Duality in Intuitionistic Propositional Logic

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Abstract

It is known that provability in propositional intuitionistic logic is PSPACE-complete. As PSPACE is closed under complements, there must exist a LOGSPACE-reduction from refutability to provability. Here we describe a direct translation: given a formula φ, we define $\overline{\varphi}$ so that $\overline{\varphi}$ is provable if and only if φ is not.

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Introduction

A dual concept to proof theory is refutation theory [9] where one asks how to refute or disprove a formula. Various refutation systems occur in the literature, e.g. [2, 4, 9] to derive formal refutations. This paper takes a look from another angle: we ask if one can internalize refutability as provability. A positive answer to this question may depend on the particular logic, the intuitionistic propositional calculus (IPC) being a most promising case. Indeed, the PSPACE-completeness of IPC means that non-provability is LOGSPACE reducible to provability and vice versa. Here we show how to construct, for a given formula φ, another formula $\overline{\varphi}$ which is provable if and only if φ is not. The construction works in logarithmic space, in particular in polynomial time.

The inspiration for our approach comes from a computational interpretation of logic, which can be seen as yet another side of Curry-Howard isomorphism, namely the equivalence:

$$\textit{Proof construction} \quad \Leftrightarrow \quad \textit{Computation}$$

This paradigm is implicitly exploited by many authors, especially in hardness and undecidability proofs, e.g. [5], but it is rarely explicitly formulated. The idea is extremely simple: the task to prove a judgment of the form

$$\Gamma \vdash \tau$$

is nothing else than a configuration of a machine, where
- τ is the present internal state, and
- Γ is the contents of memory.

The use of machine memory has to be cautious: usually assumptions are non-disposable (unless we deal with some substructural logic) and one cannot verify that an assumption is not available. On the other hand, proof search algorithms naturally use both nondeterministic choices and universal splits (recursive calls). Machines adequate for IPC should therefore be defined as alternating automata operating on write-once binary registers. Every register represents an assumption: the value true means that the assumption is available. Registers can only be accessed as positive guards: to execute an action the machine may have to check that a given register is set to true. A register cannot be checked for the value false nor unset to false. A variant of this model is mentioned in [6], an elaborated first-order version is developed in [7]. The Wajsberg/Ben-Yelles algorithm for IPC, like in [10], can easily be implemented on such monotone automata [8]. On the other hand, for every monotone automaton M one can construct a formula φ_M such that φ_M is a theorem of IPC if and only if M halts. The formula φ_M can be defined using implication as the only conjective. So it is just a simple type and only of order 3.

Under this understanding, our construction in this paper can be seen as complementation of monotone automata: given an automaton M define another automaton $\overline{M}$ so that $\overline{M}$ halts if and only if M does not halt. Such interpretation inspired our presentation below which can be easily translated to the language of automata. This could make the whole development somewhat more concise and technically direct, but we decided to remain within the language of propositional logic, to demonstrate its flexibility.

Our goal is to define a formula $\overline{\varphi}$ that has a proof when a given φ has none. What $\overline{\varphi}$ actually states is that φ has no *normal* proof *without repeated judgments* (and therefore of bounded size). To handle the first aspect we use lambda-notation for proofs and we appeal to normalization. To control proof size we found it convenient to define a restricted version of natural deduction rules (Figure 1) where additional annotations are used to disallow cycles.

Natural deduction

We consider propositional formulas built from the connectives $\wedge$, $\vee$, $\to$ and $\bot$. Variables and $\bot$ are called *atoms*. Negation $\neg\alpha$ is defined as $\alpha \to \bot$. We assume that implication is right-associative, i.e., we write $\alpha \to \beta \to \gamma$ for $\alpha \to (\beta \to \gamma)$. If $\mathcal{S} = \{\alpha_1, \ldots, \alpha_k\}$ then $\mathcal{S} \to \beta$ abbreviates any formula of the form $\alpha_1 \to \cdots \to \alpha_k \to \beta$ (disregarding the order of premises). Notation for sets of formulas is simplified, e.g. Γ, Σ stands for $\Gamma \cup \Sigma$ and Γ, α for $\Gamma \cup \{\alpha\}$.

Our natural deduction calculus (Figure 1) derives judgments of the form $\Gamma \vdash \alpha\,[\Sigma]$, where Γ and Σ are sets of formulas, and α is a formula. The meaning of $\Gamma \vdash \alpha\,[\Sigma]$ is that the ordinary judgment $\Gamma \vdash \alpha$ is provable without (directly) addressing proof goals in Σ. To see this, one reads the rules upwards, in the order of proof search. Then, at every step, the set Σ of forbidden goals is expanded by the current goal unless a new assumption is added; then Σ is reset to $\varnothing$. This protocol ensures that no judgment can be repeated on any proof branch. Note that the rules are "upward-preserving" in that all assumptions occurring in conclusion must occur in the premises as well.

A convenient proof notation for propositional intuitionistic logic is an extended lambda-calculus as e.g. in [1]. From this point of view, natural deduction becomes a type-assignment (or, perhaps more adequately, "term-assignment" or "proof-assignment") system (Figure 2), deriving judgments $\Gamma \vdash M : \alpha\,[\Sigma]$ with the additional term component M. (N.B. we identify α-convertible terms.) Strictly speaking, Γ can no longer be just a set of formulas and must be understood as a *type environment*, i.e., a set of variable declarations $(x : \alpha)$. Fortunately, we do not need to consider environments Γ involving more than one declaration of the same α. To make it precise, we say that an environment is *simple* when $(x : \alpha), (y : \alpha) \in \Gamma$ implies $x = y$. Simple environments can thus be identified with sets of formulas. In Figure 2, we assume Γ simple in all rules,[1] so the notation $\alpha \in \Gamma$ (resp. $\alpha \notin \Gamma$) can safely be read as "there is (resp. is not) a variable of type α in Γ"). Note that in rule (W$\to_2$) it is assumed that $\gamma \in \Gamma$ despite the lambda-introduction.

