No Krasnoselskii Number for General Sets

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──── **Abstract** ────

For a family $\mathcal{F}$ of non-empty sets in $\mathbb{R}^d$, the Krasnoselskii number of $\mathcal{F}$ is the smallest m such that for any $S \in \mathcal{F}$, if every m or fewer points of S are visible from a common point in S, then any finite subset of S is visible from a single point. More than 35 years ago, Peterson asked whether there exists a Krasnoselskii number for general sets in $\mathbb{R}^d$. The best known positive result is Krasnoselskii number 3 for closed sets in the plane, and the best known negative result is that if a Krasnoselskii number for general sets in $\mathbb{R}^d$ exists, it cannot be smaller than $(d+1)^2$.

In this paper we answer Peterson's question in the negative by showing that there is no Krasnoselskii number for the family of all sets in $\mathbb{R}^2$. The proof is non-constructive, and uses transfinite induction and the well-ordering theorem.

In addition, we consider Krasnoselskii numbers with respect to visibility through polygonal paths of length $\leq n$, for which an analogue of Krasnoselskii's theorem for compact simply connected sets was proved by Magazanik and Perles. We show, by an explicit construction, that for any $n \geq 2$, there is no Krasnoselskii number for the family of compact sets in $\mathbb{R}^2$ with respect to visibility through paths of length $\leq n$. (Here the counterexamples are finite unions of line segments.)

1 Introduction

▶ **Definition 1.** *For a set $S \subset \mathbb{R}^d$ and two points $x, y \in S$, we say that "x sees y through S" or "y is visible from x through S" if S includes the segment $[x, y]$ or $x = y$. A set $S \subset \mathbb{R}^d$ is "starshaped" if some point $x \in S$ sees all other points in S. A set $S \subset \mathbb{R}^d$ is "finitely starlike" if every finite subset of S is visible from a common point.*

One of the best-known applications of Helly's theorem (that is, actually, equivalent to Helly's theorem, see [1]) is the following theorem, due to Krasnoselskii [12]:

▶ **Theorem 2** (Krasnoselskii). *Let S be a non-empty compact set in $\mathbb{R}^d$. If every $d+1$ or fewer points of S are visible from a common point, then S is starshaped.*

In particular, the case $d = 2$ of the theorem asserts that if an art gallery is so shaped that for every three paintings there is a place where you can stand and see those three, then there is a place where you can stand and see all the paintings.

Krasnoselskii's theorem fails for unbounded closed sets. However, all known counterexamples satisfy the weaker requirement of being finitely starlike. This led Peterson [15] to ask the following:
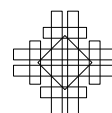
▶ **Question 3** (Peterson). *Is there a Krasnoselskii number $K(d)$ such that for any infinite set $S \subset \mathbb{R}^d$, if every $K(d)$ points of S are visible from a single point, then S is finitely starlike?*

Peterson's question appears in the Handbook of Combinatorics [9, Chapter 14, Section 6.1.2, written by Erdős and Purdy] and in a treatise on open problems in geometry by Croft, Falconer and Guy [8, Chapter E1]. It was studied in a number of works. A few positive results were obtained, for special cases of sets that satisfy additional topological conditions [2, 4, 5]. Those include Krasnoselskii number 3 for closed sets in the plane, and Krasnoselskii numbers 4 and 5 under more complex conditions on the boundary of the set. On the other hand, the complement of the closed unit cube in $\mathbb{R}^d$ shows that if a Krasnoselskii number for open sets in $\mathbb{R}^d$ exists, it must be $\geq 2d$. Breen [4] showed by an explicit example that if a Krasnoselskii number for F_σ sets in the plane exists, then it must be ≥ 9. This result was generalized to any $d \geq 2$, with the lower bound $(d + 1)^2$, by Shlezinger [16]. For other partial results and related problems, see [3] and the references therein.

In this paper we answer Peterson's question in the negative by showing that the Krasnoselskii number $K(2)$ does not exist, and consequently, $K(d)$ does not exist for any $d > 2$, since a planar construction can be embedded in $\mathbb{R}^d$ for any $d > 2$. Formally, we prove the following:

▶ **Theorem 4.** *For any $k \geq 2$, there exists a set $S \subset \mathbb{R}^2$ such that any $2k + 3$ points in S are visible from a common point, but some $2k + 4$ points in S are not simultaneously seen from any point.*

Unlike Breen's construction that leads to an F_σ-set, our proof is non-constructive and uses transfinite induction and the well-ordering theorem. We note that we do use the axiom of choice which is equivalent to the well-ordering theorem, but we do not use any additional hypothesis except for the standard ZFC system of axioms.

In the second part of the paper, we consider a generalization of the notion of visibility, namely visibility through polygonal paths.

