Can You Walk This? Eulerian Tours and IDEA Instructions

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Abstract

We illustrate and animate the classic problem of deciding whether a given graph has an Eulerian path. Starting with a collection of instances of increasing difficulty, we present a set of pictorial instructions, and show how they can be used to solve all instances. These *IDEA instructions* ("A series of nonverbal algorithm assembly instructions") have proven to be both entertaining for experts and enlightening for novices. We (w)rap up with a song and dance to Euler's original instance.

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1 Introduction

Deciding whether a connected graph $G = (V, E)$ has an Eulerian path is a natural problem of graph theory: Find a path P that contains all edges in E, starting at a suitable vertex s and ending at a vertex t. As a path, P is connected and may not contain any edge more than once; therefore, following an Eulerian path corresponds to drawing all of the edges in one continuous stroke, without duplicating any edge or lifting the pen – or crossing all bridges in a city in one contiguous walk without duplicating a bridge. First introduced by Euler in 1741 [2], it is not just a classic, but arguably *the* problem that started graph theory itself.

The algorithmic side of the problem spans several centuries, with publications in a variety of languages; see Fig. 1. In his paper (published in Latin), Euler gave a necessary condition for the existence of P: there may be at most two vertices of odd degree, which would have to be s and t. Hierholzer [6] (in an 1873 posthumous paper published in German) gave an algorithmic proof that this condition is sufficient, based on iteratively merging cycles. An alternative was given by Fleury [4] in 1883 (published in French): If the necessary condition is satisfied, it is possible to find an Eulerian path by greedily following edges, starting at an odd-degree vertex s if one exists, subject to never disconnecting the set of unused edges. From an algorithmic point of view, Fleury's algorithm is somewhat inefficient, as it requires keeping track of connected components; from an intuitive perspective, Fleury's method is quite elegant, as it does indeed provide a method for drawing the graph in one stroke, without resorting to retroactively including leftover cycles according to Hierholzer.

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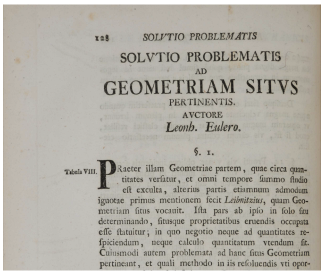


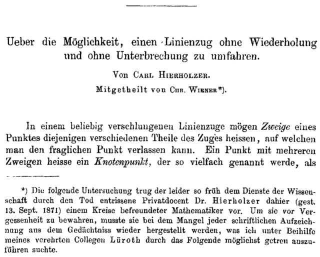
128 SOLVTIO PROBLEMATIS

SOLVTIO PROBLEMATIS AD GEOMETRIAM SITVS PERTINENTIS.

AVCTORE *Leonh. Eulero.*

§. 1.

Tabula VIII. Praeter illam Geometriae partem, quae circa quantitates verſatur, et omni tempore ſummo ſtudio eſt exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit *Leibnitzius*, quam Geometriam ſitus vocauit. Iſta pars ab ipſo in ſolo ſitu determinando, ſitusque proprietatibus eruendis occupata eſſe ſtatuitur; in quo negotio neque ad quantitates reſpiciendum, neque calculo quantitatum vtendum ſit. Cuiusmodi autem problemata ad hanc ſitus Geometriam pertineant, et quali methodo in iis reſoluendis vti opor-



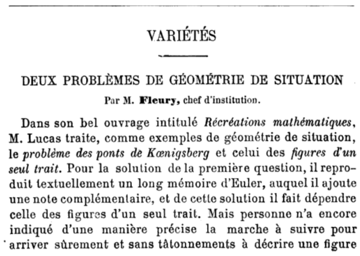
Ueber die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren.

Von Carl Hierholzer.

Mitgetheilt von Chr. Wiener*).

In einem beliebig verschlungenen Linienzuge mögen *Zweige* eines Punktes diejenigen verschiedenen Theile des Zuges heissen, auf welchen man den fraglichen Punkt verlassen kann. Ein Punkt mit mehreren Zweigen heisse ein *Knotenpunkt*, der so vielfach genannt werde, als

*) Die folgende Untersuchung trug der leider so früh dem Dienste der Wissenschaft durch den Tod entrissene Privatdocent Dr. Hierholzer dahier (gest. 13. Sept. 1871) einem Kreise befreundeter Mathematiker vor. Um sie vor Vergessenheit zu bewahren, musste sie bei dem Mangel jeder schriftlichen Aufzeichnung aus dem Gedächtniss wieder hergestellt werden, was ich unter Beihilfe meines verehrten Collegen Lüroth durch das Folgende möglichst getreu auszuführen suchte.



