Optimal Bounds for the Colorful Fractional Helly Theorem

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Abstract

The well known fractional Helly theorem and colorful Helly theorem can be merged into the so called colorful fractional Helly theorem. It states: for every $\alpha \in (0, 1]$ and every non-negative integer d, there is $\beta_{\text{col}} = \beta_{\text{col}}(\alpha, d) \in (0, 1]$ with the following property. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$, respectively. If at least $\alpha n_1 n_2 \cdots n_{d+1}$ of the colorful $(d + 1)$-tuples have a nonempty intersection, then there is $i \in [d + 1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $\beta_{\text{col}} n_i$ with a nonempty intersection. (A colorful $(d + 1)$-tuple is a $(d + 1)$-tuple $(F_1, \ldots, F_{d+1})$ such that F_i belongs to $\mathcal{F}_i$ for every i.)

The colorful fractional Helly theorem was first stated and proved by Bárány, Fodor, Montejano, Oliveros, and Pór in 2014 with $\beta_{\text{col}} = \alpha/(d + 1)$. In 2017 Kim proved the theorem with better function β_{col}, which in particular tends to 1 when α tends to 1. Kim also conjectured what is the optimal bound for $\beta_{\text{col}}(\alpha, d)$ and provided the upper bound example for the optimal bound. The conjectured bound coincides with the optimal bounds for the (non-colorful) fractional Helly theorem proved independently by Eckhoff and Kalai around 1984.

We verify Kim's conjecture by extending Kalai's approach to the colorful scenario. Moreover, we obtain optimal bounds also in a more general setting when we allow several sets of the same color.

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1 Introduction

The target of this paper is to provide optimal bounds for the colorful fractional Helly theorem first stated by Bárány, Fodor, Montejano, Oliveros, and Pór [5], and then improved by Kim [13]. In order to explain the colorful fractional Helly theorem, let us briefly survey the preceding results.

The starting point, as usual in this context, is the Helly theorem:

▶ **Theorem 1** (Helly's theorem [8]). *Let $\mathcal{F}$ be a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$. Assume that every subfamily of $\mathcal{F}$ with exactly $d+1$ members has a nonempty intersection. Then all sets in $\mathcal{F}$ have a nonempty intersection.*

Helly's theorem admits numerous extensions and two of them, important in our context, are the fractional Helly theorem and the colorful Helly theorem. The fractional Helly theorem of Katchalski and Liu covers the case when only some fraction of the $d+1$ tuples in $\mathcal{F}$ has a nonempty intersection.

▶ **Theorem 2** (The fractional Helly theorem [12]). *For every $\alpha \in (0,1]$ and every non-negative integer d, there is $\beta = \beta(\alpha, d) \in (0,1]$ with the following property. Let $\mathcal{F}$ be a finite family of $n \geq d+1$ convex sets in $\mathbb{R}^d$ such that at least $\alpha\binom{n}{d+1}$ of the subfamilies of $\mathcal{F}$ with exactly $d+1$ members have a nonempty intersection. Then there is a subfamily of $\mathcal{F}$ with at least βn members with a nonempty intersection.*

An interesting aspect of the fractional Helly theorem is not only to show the existence of $\beta(\alpha, d)$ but also to provide the largest value of $\beta(\alpha, d)$ with which the theorem is valid. This has been resolved independently by Eckhoff [7] and by Kalai [10] showing that the fractional Helly theorem holds with $\beta(\alpha, d) = 1 - (1-\alpha)^{1/(d+1)}$; yet another simplified proof of this fact has been subsequently given by Alon and Kalai [2]. It is well known that this bound is sharp by considering a family $\mathcal{F}$ consisting of $\approx (1 - (1-\alpha)^{1/(d+1)})n$ copies of $\mathbb{R}^d$ and $\approx (1-\alpha)^{1/(d+1)}n$ hyperplanes in general position; see, e.g., the introduction of [10].

The colorful Helly theorem of Lovász covers the case where the sets are colored by $d+1$ colors and only the "colorful" $(d+1)$-tuples of sets in $\mathcal{F}$ are considered. Given families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ of sets in $\mathbb{R}^d$ a family of sets $\{F_1, \ldots, F_{d+1}\}$ is a *colorful* $(d+1)$*-tuple* if $F_i \in \mathcal{F}_i$ for $i \in [d+1]$, where $[n] := \{1, \ldots, n\}$ for a non-negative integer $n \geq 1$. (The reader may think of $\mathcal{F}$ from preceding theorems decomposed into color classes $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$.)

▶ **Theorem 3** (The colorful Helly theorem [14, 4]). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$. Assume that every colorful $(d+1)$-tuple has a nonempty intersection. Then one of the families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ has a nonempty intersection.*

Both the colorful Helly theorem and the fractional Helly theorem with optimal bounds imply the Helly theorem. The colorful one by setting $\mathcal{F}_1 = \cdots = \mathcal{F}_{d+1} = \mathcal{F}$ and the fractional one by setting $\alpha = 1$ giving $\beta(1, d) = 1$.

The preceding two theorems can be merged into the following colorful fractional Helly theorem:

▶ **Theorem 4** (The colorful fractional Helly theorem [5]). *For every $\alpha \in (0,1]$ and every non-negative integer d, there is $\beta_{\mathrm{col}} = \beta_{\mathrm{col}}(\alpha, d) \in (0,1]$ with the following property. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$, respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d+1)$-tuples have a nonempty intersection, then there is an $i \in [d+1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $\beta_{\mathrm{col}} n_i$ with a nonempty intersection.*

Bárány et al. proved the colorful fractional Helly theorem with the value $\beta_{\mathrm{col}}(\alpha, d) = \frac{\alpha}{d+1}$ and they used it as a lemma [5, Lemma 3] in a proof of a colorful variant of a (p,q)-theorem. Despite this, the optimal bound for β_{col} seems to be of independent interest. In particular, the bound on β_{col} has been subsequently improved by Kim [13] who showed that the colorful fractional Helly theorem is true with $\beta_{\mathrm{col}}(\alpha, d) = \max\{\frac{\alpha}{d+1}, 1 - (d+1)(1-\alpha)^{1/(d+1)}\}$. On the other hand, the value of $\beta_{\mathrm{col}}(\alpha, d)$ cannot go beyond $1 - (1-\alpha)^{1/(d+1)}$ because essentially the same example as for the standard fractional Helly theorem applies in this setting as well – it is sufficient to set $n_1 = n_2 = \cdots = n_{d+1}$ and take $\approx (1-(1-\alpha)^{1/(d+1)})n_i$ copies of $\mathbb{R}^d$ and $\approx (1-\alpha)^{1/(d+1)}n_i$ hyperplanes in general position in each color class.[1] (Kim [13] provides a slightly different upper bound example showing the same bound.)

Coming back to the lower bound on $\beta_{\mathrm{col}}(\alpha, d)$, Kim explicitly conjectured that $1-(1-\alpha)^{1/(d+1)}$ is also a lower bound, thereby an optimal bound for the colorful fractional Helly theorem. He also provides a more refined conjecture, that we discuss slightly later (see Conjecture 8), which implies this lower bound. We prove the refined conjecture, and therefore the optimal bounds for the colorful fractional Helly theorem.