[1] The insightful reader should note that we do not claim subject reduction for the system in Figure 2, cf. e.g. [3]; the existence of normal forms is inherited from the standard system.

$$(\mathrm{Ax})\quad \Gamma \vdash \alpha\,[\Sigma]\ [\alpha \in \Gamma] \qquad\qquad (\mathrm{E}\bot)\ \frac{\Gamma \vdash \bot\,[\Sigma,\alpha]}{\Gamma \vdash \alpha\,[\Sigma]}\,[\bot \notin \Sigma]$$

$$(\mathrm{E}\wedge_1)\ \frac{\Gamma \vdash \alpha \wedge \beta\,[\Sigma,\alpha]}{\Gamma \vdash \alpha\,[\Sigma]}\,[\alpha \wedge \beta \notin \Sigma\,] \qquad (\mathrm{E}\wedge_2)\ \frac{\Gamma \vdash \beta \wedge \alpha\,[\Sigma,\alpha]}{\Gamma \vdash \alpha\,[\Sigma]}\,[\beta \wedge \alpha \notin \Sigma\,]$$

$$(\mathrm{W}\wedge)\ \frac{\Gamma \vdash \gamma\,[\Sigma,\gamma \wedge \delta] \quad \Gamma \vdash \delta\,[\Sigma,\gamma \wedge \delta]}{\Gamma \vdash \gamma \wedge \delta\,[\Sigma]}\,[\gamma,\delta \notin \Sigma]$$

$$(\mathrm{E}\vee)\ \frac{\Gamma \vdash \gamma \vee \delta\,[\Sigma,\alpha] \quad \Gamma, x:\gamma \vdash \alpha[\varnothing] \quad \Gamma, y:\delta \vdash \alpha[\varnothing]}{\Gamma \vdash \alpha\,[\Sigma]}\,[\gamma \vee \delta \notin \Sigma, \gamma,\delta \notin \Gamma]$$

$$(\mathrm{W}\vee_1)\ \frac{\Gamma \vdash \gamma\,[\Sigma,\gamma \vee \delta]}{\Gamma \vdash \gamma \vee \delta\,[\Sigma]}\,[\gamma \notin \Sigma] \qquad (\mathrm{W}\vee_2)\ \frac{\Gamma \vdash \delta\,[\Sigma,\gamma \vee \delta]}{\Gamma \vdash \gamma \vee \delta\,[\Sigma]}\,[\delta \notin \Sigma]$$

$$(\mathrm{E}{\to})\ \frac{\Gamma \vdash \beta \to \alpha\,[\Sigma,\alpha] \quad \Gamma \vdash \beta\,[\Sigma,\alpha]}{\Gamma \vdash \alpha\,[\Sigma]}\,[\beta, \beta \to \alpha \notin \Sigma]$$

$$(\mathrm{W}{\to}_1)\ \frac{\Gamma, \gamma \vdash \delta[\varnothing]}{\Gamma \vdash \gamma \to \delta\,[\Sigma]}\,[\gamma \notin \Gamma] \qquad (\mathrm{W}{\to}_2)\ \frac{\Gamma \vdash \delta\,[\Sigma, \gamma \to \delta]}{\Gamma \vdash \gamma \to \delta\,[\Sigma]}\,[\delta \notin \Sigma, \gamma \in \Gamma]$$

Figure 1 Natural deduction.

It is convenient to use term notation to express properties of proofs. But the principal use of lambda-terms is that they normalize, and thus proof search can be restricted to lambda-terms in normal form [1].

We are very relaxed regarding the notation. For example we write $\Gamma \vdash M : \alpha\,[\Sigma]$ when it is convenient to mention the proof M, and and $\Gamma \vdash \alpha\,[\Sigma]$ when M is not relevant. Lambda-terms are, for simplicity, written in Curry-style (without type decoration) but types are always implicit, and can be marked e.g. as superscripts, whenever it is useful. The notation $\Gamma \vdash \alpha$ and $\Gamma \vdash M : \alpha$ means ordinary intuitionistic provability and term-assignment as in [1]. Substitution of N for free occurrences of x in M is written $M[x := N]$.

The following definition is needed for the proof of completeness of our system (Lemma 2). Let a term M be typable in a simple environment Γ. The set $U_\Gamma(M)$ of types *directly used in* M with respect to Γ is defined by induction below. Informally, members of $U_\Gamma(M)$ are (with one exception) types of proper subterms of M, not in scope of a variable binding in M. The exception is a lambda-abstraction representing an unnecessary (duplicated) assumption.

$U_\Gamma(x) = \varnothing$;
$U_\Gamma(P^{\gamma\to\delta}M^\gamma) = U_\Gamma(P) \cup U_\Gamma(M) \cup \{\gamma \to \delta, \gamma\}$;
$U_\Gamma(\lambda x^\gamma.\, N^\delta) = U_\Gamma(N[x := y]) \cup \{\delta\}$, when $(y : \gamma) \in \Gamma$, and $U_\Gamma(\lambda x^\gamma.\, N^\delta) = \varnothing$, otherwise;
$U_\Gamma(M[\alpha]) = U_\Gamma(M) \cup \{\bot\}$;
$U_\Gamma(P^{\gamma\wedge\delta}\{1\}) = U_\Gamma(P^{\gamma\wedge\delta}\{2\}) = U_\Gamma(P) \cup \{\gamma \wedge \delta\}$;
$U_\Gamma(\langle M^\gamma, N^\delta\rangle) = U_\Gamma(M) \cup U_\Gamma(N) \cup \{\gamma, \delta\}$;
$U_\Gamma(\langle M^\gamma\rangle_1) = U_\Gamma(M) \cup \{\gamma\}$, and $U_\Gamma(\langle M^\delta\rangle_2) = U_\Gamma(M) \cup \{\delta\}$;
$U_\Gamma(P^{\gamma\vee\delta}[x.M, y.N]) = U_\Gamma(P) \cup \{\gamma \vee \delta\}$.

$$\text{(Ax)}\quad \Gamma, x:\alpha \vdash x:\alpha\,[\Sigma] \qquad\qquad \text{(E}\bot\text{)}\quad \frac{\Gamma \vdash P:\bot\,[\Sigma,\alpha]}{\Gamma \vdash P[\alpha]:\alpha\,[\Sigma]}\,[\bot \notin \Sigma]$$

$$\text{(E}\wedge_1\text{)}\quad \frac{\Gamma \vdash P:\alpha\wedge\beta\,[\Sigma,\alpha]}{\Gamma \vdash P\{1\}:\alpha\,[\Sigma]}\,[\alpha\wedge\beta \notin \Sigma] \qquad \text{(E}\wedge_2\text{)}\quad \frac{\Gamma \vdash P:\beta\wedge\alpha\,[\Sigma,\alpha]}{\Gamma \vdash P\{2\}:\alpha\,[\Sigma]}\,[\beta\wedge\alpha \notin \Sigma]$$

$$\text{(W}\wedge\text{)}\quad \frac{\Gamma \vdash M:\gamma\,[\Sigma,\gamma\wedge\delta] \quad \Gamma \vdash N:\delta\,[\Sigma,\gamma\wedge\delta]}{\Gamma \vdash \langle M,N\rangle:\gamma\wedge\delta\,[\Sigma]}\,[\gamma,\delta \notin \Sigma]$$