VARIÉTÉS

DEUX PROBLÈMES DE GÉOMÉTRIE DE SITUATION

Par M. **Fleury**, chef d'institution.

Dans son bel ouvrage intitulé *Récréations mathématiques*, M. Lucas traite, comme exemples de géométrie de situation, le *problème des ponts de Kœnigsberg* et celui des *figures d'un seul trait*. Pour la solution de la première question, il reproduit textuellement un long mémoire d'Euler, auquel il ajoute une note complémentaire, et de cette solution il fait dépendre celle des figures d'un seul trait. Mais personne n'a encore indiqué d'une manière précise la marche à suivre pour arriver sûrement et sans tâtonnements à décrire une figure

Figure 1 Three fundamental papers on the topic that describe the relationship between geometry and graph theory in three different languages: (Left) Euler [2], (Center) Hierholzer [6], (Right) Fleury [4].

2 Eulerian paths and education

Euler made use of the combination of a specific instance and a relatively accessible problem to develop the universal and deep concept of graphs: He provided a very concrete example, instead of a series of axioms and abstract definitions. This makes questions of Eulerian paths ideally suited for introducing novices to both graphs and algorithmic problems. Not only first-year undergraduates, but also high-school and even elementary school students quickly grasp the underlying concepts when challenged with a sequence of instances of increasing difficulty. Progressing from easier to harder instances ensures a good balance between pitfalls and rewards, leading to insights and generalizations, providing both motivation, a sense for solutions and difficulties, and appreciation for general methods and algorithms. Such a set of instances was used in actual work with students at all levels, starting at elementary school. The first set of twelve instances (Figure 2) focuses on the role of odd vertices, supporting the discovery of their importance by gradually increasing the level of complexity. The second set (starting with (2.1), not shown in the figure, but used in the video) provides larger instances in which odd-degree vertices do not matter, but the role of connectivity is highlighted.

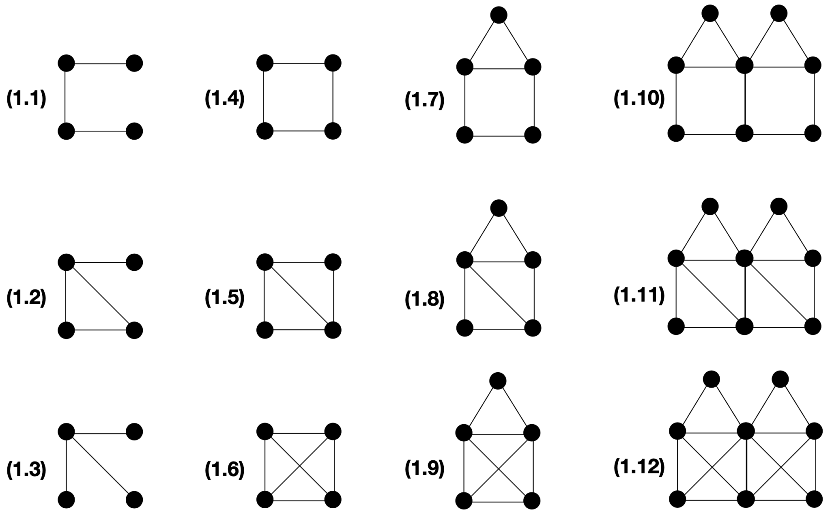


Figure 2 The set of instances used in the video.

3 IDEA instructions

One of the fundamental concepts of mathematics is abstraction: definitions, theorems and proofs ought to be valid regardless of a visual depiction. This angle is also quite natural and important when turning mathematical results into algorithms and computer code: "The computer doesn't need to understand the algorithm, its task is to run the program." (Tarjan [5]). However, the main purpose of a human brain (in particular, one in the process of being trained) is not to run code, but to develop an understanding for the underlying concepts and methods.

That is why two of us (Sándor Fekete and Sebastian Morr) have designed a series of instruction sheets that show algorithms without words, loosely inspired by furniture assembly instructions, but more challenging: While a piece of furniture is just a single, specific instance, an algorithm has to be ready for *any* instance. (See our website [3].) The absence of words forces the user to interpret the meaning of images, thereby enhancing the role of intuitive understanding. In combination with sets of instances, this encourages experimentation, so the IDEA instructions may also play the role of "cheat sheets" that provide light-bulb moments when trying to overcome a challenging difficulty. Experience shows that this strengthens both the interpretation of individual steps and the appreciation for crucial tricks.