▶ **Theorem 5** (The optimal colorful fractional Helly theorem). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$, respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d+1)$-tuples have a nonempty intersection, for $\alpha \in (0,1]$, then there is an $i \in [d+1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $(1-(1-\alpha)^{1/(d+1)})n_i$ with a nonempty intersection.*

In the proof we follow the exterior algebra approach which has been used by Kalai [10] in order to provide optimal bounds for the standard fractional Helly theorem. We have to upgrade Kalai's proof to the colorful setting. This requires guessing the right generalization of several steps in Kalai's proof (in particular guessing the statement of Theorem 10 below). However, we honestly admit that after making these "guesses" we follow Kalai's proof quite straightforwardly.

Let us also compare one aspect of our proof with the previous proof of the weaker bound by Kim [13]: Kim's proof uses the colorful Helly theorem as a blackbox while our proof includes the proof of the colorful Helly theorem.

Last but not least, the exterior algebra approach actually allows to generalize Theorem 5 in several different directions. The extension to so called d-collapsible complexes is essentially mandatory for the well working proof while the other generalizations that we will present just follow from the method. We will discuss this in detail in forthcoming subsections of the introduction.

1.1 d-representable and d-collapsible complexes

The nerve and d-representable complexes. The important information in Theorems 1, 2, 3, 4, and 5 is which subfamilies have a nonempty intersection. This information can be stored in a simplicial complex called the nerve.

A *(finite abstract) simplicial complex* is a set system K on a finite *set of vertices* N such that whenever $A \in \mathsf{K}$ and $B \subseteq A$, then $B \in \mathsf{K}$. (The standard notation for the vertex set would be V but this notation will be more useful later on when we will often use capital letters such as R for some set and the corresponding lower case letters such as r for its size.) The elements of K are *faces* (a.k.a. *simplices*) of K. The *dimension* of a face $A \in \mathsf{K}$ is defined as $\dim A = |A| - 1$; this corresponds to representing A as an $(|A|-1)$-dimensional simplex. The dimension of K, denoted $\dim \mathsf{K}$, is the maximum of the dimensions of faces in K. A face of dimension k is a *k-face* in short. *Vertices* of K are usually identified with 0-faces, that is, $v \in N$ is identified with $\{v\} \in \mathsf{K}$. (Though the definition of simplicial complex allows that $\{v\} \notin \mathsf{K}$ for $v \in N$, in our applications we will always have $\{v\} \in \mathsf{K}$ for $v \in N$.) Given a

[1] At the end of Section 3 we discuss this example in full detail in more general context. However, in this special case, it is perhaps much easier to check directly that β_{col} cannot be improved due to this example.

family of sets $\mathcal{F}$, the *nerve* of $\mathcal{F}$ is the simplicial complex whose vertex set is $\mathcal{F}$ and whose faces are subfamilies with a nonempty intersection. A simplicial complex is *d-representable* if it is the nerve of a finite family of convex sets in $\mathbb{R}^d$.

As a preparation for the d-collapsible setting, we now restate Theorem 5 in terms of d-representable complexes. For this we need two more notions. Given a simplicial complex K and a subset U of the vertex set N, the *induced subcomplex* $\mathsf{K}[U]$ is defined as $\mathsf{K}[U] := \{A \in \mathsf{K}\colon A \subseteq U\}$. Now, let us assume that the vertex set N is split into $d+1$ pairwise disjoint subsets $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ (we can think of this partition as coloring each vertex of N with one of the $d+1$ possible colors).[2] Then a *colorful d-face* is a d-face A, such that $|A \cap N_i| = 1$ for every $i \in [d+1]$.

▶ **Theorem 6** (Theorem 5 reformulated)**.** *Let K be a d-representable simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint subsets. Let $n_i := |N_i|$ for $i \in [d+1]$ and assume that K contains at least $\alpha n_1 \cdots n_{d+1}$ colorful d-faces for some $\alpha \in (0,1]$. Then there is an $i \in [d+1]$ such that $\dim \mathsf{K}[N_i] \geq (1-(1-\alpha)^{1/(d+1)})n_i - 1$.*

Theorem 6 is indeed just a reformulation of Theorem 5: Considering $\mathcal{F}$ as disjoint union[3] $\mathcal{F} = \mathcal{F}_1 \sqcup \cdots \sqcup \mathcal{F}_{d+1}$, then K corresponds to the nerve of $\mathcal{F}$, colorful d-faces correspond to colorful $(d+1)$-tuples with nonempty intersection and the dimension of $\mathsf{K}[N_i]$ corresponds to the size of largest subfamily of $\mathcal{F}_i$ with nonempty intersection minus 1. (The shift by minus 1 between size of a face and dimension of a face is a bit unpleasant; however, we want to follow the standard terminology.)

***d*-collapsible complexes.** In [17] Wegner introduced an important class of simplicial complexes, so called d-collapsible complexes. They include all d-representable complexes, which is the main result of [17], while they admit quite simple combinatorial description which is useful for induction.

Given a simplicial complex K, we say that a simplicial complex K' arises from K by an *elementary d-collapse*, if there are faces $L, M \in \mathsf{K}$ with the following properties: (i) $\dim L \leq d-1$; (ii) M is the unique inclusion-wise maximal face which contains L; and (iii) $\mathsf{K}' = \mathsf{K} \setminus \{A \in \mathsf{K}\colon L \subseteq A\}$. A simplicial complex K is *d-collapsible* if there is a sequence of simplicial complexes $\mathsf{K}_0, \dots, \mathsf{K}_\ell$ such that $\mathsf{K} = \mathsf{K}_0$; K_i arises from K_{i-1} by an elementary d-collapse for $i \in [\ell]$; and K_ℓ is the empty complex.

We will prove the following generalization of Theorem 6 (equivalently of Theorem 5).

▶ **Theorem 7** (The optimal colorful fractional Helly theorem for d-collapsible complexes)**.** *Let K be a d-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint subsets. Let $n_i := |N_i|$ for $i \in [d+1]$ and assume that K contains at least $\alpha n_1 \cdots n_{d+1}$ colorful d-faces for some $\alpha \in (0,1]$. Then there is an $i \in [d+1]$ such that $\dim \mathsf{K}[N_i] \geq (1-(1-\alpha)^{1/(d+1)})n_i - 1$.*

1.2 Kim's refined conjecture and further generalization

As a tool for a possible proof of Theorem 5, Kim [13, Conjecture 4.2] suggested the following conjecture. (The notation k_i in Kim's statement of the conjecture is our $r_i + 1$.)

[2] We use the notation $\sqcup$ to emphasize the disjoint union.

[3] If there are any repetitions of sets in $\mathcal{F}$, which we generally allow for families of sets, then each repetition creates a new vertex in the nerve.

