# Orientation Preserving Maps of the Square Grid

**Imre Bárány** ✉
Alfréd Rényi Institute of Mathematics, Budapest, Hungary
Department of Mathematics, University College London, UK

**Attila Pór** ✉
Department of Mathematics, Western Kentucky University, Bowling Green, KY, USA

**Pavel Valtr**
Department of Applied Mathematics, Faculty of Mathematics and Physics,
Charles University, Praha 1, Czech Republic

**Abstract**

For a finite set $A \subset \mathbb{R}^2$, a map $\varphi : A \to \mathbb{R}^2$ is *orientation preserving* if for every non-collinear triple $u, v, w \in A$ the orientation of the triangle $u, v, w$ is the same as that of the triangle $\varphi(u), \varphi(v), \varphi(w)$. We prove that for every $n \in \mathbb{N}$ and for every $\varepsilon > 0$ there is $N = N(n, \varepsilon) \in \mathbb{N}$ such that the following holds. Assume that $\varphi : G(N) \to \mathbb{R}^2$ is an orientation preserving map where $G(N)$ is the grid $\{(i, j) \in \mathbb{Z}^2 : -N \le i, j \le N\}$. Then there is an affine transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ and $a \in \mathbb{Z}^2$ such that $a + G(n) \subset G(N)$ and $\|\psi \circ \varphi(z) - z\| < \varepsilon$ for every $z \in a + G(n)$. This result was previously proved in a completely different way by Nešetřil and Valtr, without obtaining any bound on $N$. Our proof gives $N(n, \varepsilon) = O(n^4 \varepsilon^{-2})$.

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## 1 Introduction

This paper is about orientation preserving maps of the $n \times n$ grid. We denote by $G(N)$ the grid $\{(i, j) \in \mathbb{Z}^2 : -N \le i, j \le N\}$ and by $G^*(n)$ the grid $\{(i, j) \in \mathbb{Z}^2 : 1 \le i, j \le n\}$. A map $\varphi : G(N) \to \mathbb{R}^2$ is *orientation preserving* if for every non-collinear triple $u, v, w \in G(N)$ the orientation of the triangle $u, v, w$ is the same as that of the triangle $\varphi(u), \varphi(v), \varphi(w)$, or with a formula

$$\operatorname{sign}\det \begin{bmatrix} u & v & w \\ 1 & 1 & 1 \end{bmatrix} = \operatorname{sign}\det \begin{bmatrix} \varphi(u) & \varphi(v) & \varphi(w) \\ 1 & 1 & 1 \end{bmatrix}.$$

We are going to show that given an orientation preserving map $\varphi : G(N) \to \mathbb{R}^2$ there is a $n \times n$ subgrid of $G(N)$ whose image under $\varphi$ is very close to an affine image of the $n \times n$ grid provided $N$ is large enough (polynomial in $n$ and $1/\varepsilon$). Precisely we have the following result.

▶ **Theorem 1.** *For every $n \in \mathbb{N}$ and for every $\varepsilon > 0$ there is $N = N(n, \varepsilon)$ such that if $\varphi : G(N) \to \mathbb{R}^2$ is an orientation preserving map, then there is an affine transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ and $a \in \mathbb{Z}^2$ such that $a + G^*(n) \subset G(N)$ and for every $z \in a + G^*(n)$*

$$\|\psi \circ \varphi(z) - z\| < \varepsilon.$$

*Here $N(n, \varepsilon) = O(n^4 \varepsilon^{-2})$.*


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Theorem 1 without the explicit bound $N(n,\varepsilon) = O(n^4\varepsilon^{-2})$ was already proved by Nešetřil and Valtr [6, Lemma 10] as the key tool for proving several Ramsey-type results (see also the paper [5] for related results). However, the proof in the paper [6] relied on repeated compactness arguments, thus it gave no upper bound on $N$. Our bound $N(n,\varepsilon) = O(n^4\varepsilon^{-2})$ makes ground for giving explicit bounds for Ramsey-type results given in the paper [6]; see concluding remark (1) on page 105 of the paper [6] where the lack of an explicit bound is discussed. From the (discrete and) computational geometry point of view, the most interesting consequences of our bound $N(n,\varepsilon) = O(n^4\varepsilon^{-2})$ in Theorem 1 might be those which are connected with the study of order types, as described in the next section.

We remark that the function $N(n,\varepsilon)$ in Theorem 1 satisfies the lower bound $N(n,\varepsilon) = \Omega(n^2\varepsilon^{-1})$. The example showing this is a projective map that carries the line containing one edge of the square $[-N,N]^2$ to the line at infinity.

## 2 Connections to order types and motivation

An *order type* of size $n$ is an equivalence class of all $n$-point sets which can be mapped into each other by strongly order preserving maps, where a map $\varphi : A \to \mathbb{R}^2$ from a finite planar point set $A$ to $\mathbb{R}^2$ is *strongly orientation preserving* if it is orientation preserving and, additionally, it maps collinear triples of $A$ to collinear triples. If the sets of an order type are in general position then we say that the order type is *in general position*. Order types have been studied from various perspectives, for example, see the paper of Goodman and Pollack [1] for a classical result and the recent paper of Pilz and Welzl [4] for further references.

The *span* of a finite point set $A \subset \mathbb{R}^2$ is the ratio between the maximum distance in $A$ and the minimum distance in $A$. Note that due to projective transformations the supremum of the spans of the sets of any fixed order type (of size at least three) is $\infty$. We define the *span* of an order type $T$ as the infimum of the spans of the point sets in $T$. By famous results of Goodman, Pollack and Sturmfels [2] and of Kratochvíl and Matoušek [3], there are order types of size $n$ with double exponential span.

