes in  $\chi(\tau)$ . First, we use Lemma 9 to define  $c_\tau$  restricted to  $\chi(\tau)$ , i.e.  $c_\tau|_{\chi(\tau)}$ , and then extend the coloring to all vertices in  $\chi^{\ell+1}(\sigma)$ , to finally obtain  $c_\tau$ .

Let  $\rho = ids(\tau)$ . Note that  $\rho = \sigma$  but for clarity we use  $\rho$ .  $ids$ 's naturally induce a bijection between vertices of  $\rho$  and  $\tau$ , and  $\chi(\rho)$  and  $\chi(\tau)$ , respectively, hence any coloring (simplicial map)  $\chi(\rho) \rightarrow \rho$  induces a coloring  $\chi(\tau) \rightarrow ids(\tau)$ . Below, when we use Lemma 9 applied to  $\rho = ids(\tau)$ , we can speak about faces of  $\tau$  instead of faces of  $\rho$ .

Observe that either for every  $(n - 2)$ -face  $\tau'$  of  $\tau$ ,  $carr(\tau', \sigma, \chi^\ell) = \sigma$ , or for one  $(n - 2)$ -face  $\tau''$  of  $\tau$ ,  $carr(\tau', \chi^\ell) = ids(\tau'')$  and for every other  $(n - 2)$ -face  $\tau'$  of  $\tau$ ,  $carr(\tau', \sigma, \chi^\ell) = \sigma$ . Intuitively,  $\tau'$  is “inside”  $\chi^\ell(\sigma)$  or only one  $(n - 2)$ -face of  $\tau'$  “touches” the boundary of  $\chi^\ell(\sigma)$  (see Figure 3). We set  $c_\tau|_{\chi(\tau)}$  using a coloring of  $\chi(\tau)$  in Lemma 9, as follows. In the former case,  $c_\tau|_{\chi(\tau)}$  is obtained with a coloring of  $\chi(\tau)$ , as in Case (2.a) of Lemma 9, while in the latter case, is obtained with a coloring as in case (2.b), where  $\tau''$  is the chosen face in that case of the lemma.

We argue that Lemma 9 and the definition of  $val$  implies that for any face  $\tau'$  of  $\tau$ , it holds that  $c_\tau(\chi(\tau')) = val(\tau')$ , which is good because we want  $c_\tau$  to be consistent and complete w.r.t.  $val$ . If  $dim(\tau') \leq n - 3$ , then  $val(\tau') = ids(\tau')$ , by definition of  $val$ , and from Lemma 9(1), we know that  $c_\tau(\chi(\tau')) = ids(\tau')$ . Also, by definition of  $val$ , if  $dim(\tau') > n - 3$ , then  $val(\tau') = carr(\tau', \sigma, \chi^\ell)$ . Note that if  $dim(\tau') = n - 1$  (hence  $\tau' = \tau$ ), then  $val(\tau') = \sigma$ , and  $c_\tau(\chi(\tau')) = val(\tau')$ , by Lemma 9(3). The subcase that remains to be shown is when  $dim(\rho) = n - 2$ . Again, if  $val(\tau') = \sigma$ ,  $c_\tau(\chi(\tau')) = val(\tau')$ , by Lemma 9(3). Thus, consider

the case  $val(\tau') = carr(\tau', \sigma, \chi^\ell) \neq \sigma$ . Observe that this can only happen when  $\tau'$  is at the boundary of  $\chi^\ell(\sigma)$ , and hence  $carr(\tau', \sigma, \chi^\ell) = ids(\tau')$ . By Lemma 9(2.b),  $c_\tau(\chi(\tau')) = ids(\tau')$  ( $\tau'$  was the chosen  $(n-2)$ -face of  $\tau$  in case (2.b) of Lemma 9 when defining  $c_\tau$  on  $\chi(\tau)$ ).

We now extend the coloring  $c_\tau$  in two steps. First, for any vertex  $v \in \chi^\ell(\sigma)$  that does not belong to  $\chi(\tau)$ , we first set  $c_\tau(v) = id(v)$ . Thus, for any input simplex  $\lambda \in \chi^\ell(\sigma)$  that does not intersect  $\tau$ , we have that  $c_\tau(\chi(\lambda)) = ids(\lambda)$ . That is fine if  $dim(\lambda) \neq n-2$ , or  $dim(\lambda) = n-2$  and  $carr(\lambda, \sigma, \chi^\ell) \neq \sigma$ , because in such cases  $c_\tau(\chi(\lambda)) = val(\lambda)$ , by definition of  $val$ .