▶ **Conjecture 8** ([13]). *Let n_i be positive and r_i non-negative integers for $i \in [d+1]$ with $n_i \geq r_i + 1$. Let $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ be families of convex sets in $\mathbb{R}^d$ such that $|\mathcal{F}_i| = n_i$ and there is no subfamily of $\mathcal{F}_i$ of size $r_i + 1$ with non-empty intersection for every $i \in [d+1]$. Then the number of colorful $(d+1)$-tuples with nonempty intersection is at most*

$$n_1 \cdots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}).$$

We explicitly prove this conjecture in a slightly more general setting for d-collapsible complexes. (Note that the condition "no subfamily of size $r_i + 1$" translates as "no r_i-face", that is, "the dimension is at most $r_i - 1$".)

▶ **Proposition 9.** *Let n_i be positive and r_i non-negative integers for $i \in [d+1]$ with $n_i \geq r_i + 1$. Let K be a d-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint subsets such that $|N_i| = n_i$. Assume that $\dim \mathsf{K}[N_i] \leq r_i - 1$ for every $i \in [d+1]$. Then K contains at most*

$$n_1 \cdots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1})$$

colorful d-faces.

Our main technical result. Now, let us present our main technical tool for a proof of Proposition 9 and consequently for a proof of Theorem 7 as well.

We denote by $\mathbb{N}$ the set of positive integers whereas $\mathbb{N}_0$ is the set of non-negative integers. Let us consider $c \in \mathbb{N}$ and vectors $\mathbf{k} = (k_1, \dots, k_c), \mathbf{r} = (r_1, \dots, r_c) \in \mathbb{N}_0^c$ and $\mathbf{n} = (n_1, \dots, n_c) \in \mathbb{N}^c$ such that $\mathbf{k}, \mathbf{r} \leq \mathbf{n}$. (Here the notation $\mathbf{a} \leq \mathbf{b}$ means that $\mathbf{a}$ is less or equal to $\mathbf{b}$ in every coordinate.) We will also use the notation $k := k_1 + \cdots + k_c$, $n := n_1 + \cdots + n_c$, and $r := r_1 + \cdots + r_c$. Let N be a set with n elements partitioned as $N = N_1 \sqcup \cdots \sqcup N_c$ where $|N_i| = n_i$ for $i \in [c]$. By $\binom{N}{\mathbf{k}}$ we denote the set of all subsets A of N such that $|A \cap N_i| = k_i$ for every $i \in [c]$. Note that $\binom{N}{\mathbf{k}} \subseteq \binom{N}{k}$ where $\binom{N}{k}$ denotes the set of all subsets of N of size k.

Let K be a simplicial complex with the vertex set N as above. We say that a face A of K is $\mathbf{k}$-colorful if $A \in \binom{N}{\mathbf{k}}$, that is, $|A \cap N_i| = k_i$ for every $i \in [c]$. The earlier notion of colorful face corresponds to setting $c = d+1$ and $\mathbf{k} = \mathbf{1} := (1, \dots, 1) \in \mathbb{N}^c$. By $f_{\mathbf{k}} = f_{\mathbf{k}}(\mathsf{K})$ we denote the $\mathbf{k}$-*colorful f-vector* of K, that is, the number of $\mathbf{k}$-colorful faces in K.

Let us further assume that we are given sets $R_i \subseteq N_i$ with $|R_i| = r_i$ for every $i \in [c]$. Let $R = R_1 \sqcup \cdots \sqcup R_c$ and $\bar{R} := N \setminus R$. Then, for $\mathbf{n}, \mathbf{r}$ as above and a positive integer d, we define the set system

$$P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) = \left\{ S \in \binom{N}{\mathbf{k}} : |S \cap \bar{R}| \leq d \right\}.$$

We remark that $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$ is not a simplicial complex, as it contains only sets in $\binom{N}{\mathbf{k}}$. However, this set system is useful for estimating the number of $\mathbf{k}$-colorful faces in a d-collapsible complex.

By $p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$ we denote the size of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$, that is, $p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) := |P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})|$.

▶ **Theorem 10.** *For integers $c, d \geq 1$, let K be a d-collapsible simplicial complex with vertex partition $N = N_1 \sqcup \cdots \sqcup N_c$ and let $\mathbf{n} = (n_1, \dots, n_c) \in \mathbb{N}^c$ be the vector with $n_i = |N_i|$. For $\mathbf{r} = (r_1, \dots, r_c) \in \mathbb{N}^c$ such that $\dim \mathsf{K}[N_i] \leq r_i - 1$ for $i \in [c]$ and $\mathbf{k} \in \mathbb{N}_0^c$ such that $\mathbf{k} \leq \mathbf{n}$ it follows that*

$$f_{\mathbf{k}}(\mathsf{K}) \leq p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}).$$

$$\begin{array}{ccccc} \text{Thm. 10} & \Rightarrow \text{Prop. 9} & \Rightarrow \text{Thm. 7} & & \\ & \Downarrow & \Downarrow & & \\ & \text{Conj. 8} & \Rightarrow \text{Thm. 6} & \Leftrightarrow & \text{Thm. 5} \end{array}$$

Figure 1 The diagram of implications from Theorem 10 to Theorem 5. The top implications are proved below the statement of Theorem 10. The vertical implications follow from the main result of [17]. The equivalence at the bottom line has been explained below the statement of Theorem 6. Finally, note that we do not really need the implication Conj. 8 ⇒ Thm. 6. It comes from [13] where it also appears without explicit proof but it is easy – the interested reader may reconstruct it by checking the implication Prop. 9 ⇒ Thm. 7.

Theorem 10 is proved in Section 2. Here we show the implications Theorem 10 ⇒ Proposition 9 and Proposition 9 ⇒ Theorem 7. In addition, we advertise that Theorem 10 yields further generalizations of Theorem 7. We explain this last part in Section 3. For the convenience of the reader, we survey all the implications between the six preceding statements in Figure 1.

Proof of Proposition 9 modulo Theorem 10. We use Theorem 10 with $c = d+1$ and $\mathbf{k} = \mathbf{1}$. Then it is sufficient to compute $p_{\mathbf{1}}(\mathbf{n}, d, \mathbf{r})$. On the one hand, the size of $\binom{N}{\mathbf{1}}$ is $n_1 \dots n_{d+1}$. On the other hand, A belongs to $\binom{N}{\mathbf{1}} \setminus P_{\mathbf{1}}(\mathbf{n}, d, \mathbf{r})$ if and only if $|A \cap (N_i \setminus R_i)| = 1$ for every $i \in [d+1]$. Then, the number of such A is $(n_1 - r_1) \cdots (n_{d+1} - r_{d+1})$. Combining these observations we obtain the required formula

$$p_{\mathbf{1}}(\mathbf{n}, d, \mathbf{r}) = n_1 \dots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}).$$ ◀

Proof of Theorem 7 modulo Proposition 9. By contradiction, let us assume that for every $i \in [d+1]$ we get $\dim \mathsf{K}[N_i] < (1 - (1-\alpha)^{1/(d+1)})n_i - 1$. Let us set $r_i := \dim \mathsf{K}[N_i] + 1 < (1 - (1-\alpha)^{1/(d+1)})n_i$. Then Proposition 9 gives that the number of colorful d-faces is at most

$$\prod_{i=1}^{d+1} n_i - \prod_{i=1}^{d+1} (n_i - r_i) < \prod_{i=1}^{d+1} n_i - (1 - (1 - (1-\alpha)^{1/(d+1)}))^{d+1} \prod_{i=1}^{d+1} n_i = \alpha \prod_{i=1}^{d+1} n_i$$

which is a contradiction due to the strict inequality on the line above. ◀

A topological version? A simplicial complex K is *d-Leray* if the ith reduced homology group $\tilde{H}_i(\mathsf{L})$ (over $\mathbb{Q}$) vanishes for every induced subcomplex $\mathsf{L} \leq \mathsf{K}$ and every $i \geq d$. As we already know, every d-representable complex is d-collapsible, and in addition every d-collapsible complex is d-Leray [17]. Helly-type theorems usually extend to d-Leray complexes and such extensions are interesting because they allow topological versions of Helly-type when collections of convex sets are replaced with good covers. We refer to several concrete examples [9, 11, 3] or to the survey [16].