▶ **Definition 5.** *For a set $S \subset \mathbb{R}^d$ and $x, y \in S$, we say that "x sees y through S by a polygonal path of length n" or "y is visible from x through S by a polygonal path of length n" if there exist points $x_1, x_2, \ldots, x_{n-1}$ and a polygonal path*

$$\langle x, x_1, \ldots, x_{n-1}, y \rangle = [x, x_1] \cup [x_1, x_2] \cup \ldots \cup [x_{n-1}, y]$$

that lies entirely within S.

This definition of visibility is closely related to the notion of L_n-sets. These are sets that are connected through polygonal paths of length $\leq n$. This notion was defined by Horn and Valentine [10], and studied in numerous papers (see, e.g., [7, 14, 17]). Of course, the original notion of visibility considered above corresponds to the case of visibility through polygonal paths of length $n = 1$.

In [14, Theorem 1.2], Magazanik and Perles proved a Krasnoselskii-type theorem with respect to visibility through polygonal paths. Formally, they showed that for any n and for any compact and simply connected set $S \subset \mathbb{R}^2$, if every three points in S are visible from a common point through polygonal paths of length $\leq n$, then all points of S are visible from a single point through polygonal paths of length $\leq n$.

This raises the natural analogue of Peterson's question for this notion of visibility: Is there a number $K'(2, n)$ such that for any set $S \subset \mathbb{R}^2$, if every $K'(2, n)$ or fewer points of S are visible from a common point through polygonal paths of length $\leq n$, then every finite subset of S is visible from a common point through polygonal paths of length $\leq n$?

We answer this question in the negative. Namely, we prove, by constructing a sequence of explicit examples, the following:

▶ **Theorem 6.** *For any $n, k \geq 2$, there exists a compact set $S \subset \mathbb{R}^2$ such that every k points in S are visible from a common point through polygonal paths of length $\leq n$, but some $k + 1$ points of S are not visible from a common point through polygonal paths of length $\leq n$.*

The sets S we construct in the proof of Theorem 6 are actually finite unions of closed line segments.

A similar result with respect to the related concept of visibility through *staircase* paths of length $\leq n$ was obtained by Breen [6]. However, the construction of [6] is significantly different from ours.

This paper is organized as follows: In Section 2 we study Peterson's problem with respect to the classical definition of visibility and prove Theorem 4. In Section 3 we study visibility through polygonal paths of length $\leq n$ and prove Theorem 6.

2 Proof of Theorem 4

2.1 Some basic facts from set theory

In this subsection we recall a few basic definitions and facts from set theory that will be used in our proof. For the sake of brevity, we present the definitions in a somewhat informal way. A formal treatment can be found in standard textbooks on basic set theory, e.g., [11, 13].

Well ordering. Let S be a set. A binary relation $<$ on S is called a *well-ordering* if:
1. For any $x, y \in S$, exactly one of the following holds: either $x < y$, or $y < x$, or $x = y$.
2. ” $<$ ” is transitive.
3. Any non-empty subset S' of S has a first element x_0, namely, $\forall \emptyset \neq S' \subset S, \exists x_0 \in S' : \forall x \in S', x_0 \leq x$.

A pair $(S, <)$ where $<$ is a well-ordering on S is called a *well-ordered set*.

Transitive sets and ordinals. A set S is called *transitive* if any element of S is a subset of S, that is, if $(x \in S) \wedge (y \in x) \Rightarrow (y \in S)$. A set O is called an *ordinal* if it is transitive and $(O, \in)$ (i.e., O with the relation $(x < y) \Leftrightarrow (x \in y)$) is a well-ordered set.

Initial section. Let $(S, <)$ be a well-ordered set. An initial section of S is a subset $S' \subset S$ of the form $S' = \{x \in S : x < y\}$, for some $y \in S$.

Isomorphism. Two well-ordered sets $(S, <_S)$ and $(T, <_T)$ are *isomorphic* if there exists a bijection $f : S \rightarrow T$ such that $\forall x, y \in S : (x <_S y) \Leftrightarrow (f(x) <_T f(y))$.

Two basic facts on ordinals.
1. For any set $\mathcal{O}$ of ordinals, $(\mathcal{O}, \in)$ is a well-ordered set. In particular, any set of ordinals contains a minimal element with respect to the relation $\in$.
2. Any well-ordered set $(S, <)$ is isomorphic to a unique ordinal.

The well-ordering theorem. For any set S, there exists a relation $<$ on S such that $(S, <)$ is a well-ordered set. (This theorem may be considered an axiom, being equivalent to the Axiom of Choice).

Initial ordinal. For a set S, consider all possible well-orderings of S. Each of them is isomorphic to a unique ordinal. The corresponding set of ordinals has a minimal element (with respect to the relation $\in$). This element is called *the initial ordinal* that corresponds to S.