$$\text{(E}\vee\text{)}\quad \frac{\Gamma \vdash P:\gamma\vee\delta\,[\Sigma,\alpha] \quad \Gamma, x:\gamma \vdash M:\alpha[\varnothing] \quad \Gamma, y:\delta \vdash N:\alpha[\varnothing]}{\Gamma \vdash P[x.M,y.N]:\alpha\,[\Sigma]}\,[\gamma\vee\delta \notin \Sigma, \gamma,\delta \notin \Gamma]$$

$$\text{(W}\vee_1\text{)}\quad \frac{\Gamma \vdash M:\gamma\,[\Sigma,\gamma\vee\delta]}{\Gamma \vdash \langle M\rangle_1:\gamma\vee\delta\,[\Sigma]}\,[\gamma \notin \Sigma] \qquad \text{(W}\vee_2\text{)}\quad \frac{\Gamma \vdash M:\delta\,[\Sigma,\gamma\vee\delta]}{\Gamma \vdash \langle M\rangle_2:\gamma\vee\delta\,[\Sigma]}\,[\delta \notin \Sigma]$$

$$\text{(E}\to\text{)}\quad \frac{\Gamma \vdash P:\beta\to\alpha\,[\Sigma,\alpha] \quad \Gamma \vdash M:\beta\,[\Sigma,\alpha]}{\Gamma : PM:\alpha\,[\Sigma]}\,[\beta,\beta\to\alpha \notin \Sigma]$$

$$\text{(W}\to_1\text{)}\quad \frac{\Gamma, x:\gamma \vdash M:\delta[\varnothing]}{\Gamma \vdash \lambda x\,M:\gamma\to\delta\,[\Sigma]}\,[\gamma \notin \Gamma] \qquad \text{(W}\to_2\text{)}\quad \frac{\Gamma \vdash M:\delta\,[\Sigma,\gamma\to\delta]}{\Gamma \vdash \lambda x\,M:\gamma\to\delta\,[\Sigma]}\,[\delta \notin \Sigma, \gamma \in \Gamma]$$

Figure 2 Extended lambda-calculus.

▶ **Lemma 1.** *If $\Gamma \vdash M : \alpha$, and $\beta \in U_\Gamma(M)$, then $\Gamma \vdash N : \beta$, for some term N, shorter than M. In particular, if M is the shortest term of type α in Γ, then $\alpha \notin U_\Gamma(M)$.*

Proof. Easy induction with respect to M. ◀

▶ **Lemma 2.** *The system in Figure 2 is sound and complete in the following sense:*

- *If $\Gamma \vdash M : \alpha\,[\Sigma]$, for some Σ, then $\Gamma \vdash M : \alpha$;*
- *If $\Gamma \vdash M : \alpha$, and Γ is simple, then $\Gamma \vdash M : \alpha\,[\Sigma]$, for all Σ with $\Sigma \cap U_\Gamma(M) = \varnothing$.*

In particular, $\Gamma \vdash M : \alpha$ is equivalent to $\Gamma \vdash M : \alpha\,[\varnothing]$.

Proof. The first part follows by a simple erasure of the unnecessary annotations. The second part we prove by induction with respect to the size of a smallest possible witness M such that $\Gamma \vdash M : \alpha$ holds. From Lemma 1 we know that $\alpha \notin U_\Gamma(M)$.

If M is a variable then the claim holds trivially. Assume that $M = P[\alpha]$ with P of type $\bot$. Then $\alpha \notin U_\Gamma(M) = U_\Gamma(P) \cup \{\bot\}$, hence $(\Sigma, \alpha) \cap U_\Gamma(P) = \varnothing$. Observe that P is the shortest term of type $\bot$, hence $\Gamma \vdash P : \bot\,[\Sigma, \alpha]$ holds by induction. Also $\Sigma \cap U_\Gamma(M) = \varnothing$, so $\bot \notin \Sigma$, rule (E$\bot$) applies, and yields $\Gamma \vdash M : \alpha\,[\Sigma]$.

Consider the case $M = P[x.R, y.N]$ with P of type $\gamma \vee \delta$. Then $\Gamma \vdash P : \gamma \vee \delta\,[\Sigma, \alpha]$ holds by induction, because Σ, α is disjoint with $U_\Gamma(P)$, as $\alpha \notin U_\Gamma(M) \supseteq U_\Gamma(P)$. Now note that $\gamma, \delta \notin \Gamma$, as otherwise either R or N would make a shorter inhabitant of α than M. It follows that environments Γ, γ and Γ, δ are simple. Hence the judgments $\Gamma, x : \gamma \vdash R : \alpha[\varnothing]$ and $\Gamma, y : \delta \vdash N : \alpha[\varnothing]$ also hold by induction, because the empty set is disjoint with everything. To apply rule (E$\vee$) we check that $\gamma \vee \delta \notin \Sigma$ because $\gamma \vee \delta \in U_\Gamma(M)$.

As another example consider $\alpha = \gamma \to \delta$, and let $M = \lambda x\, N$. Then $\Gamma, x : \gamma \vdash N : \delta$, and we have two cases depending on whether $\gamma \in \Gamma$ or not. If $\gamma \notin \Gamma$ then $\Gamma, x : \gamma$ is a simple environment, and $\Gamma, x : \gamma \vdash N : \delta\,[\varnothing]$ holds by the induction hypothesis. Thus $\Gamma \vdash \lambda x\, N : \gamma \to \delta\,[\Sigma]$ using rule $(\mathrm{W}{\to}_1)$.

If $\gamma \in \Gamma$, say $(y : \gamma) \in \Gamma$, then $\Gamma, x : \gamma$ is not simple. But then $\Gamma \vdash N[x := y] : \delta$. The term $N[x := y]$ is of the same size as N, so it is still a smallest possible term of type δ. Now $U_\Gamma(N[x := y]) \cap (\Sigma, \gamma \to \delta) = \varnothing$ because $\gamma \to \delta \notin U_\Gamma(M) \supseteq U_\Gamma(N[x := y])$ and $U_\Gamma(M) \cap \Sigma = \varnothing$. So we can apply the induction hypothesis to $\Gamma \vdash N[x := y] : \delta$ and infer $\Gamma \vdash N[x := y] : \delta\,[\Sigma, \gamma \to \delta]$. Since $\delta \in U_\Gamma(M)$, we have $\delta \notin \Sigma$, so rule $(\mathrm{W}{\to}_2)$ yields $\Gamma \vdash \lambda y.\, N[x := y] : \gamma \to \delta\,[\Sigma]$ and it remains to observe that $\lambda y.\, N[x := y]$ is just the same term as $\lambda x\, N$. Other cases are similar. ◀