We conjecture that it should be possible to extend Theorem 7 to d-Leray complexes and probably Theorem 10 as well. In the full version [6], we briefly discuss a possible approach but also a difficulty in that approach.

▶ **Conjecture 11** (The optimal colorful fractional Helly theorem for d-Leray complexes)**.** *Let K be a d-Leray simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint subsets. Let $n_i := |N_i|$ for $i \in [d+1]$ and assume that K contains at least $\alpha n_1 \cdots n_{d+1}$ colorful d-faces for some $\alpha \in (0,1]$. Then there is an $i \in [d+1]$ such that* $\dim \mathsf{K}[N_i] \geq (1 - (1-\alpha)^{1/(d+1)})n_i - 1$.

2 Exterior algebra

In this section we prove Theorem 10. First we overview the required tools from exterior algebra – here we follow [10, Section 2] very closely.

Let N be a finite set of n elements (with a fixed total order $\leq$) and let $V = \mathbb{R}^N$ be the n-dimensional real vector space with standard basis vectors e_i for $i \in N$. Let $\bigwedge V$ be the 2^n dimensional exterior algebra over V with basis vectors e_S for $S \subseteq N$. The exterior product $\wedge$ on this algebra is defined so that it satisfies (i) $e_\emptyset$ is a neutral element, that is $e_\emptyset \wedge e_S = e_S = e_S \wedge e_\emptyset$; (ii) $e_S = e_{i_1} \wedge \cdots \wedge e_{i_s}$ for $S = \{i_1, \ldots, i_s\} \subseteq N$ where $i_1 < \cdots < i_s$ and we identify e_i with $e_{\{i\}}$ for $i \in N$ (iii) $e_i \wedge e_j = -e_j \wedge e_i$ for $i, j \in N$. By $\bigwedge^k V$ we denote the subspace of $\bigwedge V$ generated by $(e_S)_{S \in \binom{N}{k}}$ where $0 \leq k \leq n$. We consider the standard inner product on both V and $\bigwedge V$ so that $(e_i)_{i \in N}$ and $(e_S)_{S \subseteq N}$ are their orthonormal bases, respectively. Then $(e_S)_{S \in \binom{N}{k}}$ is also an orthonormal basis of $\bigwedge^k V$.

Given another basis $(g_i)_{i \in N}$, let $A = (a_{ij})_{i,j \in N}$ be the $N \times N$ transition matrix[4] from $(e_i)_{i \in N}$ to $(g_i)_{i \in N}$, that is, $g_i = \sum_{j \in N} a_{ij} e_j$ for any $i \in N$. The basis $(g_i)_{i \in N}$ induces a basis of $\bigwedge V$ given by $g_S = g_{i_1} \wedge \cdots \wedge g_{i_s}$ for $S = \{i_1, \ldots, i_s\} \subseteq N$. Transition from the standard basis $(e_S)_{S \in \binom{N}{k}}$ of $\bigwedge^k V$ to $(g_S)_{S \in \binom{N}{k}}$ is given by

$$g_S = \sum_{T \in \binom{N}{k}} \det(A_{S|T}) e_T \tag{1}$$

where $A_{S|T} = (a_{ij})_{i \in S, j \in T}$ for $S, T \subseteq N$.

Given an m-element set M and $M \times N$-matrix A and $k \leq m, n$, let $C_k(A)$ be the *compound* matrix $(\det A_{S|T})_{S \in \binom{M}{k}, T \in \binom{N}{k}}$.

The following lemma is implicitly contained in [10].

▶ **Lemma 12.** *If the columns of A are linearly independent, then the columns of $C_k(A)$ are linearly independent as well.*

Proof. If columns of A are linearly independent, then $n \leq m$. Consider an arbitrary square submatrix B of rank n. Considering B as a transition matrix from $(e_i)_{i \in N}$ to $(g_i)_{i \in N}$, we get that $C_k(B)$ is a transition matrix from $(e_S)_{S \in \binom{N}{k}}$ to $(g_S)_{S \in \binom{N}{k}}$, thus $C_k(B)$ has full rank. However, $C_k(B)$ is also a submatrix of $C_k(A)$ with all $\binom{n}{k}$ columns. ◀

Now, let us in addition assume that $(g_i)_{i \in N}$ is an orthonormal basis of V. As pointed out by Kalai, it follows from the Cauchy-Binet formula that $(g_S)_{S \in \binom{N}{k}}$ is also an orthonormal basis of $\bigwedge^k V$.

For $f, g \in \bigwedge V$ we define its *left interior product*, denoted by $g \llcorner f$, as the unique element in $\bigwedge V$ which satisfies that $\langle u, g \llcorner f \rangle = \langle u \wedge g, f \rangle$ for all $u \in \bigwedge V$. It turns out that $g_T \llcorner g_S$ is non-zero only if $T \subseteq S$, in which case $g_T \llcorner g_S = \pm g_{S \setminus T}$. (The sign is uniquely determined, but we do not need to express it explicitly.)

Colored exterior algebra. Now we extend the previous tools to the colored setting. From now on, let us assume that N is an n-element set decomposed into c-color classes, $N = N_1 \sqcup \cdots \sqcup N_c$. (The total order on N in this case starts with elements of N_1, then continues with elements

[4] Here we index rows and columns of a matrix by elements from some set, not necessarily integers. That is by $N \times N$ matrix we mean the matrix where both rows and columns are indexed by elements of N.

of N_2, etc.) We pick an $N \times N$-matrix A so that it is a block-diagonal matrix with blocks corresponding to individual N_i. That is, $A_{N_i|N_j}$ is a zero matrix whenever $i \neq j$. On the other hand, as shown by Kalai [10, Section 2], it is possible to pick each $A_{N_i|N_i}$ so that $(g_j)_{j \in N_i}$ is an orthonormal basis of the subspace of V generated by $(e_j)_{j \in N_i}$ and each square submatrix of $A_{N_i|N_i}$ has full rank. Therefore, from now on, we assume that we picked A and the vectors g_j this way. (Such a block matrix, for $c = 2$, is previously mentioned in [15].)