Cardinals. Informally, the cardinality of a set S measures its "size". Formally, the cardinal of S, denoted by $|S|$, is identified with the initial ordinal that corresponds to S. This allows comparing the "sizes" of any two sets, as any two ordinals are comparable by the relation $\in$. It is clear that we have $|S| \leq |T|$ if and only if there exists an injection $f : S \to T$.

Arithmetic of infinite cardinals. We use two basic facts from cardinal arithmetic. If S, T are infinite sets then $|S \times T| = |S \cup T| = \max(|S|, |T|)$. If, in addition, $|S| < |T|$, then $|T \setminus S| = |T|$.

Initial section of an initial ordinal. It follows from the above definitions that if O is the initial ordinal that corresponds to some set S and O' is an initial section of O, then $|O'| < |O|$.

Induction on ordinals and transfinite induction. The induction principle on ordinals implies the following: Let O be an ordinal, and let P be a property of ordinals. If for any $O' \in O$, we have $(\forall O'' < O' : P(O'')) \Rightarrow P(O')$, then $\forall O' \in O, P(O')$. As any well-ordered set corresponds to a unique ordinal, this method can be used to prove claims on general well-ordered sets. Its application is commonly called *transfinite induction*.

Application in our case. A classical way to use transfinite induction to prove an assertion on a general set S is to use the well-ordering theorem to define a well-ordering $<$ such that $(S, <)$ is isomorphic to the initial ordinal that corresponds to S, and then to use transfinite induction to prove the assertion for $(S, <)$. An important feature deployed in this method is that for any initial section S' of $(S, <)$, we have $|S'| < |S|$, as was explained above.

We shall apply this method to the set S all k-tuples of points in the lower half-plane (whose cardinality is clearly $\mathfrak{c} = 2^{\aleph_0}$), and the fact that the cardinality of any initial section in our ordering is strictly smaller than $\mathfrak{c}$ will play a crucial role in our proof.

2.2   Proof of the theorem

In this subsection we prove Theorem 4. Let us recall its statement.

Theorem 4. For any $2 \leq k \in \mathbb{N}$, there exists a set $T = T(k) \subset \mathbb{R}^2$ such that the following holds:
1. Any $2k + 3$ points of T are seen, through T, from a common point $x \in T$.
2. There are $2k + 4$ points in T that are not seen, through T, from any point of T.

We divide the presentation into three parts. First, we reduce the problem to proving the existence of a subset S of the real line that satisfies certain conditions, called a *k-shutter*. Then, we describe the transfinite induction argument, without getting into the geometric details. Finally, we present a formal proof that fills in the geometric part.

2.2.1   Reduction to proving the existence of a k-shutter

Let us denote by $\mathbb{R}_+^2$ the upper open half-plane $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\}$, and by $\mathbb{R}_-^2$ the lower open half-plane $\mathbb{R}_-^2 = \{(x, y) \in \mathbb{R}^2 | y < 0\}$. For two points $x \in \mathbb{R}_+^2, y \in \mathbb{R}_-^2$, and a subset S of the x-axis, $S \subset \{(x, 0) | x \in \mathbb{R}\}$, we say that x *sees* y *via* S, if x sees y through $\mathbb{R}_+^2 \cup S \cup \mathbb{R}_-^2$. (Note that this last definition is not the same as visibility *through* S defined above.)

The following definition plays a central role in the proof of Theorem 4.

▶ **Definition 7.** *A subset S of the x-axis, $S \subset \{(x, 0) | x \in \mathbb{R}\}$, is called a k-shutter, if:*
(a) *For any k-tuple $\{a_1, \ldots, a_k\} \subset \mathbb{R}_-^2$ there exists a point $z \in \mathbb{R}_+^2$ that sees each a_i $(1 \leq i \leq k)$ via S, and*
(b) *There exists a $(k+1)$-tuple $K = \{y_1, \ldots, y_{k+1}\} \subset \mathbb{R}_-^2$ such that no point in $\mathbb{R}_+^2$ sees all the points of K via S.*

Note that, by symmetry, the definition of a k-shutter is not sensitive to interchanging $\mathbb{R}_+^2$ and $\mathbb{R}_-^2$. In addition, it is easy to see that for $k \geq 2$, any k-shutter S satisfies $|S| \geq 2$ and does not include a segment.

▶ **Observation 8.** *In order to prove Theorem 4, it suffices to prove that for any $k \in \mathbb{N}$, there exists a k-shutter S.*

Proof of Observation 8. Given a k-shutter S, let $T = \mathbb{R}_+^2 \cup \mathbb{R}_-^2 \cup S$, and let T' be a set of $2k+3$ points of T. If $T' \cap S = \emptyset$, then any point of S sees all points of T' through T. If $T' \cap S$ consists of a single point x, then x sees all the points of T' through T. If $|T' \cap S| \geq 2$, then by the pigeonhole principle, either $T' \cap \mathbb{R}_-^2$ or $T' \cap \mathbb{R}_+^2$ contains at most k points. W.l.o.g., assume $|T' \cap \mathbb{R}_-^2| \leq k$. As S is a k-shutter, some point $z \in \mathbb{R}_+^2$ sees $T' \cap \mathbb{R}_-^2$ through T. The point z clearly sees all points of T' through T.

