

Classifying Convex Bodies by Their Contact and Intersection Graphs

Anders Aamand  

BARC, University of Copenhagen, Denmark

Mikkel Abrahamsen  

BARC, University of Copenhagen, Denmark

Jakob Bæk Tejs Knudsen  

BARC, University of Copenhagen, Denmark

Peter Michael Reichstein Rasmussen  

BARC, University of Copenhagen, Denmark

Abstract

Let A be a convex body in the plane and A_1, \dots, A_n be translates of A . Such translates give rise to an *intersection graph* of A , $G = (V, E)$, with vertices $V = \{1, \dots, n\}$ and edges $E = \{uv \mid A_u \cap A_v \neq \emptyset\}$. The subgraph $G' = (V, E')$ satisfying that $E' \subset E$ is the set of edges uv for which the interiors of A_u and A_v are disjoint is a *unit distance graph* of A . If furthermore $G' = G$, i.e., if the interiors of A_u and A_v are disjoint whenever $u \neq v$, then G is a *contact graph* of A .

In this paper, we study which pairs of convex bodies have the same contact, unit distance, or intersection graphs. We say that two convex bodies A and B are equivalent if there exists a linear transformation B' of B such that for any slope, the longest line segments with that slope contained in A and B' , respectively, are equally long. For a broad class of convex bodies, including all strictly convex bodies and linear transformations of regular polygons, we show that the contact graphs of A and B are the same if and only if A and B are equivalent. We prove the same statement for unit distance and intersection graphs.

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1 Introduction

Consider a convex body A , i.e., a convex, compact region of the plane with non-empty interior, and let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a set of n translates of A . Then \mathcal{A} gives rise to an *intersection graph* $G = (V, E)$, where $V = \{1, \dots, n\}$ and $E = \{uv \mid A_u \cap A_v \neq \emptyset\}$, and a *unit distance graph* $G' = (V, E')$, where $uv \in E'$ if and only if $uv \in E$ and A_u and A_v have disjoint interiors. In the special case that $G = G'$ (i.e., the convex bodies of \mathcal{A} have pairwise disjoint interiors), we say that G is a *contact graph* (also known as a *touch graph* or *tangency graph*). Thus, A defines three classes of graphs, namely the intersection graphs $I(A)$, the unit distance graphs $U(A)$, and the contact graphs $C(A)$ of translates of A .



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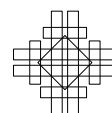
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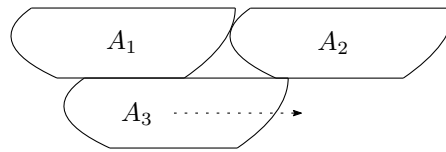
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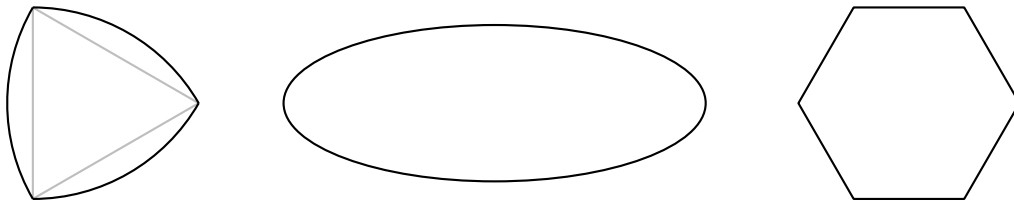
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■ **Figure 1** Translates of a convex body not having the URTC property. The disk A_3 can “slide” along A_1 and A_2 .



■ **Figure 2** Reuleaux triangle (left), ellipse (middle), and regular hexagon (right).

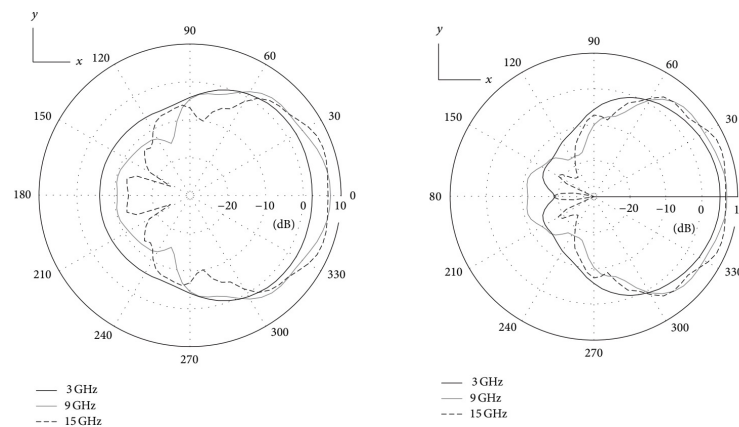
The study of intersection graphs has been an active research area in discrete and computational geometry for the past three decades. For instance, numerous papers consider the problem of solving classical graph problems efficiently on various classes of geometric intersection graphs; see Section 1.1 for some references. Meanwhile, the study of contact graphs of translates of a convex body has much older roots. It is closely related to the packings of such a body, which has a very long and rich history in mathematics going back (at least) to the seventeenth century, where research on the packings of circles of varying and constant radii was conducted and Kepler famously conjectured upon a 3-dimensional counterpart of such problems, the packing of spheres.

In this paper we investigate the question of when two convex bodies A and B give rise to the same classes of graphs. We restrict ourselves to convex bodies A that have the URTC property (*unique regular triangle constructibility*). This is the property that given two interior disjoint translates A_1, A_2 of A that touch, there are exactly two ways to place a third translate A_3 such that A_3 is interior disjoint from A_1 and A_2 , but touches both. Convex bodies with the URTC property include all linear transformations of regular polygons except squares and all strictly convex bodies [15]. Thus, almost all convex bodies in a measure theoretic sense have the property [15, 18, 30]. A convex body without the property must have a sufficiently long line segment on the boundary (to be made precise in Section 1.3); see Figure 1 for an example.

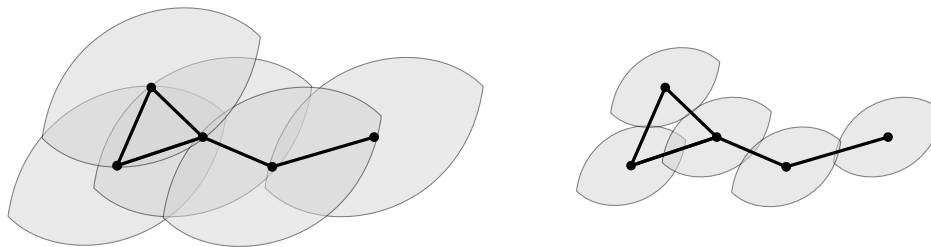
The main result of the paper is summarized in the following theorem.