Similarly as in the introduction, let us set $\mathbf{n} = (n_1, \ldots, n_c)$ so that $n_i = |N_i|$ for $i \in [c]$; for simplicity, let us assume that each N_i is nonempty – that is, $\mathbf{n}$ is a c-tuple of positive integers. Let us also consider another c-tuple $\mathbf{k} = (k_1, \ldots, k_c)$ of non-negative integers such that $\mathbf{k} \leq \mathbf{n}$ and we set $k := k_1 + \cdots + k_c$. Then by $\bigwedge^{\mathbf{k}} V$ we mean the subspace of $\bigwedge V$ generated by $(e_S)_{S \in \binom{N}{\mathbf{k}}}$; recall that $\binom{N}{\mathbf{k}}$ is the set of all subsets A of N such that $|A \cap N_i| = k_i$ and that $\binom{N}{\mathbf{k}} \subseteq \binom{N}{k}$. Thus we also get that $\bigwedge^{\mathbf{k}} V$ is a subspace of $\bigwedge^k V$. In addition, due to our choice of $(g_j)_{j \in N}$ we get that $g_S \in \bigwedge^{\mathbf{k}} V$ if $S \in \binom{N}{\mathbf{k}}$. In addition $\det A_{S|T} = 0$ if $T \in \binom{N}{k} \setminus \binom{N}{\mathbf{k}}$ because $A_{S|T}$ is in this case a block matrix such that some of the blocks is not a square. Thus the formula (1) simplifies to

$$g_S = \sum_{T \in \binom{N}{\mathbf{k}}} \det(A_{S|T}) e_T. \tag{2}$$

Proof of Theorem 10. For $\mathbf{k} \in \mathbb{N}^c$ such that $k \leq d$ we have that $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) = \binom{N}{\mathbf{k}}$, thus the theorem follows trivially. On the other hand, if $k > r$, then $k_i > r_i$ for some i and consequently $f_{\mathbf{k}}(\mathsf{K}) = 0$ due to our assumption $\dim \mathsf{K}[N_i] \leq r_i - 1$; therefore the theorem again follows trivially. From now on we assume $d + 1 \leq k \leq r$. (We also use the notation for the sets R, $\bar{R}$ and R_i with $|R_i| = r_i$ as in the definition of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$.)

Let us define the following subspaces of $\bigwedge^{\mathbf{k}} V$

$$A_{\mathbf{k}} := \left\{ m \in \bigwedge^{\mathbf{k}} V : \left(\forall T \in \binom{R}{k-d} \right) \; g_T \llcorner m = 0 \right\},$$

and

$$W_{\mathbf{k}} := \operatorname{span} \left\{ e_S \in \bigwedge^{\mathbf{k}} V : S \in \binom{N}{\mathbf{k}} \text{ and } S \in \mathsf{K} \right\},$$

from the definition it follows that the colorful f-vector and the dimension of $W_{\mathbf{k}}$ coincide, i.e. $f_{\mathbf{k}} = \dim(W_{\mathbf{k}})$.

We claim that

$$\dim(A_{\mathbf{k}}) \geq \left| \binom{N}{\mathbf{k}} \right| - p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}).$$

Indeed, if $S \in \binom{N}{\mathbf{k}}$ such that $S \notin P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$, then $|S \cap \bar{R}| > d$. As $S \subseteq R \sqcup \bar{R} = N$ and $|S| = k$ we have that $|S \cap R| < k - d$. If $T \in \binom{R}{k-d}$ we have that $S \not\supseteq T$; therefore $g_T \llcorner g_S = 0$. From this it follows that $g_S \in A_{\mathbf{k}}$ and finally the claim because $g_S \in \bigwedge^{\mathbf{k}} V$.

The core of the proof is to show $A_{\mathbf{k}} \cap W_{\mathbf{k}} = \{0\}$. Once we have this, we get $f_k(\mathsf{K}) = \dim(W_k) \leq \dim \bigwedge^{\mathbf{k}} V - \dim A_{\mathbf{k}} \leq p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$ which proves the theorem.

For contradiction, let $m \in A_k \cap W_k$ be a non-zero element. Because $m \in W_{\mathbf{k}}$, it can be written as $m = \sum \alpha_S e_S$ where the sum is over all $S \in \binom{N}{\mathbf{k}}$ such that $S \in \mathsf{K}$. Let $\mathsf{K}_0, \ldots, \mathsf{K}_\ell$ be a sequence of simplicial complexes showing d-collapsibility of K. In addition, due to [10, Lemma 3.2], it is possible to assume that K_i arises from K_{i-1} by so called *special* elementary d-collapse which is either a removal of a maximal face of dimension at most $d-1$ or the minimal face (the face L in the definition) has dimension exactly $d-1$.

Now let us consider the first step from K_{i-1} to K_i such that a face $U \in \binom{N}{\mathbf{k}}$ with non-zero α_U is eliminated. Denote by L and M the faces determining the collapse as in the definition. We have $L \subseteq U \subseteq M$, $|M| \geq |U| = k > d$ and therefore $|L| = d$ (equivalently, $\dim L = d-1$), because the collapse is special. For $T \in \binom{R}{k-d}$ let $\mathbf{t} = (t_1, \dots, t_c) \in \mathbb{N}^c$ be such that $t_i = |T \cap N_i|$. Then $g_T = \sum_{P \in \binom{N}{\mathbf{t}}} \det(A_{T|P}) e_P$ via (2). We also need to simplify the expression $\langle e_L, g_T \llcorner e_S \rangle$ for $S \in \binom{N}{\mathbf{k}}$. We obtain

$$\langle e_L, g_T \llcorner e_S \rangle = \langle e_L \wedge g_T, e_S \rangle = \sum_{P \in \binom{N}{\mathbf{t}}} \det(A_{T|P}) \langle e_L \wedge e_P, e_S \rangle \tag{3}$$

If $S \not\supseteq L$ then $\langle e_L \wedge e_P, e_S \rangle = 0$ for all P, and therefore $\langle e_L, g_T \llcorner e_S \rangle = 0$. If $S \supseteq L$ then $\langle e_L \wedge e_P, e_S \rangle = 0$ unless $P = S \setminus L$ and therefore $\langle e_L, g_T \llcorner e_S \rangle = \langle e_L \wedge e_{S \setminus L}, e_S \rangle \det(A_{T|S \setminus L})$.

Since $m \in A_k$, for arbitrary $T \in \binom{R}{k-d}$ we get

$$\begin{aligned} 0 = \langle e_L, g_T \llcorner m \rangle &= \sum_{S \in \binom{N}{\mathbf{k}} : S \in \mathsf{K}} \alpha_S \langle e_L, g_T \llcorner e_S \rangle = \sum_{S \in \binom{N}{\mathbf{k}} : S \in \mathsf{K}_{i-1}} \alpha_S \langle e_L, g_T \llcorner e_S \rangle \\ &= \sum_{S \in \binom{N}{\mathbf{k}} : S \supseteq L} \alpha_S \langle e_L, g_T \llcorner e_S \rangle = \sum_{S \in \binom{N}{\mathbf{k}} : M \supseteq S \supseteq L} \alpha_S \langle e_L \wedge e_{S \setminus L}, e_S \rangle \det(A_{T|S \setminus L}) \end{aligned}$$

where the third equality follows from the fact that $\alpha_S = 0$ for $S \in \mathsf{K} \setminus \mathsf{K}_{i-1}$ due to our choice of K_{i-1} and the last two equalities follow from our earlier simplification of $\langle e_L, g_T \llcorner e_S \rangle$. (We also use that the expressions $S \supseteq L$ and $M \supseteq S \supseteq L$ are equivalent as M is the unique maximal face containing L.)