On the other hand, take $K_1 \subset \mathbb{R}_-^2$ to be a $(k+1)$-set that is not seen via S by any point in $\mathbb{R}_+^2$, and take $K_2 \subset \mathbb{R}_+^2$ to be a $(k+1)$-set that is not seen via S by any point in $\mathbb{R}_-^2$. Since $|S| \geq 2$ and S does not include a segment, there exist two points $s_1, s_2 \in S$ that do not see each other through S. Then the set $K_1 \cup K_2 \cup \{s_1, s_2\} \subset T$ is a $(2k+4)$-sized subset of T that is not seen by any point through T. ◀

2.2.2 The transfinite induction argument

To prove the existence of a k-shutter, we use transfinite induction. Let $K = \{y_1, \ldots, y_{k+1}\} \subset \mathbb{R}_-^2$ be a fixed $(k+1)$-set of points in $\mathbb{R}_-^2$, and let $U = \{\{a_1, a_2, \ldots, a_k\} \subset \mathbb{R}_-^2\}$ be the family of all k-subsets of $\mathbb{R}_-^2$. Clearly, $|U| = \mathfrak{c}$. Let O be the initial ordinal of cardinality $\mathfrak{c}$, namely, $|O| = \mathfrak{c}$ and for each $\lambda \in O$, $|\{\alpha \in O : \alpha < \lambda\}| < \mathfrak{c}$. Let $\lambda \mapsto u_\lambda$ ($\lambda \in O$) be a bijection between O and U (whose existence follows from the well-ordering theorem). We shall prove the existence of a k-shutter S such that each k-set in U is seen via S by some $z \in \mathbb{R}_+^2$, while no $z \in \mathbb{R}_+^2$ sees the $(k+1)$-set K via S.

We construct S by a transfinite induction process, that corresponds to induction on the ordinal O. We index the steps of the process by the elements of O, and in step λ we "take care" of the k-set $a^\lambda = \{a_1^\lambda, \ldots, a_k^\lambda\}$ that corresponds (in the isomorphism) to $\lambda \in O$.

During the process, we construct two increasing sequences of subsets of the x-axis: $\{A_\lambda\}_{\lambda \in O}$ and $\{B_\lambda\}_{\lambda \in O}$ (where "increasing" means that if $\lambda_1 < \lambda_2$ then $A_{\lambda_1} \subseteq A_{\lambda_2}$ and $B_{\lambda_1} \subseteq B_{\lambda_2}$). The sets A_λ contain points that will be included in S, while the sets B_λ contain points that will not be included in S. Of course, we make sure that the B_λ's are disjoint from the A_λ's. In addition, we make sure that at each step λ, we have $|A_\lambda|, |B_\lambda| \leq \max(|\lambda|, \aleph_0)$.

At Step λ, we define A_λ and B_λ by adding points to the sets $A_\lambda^0 = \bigcup_{\lambda' < \lambda} A_{\lambda'}$ and $B_\lambda^0 = \bigcup_{\lambda' < \lambda} B_{\lambda'}$, respectively. First we add at most $\max(|\lambda|, \aleph_0)$ points to B_λ^0 to form B_λ in a certain way to be explained shortly. Then we add $k-1$ (or fewer) points to A_λ^0 to form A_λ, in such a way that the set $a^\lambda = \{a_1^\lambda, \ldots, a_k^\lambda\}$ is seen through A_λ by some $z \in \mathbb{R}_+^2$, while the set K is not seen through A_λ by any $z \in \mathbb{R}_+^2$. The points added to B_λ are responsible for the latter condition – they are chosen in such a way that forcing A_λ to avoid them will guarantee that K is not seen via A_λ from any point in the upper open half-plane.

At the end of the process, we set $S = \bigcup_{\lambda \in O} A_\lambda$. It is clear from the construction that any k-set $a^\lambda \in U$ is seen via S by some $z \in \mathbb{R}_+^2$ (since it is seen via A_λ) and also that K is not seen via S by any $z \in \mathbb{R}_+^2$ (as otherwise, assuming that for any $1 \leq i \leq k+1$, y_i is seen via some A_{λ_i}, K would be seen via A_λ for $\lambda = \max\{\lambda_1, \ldots, \lambda_{k+1}\}$, contradicting the construction).

The crucial observation that makes the definition of the sets A_λ and B_λ possible is that the cardinality of any initial section of O is smaller than $\mathfrak{c}$, and thus, at any step λ of the process we have $|A_\lambda^0|, |B_\lambda^0| < \mathfrak{c}$. This means that the sets A_λ^0 and B_λ^0 cover only a "very small" part of the x-axis, and so we have "enough room" to select the points we need to add in order to define B_λ and A_λ, as will be shown formally below.