The construction

In what follows we fix a formula φ and we define a formula $\overline{\varphi}$ to satisfy the equivalence:

$$\nvdash \varphi \qquad \Leftrightarrow \qquad \vdash \overline{\varphi}. \tag{*}$$

Let φ be of length n and let $\mathcal{S}$ be the set of all subformulas of φ. Then $\mathcal{S}$ has at most n elements. For $\alpha, \beta \in \mathcal{S}$, and $t = 0, \ldots, n$, the following propositional symbols may occur in $\overline{\varphi}$:

- $\mathsf{D}_{\alpha,t}$ – "Refute α in $n - t$ phases";
- $\mathsf{D}'_{\alpha,t}$ – "Refute α in $n - t$ phases without addressing goal α again";
- $\mathsf{P}_{\alpha,t}$ – "Assumption α present in phase t";
- $\mathsf{N}_{\alpha,t}$ – "Assumption α not added in phase t";
- $\mathsf{X}_{\alpha,t}$ – "Goal α cannot be derived in phase t using the axiom rule";
- $\mathsf{W}_{\alpha,t}$, $\mathsf{W}^1_{\alpha,t}$, $\mathsf{W}^2_{\alpha,t}$ – "Goal α cannot be derived in phase t by introduction";
- $\mathsf{E}_{\alpha,\beta,t}$ – "Goal α cannot be derived in phase t by elimination of β".

Atoms subscripted by t are called *t-atoms*. The intuitive meaning of those is given above. However, the role of $\mathsf{D}_{\alpha,t}$ is twofold and depends on whether $\mathsf{D}_{\alpha,t}$ occurs as a proof goal or as an assumption. While proving $\mathsf{D}_{\alpha,t}$ amounts to disproving α, an assumption of $\mathsf{D}_{\alpha,t}$ should be read as disqualifying α as a possible proof goal.

When $\beta \in \mathcal{S}$, $t \le n$, $\Gamma \subseteq \mathcal{S}$, we use the abbreviations:

- $\mathcal{A}_{\beta,t} = \{\mathsf{P}_{\beta,t}\} \cup \{\mathsf{N}_{\alpha,t} \mid \alpha \in \mathcal{S} \wedge \alpha \ne \beta\}$ – "The unique assumption added in phase t is β";
- $\mathcal{N}_{\beta,t\downarrow} = \{\mathsf{N}_{\beta,u} \mid u \le t\}$ – "Formula β not assumed until phase t";
- $\mathcal{N}_{\Gamma,t\downarrow} = \{\mathsf{N}_{\beta,u} \mid u \le t \wedge \beta \notin \Gamma\}$ – "No formula outside of Γ assumed until phase t";
- $\mathcal{P}_{\Gamma,t} = \{\mathsf{P}_{\gamma,t} \mid \gamma \in \Gamma\}$ – "Formulas in Γ assumed in phase t";
- $\mathcal{D}_{\Sigma,t} = \{\mathsf{D}_{\beta,t} \mid \beta \in \Sigma\}$ – "Goals in Σ are forbidden in phase t".

The formula $\overline{\varphi}$ to be defined has the form:

$$\overline{\varphi} = \Delta \to \mathsf{D}_{\varphi,0},$$

where Δ is the set consisting of the following implicational formulas (called *axioms*):

1. $\mathsf{N}_{\alpha,0}$, for all $\alpha \in \mathcal{S}$;
2. $\mathsf{P}_{\alpha,t} \to \mathsf{P}_{\alpha,t+1}$, for all $\alpha \in \mathcal{S}$, $t < n$;
3. $(\mathsf{D}_{\alpha,t} \to \mathsf{D}'_{\alpha,t}) \to \mathsf{D}_{\alpha,t}$, for all $\alpha \in \mathcal{S}$, and all $t \le n$;
4. $\mathcal{M}_{\alpha,t} \to \mathsf{D}'_{\alpha,t}$, for all $\alpha \in \mathcal{S}$, and all $t \le n$, where the set $\mathcal{M}_{\alpha,t}$ consists of the atoms:
   - $\mathsf{X}_{\alpha,t}$;
   - $\mathsf{E}_{\alpha,\perp,t}$, and $\mathsf{E}_{\alpha,\beta\wedge\alpha,t}$, $\mathsf{E}_{\alpha,\alpha\wedge\beta,t}$, $\mathsf{E}_{\alpha,\beta\vee\gamma,t}$, $\mathsf{E}_{\alpha,\beta\to\alpha,t}$, for all $\beta, \gamma \in \mathcal{S}$;
   - $\mathsf{W}_{\alpha,t}$, in case α is not an atom;

5. $\mathcal{N}_{\alpha,t\downarrow} \to \mathsf{X}_{\alpha,t}$;
6. $\mathsf{D}_{\gamma,t} \to \mathsf{W}_{\alpha,t}$, and $\mathsf{D}_{\delta,t} \to \mathsf{W}_{\alpha,t}$, for $\alpha = \gamma \wedge \delta$;
7. $\mathsf{D}_{\gamma,t} \to \mathsf{D}_{\delta,t} \to \mathsf{W}_{\alpha,t}$, for $\alpha = \gamma \vee \delta$;
8. $\mathsf{W}^1_{\alpha,t} \to \mathsf{W}^2_{\alpha,t} \to \mathsf{W}_{\alpha,t}$, $\quad \mathsf{P}_{\gamma,t} \to \mathsf{W}^1_{\alpha,t}$, $\quad (\mathcal{A}_{\gamma,t+1} \to \mathsf{D}_{\delta,t+1}) \to \mathsf{W}^1_{\alpha,t}$, $\quad \mathcal{N}_{\gamma,t\downarrow} \to \mathsf{W}^2_{\alpha,t}$, and $\mathsf{D}_{\delta,t} \to \mathsf{W}^2_{\alpha,t}$, for $\alpha = \gamma \to \delta$;
9. $\mathsf{D}_{\perp,t} \to \mathsf{E}_{\alpha,\perp,t}$;
10. $\mathsf{D}_{\alpha\wedge\beta,t} \to \mathsf{E}_{\alpha,\alpha\wedge\beta,t}$, and $\mathsf{D}_{\beta\wedge\alpha,t} \to \mathsf{E}_{\alpha,\beta\wedge\alpha,t}$, for all $\beta \in \mathcal{S}$;
11. $\mathsf{D}_{\beta,t} \to \mathsf{E}_{\alpha,\beta\to\alpha,t}$, and $\mathsf{D}_{\beta\to\alpha,t} \to \mathsf{E}_{\alpha,\beta\to\alpha,t}$, for all $\beta \in \mathcal{S}$;
12. $\mathsf{D}_{\gamma\vee\delta,t} \to \mathsf{E}_{\alpha,\gamma\vee\delta,t}$, $\mathsf{P}_{\gamma,t} \to \mathsf{E}_{\alpha,\gamma\vee\delta,t}$, $\mathsf{P}_{\delta,t} \to \mathsf{E}_{\alpha,\gamma\vee\delta,t}$, $(\mathcal{A}_{\delta,t+1} \to \mathsf{D}_{\alpha,t+1}) \to \mathsf{E}_{\alpha,\gamma\vee\delta,t}$, and $(\mathcal{A}_{\gamma,t+1} \to \mathsf{D}_{\alpha,t+1}) \to \mathsf{E}_{\alpha,\gamma\vee\delta,t}$, for all $\delta, \gamma \in \mathcal{S}$.