► **Theorem 1.** *Let A and B be convex bodies with the URTC property. Then each of the identities $I(A) = I(B)$, $U(A) = U(B)$, and $C(A) = C(B)$ holds if and only if the following condition is satisfied: there is a linear transformation B' of B such that for any slope, the longest segments contained in A and B' , respectively, with that slope are equally long.*

If the condition from the theorem is satisfied, we say that A and B are *equivalent*. The length of the longest segment with a given slope contained in a convex body A is often called the *width* of A in the corresponding direction. A circle has constant width but there are other convex bodies of constant width, the simplest example being the Reuleaux triangle; see Figure 2. As an example it follows from the theorem that circles and Reuleaux triangles have the same contact, unit distance, and intersection graphs, which in turn are the same as those for ellipses (ellipses are linear transforms of circles). It also follows that these classes are different from those of regular hexagons.



■ **Figure 3** The strength of radiation in every direction and at various frequencies for two different transmitters described in [25]. In engineering circles, this known as the *radiation pattern*.



■ **Figure 4** When the reachable region of a device is symmetric and the devices are oriented in the same way, a communication network is the intersection graph of the reachable region scaled by $1/2$. Left: A network of five identical devices with the reachable regions shown. Right: The intersection graph of the reachable regions scaled by $1/2$.

1.1 Practical Implications

From a practical point of view, the research on intersection graphs is often motivated by the applicability of these graphs when modeling wireless communication networks and facility location problems. If a device is located at some point in the plane and is able to transmit to and receive from all other devices within some distance, then the devices can be represented as unit disks in such a way that two devices can communicate if and only if their disks overlap. Many highly-cited papers gave this motivation for studying unit disk intersection graphs [9, 14, 16, 21, 29] and it remains a motivation for new research [7, 12, 13, 24].

However, it is in general not the case that a transmitter emits an equally strong signal in all directions. For a real-world example of how the signal strength may vary in different directions; see Figure 3. If the networks that can be made with devices of a given type are not the unit disk intersection graphs, the algorithms for unit disk graphs cannot be expected to work when applied to the actual networks. It is therefore necessary to study how the possible networks that can be made with devices of different types depend on the radiation pattern of the devices. See Figure 4 for a demonstration of the connection between communication networks of a device with a non-circular radiation pattern and intersection graphs of the corresponding convex body.

1.2 Other Related Work

An important notion in the area of contact graphs is that of the *Hadwiger number* of a body K , which is the maximum possible number of pairwise interior-disjoint translates K_i of K that each touch but do not overlap K . The Hadwiger number of K is thus the maximum degree of a contact graph of translates of K . In the plane, the Hadwiger number is 8 for parallelograms and 6 for all other convex bodies. We refer the reader to the books and surveys by László and Gábor Fejes Tóth [28, 10] and Böröczky [3].

Another noteworthy result on contact graphs is the Circle Packing Theorem (also known as the Koebe–Andreev–Thurston Theorem): A graph is simple and planar if and only if it is the contact graph of some set of circular disks in the plane (the radii of which need not be equal). The result was proven by Koebe in 1935 [19] (see [11] for a streamlined, elementary proof). Schramm [26] generalized the circle packing theorem by showing that if a planar convex body with smooth boundary is assigned to each vertex in a planar graph, then the graph can be realized as a contact graph where each vertex is represented by a homothet (i.e., a scaled translation) of its assigned body.

Several papers have compared classes of intersection graphs of various geometric objects, see for instance [4, 6, 8, 17, 20]. Most of the results are inclusions between classes of intersection graphs of one-dimensional objects such as line segments and curves.

A survey by Swanepoel [27] summarizes results on minimum distance graphs and unit distance graphs in normed spaces, including bounds on the minimum/maximum degree, maximum number of edges, chromatic number, and independence number.

In the area of computational geometry, Müller, van Leeuwen, and van Leeuwen [23] gave sharp upper and lower bounds on the size of an integer grid used to represent an intersection graph of translates of a convex polygon with corners at rational coordinates. Their results imply that for any convex polygon R with rational corners, the problem of recognizing intersection graphs of translates of R is in NP. On the contrary, it is open whether recognition of unit disk intersection graphs in the Euclidean plane is in NP. Indeed, the problem is $\exists\mathbb{R}$ -complete (and thus in PSPACE), and using integers to represent the center coordinates and radii of the disks in some graphs requires exponentially many bits [5, 22].

Bonnet, Grelier, and Miltzow [2] showed how well-known algorithms for the clique problem in unit disk intersection graphs and disk intersection graphs can be adjusted to work for intersection graphs of translates or homothets of an arbitrary centrally symmetric convex body.

1.3 Preliminaries

We begin by defining some basic geometric concepts and terminology.

For a subset $A \subset \mathbb{R}^2$ of the plane we denote by A° the interior of A , that is,

$$A^\circ = \{x \in A \mid \exists \text{ open } U \subset \mathbb{R}^2 \text{ such that } \{x\} \subset U \subset A\}.$$

We say that A is a *convex body* if A is compact, convex, and has non-empty interior. We say that A is *symmetric* if whenever $x \in A$, then $-x \in A$. It is well-known that if A is a symmetric convex body, then the map $\|\cdot\|_A : \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$\|x\|_A = \inf\{\lambda \geq 0 \mid x \in \lambda A\},$$

is a norm. Moreover $A = \{x \in \mathbb{R}^2 \mid \|x\|_A \leq 1\}$ and $A^\circ = \{x \in \mathbb{R}^2 \mid \|x\|_A < 1\}$.

It follows from these properties that for translates $A_1 = A + v_1$ and $A_2 = A + v_2$ it holds that $A_1 \cap A_2 \neq \emptyset$ if and only if $\|v_1 - v_2\|_A \leq 2$ and $A_1^\circ \cap A_2^\circ \neq \emptyset$ if and only if $\|v_1 - v_2\|_A < 2$. This means that when studying contact, unit distance, and intersection

graphs of a symmetric convex body A , we can shift our viewpoint from translates of A to point sets in \mathbb{R}^2 and their $\|\cdot\|_A$ -distances: If $\mathcal{A} \subset \mathbb{R}^2$ is a set of points we define $I_A(\mathcal{A})$ and $U_A(\mathcal{A})$ to be the graphs with vertex set \mathcal{A} and edge sets $\{(x, y) \in \mathcal{A}^2 \mid x \neq y \text{ and } \|x - y\|_A \leq 2\}$ and $\{(x, y) \in \mathcal{A}^2 \mid \|x - y\|_A = 2\}$, respectively. Moreover, if for all distinct points $x, y \in \mathcal{A}$ it holds that $\|x - y\|_A \geq 2$, we say that \mathcal{A} is *compatible with* A and define $C_A(\mathcal{A})$ to be the graph with vertex set \mathcal{A} and edge set $\{(x, y) \in \mathcal{A}^2 \mid \|x - y\|_A = 2\}$. Then $I_A(\mathcal{A})$, $U_A(\mathcal{A})$, and $C_A(\mathcal{A})$, respectively, are isomorphic to the intersection, unit distance, and contact graph of A realized by the translates $(A + a)_{a \in \mathcal{A}}$. When studying contact, unit distance, and intersection graphs of a symmetric, convex body A we will view them as being induced by point sets rather than by translates of A .