But if  $dim(\lambda) = n-2$  and  $carr(\lambda, \sigma, \chi^\ell) = \sigma$ , then  $c_\tau(\chi(\lambda)) = ids(\lambda) \subset val(\lambda) = \sigma$ , and then in this case  $c_\tau$  is not complete. Note that the proper contention is because there is no vertex in  $\chi(\lambda)$  that is mapped to the unique vertex in  $\sigma \setminus ids(\lambda)$ . To solve this issue, for every such input simplex  $\lambda \in \chi^\ell(\sigma)$ , we pick one vertex  $v \in \chi(\lambda)$  with  $carr(v, \lambda, \chi) = \lambda$  (which belongs to the “central”  $(n-2)$ -simplex of  $\chi(\lambda)$ ) and set  $c_\tau(v)$  to the unique vertex in  $\sigma \setminus ids(\lambda)$ . Therefore, we now have that  $c_\tau(\chi(\lambda)) = val(\lambda) = \sigma$ .

To fully prove that  $c_\tau$  is consistent and complete w.r.t.  $val$ , the only case that remains is of an input simplex  $\lambda$  that intersects  $\tau$  but is not one of its faces. Let  $\lambda'$  and  $\tau'$  be the proper faces of  $\lambda$  and  $\tau$  such that  $\lambda = \lambda' \cup \tau'$ . We already know that  $c_\tau(\chi(\lambda')) = val(\lambda')$  and  $c_\tau(\chi(\tau')) = val(\tau')$ . If  $dim(\lambda' \cup \tau') \leq n-3$ , the definition of  $val$  implies that  $val(\lambda \cup \tau') = val(\lambda') \cup val(\tau')$ , hence  $c_\tau(\chi(\lambda' \cup \tau')) = val(\lambda \cup \tau')$ . If  $dim(\lambda' \cup \tau') = n-1$ , then it must be that  $val(\lambda \cup \tau') = \sigma$ , from the definition of  $val$ , and then clearly  $c_\tau(\chi(\lambda' \cup \tau')) = val(\lambda \cup \tau')$ , by construction. If  $dim(\lambda' \cup \tau') \geq n-2$ , we have two cases,  $val(\lambda' \cup \tau')$  is either  $ids(\lambda' \cup \tau')$  or  $\sigma$ ; in any case, the very definition of  $c_\tau$  implies that  $c_\tau(\chi(\lambda' \cup \tau')) = val(\lambda \cup \tau')$ .

Therefore, so far we have a coloring  $c_\tau$  that is consistent and complete w.r.t.  $val$  and  $c_\tau(\chi(\tau))$  has no fully colored  $(n-1)$ -simplexes (since we defined  $c_\tau(\chi(\tau))$  using Lemma 9). To finally conclude that  $(\chi^\ell(\sigma), \sigma, val)$  is locally solvable in one round, we argue  $c_\tau$  is a Sperner coloring, which essentially follows because  $val$  is a Sperner-valency coloring and  $c_\tau$  is consistent and complete w.r.t.  $val$ . To prove the claim in detail, consider any vertex  $v \in \chi^\ell(\sigma)$ . If  $v \notin \chi(\tau)$ , then  $c_\tau(v) = id(v) \in carr(v, \sigma, \chi^\ell)$ . Otherwise, let  $\tau' = carr(v, \tau, \chi)$ . Note that  $ids(\tau') \subseteq carr(v, \sigma, \chi^\ell)$ . It follows from Lemma 9 that  $c_\tau(\chi(\tau'))$  is either  $ids(\tau')$  or  $\sigma$ . If  $c_\tau(\chi(\tau')) = ids(\tau')$  then  $c_\tau(v) \in carr(v, \sigma, \chi^\ell)$ . For the remaining case, note that  $c_\tau(\chi(\tau')) = \sigma$  only if  $dim(\tau') = n-1$  (hence  $carr(\tau', \sigma, \chi^\ell) = \sigma$ ), or  $dim(\tau') = n-2$  and  $carr(\tau', \sigma, \chi^\ell) = \sigma$  (i.e.  $\tau'$  is not the chosen  $(n-2)$ -face of  $\tau$  in the case (2.b) of Lemma 9); in either case we have that  $c_\tau(v) \in carr(v, \sigma, \chi^\ell)$ . We conclude that  $c_\tau$  is a Sperner coloring. ◀