The main equivalence (*) follows from Lemma 3 below, for $\Gamma = \Sigma = \varnothing$, $\alpha = \varphi$, and $t = 0$. (Note that $\mathcal{N}_{\varnothing,0\downarrow} \subseteq \Delta$, $\mathcal{P}_{\varnothing,0} = \mathcal{D}_{\varnothing,0} = \varnothing$.)

▶ **Lemma 3.** *For $|\Gamma| = t$, and $\alpha \notin \Sigma$:*

$$\Gamma \nvdash \alpha\,[\Sigma] \qquad \textit{iff} \qquad \Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,t} \vdash \mathsf{D}_{\alpha,t}.$$

Proof of Lemma 3

We begin with some additional notation and preparatory lemmas. Consider a set of atoms of shape $\Xi = \mathcal{N}, \mathcal{P}, \mathcal{D}$, where $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{D}$ consist, respectively, of atoms of the form $\mathsf{N}_{\alpha,u}$, $\mathsf{P}_{\alpha,u}$, and $\mathsf{D}_{\alpha,u}$. Write $\max\Xi$ for the largest u such that a u-atom occurs in Ξ. For $t = \max\Xi$, define $|\Xi|_t = |\mathcal{N}|_{t\downarrow}, |\mathcal{P}|_t, |\mathcal{D}|_t$, where:

$$|\mathcal{N}|_{t\downarrow} = \bigcup\{\mathcal{N}_{\alpha,t\downarrow} \mid \mathcal{N}_{\alpha,t\downarrow} \subseteq \mathcal{N}\}, \quad |\mathcal{P}|_t = \{\mathsf{P}_{\alpha,t} \mid \exists u \le t.\, \mathsf{P}_{\alpha,u} \in \mathcal{P}\}, \quad |\mathcal{D}|_t = \{\mathsf{D}_{\alpha,t} \mid \mathsf{D}_{\alpha,t} \in \mathcal{D}\}.$$

The set $|\Xi|_t$ consists of atoms (either occurring in Ξ or derivable from Ξ) that are relevant towards a t-atomic proof goal, as stated in Lemma 5.

▶ **Lemma 4.** *Let $\Xi = \mathcal{N}, \mathcal{P}, \mathcal{D}$ be as above, with $t = \max\Xi$. If $\Delta, \Xi \vdash \mathsf{P}_{\alpha,u}$, for some $u \le t$, then there is $v \le u$ such that $\mathsf{P}_{\alpha,v} \in \mathcal{P}$. In particular, $\mathsf{P}_{\alpha,t} \in |\mathcal{P}|_t$.*

Proof. Induction with respect to the size of a normal term M such that $\Delta, \Xi \vdash M : \mathsf{P}_{\alpha,u}$. If M is a variable then $v = u$. Otherwise $M = xN$, where x is a variable of type $\mathsf{P}_{\alpha,u-1} \to \mathsf{P}_{\alpha,u}$ and N has type $\mathsf{P}_{\alpha,u-1}$, as only axioms (2) have targets of the form $\mathsf{P}_{\alpha,u}$. We apply the induction hypothesis to N. ◀

We define the *weight* of a term M as the number of symbols in M, excluding parentheses and occurrences of variables of type (2).

▶ **Lemma 5.** *Let $\Xi = \mathcal{N}, \mathcal{P}, \mathcal{D}$ be as above, with $t = \max\Xi$, and let C_t be a t-atom, not of the form $\mathsf{N}_{\alpha,t}$. If $\Delta, \Xi \vdash M : \mathsf{C}_t$, and M is normal, then also $\Delta, |\Xi|_t \vdash M' : \mathsf{C}_t$, where the weight of the proof M' does not exceed the weight of M.*

Proof. Induction with respect to the weight of a normal term M such that $\Delta, \Xi \vdash M : \mathsf{C}_t$. Clearly, M must be an application of an assumption variable to zero, one or more arguments. If M is a variable (has no arguments), then $\mathsf{C}_t \in \Xi$. Then also $\mathsf{C}_t \in |\Xi|_t$, by definition (recall that atoms $\mathsf{N}_{\alpha,t}$ are excluded), and we can take $M' = M$.

Otherwise let x be the head variable of M. The type of x is one of the axioms in Δ. Assume first that $x : \mathsf{P}_{\alpha,t-1} \to \mathsf{P}_{\alpha,t}$. Then M has type $\mathsf{P}_{\alpha,t}$, whence $\mathsf{P}_{\alpha,t} \in |\mathcal{P}|_t$, by Lemma 4. We take the appropriate variable as M'.

Now $M = x\vec{N}$, for some vector $\vec{N}$ of arguments. If types of these arguments are t-atoms in Ξ, other than $\mathsf{N}_{\alpha,u}$, then we apply the induction hypothesis to each component of $\vec{N}$.

Atoms $\mathsf{N}_{\alpha,u}$ only occur as arguments in the axioms: $\mathcal{N}_{\alpha,t\downarrow} \to \mathsf{X}_{\alpha,t}$ and $\mathcal{N}_{\gamma,t\downarrow} \to \mathsf{W}^2_{\gamma\to\delta,t}$. If one of them is the type of x then $\mathcal{N}_{\alpha,t\downarrow}$ (resp. $\mathcal{N}_{\gamma,t\downarrow}$) must be a subset of Ξ, more precisely a subset of $\mathcal{N}$. But then it is actually a subset of $|\mathcal{N}|_{t\downarrow}$, so we can take $M' = M$.