We say that a (not necessarily symmetric) convex body A in the plane has the URTC property if the following holds: For any two interior disjoint translates of A , A_1 and A_2 , satisfying $A_1 \cap A_2 \neq \emptyset$, there exists precisely two vectors $v \in \mathbb{R}^2$ such that for $i \in \{1, 2\}$, $(A + v)^\circ \cap A_i^\circ = \emptyset$ but $(A + v) \cap A_i \neq \emptyset$. If A is symmetric, this amounts to saying that for any two points $v_1, v_2 \in \mathbb{R}^2$ with $\|v_1 - v_2\|_A = 2$, the set $\{v \in \mathbb{R}^2 \mid \|v - v_1\|_A = \|v - v_2\|_A = 2\}$ has size two. Gehér [15] proved that a symmetric convex body A has the URTC property if and only if the boundary ∂A does not contain a line segment of length more than 1 in the $\|\cdot\|_A$ -norm. See Figure 1 for an example of a convex body not having the URTC property.

A *drawing* of a graph $G \in I(A)$ as an intersection graph of a symmetric convex body A is a point set $\mathcal{A} \subset \mathbb{R}^2$ and a set of straight line segments \mathcal{L} such that $I_A(\mathcal{A})$ is isomorphic to G and \mathcal{L} is exactly the line segments between the points $u, v \in \mathcal{A}$ which are connected by an edge in G . We define a drawing of a graph G as a contact and unit distance graph similarly.

For a norm $\|\cdot\|$ on \mathbb{R}^2 and a line segment ℓ with endpoints a and b we will often write $\|\ell\| = \|ab\|$ instead of $\|a - b\|$. Also, if A is a symmetric convex body and $U, V \subset \mathbb{R}^2$, we define $d_A(U, V) := \inf\{\|uv\|_A \mid (u, v) \in U \times V\}$.

1.4 Structure and Techniques of the Paper

Establishing the sufficiency of the condition of Theorem 1, i.e., showing that if A and B are equivalent then $I(A) = I(B)$, $U(A) = U(B)$, and $C(A) = C(B)$, is relatively straightforward and has been deferred to the full version of the paper. It is also relatively easy to reduce Theorem 1 to the case where the convex bodies are symmetric so this too is deferred to the full version. When both A and B are symmetric, they are equivalent according to the condition of Theorem 1 if and only if they are linear transformations of each other.

Thus, left with the task of proving the necessity of the condition of Theorem 1 in the symmetric case, we proceed in two steps. First, in Section 2, we prove the following result, which for contact and unit distance graphs is a generalization of this direction of Theorem 1.

► **Theorem 2.** *Let A and B be symmetric convex bodies with the URTC property such that A is not a linear transformation of B . There exists a graph $G \in C(A)$ such that for all $H \in C(B)$ and all subgraphs $H' \subseteq H$, G is not isomorphic to H' . In particular $C(A) \setminus C(B) \neq \emptyset$.*

As we will also discuss in Section 2 the same result holds if $C(X)$ is replaced by $U(X)$ for $X \in \{A, B\}$ everywhere in the theorem above and the proof is identical.

The core idea in proving Theorem 2 is to consider a graph G satisfying that any drawing of G as a contact graph of A has certain structural properties. Concretely, we ensure that any drawing of G as a contact graph of A consists of many large hollow hexagons. In the interior of each hexagon, we force there to be a “bridge” of translates of A connecting the

sides of the hexagon. We show that if B is not a linear transformation of A , then the contact graph cannot be realized by translates of B if we make sufficiently many and sufficiently large hexagons with bridges of different slopes. See Figures 6 and 7 for illustrations.

To include intersection graphs, we proceed with the second step. In Section 3, we prove the following result which combined with Theorem 2 immediately yields the necessity of the condition of Theorem 1 for intersection graphs.

► **Theorem 3.** *Let A and B be symmetric convex bodies. If there exists a graph $G \in C(A)$ such that for all $H \in C(B)$ and all subgraphs $H' \subset H$, G is not isomorphic to H' , then $I(A) \neq I(B)$.*

This result holds for general symmetric convex bodies. An improvement of Theorem 2 to general symmetric convex bodies (not necessarily having the URTC property) would thus yield a version of Theorem 1 that also holds for general convex bodies.

In order to prove Theorem 3, we proceed as follows. For every positive integer k we construct a gadget $Q_k \in I(A)$ which contains as a subgraph a distinguished cycle $\alpha_k \subset Q_k$. We prove that in any drawing of Q_k as an intersection graph of translates of A , α_k is contained in a translation of the annulus $kA \setminus (k-1)A$ (here, $kA = \{ka \mid a \in A\}$ is the scaling of A by k). This allows us to view α_k as an upscaled copy of the boundary of A with a precision error decreasing in k . Similarly, in any drawing of the same gadget Q_k as an intersection graph of another body, B , the cycle α_k appears as an upscaled copy of B . The idea is then to simulate a contact graph $G \in C(A)$ using distinct copies of Q_k , where each copy plays the role of a single vertex in G . If A is not a linear transformation of B , we can choose G with the properties promised in Theorem 2. We are then able to prove that if we choose k sufficiently large (i.e., obtaining sufficiently high resemblance between α_k and an upscaled copy of A resp. B), then we can realize the simulation of G as an intersection graph using translates of A , but not using translates of B . This then implies $I(A) \neq I(B)$ as desired.

Beyond aiding us in the proof of our main theorem, we believe that this proof technique – the reduction from intersection to contact graphs – is of independent interest. It appears a novel approach with the potential to answer other questions on intersection graphs.

2 Contact and Unit Distance Graphs

In this section we prove Theorem 2. The proof for unit distance graphs is completely identical so we will merely provide a remark justifying this claim by the end of the section.

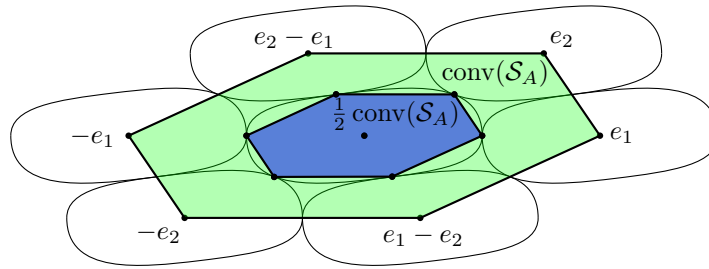
Throughout the section A and B will denote symmetric convex bodies. For $\theta \in [0, 2\pi)$ we define $e_A(\theta)$ to be the vector of argument θ and with $\|e_A(\theta)\|_A = 1$. We also define $\rho_A(\theta) = 2\|e_A(\theta)\|_2$. Then $\rho_A(\theta)$ can be thought of as the “diameter” of A in direction θ . One of our most important tools is the following lemma.