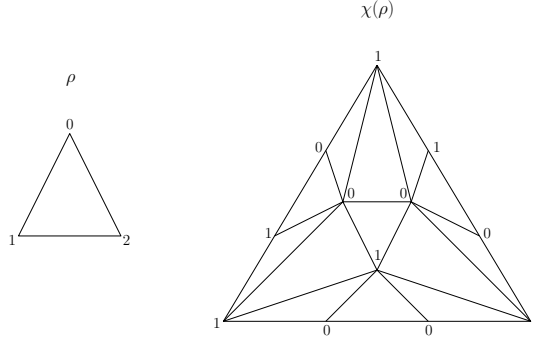
## 4.2 Weak Symmetry Breaking

► **Theorem 10.** *For any  $n \geq 3$  and  $\ell \geq 1$ , the weak symmetry breaking valency task  $\mathcal{T} = (\chi^\ell(\sigma), \sigma, val)$  in Definition 6 is locally solvable in one round.*

The proof of Theorem 10 is similar in structure to the proof for set agreement in the previous section. It relies on Lemma 11 below to produce binary colorings that are almost symmetric on the boundary and do not have monochromatic  $dim(\sigma)$ -simplexes. Figure 4 shows an example of such a coloring. In the proof of Theorem 10, we use these binary colorings to locally solve symmetric binary-valency tasks.

► **Lemma 11.** *Consider the  $(n-1)$ -dimensional simplex  $\rho = \{0, \dots, n-1\}$  with  $n \geq 3$ . There is a binary coloring (simplicial map)  $b : \chi(\rho) \rightarrow \{0, 1\}$  such that:*

1. *for every  $\rho' \subset \rho$  with  $dim(\rho') \leq n-3$ ,  $b(\chi(\rho')) = \{1\}$ ,*
2. *for every  $(n-2)$ -dimensional face  $\rho' \subseteq \rho$ ,  $b(\chi(\rho')) = \{0, 1\}$ ,*
3.  *$b(\chi(\rho)) = \{0, 1\}$  and there is no monochromatic  $(n-1)$ -simplex in  $b(\chi(\rho))$ .*



■ **Figure 4** Example for Lemma 11. Since every vertex  $v$  of  $\rho$  has dimension  $n - 3$ , the only vertex in  $\chi(v)$  has color 1, implying (1). For every  $(n - 2)$ -face  $\rho'$  of  $\rho$ , there are vertices in  $\chi(\{0, 2\})$  with color 0 and 1, implying (2). Finally,  $\chi(\rho)$  has no monochromatic  $(n - 1)$ -simplex, implying (3).

**Proof of Theorem 10.** We now show that the weak symmetry breaking valency task  $\mathcal{T} = \langle \chi^\ell(\sigma), \sigma, \text{val} \rangle$  is locally solvable in one round. We need to show that for every input simplex  $\tau \in \chi^\ell(\sigma)$ , we define a symmetric binary coloring  $b_\tau$  of  $\chi^{\ell+1}(\sigma)$  that is consistent and complete w.r.t.  $\text{val}$  and has no monochromatic  $(n - 1)$ -simplexes in  $b_\tau(\chi(\tau))$ . We focus on the case is when  $\tau$  is of dimension  $n - 1$  because for any simplex  $\tau'$  of a smaller dimension, we can just pick any  $\tau$  containing  $\tau'$ , and set  $b_{\tau'}$  to  $b_\tau$  restricted to  $\chi(\tau')$ , i.e.  $b_{\tau'}|_{\chi(\tau')}$ .

For the rest of the proof fix an  $(n - 1)$ -dimensional simplex of  $\tau \in \chi^\ell(\sigma)$ . We define a symmetric binary coloring  $b_\tau$  that is consistent and complete w.r.t.  $\text{val}$  and has no monochromatic  $(n - 1)$ -simplexes in  $\chi(\tau)$ . First, we use Lemma 11 to define  $b_\tau$  restricted to  $\chi(\tau)$ , i.e.  $b_\tau|_{\chi(\tau)}$ , and then extend the coloring to all vertices in  $\chi^{\ell+1}(\sigma)$ , to finally obtain  $b_\tau$ .

Let  $\rho = \text{id}_s(\tau)$ . Note that  $\rho = \sigma$  but for clarity we use  $\rho$ .  $\text{id}$ 's naturally induce a bijection between  $\rho$  and  $\tau$ , and  $\chi(\rho)$  and  $\chi(\tau)$ , hence any coloring (simplicial map)  $\chi(\rho) \rightarrow \rho$  induces a coloring  $\chi(\tau) \rightarrow \text{id}_s(\tau)$ . Below, when we use Lemma 11 applied to  $\rho = \text{id}_s(\tau)$ , we can speak about faces of  $\tau$  instead of faces of  $\rho$ .