We also have $U \in \binom{N}{\mathbf{k}}$ with $M \supseteq U \supseteq L$ for which $\alpha_U \neq 0$ as well as $\langle e_L \wedge e_{U \setminus L}, e_U \rangle$ is nonzero (the latter one equals ± 1). Therefore the expression above is a linear dependence of the columns of $C_{k-d}(A_{R|M \setminus L})$. However, we will also show that the columns of $C_{k-d}(A_{R|M \setminus L})$ are linearly independent, thereby getting a contradiction. Via Lemma 12, it is sufficient to check that the columns of $A_{R|M \setminus L}$ are linearly independent. Because A is a block-matrix with blocks $A_{N_i|N_i}$, we get that $A_{R|M \setminus L}$ is a block matrix with blocks $A_{R_i|(M \setminus L) \cap N_i}$. Thus it is sufficient to check that the columns are independent in each block. But this follows from our assumptions of how we picked A in each block, using that $|R_i| = r_i \geq |(M \setminus L) \cap N_i|$ as $|M \cap N_i| \leq r_i$ due to our assumption $\dim \mathsf{K}[V_i] \leq r_i - 1$. ◀

3 k-colorful fractional Helly theorem

Theorem 10 allows to generalize Theorem 7 in two more directions.

The first generalization of Theorem 7 is already touched in the introduction. We can deduce an analogy of Theorem 7 for $\mathbf{k}$-colorful faces (instead of just colorful d-faces) where $\mathbf{k} = (k_1, \dots, k_c) \in \mathbb{N}_0^c$ is some vector with $c \geq 1$. For example, if $d = 2$, $\mathbf{k} = (2, 1, 1)$ and we understand the partition of $N = N_1 \sqcup N_2 \sqcup N_3$ as coloring the vertices of K red, green, or blue. Then we seek for the number of faces that contain two red vertices, one green vertex and one blue vertex.

For the second generalization, let us first observe that in the conclusion of Theorem 7 there is the same coefficient $1 - (1-\alpha)^{1/(d+1)}$ independently of i. However, in the notation of Theorem 7, we may also seek for i such that $\dim \mathsf{K}[N_i] \geq \beta_i n_i + 1$ where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_c) \in$

$(0,1]^c$ is some fixed vector. Then for given $\boldsymbol{\beta}$, we want to find the lowest $\alpha \in (0,1]$ with which we reach the conclusion analogous as in Theorem 7. This is a natural analogy of various Ramsey type statements: for example, if the edges of a complete graph G with at least 9 vertices are colored blue or red, then the graph contains either a blue copy of the complete graph on 3 vertices or a red copy of the complete graph on 4 vertices.

For the purpose of stating the generalization, let us set

$$L_{\mathbf{k}}(d) := \{\boldsymbol{\ell} = (\ell_1, \cdots \ell_c) \in \mathbb{N}_0^c \colon \ell_1 + \cdots + \ell_c \leq d \text{ and } \ell_i \leq k_i \text{ for } i \in [c]\} \tag{4}$$

and

$$\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) := \sum_{\boldsymbol{\ell}=(\ell_1,\ldots,\ell_c)\in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{k_i}{\ell_i} (1-\beta_i)^{\ell_i} (\beta_i)^{k_i-\ell_i}. \tag{5}$$

▶ **Theorem 13.** *Let $c, d \geq 1$ and $\mathbf{k} = (k_1, \ldots, k_c) \in \mathbb{N}_0^c$ be such that $k := k_1 + \cdots + k_c \geq d+1$. Let K be a d-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_c$ divided into c disjoint subsets. Let $n_i := |N_i|$ for $i \in [c]$ and assume that K contains at least $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \left|\binom{N}{\mathbf{k}}\right|$ $\mathbf{k}$-colorful faces for some $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_c) \in (0,1]^c$. Then there is an $i \in [c]$ such that* $\dim \mathsf{K}[N_i] \geq \beta_i n_i - 1$.

The formula (5) for $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ in Theorem 13 is, unfortunately, a bit complicated. However, this is the optimal value for α in the theorem. We first prove Theorem 13 and then we will provide an example showing that for every d, $\mathbf{k}$ and $\boldsymbol{\beta}$ as in the theorem, the value for α cannot be improved. The remark below is a probabilistic interpretation of (5). (This, for example, easily reveals that $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \in (0,1]$ for given parameters and will help us with checking monotonicity in $\boldsymbol{\beta}$.)

▶ Remark 14. Consider a random experiment where we gradually for each i pick k_i numbers $x_1^i, \ldots, x_{k_i}^i$ in the interval $[0,1]$ independently at random (with uniform distribution). Let ℓ_i be the number of x_j^i which are greater than β_i and let us consider the event $A_{\mathbf{k}}(d, \boldsymbol{\beta})$ expressing that $\ell_1 + \cdots + \ell_c \leq d$. Then $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ is the probability $\mathbf{P}[A_{\mathbf{k}}(d, \boldsymbol{\beta})]$. Indeed, the probability that the number of x_j^i which are greater than β_i is exactly ℓ_i is given by the expression beyond the sum in (5). Therefore, we need to sum this over all options giving $\ell_1 + \cdots + \ell_c \leq d$ and $\ell_i \leq k_i$.

In the proof of Theorem 13 we will need the following slightly modified proposition. We relax "at least" to "more than" while we aim at a strict inequality in the conclusion – this innocent change will be a significant advantage in the proof. On the other hand, after this change we can drop the assumption $k \geq d+1$. But this is only a cosmetic change, because the proposition below is vacuous if $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) = 1$ which in particular happens if $k < d+1$.

▶ **Proposition 15.** *Let $c, d \geq 1$ and $\mathbf{k} = (k_1, \ldots, k_c) \in \mathbb{N}_0^c$. Let K be a d-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_c$ divided into c disjoint subsets. Let $n_i := |N_i|$ for $i \in [c]$ and assume that K contains more than $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \left|\binom{N}{\mathbf{k}}\right|$ $\mathbf{k}$-colorful faces for some $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_c) \in (0,1]^c$. Then there is an $i \in [c]$ such that* $\dim \mathsf{K}[N_i] > \beta_i n_i - 1$.

First we show how Theorem 13 follows from Proposition 15 by a limit transition. Then we prove Proposition 15.

Proof of Theorem 13 modulo Proposition 15. Let us consider $\varepsilon > 0$ such that $\boldsymbol{\beta} - \boldsymbol{\varepsilon} \in (0,1]^c$ for $\boldsymbol{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \in (0,1]^c$.

First, we need to check $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) > \alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$. For this we will use Remark 14 and we also use $k \geq d+1$. It is easy to check $A_{\mathbf{k}}(d, \boldsymbol{\beta}) \supseteq A_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$ which gives $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \geq \alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$. In order to show the strict inequality, it remains to show that $A_{\mathbf{k}}(d, \boldsymbol{\beta}) \setminus A_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$ has positive probability. Consider the output of the experiment when each $x_i^j \in (\beta_i - \varepsilon, \beta_i)$. This output has positive probability ε^k. In addition, this output belongs to $A_{\mathbf{k}}(d, \boldsymbol{\beta})$ whereas it does not belong to $A_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$ (because $k \geq d+1$) as required.