► **Lemma 4.** *Let A, B be symmetric convex bodies in \mathbb{R}^2 . If for every finite set $\Theta \subset [0, \pi)$ and for every $\varepsilon > 0$, there exists a linear map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying that $|\rho_{T(B)}(\theta) - \rho_A(\theta)| < \varepsilon$ for all $\theta \in \Theta$, then there exists a linear map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $T(B) = A$.*

Due to space limitations we have left out the proof, but it can be found in the full version.

We proceed to describe certain lattices which give rise to contact graphs that can only be realized in an essentially unique way. We start with the following definition.

► **Definition 5.** *Let $A \subset \mathbb{R}^2$ be a symmetric convex body, and $\|\cdot\|_A$ the associated norm. Let $e_1, e_2 \in \mathbb{R}^2$ be such that $\|e_1\|_A = \|e_2\|_A = \|e_1 - e_2\|_A = 2$. We define the lattice $\mathcal{L}_A(e_1, e_2) = \{a_1e_1 + a_2e_2 \mid (a_1, a_2) \in \mathbb{Z}^2\}$.*



■ **Figure 5** A symmetric convex body A and the sets $\frac{1}{2} \text{conv}(\mathcal{S}_A)$ and $\text{conv}(\mathcal{S}_A)$ (blue and green respectively) which satisfies $\frac{1}{2} \text{conv}(\mathcal{S}_A) \subseteq A \subseteq \text{conv}(\mathcal{S}_A)$.

The left hand side of figure Figure 6 illustrates the lattice structure. Let us assume that A has the URTC property and describe a few properties of the lattice $\mathcal{L}_A(e_1, e_2)$. After choosing e_1 with $\|e_1\|_A = 2$, there are precisely two vectors v with $\|v\|_A = \|v - e_1\|_A = 2$, using the URTC property. If one is v_2 the second is $e_1 - v_2$ so regardless how we choose e_2 we obtain the same lattice. Using the triangle inequality and the URTC property of A it is easily verified that for distinct $x, y \in \mathcal{L}_A(e_1, e_2)$, $\|x - y\|_A \geq 2$ with equality holding exactly if $x - y \in \mathcal{S}_A := \{e_1, e_2, e_2 - e_1, -e_1, -e_2, e_1 - e_2\}$. This implies that the contact graph $G_0 := C_A(\mathcal{L}_A(e_1, e_2))$ is in fact (isomorphic to) an infinite triangular grid.

Another useful fact is the following:

► **Lemma 6.** *With \mathcal{S}_A as above it holds that $\frac{1}{2} \text{conv}(\mathcal{S}_A) \subset A \subset \text{conv}(\mathcal{S}_A)$. Here $\text{conv}(\mathcal{S}_A)$ is the convex hull of \mathcal{S}_A . If in particular B is another symmetric convex body for which $\|e_1\|_B = \|e_2\|_B = \|e_1 - e_2\|_B = 2$, then for all $x \in \mathbb{R}^2$ it holds that $\frac{1}{2} \|x\|_A \leq \|x\|_B \leq 2 \|x\|_A$.*

Proof. See Figure 5. As $\frac{1}{2} \mathcal{S}_A \subset A$ and A is convex the first inclusion is clear. For the second inclusion we note that all points y on the hexagon connecting the points $e_1, e_2, e_2 - e_1, -e_1, -e_2, e_1 - e_2$ of \mathcal{S}_A in this order have $\|y\|_A \geq 1$ by the triangle inequality and so $A \subset \text{conv}(\mathcal{S}_A)$.

For the last statement of the lemma note that if $x \in \mathbb{R}^2$ then

$$\|x\|_B \geq \inf_{\lambda \geq 0} \{x \in \lambda \text{conv}(\mathcal{S}_B)\} = \inf_{\lambda \geq 0} \{x \in \lambda \text{conv}(\mathcal{S}_A)\} \geq \inf_{\lambda \geq 0} \{x \in 2\lambda A\} = \frac{1}{2} \|x\|_A,$$

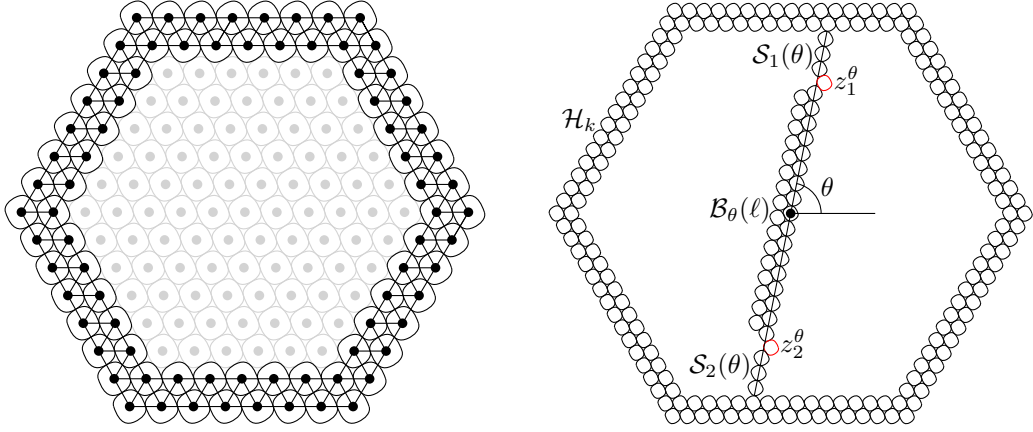
and similarly $\|x\|_A \geq \frac{1}{2} \|x\|_B$. ◀

► **Definition 7.** *We say that a graph $G = (V, E)$ is lattice unique if $|V| = n \geq 3$ and there exists an enumeration of its vertices v_1, \dots, v_n such that*

- *The vertex induced subgraph $G[v_1, v_2, v_3] \simeq K_3$ is a triangle.*
- *For $i > 3$ there exists distinct $j, k, l < i$ such that $G[v_j, v_k, v_l] \simeq K_3$ and both (v_i, v_j) and (v_i, v_k) are edges of G .*

Suppose that A is a symmetric convex body with the URTC property, that $\mathcal{A} \subset \mathbb{R}^2$ is compatible with A , and that $G = C_A(\mathcal{A})$ is lattice unique. Enumerate the points of $\mathcal{A} = \{v_1, \dots, v_n\}$ according to the definition of lattice uniqueness. Without loss of generality assume that $v_1 = 0$. Then the URTC property of A combined with the lattice uniqueness of G gives that v_4, \dots, v_n are uniquely determined from v_2 and v_3 and all contained in $\mathcal{L}_A(v_2, v_3)$. If moreover B is another convex body with the URTC property, $\mathcal{B} = \{v'_1, \dots, v'_n\} \subset \mathbb{R}^2$ has $v'_1 = 0$ and is compatible with B , and $C_B(\mathcal{B}) \simeq C_A(\mathcal{A})$ via the graph isomorphism $\varphi : v'_i \mapsto v_i$, then the linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T : a_1 v'_2 + a_2 v'_3 \mapsto a_1 v_2 + a_2 v_3$ satisfies that $T|_{\mathcal{B}} = \varphi$.