First, we set  $b_\tau|_{\chi(\tau)}$  using a coloring of  $\chi(\tau)$  in Lemma 11. We have that  $b_\tau|_{\chi(\tau)}$  is consistent and complete with respect to  $\text{val}$ : for every face  $\tau'$  of  $\tau$ , if  $\dim(\tau') \leq n - 3$ ,  $\text{val}(\tau') = \{1\}$ , by definition of  $\text{val}$ , and  $b_\tau(\chi(\tau')) = \{1\}$ , by Lemma 11(1); and if  $\dim(\tau') \geq n - 2$ ,  $\text{val}(\tau') = \{0, 1\}$ , by definition of  $\text{val}$ , and  $b_\tau(\chi(\tau')) = \{0, 1\}$ , by Lemma 11(2-3).

We extend  $b_\tau$  in two steps. In the first step, we pick any vertex  $v \in \chi^{\ell+1}(\sigma)$  that does not belong to  $\chi(\tau)$  (which is uncolored yet). If there are faces  $\sigma', \sigma''$  of  $\sigma$  with the same dimension such that  $v \in \chi^{\ell+1}(\sigma')$ ,  $\phi(v) \in \chi^{\ell+1}(\sigma'')$  and  $\phi(v) \in \chi(\tau)$ , where  $\phi$  is the simplicial bijection between  $\chi^{\ell+1}(\sigma')$  and  $\chi^{\ell+1}(\sigma'')$  that maps vertices preserving order, then set  $b_\tau(v) = b_\tau(\phi(v))$ ; otherwise, set  $b_\tau(v) = 1$ . In words: if  $\chi(\tau)$  “touches” the boundary of  $\chi^{\ell+1}(\sigma)$ , we replicate that “part” of the coloring in its symmetric “counterparts” in the boundary. Observe that  $b_\tau$  is well defined because, since  $\ell \geq 1$ , there are no two vertices of  $u, v \in \chi(\tau)$  such that there are two distinct faces  $\sigma', \sigma''$  of  $\sigma$  of same dimension such that  $u \in \chi^{\ell+1}(\sigma')$  and  $v \in \chi^{\ell+1}(\sigma'')$ ; intuitively,  $\chi(\tau)$  can “touch” either  $\chi^{\ell+1}(\sigma')$  or  $\chi^{\ell+1}(\sigma'')$  but not both. Note that  $b_\tau$  is symmetric.

It is not hard to see that for any input simplex  $\lambda \in \chi^\ell(\sigma)$  with  $\dim(\lambda) \leq n - 3$ ,  $b_\tau(\chi(\lambda)) = \text{val}(\lambda)$ . First, if  $\lambda$  is a face of  $\tau$ , we have already saw that this is true. Second, if  $\lambda$  is a face of  $\tau$ , then  $b_\tau(\chi(\lambda)) = \{1\}$  because even if a vertex  $v \in \chi(\lambda)$  is in the boundary of  $\chi^{\ell+1}(\sigma)$  and gets its color from a vertex  $u \in \chi(\tau)$  (i.e.  $b_\tau(v) = b_\tau(u)$ ), it must be that  $b_\tau(u) = 1$  because there must be a face  $\tau'$  of  $\tau$  of dimension  $\dim(\lambda)$  such that  $u \in \chi(\tau')$ , and by Lemma 11(1),  $b_\tau(\chi(\tau')) = \{1\}$ ; and finally, by definition,  $\text{val}(\lambda) = \{1\}$ .

However, we cannot say that same for any input simplex  $\lambda \in \chi^\ell(\sigma)$  with  $\dim(\lambda) \geq n - 2$ . Consider the case that  $\chi(\lambda)$  does not intersect  $\chi(\tau)$  and the boundary of  $\chi^{\ell+1}(\sigma)$ ; in that case  $b_\tau(\chi(\lambda)) = 1$ , by definition of  $b_\tau$ , but  $\text{val}(\lambda) = \{0, 1\}$ , by definition of  $\text{val}$ . We fix this issue in the second step of the construction: for any  $\lambda \in \chi^\ell(\sigma)$  with  $\dim(\lambda) = n - 2$  and  $b_\tau(\chi(\lambda)) = \{1\}$ , pick the vertex  $v \in \chi(\lambda)$  with smallest id among the vertices with  $\text{carr}(v, \lambda, \chi) = \lambda$  (namely,  $v$  is a vertex with smallest id of the “central”  $(n - 2)$ -simplex of  $\chi(\lambda)$ ), and set  $b_\tau(v) = 0$ .