There are four cases when an assumption of an axiom is not an atom. The case of axiom (3) is simple: then $M = x(\lambda y\, N)$ with $y : \mathsf{D}_{\alpha,t}$ and $N : \mathsf{D}'_{\alpha,t}$. In this case we apply the induction hypothesis to $\Delta, \Xi, \mathsf{D}_{\alpha,t} \vdash N : \mathsf{D}'_{\alpha,t}$.

A less obvious case is when the head variable of M has e.g. type $(\mathcal{A}_{\gamma,t+1} \to \mathsf{D}_{\delta,t+1}) \to \mathsf{W}^1_{\alpha,t}$. Then we have $\Delta, \Xi, \mathcal{A}_{\gamma,t+1} \vdash N : \mathsf{D}_{\delta,t+1}$, for a subterm N of M, to which we apply the induction hypothesis. This yields

$$\Delta, |\mathcal{N}, \mathcal{N}'|_{(t+1)\downarrow}, |\mathcal{P}, \mathsf{P}_{\gamma,t+1}|_{t+1}, |\mathcal{D}|_{t+1} \vdash N' : \mathsf{D}_{\delta,t+1}$$

where $\mathcal{N}' = \{\mathsf{N}_{\alpha,t+1} \mid \alpha \in \mathcal{S} \wedge \alpha \neq \gamma\}$. We want to show $\Delta, |\Xi|_t, \mathcal{A}_{\gamma,t+1} \vdash N'' : \mathsf{D}_{\delta,t+1}$, that is,

$$\Delta, |\mathcal{N}|_{t\downarrow}, |\mathcal{P}|_t, |\mathcal{D}|_t, \mathcal{A}_{\gamma,t+1} \vdash N'' : \mathsf{D}_{\delta,t+1}\,. \qquad (\dagger)$$

First observe that $|\mathcal{D}|_{t+1} = \varnothing \subseteq |\mathcal{D}|_t$. We also have $|\mathcal{N}, \mathcal{N}'|_{(t+1)\downarrow} \subseteq |\mathcal{N}|_{t\downarrow} \cup \mathcal{A}_{\gamma,t+1}$. Indeed, if $\mathsf{N}_{\alpha,u} \in |\mathcal{N}, \mathcal{N}'|_{(t+1)\downarrow}$ then the whole set $\mathcal{N}_{\alpha,(t+1)\downarrow}$ is contained in $\mathcal{N} \cup \mathcal{N}'$. Thus, for all v, if $v \leq t$ then $\mathsf{N}_{\alpha,v} \in \mathcal{N}$, and if $v = t+1$ then $\mathsf{N}_{\alpha,v} \in \mathcal{N}'$, in particular $\alpha \neq \gamma$. It follows that $\mathsf{N}_{\alpha,u} \in |\mathcal{N}|_t$, for $u \leq t$, and $\mathsf{N}_{\alpha,u} \in \mathcal{A}_{\gamma,t+1}$ when $u = t+1$.

However, it is not the case that $|\mathcal{P}, \mathsf{P}_{\gamma,t+1}|_{t+1} \subseteq |\mathcal{P}|_t, \mathcal{A}_{\gamma,t+1}$, so we cannot take $N'' = N'$. But the missing atoms $\mathsf{P}_{\beta,t+1}$ are derivable (for free) from $\mathsf{P}_{\beta,t} \in |\mathcal{P}|_t$. That is, we can replace in N' any occurrence of a variable of type $\mathsf{P}_{\beta,t+1}$ by an application of one or more axioms of type (2) without adding more weight. We conclude that the judgment $(\dagger)$ holds with some term N'' of the same weight as N'. ◀

Proof of Lemma 3 ("if" part). Assume $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,t} \vdash \mathsf{D}_{\alpha,t}$, where $|\Gamma| = t$, and $\alpha \notin \Sigma$. Suppose towards contradiction that $\Gamma \vdash \alpha\,[\Sigma]$. We proceed by induction with respect to the weight of a normal proof of $\mathsf{D}_{\alpha,t}$. Of course $\mathsf{D}_{\alpha,t} \notin \mathcal{D}_{\Sigma,t}$, so the proof is an application of the axiom $(\mathsf{D}_{\alpha,t} \to \mathsf{D}'_{\alpha,t}) \to \mathsf{D}_{\alpha,t}$. Hence $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,t}, \mathsf{D}_{\alpha,t} \vdash \mathsf{D}'_{\alpha,t}$, and this means that $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,\alpha,t} \vdash \mathsf{C}_t$, for every atom $\mathsf{C}_t \in \mathcal{M}_t$. Indeed, no other axiom targets $\mathsf{D}'_{\alpha,t}$, and we have $\mathcal{D}_{\Sigma,t}, \mathsf{D}_{\alpha,t} = \mathcal{D}_{\Sigma,\alpha,t}$.

In particular, $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,\alpha,t} \vdash \mathsf{X}_t$, and that can only happen when $\mathcal{N}_{\alpha,t\downarrow} \subseteq \mathcal{N}_{\Gamma,t\downarrow}$, i.e., when $\alpha \notin \Gamma$. It follows that the judgment $\Gamma \vdash \alpha\,[\Sigma]$ is not an axiom.

Since $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,\alpha,t} \vdash \mathsf{E}_{\alpha,\bot,t}$, it must be the case that $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,\alpha,t} \vdash \mathsf{D}_{\bot,t}$, whence $\Gamma \nvdash \bot\,[\Sigma, \alpha]$ by induction. Thus, $\Gamma \vdash \alpha\,[\Sigma]$ cannot be obtained from rule (E$\bot$).

Suppose $\Gamma \vdash \alpha\,[\Sigma]$ is derived using (E$\wedge_1$) in the last step. Then $\Gamma \vdash \alpha \wedge \beta\,[\Sigma, \alpha]$, for some β such that $\alpha \wedge \beta \notin \Sigma$. But $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,\alpha,t} \vdash \mathsf{E}_{\alpha,\alpha\wedge\beta,t}$ implies that $\mathsf{D}_{\alpha\wedge\beta,t}$ is derivable, whence $\Gamma \nvdash \alpha \wedge \beta\,[\Sigma, \alpha]$, by the induction hypothesis.

In a similar fashion we exclude all rules where Γ remains unchanged in the premises. Let us consider rule (E$\vee$). For any γ, δ we have a proof of $\mathsf{E}_{\alpha,\gamma\vee\delta}$ using one of the five available axioms (12) targetting this atom.

Suppose that $\mathsf{E}_{\alpha,\gamma\vee\delta}$ was derived using the axiom $\mathsf{D}_{\gamma\vee\delta,t} \to \mathsf{E}_{\alpha,\gamma\vee\delta,t}$. This means that $\mathsf{D}_{\gamma\vee\delta,t}$ was proved in the same environment. If $\gamma \vee \delta \notin \Sigma$ then $\Gamma \nvdash \gamma \vee \delta\,[\Sigma, \alpha]$, by induction, and rule (E$\vee$) is not applicable. This rule is also excluded when $\gamma \vee \delta \in \Sigma$.