▶ **Theorem 2.** *For $n > 1$, let $f(n)$ be the smallest real number such that, for any order type $T$ of size $n$ in general position and for any $\delta > 0$, there exists a set $A$ in $T$ having the span smaller than $f(n) + \delta$. Then there are two positive constants $c_1$ and $c_2$ such that, for any integer $n > 3$,*

$$2^{2^{c_1 n}} \le f(n) \le 2^{2^{c_2 n}}.$$

Our Theorem 1 considers subsets of sets of some order type with a small span. In particular, an immediate consequence of Theorem 1 says that some order types have the property that any set of this order type contains a rather large subset whose order type has a very small span (asymptotically as small as possible for the given size).

▶ **Theorem 3.** *For any $N \ge 2$, there is an order type $T_N$ of size $N$ in general position such that any set $A$ of $T_N$ contains a subset $B$ of size $n = \Omega(N^{1/3})$ which is an affine transform of a set having span $O(\sqrt{n})$.*

We remark that due to a simple packing argument the span of any set (or order type) of size $n \ge 2$ is at least $\Omega(\sqrt{n})$.

Another (almost immediate) consequence of Theorem 1 says that there are order types $T$ of arbitrary size $n \ge 2$ in general position such that any set $A$ of order type $T$ contains a quite large subset of points which lie, one by one, in small neighborhoods of equidistantly distributed points along some line.

▶ **Theorem 4.** *For any $N \geq 2$ and any $\varepsilon > 0$, there is an order type $T_N$ of size $N$ in general position such that any set $A$ of $T_N$ contains a subset $B$ of size $n = \Omega(N^{1/4}\varepsilon^{1/2})$ such that for some line $\ell$ and for some $n$ equally distributed points $p_1, \dots, p_n$ on $\ell$ where the distance between $p_i$ and $p_{i+1}$ is exactly $d$ for some fixed $d > 0$ and for each $i = 1, \dots, n-1$, the following holds. There is exactly one point of $B$ in the $(\varepsilon d)$-neighborhood of $p_i$ for each $i = 1, \dots, n$.*

Since some of the ratios of distances among sufficiently many equidistantly distributed points on a line approximate (with any prescribed precision) $l$ prescribed distance ratios, Theorem 4 immediately implies the following result of Nešetřil and Valtr [6, Theorem 6].

▶ **Theorem 5** (Nešetřil and Valtr [6])**.** *For any positive integer $l > 0$ and for any $l+1$ positive real numbers $\varepsilon, r_1, r_2, \dots, r_l > 0$, there exists a (finite) order type $T$ in general position such that any set of order type $T$ determines $l+1$ distances $d_i, i = 0, 1, 2, \dots, l$, such that $|\frac{d_i}{d_0} - r_i| < \varepsilon$ $(i = 1, 2, \dots, l)$.*

## 3 Preparations and sketch of proof

We start with introducing basic notation and definitions. For distinct $u, v \in \mathbb{R}^2$, $L(u,v)$ denotes the line they span. The angle $\alpha(u,v)$ is defined as the angle the vector $v - u$ and the positive half of the $x$ axis make. It is understood mod $2\pi$. Assume $\varphi_0 : G^*(n) \to \mathbb{R}^2$ is an orientation preserving map on non-collinear triples, and it satisfies the conditions $\varphi_0(1,1) = (1,1), \varphi_0(n,1) = (n,1), \varphi_0(1,n) = (1,n)$. Suppose further that for all $u, v \in G^*(n)$ with $\alpha(u,v) \in \{0, \pi/4, \pi/2\}$

$$|\alpha(u,v) - \alpha(\varphi_0(u), \varphi_0(v))| < \gamma,$$

where $\gamma > 0$. In the last step of the proof of Theorem 1 we need the following lemma.

▶ **Lemma 6.** *Assume $\gamma = O(n^{-2})$. Then, under the above conditions for every $z \in G^*(n)$ we have*

$$\|\varphi_0(z) - z\| < 20\gamma n^2.$$

The proof is given in the last section.

Another important notion is that of a *block* of an $m \times m$ grid. The horizontal, resp. vertical blocks of $G^*(m)$ are the sets (where $i, j \in [m]$)

$$H_i = \{(1,i), (2,i), \dots, (m,i)\} \text{ and } V_j = \{(j,1), (j,2), \dots, (j,m)\},$$

we will call $(1,i)$ resp. $(m,i)$ the first and last point of the block $H_i$, and similarly $(j,1)$ and $(j,m)$ are the first and last points of $V_j$. Similarly the plus and minus diagonal blocks of $G^*(m)$ are

$$\begin{aligned} D_i^+ &= \{(x,y) \in G^*(m) : x - y = i\}, \\ D_j^- &= \{(x,y) \in G^*(m) : x + y = j\}, \end{aligned}$$

here $i = 0, \pm 1, \dots, \pm(m-1)$ and $j = 2, \dots, 2m$. Their first and last points are $(i+1, 1)$ and $(m, i-m)$ for $D_i^+$ and $(1, j-1)$ and $(j-1, 1)$ for $D_j^-$. Two blocks are neighbourly if they lie in consecutive parallel lattice lines.