By construction, we have that  $b_\tau(\chi(\lambda)) = \{0, 1\} = \text{val}(\lambda)$ . Note that  $\lambda$  is not face of  $\tau$  because initially we had  $b_\tau(\chi(\lambda)) = \{1\}$ , which is not true for  $(n - 2)$ -dimensional faces of  $\tau$ , by Lemma 11(2), and if  $\lambda$  intersects  $\tau$ , then  $v \notin \chi(\tau)$  because  $v$  is an “internal” vertex of  $\chi(\lambda)$ . Therefore,  $b_\tau|_{\chi(\tau)}$  remains the same after the second step. Moreover, since we pick vertices with smallest id,  $b_\tau$  remains symmetric. Finally, for any  $\lambda \in \chi^\ell(\sigma)$  with  $\dim(\lambda) = n - 1$ , if  $\lambda = \tau$ , we know already that  $b_\tau(\chi(\lambda)) = \{0, 1\}$ , and if  $\lambda \neq \tau$ , we already saw that for every  $(n - 2)$ -face  $\tau'$  of  $\lambda$ ,  $b_\tau(\chi(\tau')) = \{0, 1\}$ , and thus  $b_\tau(\chi(\lambda)) = \{0, 1\}$ . Therefore, we conclude that  $b_\tau(\chi(\lambda)) = \text{val}(\lambda)$ , for every input simplex  $\tau \in \chi^\ell(\sigma)$ .

Thus, we have shown that  $b_\tau$  is a symmetric binary coloring that is consistent and complete w.r.t.  $\text{val}$ . Also, by Lemma 11,  $b_\tau(\chi(\tau))$  has no monochromatic simplexes of dimension  $n - 1$ . Therefore,  $\langle \chi^\ell(\sigma), \{0, 1\}, \text{val} \rangle$  is locally solvable in one round. ◀

## 5 Local Valency Impossibility Proofs

Here we make precise our notion of “impossibility proof in the FLP style,” and use Theorems 8 and 10 to argue that such impossibility proofs do not exist for  $(n - 1)$ -set agreement and weak symmetry breaking in the IIS model.

In a *local valency impossibility proof* for say, set agreement, one assumes by way of contradiction a hypothetical  $R$ -round protocol solving the task. Recall that the protocol determines valencies, for all simplexes in all rounds, starting with those of the initial configuration  $\sigma$ . The valencies must respect the task specification, since we assume the protocol solves the task. For example,  $\text{val}(P_i, i) = \{i\}$ , where  $(p_i, i) \in \sigma$  is the initial state of  $P_i$  (in an execution where  $P_i$  sees only itself, it must decide its own input value). A crucial observation is that what we are given in a local valency impossibility proof are only the valencies, and there are many protocols that could produce the same valencies (i.e., many simplicial maps  $c$  assigning decisions to  $\chi^R(\sigma)$ , yielding the same valencies).

The proof consists of  $R - 1$  phases to select a sequence of simplexes  $\sigma_0, \sigma_1, \dots, \sigma_{R-1}$ , starting with  $\sigma_0 = \sigma$  and such that  $\sigma_\ell \in \chi^\ell(\sigma)$  for all  $1 \leq \ell \leq R - 1$ , extending the sequence by one at each phase.

Assume we have selected the sequence  $\sigma_0, \dots, \sigma_\ell$ , for some  $\ell \geq 1$ . To select  $\sigma_{\ell+1} \in \chi(\sigma_\ell) \subset \chi^{\ell+1}(\sigma)$ , one considers all simplexes in  $\sigma' \in \chi(\sigma_\ell)$ , together with their valencies,  $\text{val}(\sigma')$ . When we reach phase  $R - 1$ , and we have selected  $\sigma_{R-1} \in \chi(\sigma_{R-2})$ , the protocol reveals all decisions in  $\chi(\sigma_{R-1}) \subset \chi^R(\sigma)$  (and *only* those decisions). Namely, a simplicial map  $c$  assigning a decision to each vertex of  $\chi(\sigma_{R-1})$ , respecting all previously observed valencies, namely, all those in each  $\chi(\sigma_\ell)$ .

There is a *local valency impossibility proof* for the task if and only if one can select a sequence  $\sigma_0, \sigma_1, \dots, \sigma_{R-1}$  such that the task is *not* locally solvable in one round at  $\sigma_{R-1}$ . Namely, if there is no decision function  $c$ , that respects the valencies and is consistent with the task specification. In the case of set agreement, at least one simplex must have  $n$  different decisions, for any  $c$  that respects the valencies.