Axiom $\mathsf{P}_{\gamma,t} \to \mathsf{E}_{\alpha,\gamma\vee\delta,t}$ can be used only if $\mathsf{P}_{\gamma,t} \in \mathcal{P}_{\Gamma,t}$, i.e., when $\gamma \in \Gamma$. Then rule (E$\vee$) is not applicable too (and similarly for δ). So $\gamma, \delta \notin \Gamma$ and we must have used e.g. the axiom $(\mathcal{A}_{\delta,t+1} \to \mathsf{D}_{\alpha,t+1}) \to \mathsf{E}_{\alpha,\gamma\vee\delta,t}$. It follows that M is of shape $y(\lambda\vec{x}\, N)$, where

$$\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,t}, \mathsf{D}_{\alpha,t}, \mathcal{A}_{\delta,t+1} \vdash N : \mathsf{D}_{\alpha,t+1}\,.$$

From Lemma 5 we obtain:

$$\Delta, |\mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,t}, \mathsf{D}_{\alpha,t}, \mathcal{A}_{\delta,t+1}|_{t+1} \vdash N' : \mathsf{D}_{\alpha,t+1}\,,$$

that is:

$$\Delta, |\mathcal{N}_{\Gamma,t\downarrow} \cup \{\mathsf{N}_{\beta,t+1} \mid \beta \neq \delta\}|_{(t+1)\downarrow}, |\mathcal{P}_{\Gamma,t}, \mathsf{P}_{\delta,t+1}|_{t+1}, |\mathcal{D}_{\Sigma,t}, \mathsf{D}_{\alpha,t}|_{t+1} \vdash N' : \mathsf{D}_{\alpha,t+1}\,.$$

Now we should observe that:

- $|\mathcal{D}_{\Sigma,t}, \mathsf{D}_{\alpha,t}|_{t+1} = \varnothing = \mathcal{D}_{\varnothing,t+1}$;
- $|\mathcal{N}_{\Gamma,t\downarrow} \cup \{\mathsf{N}_{\beta,t+1} \mid \beta \neq \delta\}|_{(t+1)\downarrow} \subseteq \mathcal{N}_{\Gamma,\delta,(t+1)\downarrow}$;
- $|\mathcal{P}_{\Gamma,t}, \mathsf{P}_{\delta,t+1}|_{t+1} = \mathcal{P}_{\Gamma,\delta,t+1}$.

Therefore, we can write:

$$\Delta, \mathcal{N}_{\Gamma,\delta,(t+1)\downarrow}, \mathcal{P}_{\Gamma,\delta,t+1}, \mathcal{D}_{\varnothing,t+1} \vdash N' : \mathsf{D}_{\alpha,t+1}\,.$$

Since N' is smaller than M in weight, we have $\Gamma, \delta \nvdash \alpha\,[\varnothing]$ by the induction hypothesis, and rule (E$\vee$) is now excluded too. In other cases we proceed in an analogous way. ◀

Proof of Lemma 3 ("only if" part). Let $\Gamma \nvdash \alpha\,[\Sigma]$. We prove $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,t} \vdash \mathsf{D}_{\alpha,t}$, by induction with respect to two parameters:

(1) the cardinality $n - t$ of $\mathcal{S} - \Gamma$, (2) the cardinality of $\mathcal{S} - \Sigma$.

We need to show that all atoms in $\mathcal{M}_{\alpha,t}$ can be derived from $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,t}, \mathsf{D}_{\alpha,t}$, which is the same as $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,\alpha,t}$. Then we can obtain $\mathsf{D}_{\alpha,t}$ using the axioms $\mathcal{M}_{\alpha,t} \to \mathsf{D}'_{\alpha,t}$ and $(\mathsf{D}_{\alpha,t} \to \mathsf{D}'_{\alpha,t}) \to \mathsf{D}_{\alpha,t}$.

We begin with $\mathsf{X}_{\alpha,t}$. From $\Gamma \nvdash \alpha\,[\Sigma]$ it follows that $\alpha \notin \Gamma$, hence all formulas $\mathsf{N}_{\alpha,u}$, for $u \leq t$, are in $\mathcal{N}_{\Gamma,t\downarrow}$, and the axiom $\mathcal{N}_{\alpha,t\downarrow} \to \mathsf{X}_{\alpha,t}$ can be used to prove $\mathsf{X}_{\alpha,t}$.

In order to prove $\mathsf{E}_{\alpha,\bot,t}$, we consider two cases. If $\bot \notin \Sigma$ then we observe that $\Gamma \nvdash \bot[\Sigma, \alpha]$, as otherwise rule (E$\bot$) could be used to derive $\Gamma \vdash \alpha\,[\Sigma]$. By the induction hypothesis we have $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,\alpha,t} \vdash \mathsf{D}_{\bot,t}$, because the cardinality of Γ is unchanged and the cardinality of Σ, α is greater by one than that of Σ. Our goal is accomplished with the axiom $\mathsf{D}_{\bot,t} \to \mathsf{E}_{\alpha,\bot,t}$. The case $\bot \in \Sigma$ is simpler, because then $\mathsf{D}_{\bot,t}$ just belongs to $\mathcal{D}_{\Sigma,t}$.

Consider a constant $\mathsf{E}_{\alpha,\beta\to\alpha,t}$. If $\beta \in \Sigma$ or $\beta \to \alpha \in \Sigma$ then $\mathsf{D}_{\beta,t} \in \mathsf{D}_{\Sigma,t}$ or $\mathsf{D}_{\beta\to\alpha,t} \in \mathsf{D}_{\Sigma,t}$, and $\mathsf{E}_{\alpha,\beta\to\alpha,t}$ follows easily. Otherwise, one of the premises of rule (E$\to$) does not hold, and one can apply the induction hypothesis to derive either $\mathsf{D}_{\beta,t}$ or $\mathsf{D}_{\beta\to\alpha,t}$. The induction could fail when $\beta = \alpha$, in which case $\beta \in \Sigma, \alpha$. But then we already have $\mathsf{D}_{\beta,t} = \mathsf{D}_{\alpha,t} \in \mathsf{D}_{\Sigma,t}$.

As the next example we consider the atom W_{α}, where $\alpha = \gamma \to \delta$. We should derive the two atoms $\mathsf{W}^1_{\alpha,t}$ and $\mathsf{W}^2_{\alpha,t}$.