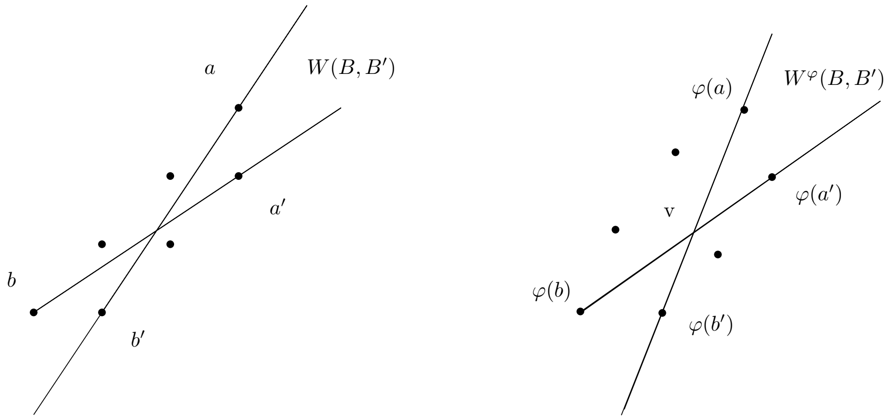


**Figure 1** Neighbourly blocks and $\varphi$ blocks separated.

Given an orientation preserving map $\varphi : G^*(m) \to \mathbb{R}^2$ the image $\varphi(B)$ of a block $B$ is called a $\varphi$ block. We need separation properties of blocks and $\varphi$ blocks. Let $B$ and $B'$ be two neighbourly blocks with first and last point $a, b$ resp. $a', b'$. Here $b - a$ and $b' - a'$ are parallel and point in the same direction, see Figure 1. It is clear that both $L(a, b')$ and $L(a', b)$ separate $B$ and $B'$. The orientation preserving properties of $\varphi$ imply that the lines

$$L_1 = L(\varphi(a), \varphi(b')) \text{ and } L_2 = L(\varphi(a'), \varphi(b))$$

also separate $\varphi(B)$ and $\varphi(B')$, or, what is the same, conv $\varphi(B)$ and conv $\varphi(B')$. The lines $L_1$ and $L_2$ define a double cone $W^\varphi(B, B')$ with apex $v = L_1 \cap L_2$ which is the double cone not containing $\varphi(B)$ and $\varphi(B')$. Similarly, let $W(B, B')$ be the double cone determined by $L(a, b')$ and $L(a', b)$, again the one not containing $B$ and $B'$. The following facts are well known.

▶ **Fact 1.** *If $u \in W^\varphi(B, B')$, then $L(u, v)$ separates $\varphi(B)$ and $\varphi(B')$.*

▶ **Fact 2.** *For any point $z \in W(B, B') \cap G^*(m)$ the line $L(\varphi(z), v)$ separates $\varphi(B)$ and $\varphi(B')$.*

We say that a point $z \in \mathbb{R}^2$ is a *separator* for the horizontal blocks $H_1, \dots, H_m$ if there are lines $L_1, \dots, L_{m-1}$, all passing through $z$ such that $L_i$ separates $H_i$ and $H_{i+1}$ for all $i$. Separator points for a set of vertical and diagonal blocks, and for $\varphi$-blocks, are defined analogously.

▶ **Fact 3.** *If $z \in G(N)$ is a separator for the horizontal (or vertical, diagonal) blocks of $G^*(m)$, then so is $\varphi(z)$ for the corresponding $\varphi$ blocks.*

Here is a quick sketch of the proof of Theorem 1. First we find a small subgrid, $G_1$, of $G(N)$. ($G_1$ lies in the upper halfplane and we ignore the part of $G(N)$ that is in the lower halfplane.) The points $z_h = (N, 0)$ resp. $z_- = (-N, N)$ are separators for the horizontal and minus diagonal blocks of $G_1$. The points $v^+, v^-$ and $w^+, w^-$ are separators for the vertical and plus diagonal blocks of $G_1$, see Figure 2. These four points are very close to the line $L(z_h, z_-)$. From the $\varphi$-image of these points we construct four collinear points that are separators for the corresponding $\varphi$ blocks of $G_1$. A projective transformation that carries the line containing these separators to infinity can be chosen so that the horizontal resp vertical $\varphi$ blocks of $G_1$ are separated by horizontal and vertical lines. This way we create a grid like structure. A small subgrid of $G_1$ can be found which satisfies the conditions of Lemma 6. The resulting map is only projective and not affine. This is to be fixed in the end.

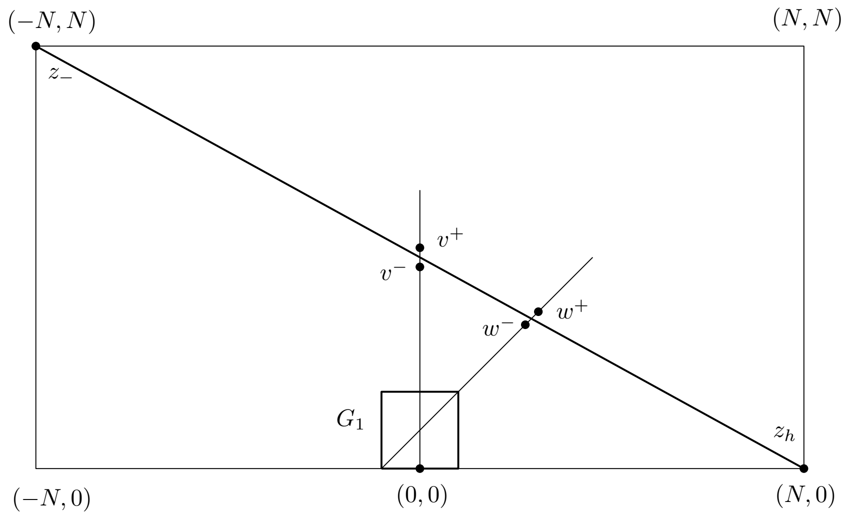


**Figure 2** $G(N)$ and $G_1$.

## 4 Finding a smaller grid and a projective transformation

We set $N = 6m^2$ and define $G_1 = \mathbb{Z}^2 \bigcap([-m,m] \times [0,2m]$, see Figure 2. The horizontal blocks of $G_1$ are $H_0, H_1, \dots, H_{2m}$ and any point in $\bigcap_0^{2m-1} W(H_i, H_{i+1})$ is a separator for these blocks. An elementary calculation shows that any point $(x,0)$ with $x > 4m^2 - m$ is a separator. In particular, $z_h = (6m^2, 0)$ is a separator. Similarly, any point $(0,y)$ with $y > 2m^2$ is a separator point for the vertical blocks of $G_1$. Another calculation shows that any point $(x,x)$ resp. $(-x,x)$ with $x > m^2$ is a separator point for the plus diagonal and the minus diagonal blocks of $G_1$. We fix $z_- = (-6m^2, 6m^2)$ as a separator point for the minus diagonal blocks.