2.2.3   Formal proof

We prove the following proposition.

▶ **Proposition 9.** *There exist increasing sequences of sets $\{A_\lambda\}_{\lambda \in O}$ and $\{B_\lambda\}_{\lambda \in O}$, such that, for any $\lambda \in O$, we have:*

1. $A_\lambda, B_\lambda \subset \{(x,0)|x \in \mathbb{R}\}$;
2. $A_\lambda \cap B_\lambda = \emptyset$;
3. $|A_\lambda|, |B_\lambda| \leq \max(|\lambda|, \aleph_0) < \mathfrak{c}$;
4. *The k-set $a^\lambda = \{a_1^\lambda, \ldots, a_k^\lambda\}$ is seen via A_λ by some $z \in \mathbb{R}_+^2$, while the $(k+1)$-set $K = (y_1, \ldots, y_{k+1})$ is not seen via A_λ by any $z \in \mathbb{R}_+^2$.*
5. $\forall 1 \leq i < j \leq k+1$, $\ell(y_i, y_j) \cap \{(x,0)|x \in \mathbb{R}\} \subset B_\lambda$, *where $\ell(y_i, y_j)$ is the line spanned by y_i and y_j.*
6. *For any $z \in \mathbb{R}_+^2$ such that z sees at least two points of K via $\bigcup_{\lambda' < \lambda} A_{\lambda'}$, there exists $1 \leq i \leq k+1$ such that $\{(x,0)|x \in \mathbb{R}\} \cap [z, y_i] \in B_\lambda$.*

Proposition 9 implies Theorem 4, since the set $S = \bigcup_{\lambda \in O} A_\lambda$ is a k-shutter, as was explained in Section 2.2.2. Note that the inductive step is performed for any $\lambda \in O$, namely, for any $\lambda < \mathfrak{c}$ (and not for $\lambda = O$), and the set S is a union of $\mathfrak{c}$ sets.

Proof. The proof is by transfinite induction on the elements of O.

Induction basis. We start by defining A_0 and B_0. (Note that there is no formal need to present the initial step separately. However, in our case this step is somewhat different from the other ones, and thus we have to present it.)

Consider all lines that are spanned by two points $y_i, y_j \in K$. Let B_0 be the set of all intersection points of these lines with the x-axis. Clearly, $|B_0| \leq \binom{k+1}{2}$, and B_0 satisfies condition 5. Condition 6 is satisfied vacuously, since $\bigcup_{\lambda' < 0} A_{\lambda'} = \emptyset$.

In order to define A_0, consider all lines that pass through at least one point of $a^0 = \{a_1^0, \ldots, a_k^0\}$ and at least one point of B_0. The union of these lines does not cover the upper half-plane (being a finite union of lines), hence, there exists a point $z \in \mathbb{R}_+^2$ that is not incident with any such line. Denote the intersections of the segments $[z, a_1^0], \ldots, [z, a_k^0]$ with the x-axis by $s_1, \ldots, s_t$ for some $t \leq k$, and set $A_0 = \{s_1, \ldots, s_t\}$. Note that by the construction of B_0, $A_0 \cap B_0 = \emptyset$.

Both A_0 and B_0 are finite, and so we have $|A_0|, |B_0| \leq \max(0, \aleph_0)$. The point z sees all k points in $a^0 = \{a_1^0, \ldots a_k^0\}$ via A_0. Furthermore, no point in $\mathbb{R}_+^2$ can see K via A_0. Indeed, $|A_0| \leq k$, and thus, by the pigeonhole principle, if some $z \in \mathbb{R}_+^2$ sees K via A_0, it must see two points y_i, y_j via the same point $s_\ell \in A_0$. In such a case, $s_\ell \in B_0$ by the definition of B_0, contradicting the construction which assures that $A_0 \cap B_0 = \emptyset$. Therefore, A_0 and B_0 satisfy the assertion of the proposition.

Induction step. For $\lambda \in O$, we describe the definition of A_λ and B_λ, assuming that for any $\lambda' < \lambda$, $A_{\lambda'}$ and $B_{\lambda'}$ have been defined and satisfy the assertion. As described above, we denote

$$A_\lambda^0 = \cup_{\lambda' < \lambda} A_{\lambda'} \qquad \text{and} \qquad B_\lambda^0 = \cup_{\lambda' < \lambda} B_{\lambda'}.$$

(Note that if $\lambda = \bar{\lambda} + 1$ is a successor ordinal, then we have $A_\lambda^0 = A_{\bar{\lambda}}$ and $B_\lambda^0 = B_{\bar{\lambda}}$.)