This means, that we can apply Proposition 15 with $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$ as we know that K has at least $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})\left|\binom{N}{\mathbf{k}}\right|$ $\mathbf{k}$-colorful faces by assumptions of Theorem 13 which is more than $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \varepsilon)\left|\binom{N}{\mathbf{k}}\right|$. We obtain $\dim \mathsf{K}[N_i] > (\beta_i - \varepsilon)n_i - 1$. By letting ε tend to 0, we obtain the required $\dim \mathsf{K}[N_i] \geq \beta_i n_i - 1$. ◀

Boosting the complex. In the proof of Proposition 15, we will need the following procedure for boosting the complex. For a given complex K with vertex set $N = N_1 \sqcup \cdots \sqcup N_c$ partitioned as usual, and a non-negative integer m we define the complex $K_{\langle m \rangle}$ as a complex with the vertex set $N \times [m] = N_1 \times [m] \sqcup \cdots \sqcup N_c \times [m]$ whose maximal faces are of the form $S \times [m]$, where S is a maximal face of K. We will also use the notation $\delta_{\mathbf{k}}(\mathsf{K}) := f_{\mathbf{k}}(K)/|\binom{N}{\mathbf{k}}|$ for the density of $\mathbf{k}$-colorful faces of K. The item (ii) of the following lemma is the contents of [1, Proposition 2.1]; we thank an anonymous referee for pointing out this reference to us.

▶ **Lemma 16.** *Let K be a simplicial complex with vertex partition $N = N_1 \sqcup \cdots \sqcup N_c$ and $\mathbf{k} = (k_1, \ldots, k_c) \in \mathbb{N}_0^c$, then*

(i) $\delta_{\mathbf{k}}(\mathsf{K}_{\langle m \rangle}) \geq \delta_{\mathbf{k}}(\mathsf{K})$*; and*

(ii) *if K is d-collapsible, then $\mathsf{K}_{\langle m \rangle}$ is d-collapsible as well.*

Proof. We prove only (i) as (ii) is the contents of [1, Proposition 2.1].

If $\delta_{\mathbf{k}}(\mathsf{K}) = 0$ there is nothing to prove. Thus we may assume that $\delta_{\mathbf{k}}(\mathsf{K}) > 0$ (equivalently $f_{\mathbf{k}}(\mathsf{K}) > 0$) and consequently we have that $|N_i| \geq k_i$. Let us interpret $\delta_{\mathbf{k}}(\mathsf{K})$ as the probability that a random $\mathbf{k}$-tuple of vertices in N is a simplex of K, and we interpret $\delta_{\mathbf{k}}(\mathsf{K}_{\langle m \rangle})$ analogously. Let $\pi\colon N \times [m] \to N$ be the projection to the first coordinate. Now, let U be a $\mathbf{k}$-tuple of vertices in $N \times [m]$ taken uniformly at random. Considering the set $\pi(U) \subseteq N$, it need not be a $\mathbf{k}$-tuple (this happens exactly when two points in U have the same image under π) but it can be extended to a $\mathbf{k}$-tuple W using that $|N_i| \geq k_i$ for every i. Let W be an extension of $\pi(U)$ to a $\mathbf{k}$-tuple, taken uniformly at random among all possible choices. Because of the choices we made, W is in fact a $\mathbf{k}$-tuple of vertices in N taken uniformly at random. (Note that the choices done in each N_i or $N_i \times [m]$ are independent of each other.) Altogether, using $\mathbf{P}$ for probability, we get

$$\delta_{\mathbf{k}}(\mathsf{K}_{\langle m \rangle}) = \mathbf{P}[U \in \mathsf{K}_{\langle m \rangle}] = \mathbf{P}[\pi(U) \in \mathsf{K}] \geq \mathbf{P}[W \in \mathsf{K}] = \delta_{\mathbf{k}}(\mathsf{K}).$$ ◀

Density of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$. Now, we will provide a formula for the density of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$. In the following computations we also set $\delta_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) = p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})/\left|\binom{N}{\mathbf{k}}\right|$ using the notation from the definition of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$. We get

$$\begin{aligned} p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) &= \left| \left\{ S \in \binom{N}{\mathbf{k}} : |S \cap \bar{R}| \leq d \right\} \right| \\ &= \sum_{\boldsymbol{\ell} = (\ell_1, \ldots, \ell_c) \in L_{\mathbf{k}}(d)} \left| \left\{ S \in \binom{N}{\mathbf{k}} : |S_i \cap \bar{R}_i| = l_i \right\} \right| \\ &= \sum_{\boldsymbol{\ell} = (\ell_1, \ldots, \ell_c) \in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{n_i - r_i}{l_i} \binom{r_i}{k_i - l_i}. \end{aligned}$$

Then, using $(x)_m := x \cdot (x-1) \cdots (x-(m-1))$, the density is given by

$$\delta_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) = \frac{p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})}{\prod_{i=1}^{c} \binom{n_i}{k_i}} = \frac{\sum\limits_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod\limits_{i=1}^{c} \binom{k_i}{\ell_i} (n_i - r_i)_{\ell_i} (r_i)_{k_i-\ell_i}}{\prod_{i=1}^{c} (n_i)_{k_i}}. \tag{6}$$

Proof of Proposition 15. For contradiction, let us assume that for every $i \in [c]$ we have that $\dim(\mathsf{K}[V_i]) \leq \beta_i n_i - 1$. Let us set $r_i := \dim(\mathsf{K}[V_i]) + 1 \leq \beta_i n_i$. Note that the conclusion of Theorem 10 can be restated as $\delta_{\mathbf{k}}(\mathsf{K}) \leq \delta_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$.

Now we get

$$\begin{aligned}
\delta_{\mathbf{k}}(\mathsf{K}) &\leq \liminf_{m\to\infty} \delta_{\mathbf{k}}(\mathsf{K}_{\langle m\rangle}) \text{ by Lemma 16(i)}\\
&\leq \liminf_{m\to\infty} \delta_{\mathbf{k}}(m\mathbf{n}, d, m\mathbf{r}) \text{ by Theorem 10 using Lemma 16(ii)}\\
&\leq \liminf_{m\to\infty} \delta_{\mathbf{k}}(m\mathbf{n}, d, \lfloor mn_i\beta_i \rfloor) \text{ using } r_i \leq \beta_i n_i \text{ and monotonicity of } p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) \text{ in } \mathbf{r}\\
&= \liminf_{m\to\infty} \frac{\sum\limits_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod\limits_{i=1}^{c} \binom{k_i}{\ell_i} (mn_i - \lfloor mn_i\beta_i \rfloor)_{\ell_i} (\lfloor mn_i\beta_i \rfloor)_{k_i-\ell_i}}{\prod_{i=1}^{c} (mn_i)_{k_i}} \text{ by (6)}\\
&= \sum_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{k_i}{\ell_i} (1-\beta_i)^{\ell_i} (\beta_i)^{k_i-\ell_i}\\
&= \alpha_{\mathbf{k}}(d, \boldsymbol{\beta})
\end{aligned}$$

which is a contradiction with the assumptions. ◀

▶ Remark 17. It would be much more natural to try to avoid boosting the complex and show directly $\delta_k(\mathsf{K}) \leq \delta_k(\mathbf{n}, d, \mathbf{r}) \leq \alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ in the proof of Proposition 15. The former inequality follows from Theorem 10. However, the latter inequality turned out to be somewhat problematic for us when we attempted to show it directly from the definition of $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ and from (6). Thus, in our computations, we take an advantage of the fact that the computations in the limit are easier.