## 18:14 Locally Solvable Tasks

Therefore, there is no such a proof if we are able to exhibit valencies of a hypothetical protocol such that, for any selection  $\sigma_0, \sigma_1, \dots, \sigma_{R-1}$ , there is a decision function  $c$  corresponding to those valencies that locally solves  $(n-1)$ -set agreement (the argument for weak symmetry breaking is analogous):

- Fix any  $R \geq 2$ .
- For the input complex, the valency of each  $\sigma' \subseteq \sigma$  is  $val(\sigma') = \sigma'$ .
- In phase  $\ell \in \{0, \dots, R-2\}$ , the valency of each simplex  $\tau \in \chi(\sigma_\ell) \subset \chi^{\ell+1}(\sigma)$  is the valency of the simplex in the valency task  $\mathcal{T}^{\ell+1} = \langle \chi^{\ell+1}(\sigma), \sigma, val \rangle$  in Definition 3, namely,  $val(\tau)$ .
- In phase  $R-1$ , the protocol picks a decision map  $c: \chi^R(\sigma) \rightarrow \sigma$  that is consistent and complete w.r.t  $\mathcal{T}^{R-1} = \langle \chi^{R-1}(\sigma), \sigma, val \rangle$  and does not have fully colored  $(n-1)$ -simplexes in  $\chi(\sigma_{R-1}) \subset \chi^R(\sigma)$ , and provides only the decisions  $c(\chi(\sigma_{R-1}))$ . Such a mapping exists since  $\mathcal{T}^{R-1}$  is  $(n-1)$ -locally solvable, due to Theorem 8.

Notice that no matter the simplex  $\sigma_\ell$  we chose in each phase, we cannot find a contradiction in the decisions of  $\chi(\sigma_{R-1})$ . The only thing that remains to be argued is that the valencies are consistent during all phases. More specifically, valencies preserve containment in the same phase and can only shrink as the phases go by, and additionally they do not contradict validity, i.e., the valency of a simplex is a subset of its carrier. Thus, for any  $R$ , there are valencies that could be produced by a hypothetical set agreement protocol. This is implied by the three properties below that are satisfied for every valency task  $\mathcal{T}^\ell = \langle \chi^\ell(\sigma), \sigma, val \rangle$ ,  $\ell \in \{1, \dots, R-1\}$ , and whose proof is based on Observation 12. These properties also show that the decisions of  $\chi^{R-1}(\sigma_{R-1})$  revealed by the protocol are consistent with all valencies in all phases.

► **Observation 12.** *For  $\ell \geq 0$ , for every  $\gamma \in \chi^\ell(\sigma)$ ,  $ID(\gamma) \subseteq carr(\gamma, \sigma, \chi^\ell)$ . Furthermore, if  $dim(carr(\gamma, \sigma, \chi^\ell)) = dim(\gamma)$ , then  $carr(\gamma, \sigma, \chi^\ell) = ID(\gamma)$ .*

**Containment** For  $\tau, \tau' \in \chi^\ell(\sigma)$  with  $\tau' \subset \tau$ , we have  $val(\tau') \subseteq val(\tau)$ . By Observation 12, for every  $\gamma \in \chi^\ell(\sigma)$ ,  $ID(\gamma) \subseteq carr(\gamma, \sigma, \chi^\ell)$ . Since  $\tau' \subset \tau$ , we have  $carr(\tau', \sigma, \chi^\ell) \subset carr(\tau, \sigma, \chi^\ell)$ . Depending on the dimension of  $\tau$ ,  $val(\tau)$  is either  $ID(\tau)$  or  $carr(\tau, \sigma, \chi^\ell)$ ; and similarly for  $\tau'$ . Therefore,  $val(\tau') \subseteq val(\tau)$ .

**Valencies shrink** Consider any  $m > \ell$  with  $0 \leq \ell + m \leq R-1$  and the valency task  $\mathcal{T}^{\ell+m} = \langle \chi^{\ell+m}(\sigma), \sigma, val' \rangle$ . For  $\tau \in \chi^\ell(\sigma)$  and  $\tau' \in \chi^m(\tau)$ ,  $val'(\tau') \subseteq val(\tau)$ . The argument is very similar to the previous one. Since  $\tau' \in \chi^m(\tau) \subset \chi^\ell(\sigma)$ , we have that  $carr(\tau', \sigma, \chi^{\ell+m}) \subseteq carr(\tau, \sigma, \chi^\ell)$ . Depending on the dimension of  $\tau$ ,  $val(\tau)$  is either  $ID(\tau)$  or  $carr(\tau, \sigma, \chi^\ell)$ , and we have that  $ID(\tau) \subseteq carr(\tau, \sigma, \chi^\ell)$ , by Observation 12; and similarly for  $\tau'$ . Therefore,  $val'(\tau') \subseteq val(\tau)$ .