By Fact 3, $\varphi(z_h)$ resp. $\varphi(z_-)$ is a separator point for the horizontal and minus diagonal $\varphi$ blocks of $G_1$. Moreover $v^+ = (3m^2 + 1, 0)$ and $v^- = (3m^2 - 1, 0)$ are both separators for the vertical blocks of $G_1$. Then so are $\varphi(v^+)$ and $\varphi(v^-)$ for the vertical $\varphi$ blocks. These points lie on opposite sides of the line $L^\varphi = L(\varphi(z_h), \varphi(z_-))$. Consequently the intersection point, $z_v$, of $L^\varphi$ and the segment $[\varphi(v^+), \varphi(v^-)]$ is a separator for the vertical $\varphi$ blocks. Completely analogously, we find a separator $z_+ \in L^\varphi$ for the plus diagonal $\varphi$ blocks of $G_1$. Namely, both $w^+ = (2m^2 + 1, 2m^2 + 1)$ and $w^- = (2m^2 - 1, 2m^2 - 1)$ are separators for the plus diagonal blocks of $G_1$, their $\varphi$-images lie on opposite sides of $L^\varphi$, so the intersection point, $z_+$, works again. Here is what we have established so far.

▶ **Lemma 7.** *The line $L^\varphi$ contains four points $\varphi(z_h), \varphi(z_-), z_v, z_+$ that are separators for the horizontal, minus diagonal, vertical and plus diagonal $\varphi$ blocks of $G_1$.*

Now apply a projective transformation $\psi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $L^\varphi$ to the line at infinity so that the horizontal resp. vertical separating lines of the corresponding $\varphi$ blocks are mapped to horizontal and vertical lines of the form

$$L(b_j)^h = \{(x,y) : y = b_j\} \text{ and } L(a_i)^v = \{(x,y) : x = a_i\},$$

here $i, j \in [2m]$ and $a_1 < a_2 < \ldots < a_{2m}$ and $b_1 < b_2 < \ldots < b_{2m}$. From this point onward we only work on points of the grid that are in the lower triangular half, so that there is no reason to worry that this projective transformation might modify orientations.

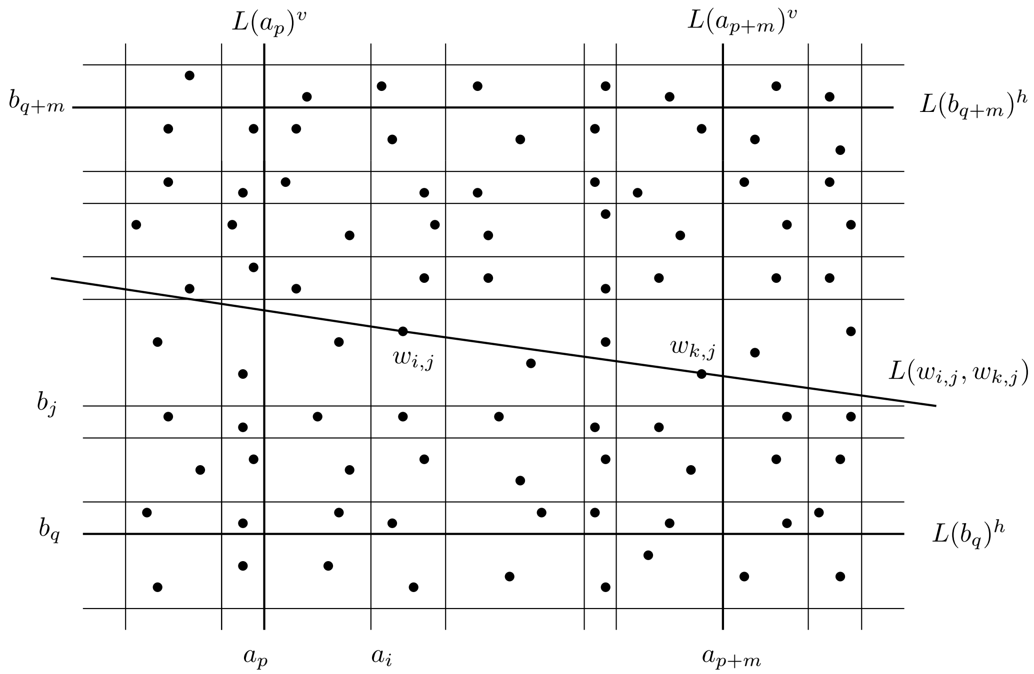


**Figure 3** The grid-like structure and the line $L(w_{i,j}, w_{k,j})$.

We still have some freedom to define $\psi_1$ more precisely. That will come a little later. Set $\varphi_1 = \psi_1 \circ \varphi$ and note that the plus and minus diagonal $\varphi_1$ blocks of $G_1$ are separated by parallel lines because the corresponding separator points are at infinity. Of course, the vertical resp. horizontal $\varphi_1$ blocks are separated by the vertical and horizontal lines $L(a_i)^v$ and $L(b_j)^h$.

Observe now that we have a grid-like structure (see Figure 3): the lines $L(a_i)^v$ and $L(b_j)^h$ determine $(2m-1)^2$ rectangular cells and each such cell contains the $\varphi_1$ image of a unique point from $G_1$. Precisely, the cell $C(i,j)$ is just the rectangle $[a_i, a_{i+1}] \times [b_j, b_{j+1}]$. It contains the point $w_{i,j}$ which is the $\varphi_1$ image of a unique point in $G_1$.