Defining B_λ. Consider the family $\bar{\mathcal{L}}$ of lines that pass through at least one point of A_λ^0 and at least one of the points of $K = \{y_1, \ldots, y_{k+1}\}$. Let $Z \subset \mathbb{R}_+^2$ be the set of all intersection points of two lines in $\bar{\mathcal{L}}$ that lie in $\mathbb{R}_+^2$. Fix $z \in Z$. Since z does not see all $k+1$ points of K via A_λ^0 (by the induction hypothesis), there exists $1 \le i \le k+1$ such that the intersection point between $[z, y_i]$ and the x-axis is not in A_λ^0. We add such an intersection point to B_λ^0 (if there are several such points, we just add one of them arbitrarily). We do this process for all $z \in Z$, thus condition 6 is satisfied. The resulting set is B_λ.

Note that since $|A_\lambda^0|, |B_\lambda^0| \le \max\{|\lambda|, \aleph_0\}$, and since $|\lambda|^2 \le \max\{|\lambda|, \aleph_0\}$, it follows that $|B_\lambda| \le \max\{|\lambda|, \aleph_0\}$. Furthermore, by construction, $B_\lambda \cap A_\lambda^0 = \emptyset$ and $B_0 \subset B_\lambda^0 \subset B_\lambda$, thus condition 5 is satisfied.

Motivation behind the definition of B_λ. In the definition of A_λ to be presented below, we add at most $k-1$ points to A_λ^0. As no point $z \in \mathbb{R}_+^2$ can see more than a single point of K via the same point in A_λ (due to the definition of B_0), a point $z \in \mathbb{R}_+^2$ can see K via A_λ only if it sees at least two points of K via A_λ^0. Z is the set of all "dangerous" points in the upper half-plane, that see via A_λ^0 at least two points of K. By our definition of B_λ, we ensure that once we define A_λ such that $A_\lambda \cap B_\lambda = \emptyset$, no point in Z will see via A_λ all $k+1$ points of K. This guarantees that no point $z \in \mathbb{R}_+^2$ can see K via A_λ.

Defining A_λ. A_λ is obtained by adding at most $k-1$ points to A_λ^0. These points are chosen such that $a^\lambda = \{a_1^\lambda, \ldots, a_k^\lambda\}$ will be seen via A_λ by some point z in the upper open half-plane. Furthermore, the point z will be a point that sees a_1^λ via A_λ^0. (This condition allows us to add only $k-1$ points, rather than k points.)

Let $x \in A_\lambda^0$ be chosen arbitrarily, and let ℓ be the line that passes through x and a_1^λ. Let

$$\mathcal{L} = \{\ell(a_i^\lambda, b) | 2 \le i \le k, b \in B_\lambda\}$$

be the set of all lines $\ell(a_i^\lambda, b)$ that pass through some point $a_i^\lambda \in a^\lambda$, $(2 \le i \le k)$, and some point $b \in B_\lambda$.

Since $|B_\lambda| \le \max(|\lambda|, \aleph_0) < \mathfrak{c}$, and since any line in $\mathcal{L}$ intersects ℓ in at most one point in the upper open half-plane, it follows that there exists a point $z \in \ell \cap \mathbb{R}_+^2$ that is not covered by any line of $\mathcal{L}$. (Recall that no line in $\mathcal{L}$ coincides with ℓ, since ℓ intersects the x-axis in $x \in A_\lambda^0$, while any line in $\mathcal{L}$ intersects the x-axis in some point of B_λ.)

The set A_λ is obtained from A_λ^0 by adding the intersections of the segments $[z, a_i^\lambda]$, for $2 \le i \le k$, with the x-axis. Note that the intersection of $[z, a_1^\lambda]$ with the x-axis is included in A_λ^0 so there is no need to add it. Also note that it is possible that some of the $k-1$ "added" points already belong to A_λ^0, and so fewer than $k-1$ points are added.

Obviously, $A_\lambda^0 \subset A_\lambda$ and $|A_\lambda| \le \max(|\lambda|, \aleph_0)$. By the construction, the added points are not in B_λ, and hence, $A_\lambda \cap B_\lambda = \emptyset$. Furthermore, the point z sees $a_1^\lambda, \ldots, a_k^\lambda$ via A_λ. Finally, the definition of B_λ guarantees that no $z \in \mathbb{R}_+^2$ sees K via A_λ, thus condition 4 is also satisfied. (See the motivation above.) Therefore, A_λ and B_λ satisfy the assertion.

By transfinite induction, this completes the proof of the proposition, and thus, also of the theorem. ◀

▶ **Remark 10.** One can strengthen the assertion of Theorem 4 by considering a countable dense collection of "forbidden" $(k+1)$-sets, instead of a single $(k+1)$-set K. No significant change in the proof method is needed.