**Validity** For  $\tau \in \chi^\ell(\sigma)$ , we have  $val(\tau) \subseteq carr(\tau, \sigma, \chi^\ell)$ . If  $dim(\tau)$  is  $n-2$  or  $n-1$ ,  $val(\tau) = carr(\tau, \sigma, \chi^\ell)$ , and if  $dim(\tau) \leq n-3$ ,  $val(\tau) = ID(\tau) \subseteq carr(\tau, \sigma, \chi^\ell)$ , where the last containment follows from Observation 12.

The previous properties hold for all simplexes of the valency tasks in Definition 3, and thus we conclude that there is no contradiction on the valencies provided during the phases.

## 6 Discussion

This paper argues that the  $(n-1)$ -set agreement and weak symmetry breaking (and hence  $(2n-2)$ -renaming) impossibilities in the wait-free read/write shared memory model cannot be proved using local arguments, in the style of FLP. We introduced the notions of *valency task* and *local solvability* for set agreement and weak symmetry breaking. We formalized

the notion of *local-valency impossibility proof* for these tasks, where a presumptive protocol for these tasks can always hide erroneous results, even after committing to valencies one round before termination. We showed that there are no local-valency impossibility proofs for  $(n - 1)$ -set agreement and weak symmetry breaking in the wait-free read/write shared memory model.

Alistarh, Aspnes, Ellen, Gelashvili and Zhu [2] studied a similar question by defining a game between a *prover* and a *protocol*, as a way to represent extension-based techniques for proving impossibility results. They have shown that, for set agreement, a protocol can win this game against any prover, thus showing extension-based techniques do not suffice for proving the impossibility of solving set agreement. Their approach is restricted to *unbounded* protocols. This also complicates the argument, since they need to work with *non-uniform* simplicial subdivisions. In contrast, we consider *bounded wait-free*. This allows to assume that all processes decide at the same round,  $R$  (hence giving more information and power to the prover), leading to simpler *uniform* subdivisions. We stress that there is no loss of generality in this assumption, since a task is wait-free solvable if and only if it is wait-free solvable by a protocol where all processes decide at the same round. Furthermore, while Alistarh et al. study only  $k$ -set agreement, we also investigate weak symmetry breaking, and by reduction, renaming.

Looking forward, it would be interesting to define a notion of valency task that can be applied to any task, such as approximate agreement. Also, in the context of randomized and non-deterministic protocols. Another interesting question is *how much* of the final decisions the protocol can reveal; for example, revealing consistent decisions even if several configurations are chosen instead of only one. Finally, we would like to explore local-valency proofs beyond our wait-free setting, in models that are not round-based [28] or non-compact [30], like  $t$ -resilient models.

---

## References

- 1 M. K. Aguilera and S. Toueg. A simple bivalency proof that  $t$ -resilient consensus requires  $t + 1$  round. *Information Processing Letters*, 71(3-4):155–158, 1999.
- 2 Dan Alistarh, James Aspnes, Faith Ellen, Rati Gelashvili, and Leqi Zhu. Why extension-based proofs fail. In *STOC*, pages 986–996, 2019.
- 3 James Aspnes. Lower bounds for distributed coin-flipping and randomized consensus. *Journal of the ACM*, 45(3):415–450, 1998.
- 4 H. Attiya, A. Bar-Noy, D. Dolev, D. Peleg, and R. Reischuk. Renaming in an asynchronous environment. *Journal of the ACM*, 37(3):524–548, 1990.
- 5 Hagit Attiya and Armando Castañeda. A non-topological proof for the impossibility of  $k$ -set agreement. *Theoretical Computer Science*, 512:41–48, 2013.
- 6 Hagit Attiya, Armando Castañeda, Danny Hendler, and Matthieu Perrin. Separating lock-freedom from wait-freedom. In *PODC*, pages 41–50, 2018. doi:10.1145/3212734.3212739.
- 7 Hagit Attiya and Keren Censor. Tight bounds for asynchronous randomized consensus. *Journal of the ACM*, 55(5):1–26, 2008.
- 8 Hagit Attiya, Nancy Lynch, and Nir Shavit. Are wait-free algorithms fast? *Journal of the ACM*, 41(1):725–763, 1994.
- 9 Hagit Attiya and Ami Paz. Counting-based impossibility proofs for set agreement and renaming. *Journal of Parallel and Distributed Computing*, 87:1–12, 2016.
- 10 E. Borowsky and E. Gafni. Generalized FLP impossibility result for  $t$ -resilient asynchronous computations. In *ACM Symposium on Theory of Computing*, pages 91–100, 1993.
- 11 Elizabeth Borowsky and Eli Gafni. Immediate atomic snapshots and fast renaming. In *PODC*, pages 41–51, 1993.