Suppose that $m$ is large, $m > 10^5$ say, and let $a_{p+5} - a_p$ resp. $b_{q+5} - b_q$ be the minimal among the numbers $a_8 - a_3, a_9 - a_4, \ldots, a_{2m-2} - a_{2m-7}$ and $b_8 - b_3, b_9 - b_4, \ldots, b_{2m-2} - b_{2m-7}$. Note that the cells in the first and last two rows and columns are not used, this "double frame" will be needed later. Here either $p < m$ or $p + 5 > m$. Similarly, either $q < m$ or $q + 5 > m$. We can assume by symmetry that $p, q < m$. We now fix $\psi_1$ (and so $\varphi_1$ as well) by requiring that $a_p = b_q = 0$ and $a_{p+m} = b_{q+m} = m$. It follows then that $0 < a_{p+5}, b_{q+5} \leq 5$.

We are going to show that, with $\varphi_1$ fixed this way, the angles of the plus diagonal separators are very close to $\pi/4$. A similar statement holds for the minus diagonal separators but we do not need that. We have the following lemma.

▶ **Lemma 8.** *If $m$ is large enough, then $0 < a_{p+k+1} - a_{p+k} < 11$ and $0 < b_{q+k+1} - b_{q+k} < 11$ for all $k = -1, 0, 1, \ldots, m-1, m$.*

**Proof.** Let $R$ be the rectangle $[a_p, a_{p+m}] \times [b_q, b_{q+m}]$. Define $G_2$ as the $m \times m$ subgrid of $G_1$ whose $\varphi_1$-image lies in $R$. Horizontal, vertical, plus and minus diagonal blocks of $G_2$ are defined the same way as those of $G_1$. Let $B_i$ be the plus diagonal $\varphi_1$ blocks of $G_2$ that contains the point $w_{p,q+i}$ for $i = 0, 1, 2, 3, 4$ and let $B_{-i}$ be the one containing $w_{p+i,q}$ for $i = 1, 2, 3, 4$.

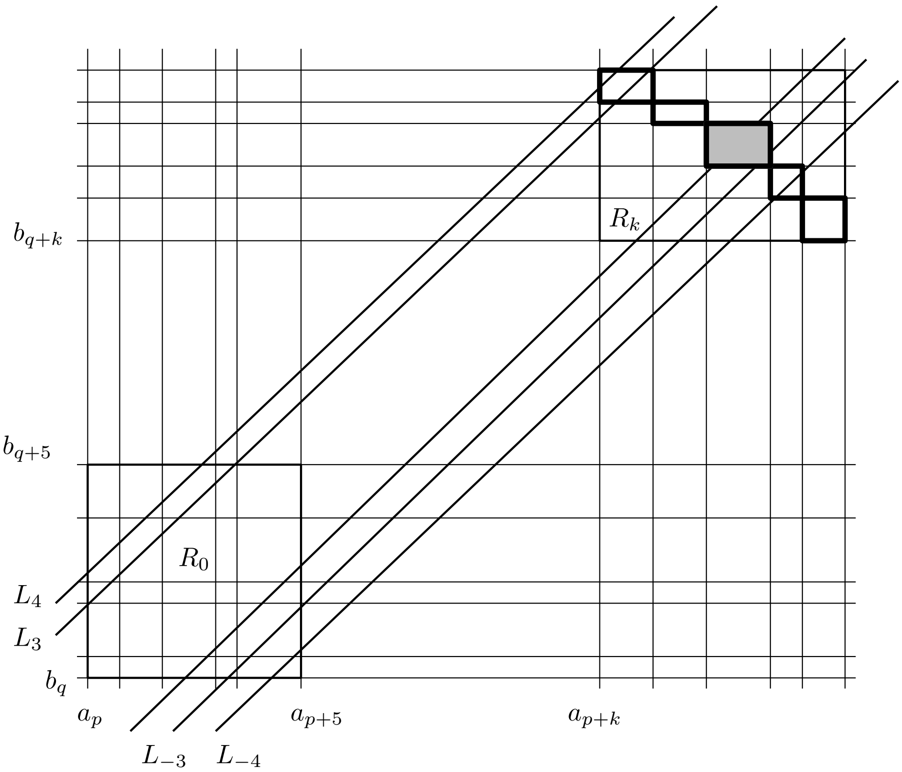


**Figure 4** Only some of the lines $L_i$ are shown, the central cell of $R_k$ is shaded.

These diagonals are separated by parallel lines $L_4, L_3, \dots, L_{-4}$ in this order. So for instance $L_1$ separates $B_1$ and $B_0$, see Figure 4. Note that each such line intersects the rectangle $R_0 = [a_p, a_{p+5}] \times [b_q, b_{q+5}]$ as otherwise $L_{-4}$ (say) would avoid it and then it cannot separate two points inside this rectangle. This implies that the distance between $L_4$ and $L_{-4}$ is less then the sum of two neighbouring sides of $R_0$, which is at most 10.

Note further that the lines $L_{-4}, \dots, L_4$ are parallel and their slope is a positive number. Consequently the angle $\beta$ these lines make with the positive half of the $x$ axis is strictly between 0 and $\pi/2$.