3   Proof of Theorem 6

In this section we prove that no Krasnoselskii-type theorem exists for the notion of visibility through a polygonal path of length $\leq n$ (in short, an n-*path*). By constructing a sequence of planar sets $S_{n,k}$, $n, k \geq 2$, each consisting of a finite union of closed segments, we prove:

Theorem 6 (Restatement). For any $n, k \geq 2$, there exists a set $S = S_{n,k} \subset \mathbb{R}^2$ such that every k points in S are visible from a common point through n-paths in S, but some $k + 1$ points of S are not visible from a common point through n-paths in S.

Proof. The proof consists of two steps. In the first step we prove the existence of $S_{2,k}$ for any $k \geq 2$, and in the second step we extend the construction of the first step to obtain $S_{n,k}$ for $n > 2$.

Step 1: Constructing $S_{2,k}$, $k \geq 2$.   We obtain $S_{2,k}$ by an appropriate planar embedding of the complete bipartite graph $K_{k+1,k+1}$ with one perfect matching removed. Let $P \subset \mathbb{R}^2$ be a convex $(2k + 2)$-gon. We assume that no three diagonals of P have a common point inside P. (This can be achieved by slightly perturbing the vertices of P.)

Label the vertices of P cyclically (clockwise) $a_0, b_0, a_1, b_1, \ldots, a_k, b_k$. We regard the indices as numbers modulo $k + 1$ (see Figure 1). Define $\kappa = \lfloor \frac{k}{2} \rfloor$.

The matching to be removed will be

$$\{[a_i, b_{i+\kappa}] : i = 0, 1, \ldots, k\} = \{[a_{i-\kappa}, b_i] : i = 0, 1, \ldots, k\}.$$

Note that the segments $[a_i, b_{i+\kappa}]$ are diagonals, not boundary edges of P.

Define, therefore, for $i = 0, 1, \ldots, k$,

$$B_i = \bigcup \{[a_\nu, b_i] : \nu = 0, 1, \ldots, k, \qquad \nu \neq i - \kappa\},$$

and let

$$S_{2,k} = \bigcup_{i=0}^{k} B_i.$$

$S_{2,2}$ is just a hexagon. $S_{2,3}$ is an octagon with four vertex-disjoint diagonals of order 3. See Figure 1 for $S_{2,4}$ and $S_{2,5}$.

Claims 11 and 12 below assert that in $S_{2,k}$, every k points are visible from a common point through a 2-path, but there exist $k + 1$ points in $S_{2,k}$ that are not visible from any common point through 2-paths.

▷ **Claim 11.**   For any k points $x_1, \ldots, x_k \in S_{2,k}$ there exists a point $z \in S_{2,k}$ that sees all of them through 2-paths in $S_{2,k}$.

Proof. Each point x_i $(1 \leq i \leq k)$ belongs to some B_j, $0 \leq j \leq k$. (If x_i lies in two B_j's, choose one of them arbitrarily.) Hence we may assume, without loss of generality, that $x_1, \ldots, x_k$ are all contained in $B_1 \cup \ldots \cup B_k$. It follows that $a_{k+1-\kappa}$ sees each x_i through a 2-path. Indeed, $a_{k+1-\kappa}$ is connected by a segment in $S_{2,k}$ to each of $b_1, \ldots, b_k$, and each b_i is connected by a segment in $S_{2,k}$ to every point in B_i.   ◁

▷ **Claim 12.**   For $i = 0, 1, \ldots, k$, let c_i be the midpoint of $[b_i, a_{i+1}]$, where the indices are taken modulo $k + 1$. Then $c_0, \ldots, c_k$ are not visible from any common point in $S_{2,k}$ through 2-paths.
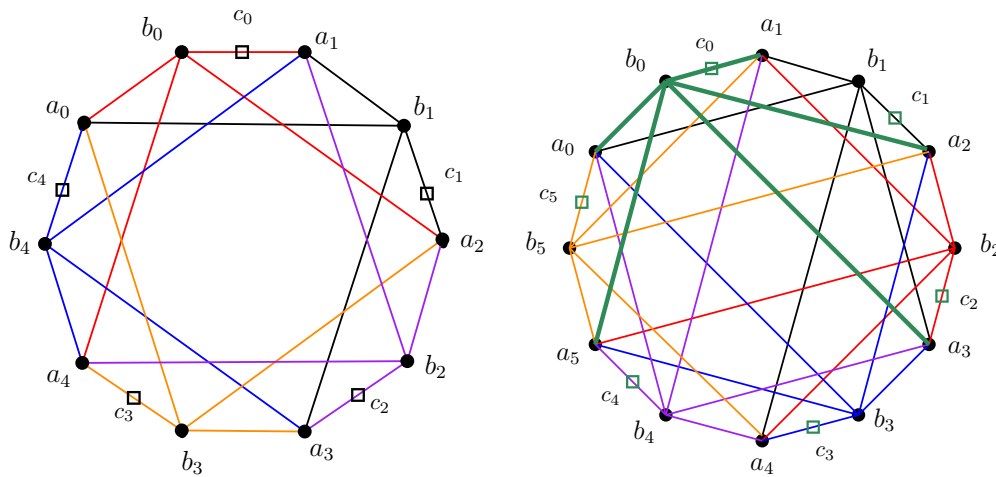
Figure 1 The left figure is $S_{2,4}$ and the right figure is $S_{2,5}$. Each B_i is colored in a different color.