We will use this observation in the proof of Theorem 2 which we now provide.



■ **Figure 6** Left: The points of \mathcal{H}_6 along with the corresponding lattice unique subgraph $G_0[\mathcal{H}_6]$. Right: The attachment of the beam $\mathcal{B}_\theta(\ell)$.

Proof of Theorem 2. We choose $e_1, e_2 \in \mathbb{R}^2$ be such that $\|e_1\|_A = \|e_2\|_A = \|e_1 - e_2\|_A = 2$ and define the lattice $\mathcal{L} := \mathcal{L}_A(e_1, e_2)$. We also define the infinite graph $G_0 := C_A(\mathcal{L})$ which by the remarks following Definition 5 is isomorphic to the infinite triangular grid. Without loss of generality we can assume that e_1 and e_2 satisfy that $\|e_1\|_2 = \|e_2\|_2 = \|e_1 - e_2\|_2 = 2$, since there exists a non-singular linear transformation T such that $\|T(e_1)\|_2 = \|T(e_2)\|_2 = \|T(e_1) - T(e_2)\|_2 = 2$, and $C(A) = C(T(A))$. Note that in this setting we can use Lemma 6 to compare A to the disk of radius 1 and obtain $\frac{1}{2}\|x\|_2 \leq \|x\|_A \leq 2\|x\|_2$ for every $x \in \mathbb{R}^2$.

We will construct G by specifying a finite point set $\mathcal{A} \subset \mathbb{R}^2$ compatible with A and define $G = C_A(\mathcal{A})$. The construction of \mathcal{A} can be divided into several sub-constructions. We start by describing a hexagon of points \mathcal{H}_k for $k \in \mathbb{N}$ which satisfies that $C_A(\mathcal{H}_k)$ is lattice unique.

► **Construction 8** (\mathcal{H}_k). For an illustration of the construction see the left-hand side of Figure 6. For $x, y \in \mathcal{L}$ we write $d(x, y)$ for the distance between x and y in the graph G_0 , and for $k \in \mathbb{N}$ we define $\mathcal{H}_k = \{x \in \mathcal{L} \mid d(x, 0) \in \{k, k+1\}\}$.

Using that G_0 is the infinite triangular grid, it is easy to check that $G_0[\mathcal{H}_k]$ is a lattice unique graph by specifying an enumeration of its vertices satisfying the condition in Definition 7. Moreover, using that e_1 and e_2 satisfy $\|e_1\|_2 = \|e_2\|_2 = \|e_1 - e_2\|_2 = 2$ it follows that the points $\{x \in \mathcal{L} \mid d(x, 0) = k\} \subset \mathcal{H}_k$ lie on a regular hexagon H_k whose corners have Euclidean distance exactly $2k$ to the origin. In particular any point $p \in \mathcal{H}_k$ has $\|p\|_2 \geq \sqrt{3}k$, and thus $\|p\|_A \geq \frac{\sqrt{3}}{2}k$ by Lemma 6.

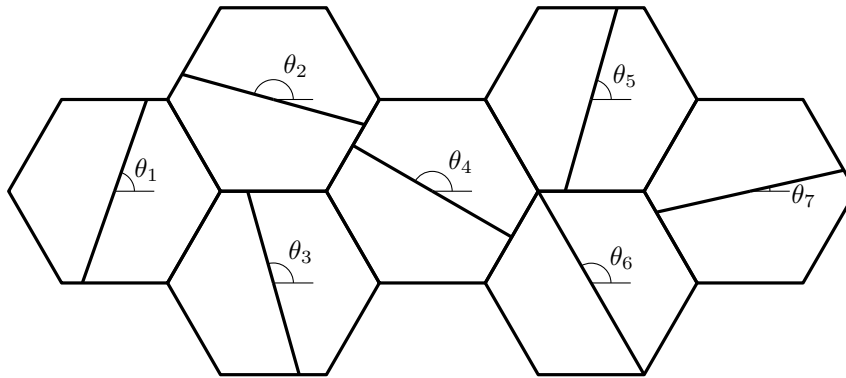
For a given $\theta \in [0, \pi)$ and $\ell \in \mathbb{N}$ we will construct a set of points $\mathcal{B}_\theta(\ell) \subset \mathbb{R}^2$ compatible with A which constitute a “beam” of argument θ :

► **Construction 9** ($\mathcal{B}_\theta(\ell)$). See Figure 6 (right). Let $e_\theta \in \mathbb{R}^2$ be the vector of argument θ with $\|e_\theta\|_A = 2$, and let $f_\theta \in \mathbb{R}^2$ be such that $\|f_\theta\|_A = \|f_\theta - e_\theta\|_A = 2$ (by the URTC property we have two choices for f_θ). For a given $\ell \in \mathbb{N}$ we define

$$\mathcal{B}_\theta(\ell) = \{ae_\theta \mid a \in \{-\ell, \dots, \ell\}\} \cup \{ae_\theta + f_\theta \mid a \in \{-\ell, \dots, \ell-1\}\}$$

As $\mathcal{B}_\theta(\ell) \subset \mathcal{L}_A(e_\theta, f_\theta)$ it is compatible with A . Moreover it is easy to specify an enumeration of the vertices of $C(\mathcal{B}_\theta(\ell))$ showing that it is lattice unique.

For a given k we want to choose ℓ as large as possible such that $\mathcal{B}_\theta(\ell)$ “fits inside” $G_0[\mathcal{H}_k]$. We then wish to “attach” $\mathcal{B}_\theta(\ell)$ to $G_0[\mathcal{H}_k]$ with extra points \mathcal{S} , the number of which does neither depend on k nor on θ . We wish to do it in such a way that $\mathcal{A}_1^k(\theta) := \mathcal{B}_\theta(\ell) \cup G_0[\mathcal{H}_k] \cup \mathcal{S}$ is compatible with A . The precise construction is as follows:



■ **Figure 7** The final point set \mathcal{A} where the point sets $\mathcal{C}_k(\theta)$ are “glued” together by translating them such that the contact graph realized by the union of the subsets $\mathcal{H}_k \subset \mathcal{C}_k(\theta)$ is lattice unique.