Consider the rectangle $R_k = [a_{p+k}, a_{p+k+5}] \times [b_{q+k}, b_{q+k+5}]$ where $k = -2, -1, 0, \dots, m-5$. (This is the point where we use the double frame.) It contains $5^2$ cells. We claim that its middle cell, $C(p+k+2, q+k+2)$, lies between the lines $L_4$ and $L_{-4}$, see Figure 4. Indeed, if it did not, then either the point $(a_{p+k+3}, b_{q+k+2})$ is below the line $L_{-4}$, or the point $(a_{p+k+2}, b_{q+k+3})$ is above the line $L_4$. In the former case the cells $C(p+k+3, q+k+1)$ and $C(p+k+4, q+k)$ also lie below the line $L_{-4}$. But then $L_{-4}$ cannot separate the points $w_{p+k+3,q+k+1} \in \varphi_1(B_{-3})$ and $w_{p+k+4,q+k} \in \varphi_1(B_{-4})$, yet $L_{-4}$ separates these two $\varphi_1$ blocks. A similar argument works when the point $(a_{p+k+2}, b_{q+k+3})$ is not below the line $L_4$.

The line $L_4$ intersects $L(a_p)^v$ below the point $(a_p, b_{q+5})$, and intersects $L(a_{p+m})^v$ above the point $(a_{p+m}, b_{q+m})$, so its slope is least $\frac{m-5}{m}$. Similar argument shows that the slope of the line $L_{-4}$ is at most $\frac{m}{m-5}$. As both slopes are equal to $\tan\beta$ we have

$$\frac{m-5}{m} \le \tan\beta \le \frac{m}{m-5}. \tag{1}$$

So for $m$ large, $\beta$ is very close to $\pi/4$ and the strip between $L_4$ and $L_{-4}$ (whose width is at most 10) intersects both axes in a segment of length shorter than 11. This and the fact that the central cell $C(p+k+2, q+k+2)$ lies between the lines $L_4$ and $L_{-4}$ finish the proof. ◀

The next target is to show that if $w^1, w^2 \in R$ belong to the same horizontal, vertical, or plus diagonal $\varphi_1$ block, then the angle of the line $L(w^1, w^2)$ is very close to $0, \pi/2, \pi/4$. We need some preparations.

Assume that $L$ and $L'$ are consecutive parallel separating lines between three plus diagonal $\varphi_1$ blocks of $G_2$ that have points in $R$. Then there are four cells $C(p+k, q+h)$, $C(p+k+1, q+h)$, $C(p+k, q+h+1)$, and $C(p+k+1, q+h+1)$ so that $L$ separates $w_{p+k,q+h+1}$ from $w_{p+k,q+h}$ and $w_{p+k+1,q+h+1}$, and $L'$ separates $w_{p+k+1,q+h}$ from $w_{p+k,q+h}$ and $w_{p+k+1,q+h+1}$. Then both lines $L$ and $L'$ have to intersect the rectangle $[a_{p+k}, a_{p+k+2}] \times [b_{q+h}, b_{h+h+2}]$. The sides of this rectangle have length at most 22.

▶ **Corollary 9.** *If $L$ and $L'$ are consecutive parallel separating lines between three (plus or minus) diagonals $\varphi_1$ blocks of $G_2$, then the strip between them intersects both axes in a segment of length at most 44.*

We show next that if $w^1$ and $w^2$ belong to the same horizontal (or vertical) $\varphi_1$ block, then their line $L(w^1, w^2)$ is almost horizontal (vertical). This is quite easy now. Recall the notation $\alpha(w^1, w^2)$ for the angle of the line $L(w^1, w^2)$.

▶ **Lemma 10.** *Assume $p \le i < k \le p+m$ and $q \le j \le q+m$. Then $|\tan \alpha(w_{i,j}, w_{k,j})| < \frac{33}{m}$. Similarly $p \le i \le p+m$ and $q \le j < k \le q+m$ imply that $|\cot \alpha(w_{i,j}, w_{i,k})| < \frac{33}{m}$.*

**Proof for the horizontal case.** The line $L(w_{i,j}, w_{k,j})$ (see Figure 3) intersects the line $L(a_p)^v$ on the interval $[(a_p, b_{j-1}), (a_p, b_{j+2})]$, as otherwise the cell $C(p-1, j-1)$ or $C(p-1, j+1)$ from the double frame would be on the wrong side of $L(w_{i,j}, w_{k,j})$, contradicting the orientation preserving property of $\varphi_1$. Same way, the line $L(w_{i,j}, w_{k,j})$ intersects $L(a_{p+m})^v$ on the interval $[(a_{p+m}, b_{j-1}), (a_{p+m}, b_{j+2})]$. The length of both intervals is at most 33 by Lemma 8. Same proof applies in the vertical case. ◀

We want to show the analogue of Lemma 10 for plus diagonal $\varphi_1$ blocks. For that purpose we have to consider a smaller subgrid of $G_2$. Namely, let $R'$ be the rectangle $[a_{p'}, a_{p'+m'}] \times [b_{q'}, b_{q'+m'}]$ anywhere near the middle of $R$ with $m'$ much shorter than $m$, $m' < \frac{m}{110}$, say. To make anywhere near more concrete choose the position of $R'$ so that centre of $R$ lies in $R'$.

▶ **Lemma 11.** *Assume $w^1, w^2 \in R'$ belong to the same plus diagonal $\varphi_1$ block of $G_2$, and $\alpha(w^1, w^2)$ differs from $\pi/4$ by $\delta$. Then $|\tan \delta| < \frac{K}{m}$ where $K$ is a constant, for instance $K = 400$ will do.*

**Proof.** Let $B_0$ be the plus diagonal $\varphi_1$ block containing $w^1, w^2$, and let $B_{-2}, B_{-1}, B_0, B_1, B_2$ the neighbouring diagonal $\varphi_1$ blocks, with parallel separating lines $L_{-2}, L_{-1}, L_1, L_2$. The strip between the lines $L_{-2}$ and $L_2$ intersects the lines $L(a_{p+m/10})^v$ and $L(a_{p+9m/10})^v$ in two segments that lie in the rectangle $R$, and have length at most $3 \cdot 44$ with 44 coming from Corollary 9. Same way as in the proof of Lemma 10, the line $L(w^1, w^2)$ has to intersect these two segments. A straightforward computation, using this fact and (1), finishes the proof. ◀

▶ Remark. In this proof we use the line $L(a_{p+m/10})^v$ (instead of $L(a_p)^v$) because its intersection with the strip between $L_{-2}$ and $L_2$ should lie inside $R$. The same reason explains the line $L(a_{p+9m/10})^v$.