For the specific problem of Eulerian paths, our instruction sheet (shown in Figure 3) provides a visual description of Fleury's algorithm. The setup proceeds in the usual algorithmic manner (given/wanted, input/output). The first step describes identifying the odd-degree vertices, while the second step instructs the user to mark them in the actual instance. Just like furniture instructions or mathematical structures, this comes with a minor puzzle for the user to figure out: Can there be instances with only a single odd-degree vertex? This has turned out to be a tantalizing question for curious students of all ages, and serves as a bridge (no pun intended) to further mathematical explorations.

The second part of the instruction sheet carries out the actual algorithm. This is split up between carrying out specific steps (left column) and figuring out a suitable next edge (right column). The latter is easily the most challenging step, in particular in a general, abstract manner, as it requires discovering the concepts of connected components and k-edge connectivity. However, given the context of bridges between islands, students are usually able to figure out this step with a combination of studying instances and referring to the instructions. This also highlights the seam between intuition and formalism, driving home the importance of clear definitions and setting the stage for universal notation and code: How can these picture puzzle steps be formalized when working towards actual implementations?

4 The Video

The video opens with the original problem in Königsberg, followed by a young boy presented with the problem set shown in Figure 2. After a sequence of successes and failures, the scene switches to the IDEA instruction sheet shown in Figure 3. This is applied to the set of instances, animating the algorithmic progress on each instance and the visual code. The video switches back to the young boy solving a large, exam-level problem for university students. The video concludes with a rap performance based on the following poem by Bill Tutte, under the pen name Blanche Descartes [1].

Some citizens of Königsberg
Were walking on the strand
Beside the river Pregel
With its seven bridges spanned.

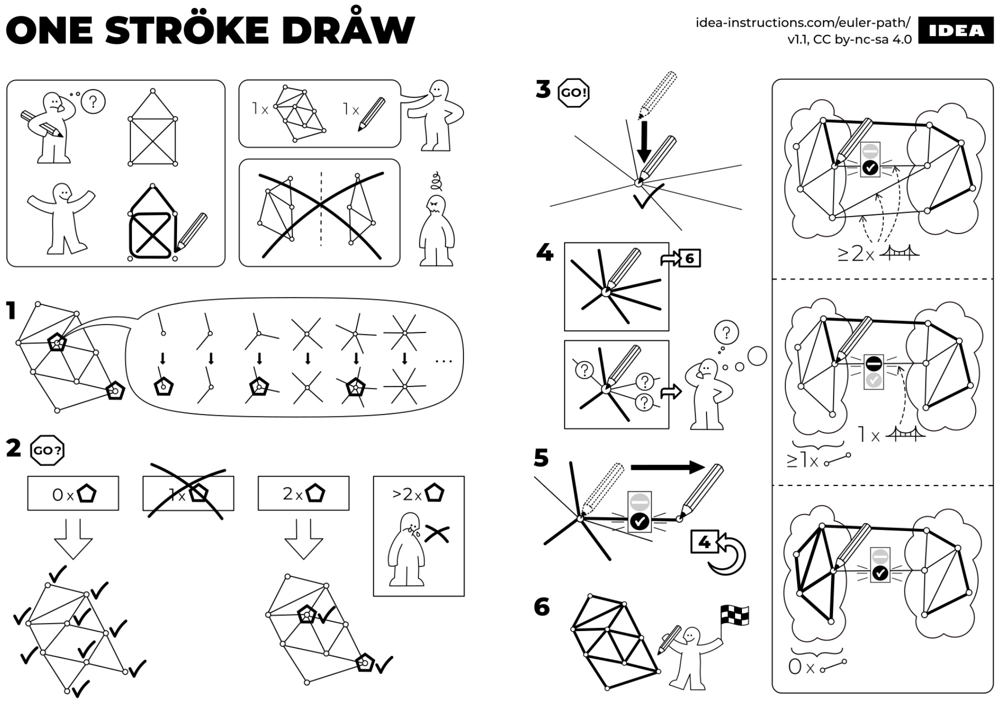


Figure 3 IDEA instructions for finding an Eulerian path: algorithmic description without words.

"O Euler, come and walk with us,"
Those burghers did beseech.
"We'll walk the seven bridges o'er,
And pass but once by each."

"It can't be done," thus Euler cried.
"Here comes the Q.E.D.
Your islands are but vertices
And four have odd degree."

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5 Karen A Frenkel. An interview with the 1986 ACM Turing award recipients – John E. Hopcroft and Robert E. Tarjan. *Communications of the ACM*, 30(3):214–222, 1987.

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