► **Construction 10** ($\mathcal{C}_k(\theta)$). See Figure 6 (right). Consider the open line segment $L_\theta = \{re_\theta \mid r \in (-r_{\max}, r_{\max})\}$ where r_{\max} is maximal with the property that for all points $x \in L_\theta$ and all $y \in H_k$ it holds that $\|x - y\|_A > 4$. Also let $\ell \in \mathbb{N}$ be maximal such that $\{ae_\theta \mid a \in \{-\ell, \dots, \ell\}\} \subset L_\theta$. Note that

$4 < d_A(\{\ell e_\theta\}, H_k) \leq 6$. Observe moreover that $\ell \geq \frac{\sqrt{3}}{4}k - 3$ as the points $p \in H_k$ have $\|p\|_A \geq \frac{\sqrt{3}}{2}k$. In particular we have the following property which we highlight for later use:

$$\text{If } k > \frac{12}{\sqrt{3} - 1} \text{ it holds that } \ell > \frac{k}{4}. \tag{1}$$

When ℓ is chosen in this fashion, we have that $\mathcal{B}_\theta(\ell)$ is contained in the interior of H_k . Now, $\mathcal{B}_\theta(\ell)$ will constitute our beam in direction θ and we will proceed to show that we can attach it to \mathcal{H}_k , as illustrated, using only a constant number of extra points. That this can be done is conceptually unsurprising but requires a somewhat technical proof.

We define $\mathcal{S}_1(\theta)$ to be extra points going from the boundary of H_k and z_1^θ to be the extra point which connects $\mathcal{B}_\theta(\ell)$ and $\mathcal{S}_1(\theta)$. This attaches one end of the beam, $\mathcal{B}_\theta(\ell)$, to H_k , and we similarly define $\mathcal{S}_2(\theta)$ and z_2^θ to attach the other end. See Figure 6 (right). In the full version we show that $|\mathcal{S}_i(\theta)| \leq 13$ for $i \in \{1, 2\}$. Letting $\mathcal{C}_k(\theta) = \mathcal{H}_k \cup \mathcal{B}_\theta(\ell) \cup \mathcal{S}_1(\theta) \cup \mathcal{S}_2(\theta) \cup \{z_1^\theta, z_2^\theta\}$ be the combination of the components completes the construction.

We are now ready to construct \mathcal{A} which will consist of several translated copies $\mathcal{C}_k(\theta)$.

► **Construction 11** (\mathcal{A}). By Lemma 4 we can find an $\varepsilon \in (0, 1)$ and a finite set of directions $\Theta \subset [0, \pi)$ such that for all linear maps $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ there exists $\theta \in \Theta$ such that

$$\left| \frac{\rho_A(\theta)}{\rho_{T(B)}(\theta)} - 1 \right| \geq \varepsilon. \tag{2}$$

That we can scale the deviation to be multiplicative rather than additive is possible because $0 < \inf_{\theta \in [0, \pi)} \rho_A(\theta) \leq \sup_{\theta \in [0, \pi)} \rho_A(\theta) < \infty$.

For each $\theta \in \Theta$ we construct a copy of $\mathcal{C}_k(\theta) = \mathcal{H}_k \cup \mathcal{B}_\theta(\ell) \cup \mathcal{S}_1(\theta) \cup \mathcal{S}_2(\theta) \cup \{z_1^\theta, z_2^\theta\}$ where k is yet to be fixed (ℓ is of course determined by k and θ). We then choose translations $t_\theta \in \mathbb{R}^2$ for each $\theta \in \Theta$ such that the sets $(\mathcal{H}_k + t_\theta)_{\theta \in \Theta}$ are pairwise disjoint, and $\bigcup_{\theta \in \Theta} (\mathcal{H}_k + t_\theta) \subset \mathbb{R}^2$ is compatible with A and induces a lattice unique contact graph. We can choose $(t_\theta)_{\theta \in \Theta}$ in numerous ways to satisfy this. One is depicted in Figure 7. Another is obtained by

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enumerating $\Theta = \{\theta_1, \dots, \theta_q\}$ and defining $t_{\theta_i} = ((2k+3)e_1 - (k+1)e_2) \times (i-1)$. The exact choice is not important and picking one, we define $\mathcal{A}(k) = \bigcup_{\theta \in \Theta} (C_k(\theta) + t_\theta)$ which is a point set compatible with A . Lastly, we set $\mathcal{A} = \mathcal{A}(\lceil \frac{180}{\varepsilon} \rceil)$.

We are now ready for the final step of the proof:

Proving that no graph in $C(B)$ contains a subgraph isomorphic to $G = C_A(\mathcal{A})$.

Suppose for contradiction that there exists a set of points $\mathcal{B} \subset \mathbb{R}^2$ such that G is isomorphic to a subgraph of $C_B(\mathcal{B})$. We may clearly assume that $|\mathcal{A}| = |\mathcal{B}|$ and we let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a bijection which is also a graph homomorphism when considered as a map $C_A(\mathcal{A}) \rightarrow C_B(\mathcal{B})$. The points $\bigcup_{\theta \in \Theta} (\mathcal{H}_k + t_\theta)$ induce a lattice unique contact graph of A . Thus, we may write $\bigcup_{\theta \in \Theta} (\mathcal{H}_k + t_\theta) = \{p_1, \dots, p_n\}$ such that p_1, p_2 and p_3 induce a triangle of G and such that for $i > 3$ there exist distinct $j, k, l < i$ such that p_j, p_k and p_l induce a triangle and such that (p_i, p_k) and (p_i, p_l) are edges of G . By translating the point sets \mathcal{A} and \mathcal{B} we may assume that $\varphi(p_1) = p_1 = 0$. Then applying an appropriate linear transformation T , thus replacing B by $T(B)$, we may assume that $\varphi(p_2) = p_2$ and $\varphi(p_3) = p_3$. Finally, the discussion succeeding Definition 7 implies that in fact $\varphi|_{\bigcup_{\theta \in \Theta} (\mathcal{H}_k + t_\theta)}$ is the identity.

As noted in Construction 11, there exists $\theta \in \Theta$ such that $\left| \frac{\rho_A(\theta)}{\rho_{T(B)}(\theta)} - 1 \right| \geq \varepsilon$. The outline of the remaining argument is as follows: The Euclidean distance between the ‘‘endpoints’’ of the beam $\mathcal{B}_\theta(\ell)$ is $2\ell\rho_A(\theta)$, but the rigidity of $\bigcup_{\theta \in \Theta} (\mathcal{H}_k + t_\theta)$ means that it is also $2\ell\rho_{T(B)}(\theta) + O(1)$. When k (and hence ℓ) is large, this will contradict the inequality above.

We refer the reader to the full version of the paper for the technical details. ◀

► **Remark 12.** We claimed that the proof of the part of Theorem 1 concerning unit distance graphs is identical to the proof above. In fact, if we replace $C(X)$ by $U(X)$ for $X \in \{A, B\}$ in the statement of Theorem 2, the result remains valid. To prove it we would construct \mathcal{A} in precisely the same manner. The important point is then that the comments immediately prior to Theorem 1 concerning the rigidity of the realization of lattice unique graphs remains valid. If in particular $\mathcal{B} \subset \mathbb{R}^2$ satisfies that $U_A(\mathcal{A}) \simeq U_A(\mathcal{B})$ via the isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$, we may assume that $\varphi|_{\bigcup_{\theta \in \Theta} (\mathcal{H}_k + t_\theta)}$ is the identity as in the proof above. The remaining part of the argument comparing the lengths of the beams then carries through unchanged. In conclusion, we are only left with the task of proving Theorem 1 for intersection graphs.