## 5 Finding an even smaller subgrid

We set $m = Cn^2\varepsilon^{-1}$ where $C > 0$ will be specified later. Let $G_3$ be the subgrid of $G_2$, a translate of the set of grid points in $[0, n]^2$ such that $\varphi_1$ maps the bottom left corner of $G_3$ to the point $(a_{p'}, b_{q'})$. Note that $n < m'$, in fact much smaller. The set $\varphi_1(G_3)$ is contained in the rectangle $R^* = [a_{p'}, a_{p'+n}] \times [b_{q'}, b_{q'+n}]$ whose sides are shorter than $11n$.

We define an affine map $\psi_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by requiring $\psi_2(w_{p',q'}) = (0, 0), \psi_2(w_{p'+n,q'}) = (n, 0)$ and $\psi_2(w_{p',q'+n}) = (0, n)$. Then $\varphi_2 = \psi_2 \circ \varphi_1$ is well-defined on $G_3$. The map $\psi_2$ hardly changes any direction. More precisely, Lemmas 10 and 11 imply the following.

▶ **Fact 4.** *If $z^1, z^2$ belong to the same horizontal, plus diagonal, vertical block of $G_3$, then $\alpha(\varphi_2(z^1), \varphi_2(z^2))$ deviates from $0, \pi/4, \pi/2$ by at most $2\delta$ where $|\tan\delta| < K/m$.*

The conditions of Lemma 6 are satisfied with $\gamma = 2 \arctan K/m$. Thus its conclusion holds: for every $z \in G_3$

$$\|\varphi_2(z) - z\| < 20\gamma n^2 \leq 40n^2 \arctan \frac{2K\varepsilon}{Cn^2} < \frac{80K}{C}\varepsilon.$$

We are almost finished, except that $\psi_3 = \psi_2 \circ \psi_1$ is not an affine but a projective transformation. It is of the form

$$\psi_3(x) = \frac{Ax}{\ell(x)}$$

where $A$ is an orientation preserving affine map, and $\ell(x)$ is the equation of the line $L^\varphi$, normalized so that $\ell(x)$ is the signed distance of $x$ from the line $L^\varphi$. This line goes to infinity under $\psi_1$ and is disjoint from $R$ and then is far from $R^*$; let $d$ denote their distance. As $n, m' < \frac{m}{110}$, the side length of $R^*$ is at most $11m' < \frac{m}{10}$. Since $R^*$ is in the middle of $R$, this implies that $d > \frac{4m}{10}$. The diameter of $R^*$ is at most $11n\sqrt{2}$, very small compared to $m$. Then for every $x \in R^*$, $d \leq \ell(x) \leq d + 9n\sqrt{2}$. Consequently, using $m = Cn^2\varepsilon^{-1}$

$$1 \leq \frac{\ell(x)}{d} \leq 1 + \frac{11n\sqrt{2}}{.4Cn^2\varepsilon^{-1}} < 1 + \frac{40\varepsilon}{Cn}.$$

The map $\psi(x) = Ax/d$ is affine and satisfies

$$\|\psi(z) - z\| \leq \|\psi(z) - \psi_3(z)\| + \|\psi_3(z) - z\|.$$

Here $Az/\ell(z)$ is inside the square $[0, n]^2$ or very close to it, so its norm is at most $2n$. Then

$$\|\psi(z) - \psi_3(z)\| = \left\| \frac{Az}{\ell(z)} \right\| \frac{\ell(z) - d}{d} \leq 2n\frac{40\varepsilon}{Cn} = \frac{80\varepsilon}{C}.$$

Thus

$$\|\psi(z) - z\| < \frac{80\varepsilon}{C} + \frac{80K\varepsilon}{C} < \varepsilon,$$

when $C$ is chosen larger than $80K + 80$. ◀

## 6 Proof of Lemma 6

**Proof.** Consider the quadrilateral $Q = \text{conv}\,\{A, B, C, D\}$ as in Figure 5. Assume that

$$\begin{aligned}|\alpha(A,B)|, |\alpha(D,C)| < \gamma, |\alpha(A,C) - \pi/4| &< \gamma \\ |\alpha(A,D) - \pi/2|, |\alpha(B,C) - \pi/2| &< \gamma.\end{aligned}$$

The sine theorem shows that, with the notation of Figure 5,

$$\frac{d}{\sqrt{2}}(1 - \tan 2\gamma) < a, a', b, b' < \frac{d}{\sqrt{2}}(1 + \tan 2\gamma).$$

Setting $M = \frac{1+\tan 2\gamma}{1-\tan 2\gamma}$ it follows that

$$M^{-1} < \frac{a}{a'}, \frac{b}{a}, \frac{b'}{a}, \frac{a}{b'} < M$$

We are going to use these inequalities in the quadrilaterals whose vertices are $\varphi_0(i,j), \varphi_0(i+1,j), \varphi_0(i,j+1), \varphi_0(i+1,j+1)$. We define $a_{i,j} = \varphi_0(i+1,j) - \varphi_0(i,j)$ and $b_{i,j} = \varphi_0(i,j+1) - \varphi_0(i,j)$. We write $a^x, a^y$ for the $x$ and $y$ components of the vector $a \in M^2$.