Proof. The cases $k = 2, 3$ are easy to verify. Let $k \geq 4$ and assume to the contrary that some point $z \in S_{2,k}$ sees all the points $c_0, \ldots, c_k$ through 2-paths in $S_{2,k}$. It follows that for each $0 \leq i \leq k$, z sees b_i or a_{i+1} through a 1-path, and in particular, the set of vertices of P that z sees through a 1-path is of size $\geq k + 1 \geq 5$. We reach a contradiction, by considering three cases:

Case 1: z is a vertex of P. Thus $z = a_i$ or $z = b_i$ for some i, $0 \leq i \leq k$. But a_i sees neither $b_{i+\kappa}$, nor $a_{i+\kappa+1}$ through a 1-path in $S_{2,k}$, and b_i sees neither $b_{i-\kappa-1}$, nor $a_{i-\kappa}$ through a 1-path in $S_{2,k}$.

Case 2: z is not a vertex of P, but z lies on a boundary edge of P. In this case, z sees through a 1-path only two vertices of P, a contradiction.

Case 3: z is not a vertex of P, but z lies on a diagonal of P. In this case, z lies on at most two diagonals of P. (Here we use the assumption that no three diagonals of P have a common point inside P.) Thus, z sees through a 1-path at most four vertices of P. This is a contradiction, as z must see at least 5 vertices of P, as was explained above.

This completes the proof. ◁

Step 2: Construction of $S_{n,k}$ for $n > 2, k \geq 2$. Given $n > 2$ and $k \geq 2$, we modify the construction of $S_{2,k}$ to obtain $S_{n,k}$, in the following way: We consider $S_{2,k}$, and for each $0 \leq i \leq k$ we add a simple polygonal path Γ_i of length $n - 2$ (with no two consecutive edges collinear) that starts at c_i and lies outside the $(2k + 2)$-gon P, such that the paths Γ_j ($0 \leq j \leq k$) are pairwise disjoint (see Figure 2). For $0 \leq i \leq k$, define $C_i = B_i \cup \Gamma_i$, and let

$$S_{n,k} = \bigcup_{i=0}^{k} C_i.$$

▷ **Claim 13.** $S_{n,k}$ satisfies the assertion of Theorem 6.

Proof. Same as in the proof of Claim 11, given $x_1, \ldots, x_k \in S_{n,k}$ we may assume that $\{x_1, \ldots, x_k\} \subset C_1 \cup \ldots \cup C_k$. In this case, $a_{k+1-\kappa}$ sees each of the points $b_1, \ldots, b_k$ through a 1-path, and thus, sees every point in $C_1 \cup \ldots \cup C_k$ through an n-path.
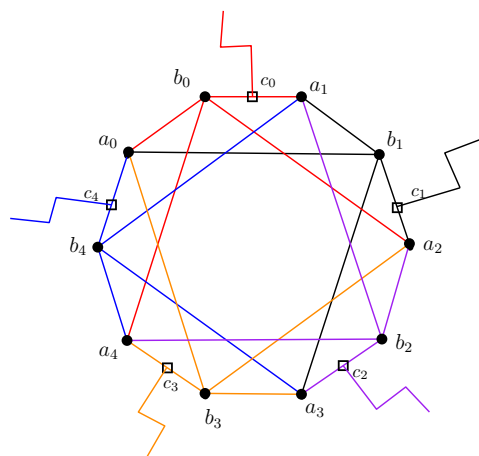
Figure 2 An illustration of $S_{5,4}$. Each C_i is colored in a different color.

On the other hand, we claim that the $k+1$ outer endpoints of $\Gamma_0, \ldots, \Gamma_k$ are not seen by any common point x through n-paths. Indeed, assume on the contrary that some $x \in S_{n,k}$ sees all these points through n-paths. If $x \in S_{2,k}$, then this assumption clearly implies that x sees all the points $c_0, \ldots, c_k$ through 2-paths, contradicting Claim 12. If $x \notin S_{2,k}$ then $x \in \Gamma_i$ for some i. In this case, our assumption implies that even c_i sees the $k+1$ outer endpoints of $\Gamma_0, \ldots, \Gamma_k$ through n-paths. However, $c_i \in S_{2,k}$, and thus we obtain a contradiction, same as in the first case. ◁

This completes the proof of Theorem 6. ◀

4 Open problems

The proof method of Theorem 4 is not constructive, and the resulting set T does not admit any "nice" topological structure (e.g., being a Borel set). Thus, it will be interesting to determine whether one may add some topological restriction on T to the conditions of Theorem 4.

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