3 Intersection Graphs

In this section we prove Theorem 3. Consider two convex bodies A and B . We are going to prove that if $I(A) = I(B)$, then for every graph $G \in C(A)$, there exists a graph $H_k(G) \in I(A)$ with properties as stated in the following lemma.

► **Lemma 13.** *Assume that $I(A) = I(B)$. For any $G \in C(A)$ and $k \geq 7$, there exists a graph $H_k(G) \in I(A)$ satisfying the following: Let $X \in \{A, B\}$. For any vertex w of G , there is a corresponding vertex $s_0(w)$ of $H_k(G)$ with the following properties. Consider an arbitrary drawing of $H_k(G)$ as in intersection graph of X and any two vertices w, w' of G and let $s_0 := s_0(w)$ and $s'_0 := s_0(w')$. Then $\|s_0 s'_0\|_X \geq 4k - 18$. Furthermore, if ww' is an edge of G , then $\|s_0 s'_0\|_X \leq 4k + 2$.*

As is evident from the lemma, the vertices $(s_0(u))_{u \in V(G)}$, of any drawing of $H_k(G)$ as an intersection graph of X , are placed approximately as the vertices of a drawing of G as a contact graph of scaled convex body $2kX$. To capture the uncertainty, we introduce the concept of ε -overlap graphs.

► **Definition 14** (ε -overlap Graph). *Let $\varepsilon > 0$ and $A \subset \mathbb{R}^2$ be a symmetric convex body, and let $v_1, \dots, v_n \in \mathbb{R}^2$ be n points in the plane. Suppose that for any $i, j \in [n]$, $\|v_i v_j\|_A \geq 2 - \varepsilon$. A graph G with vertex set $[n]$ and edge set satisfying $E(G) \subseteq \{(i, j) \in [n]^2 \mid \|v_i v_j\|_A \leq 2\}$ is called an ε -overlap graph of A . We say that $\{v_1, \dots, v_n\}$ realize the graph G as an ε -overlap graph of A . Further, we denote by $C_\varepsilon(A)$ the set of graphs that can be realized as ε -overlap graphs of A .*

We next show how Lemma 13 leads to a proof of Theorem 3. First, the following lemma provides a reduction from ε -overlap graphs to contact graphs. The proof is a standard compactness argument and can be found in the full version of the paper.

► **Lemma 15.** *Let $G_1 = (V, E_1)$ be a graph and A a convex body. If for every $\varepsilon > 0$, it holds that $G_1 \in C_\varepsilon(A)$, then there is a graph $G_2 = (V, E_2) \in C(A)$ such that $E_1 \subseteq E_2$.*

The following lemma uses Lemma 13 to show that if $I(A) = I(B)$, then any $G \in C(A)$ is an ε -overlap graph of B for all $\varepsilon > 0$.

► **Lemma 16.** *Assume that $I(A) = I(B)$. For any $G \in C(A)$, and any $\varepsilon > 0$, $G \in C_\varepsilon(B)$.*

Proof. Write $G = (V, E)$ and let $k \geq 7$ be arbitrary. The assumption $I(A) = I(B)$ in particular implies that $H_k(G) \in I(B)$. Consider a drawing of $H_k(G)$ as an intersection graph of B and define $\mathcal{B} := \left\{ \frac{s_0^u}{2k+1} \mid u \in V \right\}$. It follows from Lemma 13 that $I_B(\mathcal{B})$ is a drawing of G as a $\left(2 - \frac{4k-18}{2k+1}\right)$ -overlap graph of B . Since $\frac{4k-18}{2k+1} \geq 2 - 10/k$, it follows that G is an $10/k$ -overlap graph of B . As $k \geq 7$ was arbitrary, the desired result follows. ◀

Theorem 3 is an easy consequence of Lemma 15 and Lemma 16:

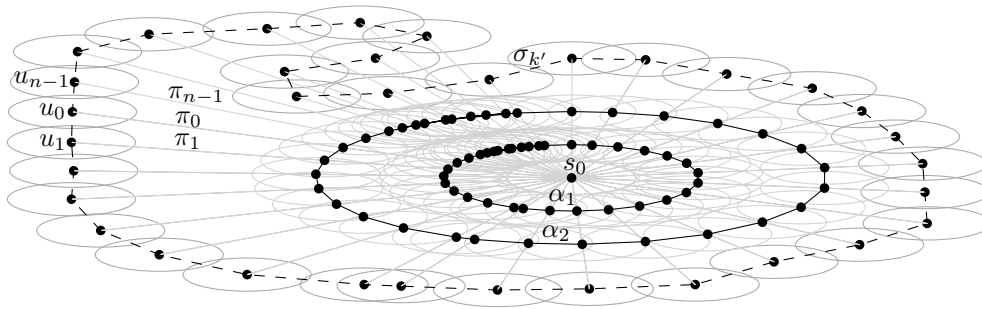
Proof of Theorem 3. Let the graph $G \in C(A)$ have the properties of the theorem, i.e., for all $H \in C(B)$, G is not isomorphic to a subgraph of H . Suppose that $I(A) = I(B)$. By Lemma 15 and 16, there is a graph $H = (V, E) \in C(B)$ such that $E' \subseteq E$, which is a contradiction. ◀

It remains to prove Lemma 13. We will proceed to describe the construction of $H_k(G)$ and provide several lemmas needed in order to prove that it satisfies the desired properties.

The proofs of these lemmas and of Lemma 13 are deferred to the full version of the paper.

For each vertex $u \in V(G)$, we make a copy of a graph Q_k to be defined in the following. The vertices of $H_k(G)$ will in turn be the union of the vertices of these copies. We will construct Q_k to have a designated vertex s_0 and a cycle α_k with the property that for every drawing of Q_k as an intersection graph of $X \in \{A, B\}$, the cycle α_k is contained in (and winds all the way around) the annulus $\{x \in \mathbb{R}^2 \mid \|s_0 x\|_X \in (2k - 3, 2k]\}$. We may then informally view α_k as an upscaled copy of X up to a slight imprecision that, compared to the size, decreases in k . In order to construct Q_k , we first define another graph P_k (which will be contained in Q_k) with a vertex s_0 such that in every drawing of P_k as an intersection graph, s_0 is contained in k nested disjoint cycles. A priori, it is not clear what it means for s_0 to be contained in a cycle of the graph in every drawing, since the drawing is not necessarily a plane embedding of the graph. However, as the following lemma shows, it is well-defined if P_k is triangle-free. We believe the result to be well-known but have been unable to find the exact formulation that we require in the literature.