The above inequalities show that in the triangle with sides $a_{i-1,1}$ and $b_{i,1}$ (see Figure 5), and in the triangle with sides $b_{i,1}$ and $a_{i,1}$

$$M^{-1} < \frac{\|b_{i,1}\|}{\|a_{i-1,1}\|} < M \text{ and } M^{-1} < \frac{\|a_{i,1}\|}{\|b_{i,1}\|} < M.$$

Consequently

$$M^{-2} < \frac{\|a_{i,1}\|}{\|a_{i-1,1}\|} < M^2 \text{ and so } \max \|a_{i,1}\| \le \min \|a_{i,1}\| M^{2(n-1)}.$$

As $a^x_{i,1} > 0$ follows from the conditions, and $a^x_{i,1} \ge \|a_{i,1}\| \cos\gamma$, we have

$$\frac{\max a^x_{i,1}}{\min a^x_{i,1}} \le \frac{\|\max a_{i,j}\|}{\|\min a_{i,j}\| \cos\gamma} < \frac{M^{2(n-1)}}{\cos\gamma} =: 1 + \Delta.$$

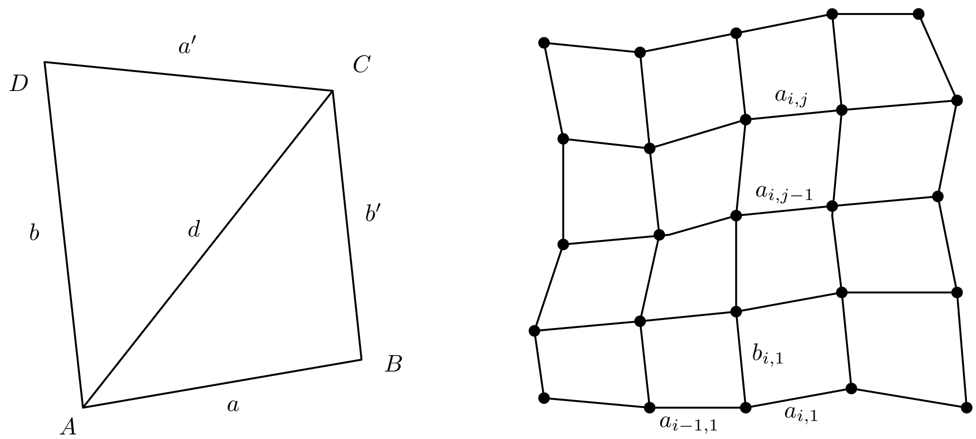


**Figure 5** The quadrilateral $Q$ and a piece of the $\varphi_0$-grid.

The average of the $a^x_{i,1}$ for $i \in [n-1]$ is 1 because $\sum_1^{n-1} a^x_{i,1} = n-1$, so $\min a^x_{i,1} \le 1 \le \max a^x_{i,x}$ implying that $\max a^x_{i,x} < (1+\Delta)\min a^x_{i,1} \le 1+\Delta$, and $\min a^x_{i,1} > \max a^x_{i,1}/(1+\Delta) \ge 1-\Delta$. Consequently

$$|a^x_{i,1} - 1| \le \Delta \text{ for all } i \in [n-1].$$

We need to estimate $\Delta$:

$$\begin{aligned}\Delta &= \frac{M^{2(n-1)}}{\cos\gamma} - 1 = \left(\frac{1+\tan 2\gamma}{1-\tan 2\gamma}\right)^{2n} \frac{1}{\cos\gamma} - 1 \\ &\le \left(1 + \frac{2\tan 2\gamma}{1-\tan 2\gamma}\right)^{2(n-1)} (1-\gamma^2)^{-1/2} - 1 \\ &\le \exp\left\{2(n-1)\frac{2\tan 2\gamma}{1-\tan 2\gamma}\right\}(1+\gamma) - 1 \le 10n\gamma.\end{aligned}$$

The last inequality follows from $e^t \le 1 + 1.1t$ which is true if $t > 0$ is small enough. This is the case as $t = 2(n-1)\frac{2\tan 2\gamma}{1-\tan 2\gamma} \approx 8n\gamma$. Consequently

$$|a^x_{i,1} - 1| \le 10n\gamma \text{ and similarly } |b^y_{1,j} - 1| \le 10n\gamma. \tag{2}$$

In the quadrilateral with sides $a_{i-1,j}$ and $a_{i,j}$ (see Figure 5 again) we have, the same way as in $Q$ above, that

$$\frac{\|a_{i,j}\|}{\|a_{i,j-1}\|} < M \text{ and so } \|a_{i,j}\| \le M^{n-1}\|a_{i,1}\|.$$

Since $M^{n-1}$ is only slightly larger than 1 and $|a^y_{i,j}| \le a^x_{i,j}\sin\gamma$ we have

$$|a^y_{i,1}| \le 2\gamma \text{ and similarly } |b^y_{1,j}| \le 2\gamma. \tag{3}$$

Finally we estimate the difference $\varphi_0(i,j) - (i,j)$. The absolute value of the $x$ component of this vector is

$$\begin{aligned} &= |1 + a^x_{1,1} + \ldots + a^x_{i-1,1} + b^x_{i,1} + \ldots b^x_{i,j-1} - i| \\ &\le |a^x_{1,1} - 1| + \ldots + |a^x_{i-1,1} - 1| + |b^x_{i,1}| + \ldots + |b^x_{i,j-1}| \\ &\le (i-1)10n\gamma + (j-1)2\gamma \le (n-1)(10n\gamma + 2\gamma) < 10n^2\gamma,\end{aligned}$$

where we used (2) and (3). Estimating the $y$ component of the vector $\varphi_0(i,j) - (i,j)$ is similar but starts with writing this vector as

$$b_{1,1} + \ldots + b_{1,j-1} + a_{j,1} + \ldots a_{j,i-1}.$$ ◀

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