- 12 Armando Castañeda and Sergio Rajsbaum. New combinatorial topology bounds for renaming: The lower bound. *Distributed Computing*, 22(5-6):287–301, 2010. doi:10.1007/s00446-010-0108-2.
- 13 Armando Castañeda and Sergio Rajsbaum. New combinatorial topology bounds for renaming: The upper bound. *J. ACM*, 59(1):3:1–3:49, 2012. doi:10.1145/2108242.2108245.
- 14 Armando Castañeda, Sergio Rajsbaum, and Michel Raynal. Unifying concurrent objects and distributed tasks: Interval-linearizability. *J. ACM*, 65(6):45:1–45:42, 2018. doi:10.1145/3266457.
- 15 Soma Chaudhuri. More *choices* allow more *faults*: Set consensus problems in totally asynchronous systems. *Information and Computation*, 105(1):132–158, 1993.
- 16 Danny Dolev, Cynthia Dwork, and Larry Stockmeyer. On the minimal synchronism needed for distributed consensus. *Journal of the ACM*, 34(1):77–97, January 1987.
- 17 Michael J. Fischer, Nancy A. Lynch, and Michael S. Paterson. Impossibility of distributed consensus with one faulty process. *Journal of the ACM*, 32(2):374–382, 1985.
- 18 Eli Gafni and Sergio Rajsbaum. Distributed programming with tasks. In *OPODIS*, pages 205–218, 2010. doi:10.1007/978-3-642-17653-1\_17.
- 19 Eli Gafni, Sergio Rajsbaum, and Maurice Herlihy. Subconsensus tasks: Renaming is weaker than set agreement. In *DISC*, pages 329–338, 2006. doi:10.1007/11864219\_23.
- 20 M. Herlihy, D. Kozlov, and S. Rajsbaum. *Distributed Computing Through Combinatorial Topology*. Elsevier-Morgan Kaufmann, 2013. doi:10.1016/C2011-0-07032-1.
- 21 Maurice Herlihy. Impossibility results for asynchronous PRAM. In *SPAA*, pages 327–336, 1991. doi:10.1145/113379.113409.
- 22 Maurice Herlihy and Sergio Rajsbaum. The topology of distributed adversaries. *Distrib. Comput.*, 26(3):173–192, June 2013. doi:10.1007/s00446-013-0189-9.
- 23 Maurice Herlihy and Nir Shavit. The topological structure of asynchronous computability. *Journal of the ACM*, 46(6):858–923, 1999.
- 24 Maurice P. Herlihy. Wait-free synchronization. *ACM Transactions on Programming Languages and Systems*, 13(1):123–149, 1991.
- 25 Idit Keidar and Sergio Rajsbaum. A simple proof of the uniform consensus synchronous lower bound. *Information Processing Letters*, 85(1):47–52, 2003. doi:10.1016/S0020-0190(02)00333-2.
- 26 Wai-Kau Lo and Vassos Hadzilacos. All of us are smarter than any of us: Nondeterministic wait-free hierarchies are not robust. *SIAM J. Comput.*, 30(3):689–728, 2000. doi:10.1137/S0097539798335766.
- 27 M. C. Loui and H. A. Abu-Amara. Memory requirements for agreement among unreliable asynchronous processes. *Advances in Computing Research*, 4:163–183, 1987.
- 28 Ronit Rubinfeld and Shlomo Moran. Closed schedulers: a novel technique for analyzing asynchronous protocols. *Distributed Computing*, 8(4):203–210, 1995. doi:10.1007/BF02242738.
- 29 Yoram Moses and Sergio Rajsbaum. A layered analysis of consensus. *SIAM Journal on Computing*, 31(4):989–1021, 2002.
- 30 Thomas Nowak, Ulrich Schmid, and Kyrill Winkler. Topological characterization of consensus under general message adversaries. In *PODC*, page 218–227, 2019. doi:10.1145/3293611.3331624.
- 31 S. Rajsbaum. Iterated shared memory models. In *LATIN*, pages 407–416, 2010. doi:10.1007/978-3-642-12200-2\_36.
- 32 Michael Saks and Fotios Zaharoglou. Wait-free k-set agreement is impossible: The topology of public knowledge. *SIAM Journal on Computing*, 29(5):1449–1483, 2000.
- 33 E. Sperner. Neuer beweis für die invarianz der dimensionszahl und des gebietes. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 6(1):265–272, 1928. doi:10.1007/BF02940617.