► **Lemma 17.** *If G is a triangle-free graph then every drawing of G as an intersection graph is a plane embedding.*



■ **Figure 9** A part of a graph Q_k . The vertices $v_i(j)$ are only shown for $j \in \{1, 2\}$, and only edges on paths π_i and cycles $\alpha_1, \alpha_2, \sigma_{k'}$ are shown.

We want to minimize the vector of these values d with respect to each vertex $u_i \in \sigma_{k'}$. To be precise, we define

$$(d_0, \dots, d_{n-1}) := \min \left(\left\lceil \frac{\|s_0 u_0\|_A - 2}{2} \right\rceil, \dots, \left\lceil \frac{\|s_0 u_{n-1}\|_A - 2}{2} \right\rceil \right),$$

where the minimum is with respect to the lexicographical order and taken over all drawings of $P_{k'}$ as an intersection graph. Consider a drawing of $P_{k'}$ as an intersection graph realizing the minimum and let \mathcal{P} be the set of vertices in the drawing. For each vertex u_i , we create a path π_i from s_0 to u_i as follows. Let \mathbf{v}_i be the unit-vector in direction $u_i - s_0$. We add new vertices placed at the points $v_i(j) := s_0 + 2j\mathbf{v}_i$ for $j \in \{1, \dots, d_i\}$. We now define the vertices of Q_k as $\mathcal{Q} := \mathcal{P} \cup \bigcup_{i=0}^{n-1} \{v_i(1), \dots, v_i(d_i)\}$ and define $Q_k = I_A(\mathcal{Q})$. See Figure 9.

► **Remark 22.** By construction, there exists a drawing of Q_k as an intersection graph of A . If there does not exist one of B , we are done, since we then clearly have that $I(A) \neq I(B)$. Now suppose that there exists a drawing of $P_{k'}$ as an intersection graph of B such that

$$\left(\left\lceil \frac{\|s_0 u_0\|_B - 2}{2} \right\rceil, \dots, \left\lceil \frac{\|s_0 u_{n-1}\|_B - 2}{2} \right\rceil \right) \prec (d_0, \dots, d_{n-1}), \tag{3}$$

where \prec denotes the lexicographical order. We can now define a graph $Q_k^B \in I(B)$ from $P_{k'}$ in a similar way as we defined Q_k by adding $\left\lceil \frac{\|s_0 u_i\|_B - 2}{2} \right\rceil$ vertices to form a path from s_0 to each u_i . It then follows from (3) that $Q_k^B \notin I(A)$, so in this case we have likewise succeeded in proving $I(A) \neq I(B)$. In the following, we therefore assume that $Q_k \in I(B)$ for any k and that no drawing of $P_{k'}$ as an intersection graph of B satisfying (3) exists.

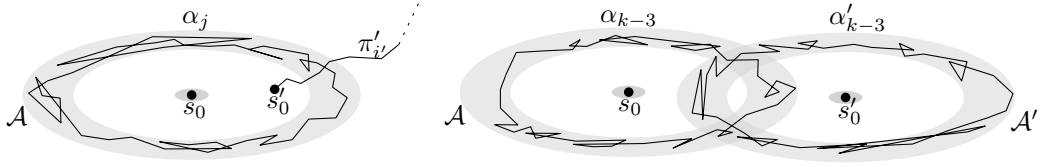
First we need to show that Q_k contains a cycle α_k as described earlier.

► **Lemma 23.** *The set of edges of Q_k contains the pairs $v_i(j)v_{i+1}(j)$ for any $i \in [n]$ and $j \in \{1, \dots, k\}$, and for each $j \in \{1, \dots, k\}$, these edges thus form a cycle α_j . In the specific drawing of Q_k as an intersection graph defined in Construction 21, the cycle α_j is contained in the annulus $\{x \in \mathbb{R}^2 \mid \|s_0 x\|_A \in [2j - 1, 2j]\}$.*

Proof. See the full version of the paper. ◀

The above lemma shows that the cycle α_k behaves nicely in one particular drawing of Q_k as an intersection graph. To see that something similar holds for every drawing, we refer the reader to the full version.

We now provide the definition of the graph $H_k(G)$, as mentioned in the beginning of this section.



■ **Figure 10** From the proof of Lemma 13 (notation as in Lemma 25). There is either an intersection between α_j and $\pi'_{i'}$ (left) or between α_{k-3} and α'_{k-3} (right) that violates Lemma 25.

► **Construction 24** ($H_k(G)$). For any $G \in C(A)$, consider a fixed drawing of G as a contact graph of A . For each vertex w of G , we make a copy of the drawing of Q_k as an intersection graph as defined in Construction 21 which we translate so that s_0 is placed at $s_0^w := (2k-2)w$. We then add all edges induced by the vertices, and the result is denoted as $H_k(G)$.

The following lemma characterizes some of the edges of $H_k(G)$ and will be crucial in the proof of Lemma 13.

► **Lemma 25.** Consider two vertices w, w' of a drawing of a graph G as a contact graph. Denote by Q and Q' the copies of Q_k in $H_k(G)$ corresponding to w and w' , respectively, such that $s_0, \pi_i, \alpha_j, v_i(j)$ denote objects in Q and $s'_0, \pi'_i, \alpha'_j, v'_i(j)$ denote objects in Q' . If $v_i(j)v'_{i'}(j')$ is an edge of $H_k(G)$, then $j + j' \geq 2k - 4$.

If ww' is an edge of G , then there is an edge $v_i(k)v'_{i'}(k)$ in $H_k(G)$.

Proof. See the full version of the paper. ◀

As mentioned, the final proof of Lemma 13 is deferred to the full version, but we can now provide the main ideas. We first need to prove that in any drawing of Q_k as an intersection graph with respect to $X \in \{A, B\}$, any cycle α_j is contained in an annulus only slightly wider than as stated in Lemma 23. Furthermore, α_j winds around s_0 in the sense that if we trace the full curve α_j , the change of argument with respect to s_0 will be $\pm 2\pi$. To prove the lower bound $\|s_0s'_0\|_X \geq 4k - 18$ in Lemma 13, we exclude that the distance is smaller by dividing into two cases depending on the actual distance. Figure 10 depicts the two cases for each of which we prove that there would be an edge in $H_k(G)$ violating Lemma 25. The upper bound $\|s_0s'_0\|_X \leq 4k + 2$ when ww' is an edge of G is likewise an easy consequence of Lemma 25, as otherwise, an edge $v_i(k)v'_{i'}(k)$ would be missing from $H_k(G)$.

4 Concluding remarks

It is natural to investigate the special case of convex bodies with the URTC property. Here our proof of Theorem 2 fails since the hexagons are not rigid structures. Together with Konrad Swanepoel, we have promising progress in generalizing Theorem 1 to also handle this case.

Another interesting direction is to consider convex bodies in three and higher dimensions. Already in three dimensions, it appears to be very difficult to characterize when two bodies induce the same graph classes.

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