Tightness of Theorem 13. We conclude this section by showing that the bound given in Theorem 13 is tight.

Let us fix $c, d \in \mathbb{N}$, $\mathbf{k} = (k_1, \dots, k_c) \in \mathbb{N}_0^c$ with $k := k_1 + \cdots + k_c \geq d + 1$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_c) \in (0, 1]^c$ as in the statement of Theorem 13. Let $0 \leq \alpha' < \alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$. We will find a complex K which contains at least $\alpha' |\binom{N}{\mathbf{k}}|$ $\mathbf{k}$-colorful faces while $\dim \mathsf{K}[N_i] < \beta_i n_i - 1$ for every $i \in [c]$ (using the notation from the statement of Theorem 13).

Similarly as in the proof of Theorem 13 let us consider $\varepsilon > 0$ such that $\boldsymbol{\beta} - \boldsymbol{\varepsilon} \in (0, 1]^c$ for $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in (0, 1]^c$. In addition, because $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ is continuous in $\boldsymbol{\beta}$ due to its definition (5), we may pick ε such that $\alpha' < \alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$. For simplicity of notation, let $\boldsymbol{\beta}' = (\beta_1', \dots, \beta_c') := \boldsymbol{\beta} - \boldsymbol{\varepsilon}$.

Now we pick a positive integer m and set $\mathbf{n} = (m, \dots, m) \in \mathbb{N}^c$, that is, $n_1 = \cdots = n_c = m$ and $n = cm$ in our standard notation. We also set $\mathbf{r} = (r_1, \dots, r_c)$ so that $r_i := \lfloor \beta_i' m \rfloor$.[5] We assume that m is large enough so that $r_i \geq k_i$ for each $i \in [c]$. We define families N_i of convex sets in $\mathbb{R}^d$ so that each N_i contains r_i copies of $\mathbb{R}^d$ and $m - r_i$ hyperplanes in

[5] This choice of $\mathbf{n}$ will yield a counterexample where each color class has equal size. It would be also possible to vary the sizes.

general position. We also assume that the collection of all hyperplanes in $N_1, \dots, N_c$ is in general position. We set K to be the nerve of the family $N = N_1 \sqcup \dots \sqcup N_c$. In particular K is d-representable (therefore d-collapsible as well).

First, we check that $\dim \mathsf{K}[N_i] < \beta_i m - 1$ provided that m is large enough. A subfamily of N_i with nonempty intersection contains at most d hyperplanes from N_i. Therefore $\dim \mathsf{K}[N_i] < r_i + d = \lfloor \beta'_i m \rfloor + d < \beta_i m - 1$ for m large enough.

Next we check that K contains at least $\alpha' |\binom{N}{\mathbf{k}}|$ $\mathbf{k}$-colorful faces provided that m is large enough. Partitioning N_i so that R_i is the subfamily of the copies of $\mathbb{R}^d$ and $\bar{R}_i$ is the subfamily of hyperplanes, we get

$$f_{\mathbf{k}}(K) = p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$$

from the definition of $p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$. Therefore (6) gives

$$\delta_{\mathbf{k}}(\mathsf{K}) = \frac{\sum\limits_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod\limits_{i=1}^{c} \binom{k_i}{\ell_i} (m - \lfloor \beta'_i m \rfloor)_{\ell_i} (\lfloor \beta'_i m \rfloor)_{k_i - \ell_i}}{\prod_{i=1}^{c} (m)_{k_i}}.$$

Passing to the limit (considering the dependency of K on m), we get

$$\lim_{m\to\infty} \delta_{\mathbf{k}}(\mathsf{K}) = \sum_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{k_i}{\ell_i} (1-\beta'_i)^{\ell_i} (\beta'_i)^{k_i-\ell_i} = \alpha_{\mathbf{k}}(d, \boldsymbol{\beta}').$$

Therefore, for m large enough K contains at least $\alpha' |\binom{N}{\mathbf{k}}|$ $\mathbf{k}$-colorful as $\alpha' < \alpha_{\mathbf{k}}(d, \boldsymbol{\beta}')$.

References

1 R. Aharoni, R. Holzman, and Z. Jiang. Rainbow fractional matchings. *Combinatorica*, 39(6):1191–1202, 2019.
2 N. Alon and G. Kalai. A simple proof of the upper bound theorem. *European J. Combin.*, 6(3):211–214, 1985.
3 N. Alon, G. Kalai, J. Matoušek, and R. Meshulam. Transversal numbers for hypergraphs arising in geometry. *Adv. in Appl. Math.*, 29(1):79–101, 2002.
4 I. Bárány. A generalization of Carathéodory's theorem. *Discrete Math.*, 40(2-3):141–152, 1982.
5 I. Bárány, F. Fodor, L. Montejano, D. Oliveros, and A. Pór. Colourful and fractional (p, q)-theorems. *Discrete Comput. Geom.*, 51(3):628–642, 2014.
6 D. Bulavka, A. Goodarzi, and M. Tancer. Optimal bounds for the colorful fractional Helly theorem. Preprint, 2020. `arXiv:2010.15765`.
7 J. Eckhoff. An upper-bound theorem for families of convex sets. *Geom. Dedicata*, 19(2):217–227, 1985.
8 E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresber. Deustch. Math.-Verein.*, 32:175–176, 1923.
9 E. Helly. Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten. *Monaths. Math. und Physik*, 37:281–302, 1930.
10 G. Kalai. Intersection patterns of convex sets. *Israel J. Math.*, 48(2-3):161–174, 1984.
11 G. Kalai and R. Meshulam. A topological colorful Helly theorem. *Adv. Math.*, 191(2):305–311, 2005.
12 M. Katchalski and A. Liu. A problem of geometry in $\mathbf{R}^n$. *Proc. Amer. Math. Soc.*, 75(2):284–288, 1979.
13 M. Kim. A note on the colorful fractional Helly theorem. *Discrete Math.*, 340(1):3167–3170, 2017.
14 L. Lovász. Problem 206. *Matematikai Lapok*, 25:181, 1974.

15 E. Nevo. Algebraic shifting and basic constructions on simplicial complexes. *J. Algebraic Combin.*, 22(4):411–433, 2005.
16 M. Tancer. Intersection patterns of convex sets via simplicial complexes: A survey. In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 521–540. Springer New York, 2013.
17 G. Wegner. d-collapsing and nerves of families of convex sets. *Arch. Math. (Basel)*, 26:317–321, 1975.