If $\gamma \in \Gamma$ then $\mathsf{P}_{\gamma,t} \in \mathcal{P}_{\alpha,t}$ and $\mathsf{P}_{\gamma,t} \to \mathsf{W}^1_{\alpha,t}$ implies $\mathsf{W}^1_{\alpha,t}$. Otherwise $\Gamma, \gamma \nvdash \delta\,[\varnothing]$, as $\gamma \to \delta$ should not be derivable using rule $(W{\to}_1)$. We can apply the induction hypothesis, because the set Γ, γ is larger than Γ. Thus, $\Delta, \mathcal{N}_{\Gamma,\gamma,(t+1)\downarrow}, \mathcal{P}_{\Gamma,\gamma,t+1}, \mathcal{D}_{\varnothing,t+1} \vdash \mathsf{D}_{\delta,t+1}$. Observe that $\mathcal{N}_{\Gamma,\gamma,(t+1)\downarrow} \subseteq \mathcal{N}_{\Gamma,t\downarrow} \cup \mathcal{A}_{\gamma,t+1}$, and that all atoms $\mathsf{P}_{\sigma,t+1} \in \mathcal{P}_{\Gamma,\gamma,t+1}$ can be derived from the set $\mathcal{P}_{\Gamma,t}, \mathcal{A}_{\gamma,t+1}$, because $\mathsf{P}_{\gamma,t+1} \in \mathcal{A}_{\gamma,t+1}$. Therefore $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,t}, \mathsf{D}_{\alpha,t}, \mathcal{A}_{\gamma,t+1} \vdash \mathsf{D}_{\delta,t+1}$. Using the axiom $(\mathcal{A}_{\gamma,t+1} \to \mathsf{D}_{\delta,t+1}) \to \mathsf{W}^1_{\alpha,t}$ we obtain what we need.

The atom $\mathsf{W}^2_{\gamma\to\delta,t}$ is easily derived from $\mathsf{D}_{\delta,t}$ in case $\delta \in \Sigma$. Similarly, if $\gamma \notin \Gamma$ then we can use the axiom $\mathcal{N}_{\gamma,t\downarrow} \to \mathsf{W}^2_{\alpha,t}$. So we assume $\delta \notin \Sigma$, $\gamma \in \Gamma$ and we apply induction to $\Gamma \nvdash \delta\,[\Sigma, \gamma \to \delta]$. This yields $\Delta, \mathcal{N}_{\Gamma,t\downarrow}, \mathcal{P}_{\Gamma,t}, \mathcal{D}_{\Sigma,\gamma\to\delta,t} \vdash \mathsf{D}_{\delta,t}$, and we use the axiom $\mathsf{D}_{\delta,t} \to W^2_{\alpha,t}$ to complete the job. Other cases are similar. ◀

A simple example

For a formula φ of length n, the "dual" formula $\overline{\varphi}$ is of size $\mathcal{O}(n^3)$ with a decently large constant, and may be quite incomprehensible even for short φ. We therefore consider an extremely simple example $\varphi = (p \to p) \to p$. By our definition we have $\overline{\varphi} = \vec{\Delta} \to \mathcal{D}_{\varphi,0}$, where $\vec{\Delta}$ abbreviates the sequence of all axioms (1–12). Not all of them are actually needed, and some can be simplified in this case. For example, a normal proof of φ itself cannot be an elimination, because only subformulas of φ are used and neither $\bot$ or $\vee$ occurs in φ. Hence the set $\mathcal{M}_{\varphi,t}$ in (4) reduces to two elements (cf. type of X_4 below). Here we only list the relevant part of $\vec{\Delta}$, in the form of variable declarations. We use the abbreviation $\alpha = p \to p$.

1. $X_1 : \mathsf{N}_{\varphi,0},\ Z_1 : \mathsf{N}_{\alpha,0},\ Y_1 : \mathsf{N}_{p,0}$;
3. $X_3 : (\mathsf{D}_{\varphi,0} \to \mathsf{D}'_{\varphi,0}) \to \mathsf{D}_{\varphi,0},\ Y_3 : (\mathsf{D}_{p,1} \to \mathsf{D}'_{p,1}) \to \mathsf{D}_{p,1},\ U_3 : (\mathsf{D}_{\varphi,1} \to \mathsf{D}'_{\varphi,1}) \to \mathsf{D}_{\varphi,1}$;
4. $X_4 : \mathsf{X}_{\varphi,0} \to \mathsf{W}_{\varphi,0} \to \mathsf{D}'_{\varphi,0},\ Y_4 : \mathsf{X}_{p,1} \to \mathsf{E}_{p,\varphi,1} \to \mathsf{E}_{p,\alpha,1} \to \mathsf{D}'_{p,1},\ U_4 : \mathsf{X}_{\varphi,1} \to \mathsf{W}_{\varphi,1} \to \mathsf{D}'_{\varphi,1}$;
5. $X_5 : \mathsf{N}_{\varphi,0} \to \mathsf{X}_{\varphi,0},\ Y_5 : \mathsf{N}_{p,0} \to \mathsf{N}_{p,1} \to \mathsf{X}_{p,1},\ U_5 : \mathsf{N}_{\varphi,0} \to \mathsf{N}_{\varphi,1} \to \mathsf{X}_{\varphi,1}$;
8. $X_8^1 : \mathsf{W}^1_{\varphi,0} \to \mathsf{W}^2_{\varphi,0} \to \mathsf{W}_{\varphi,0},\ X_8^3 : (\mathsf{P}_{\alpha,1} \to \mathsf{N}_{\varphi,1} \to \mathsf{N}_{p,1} \to \mathsf{D}_{p,1}) \to \mathsf{W}^1_{\varphi,0},\ X_8^4 : \mathsf{N}_{\alpha,0} \to \mathsf{W}^2_{\varphi,0}$,
$U_8^1 : \mathsf{W}^1_{\varphi,1} \to \mathsf{W}^2_{\varphi,1} \to \mathsf{W}_{\varphi,1},\ U_8^2 : \mathsf{P}_{\alpha,1} \to \mathsf{W}^1_{\varphi,1},\ U_8^5 : \mathsf{D}_{p,1} \to \mathsf{W}^2_{\varphi,1}$.
11. $Y_{11}^1 : \mathsf{D}_{p,1} \to \mathsf{E}_{p,\alpha,1},\ Y_{11}^2 : \mathsf{D}_{\varphi,1} \to \mathsf{E}_{p,\varphi,1}$.

A proof of $\mathsf{D}_{\varphi,0}$ can now be presented as the lambda-term:

$$X_3(\lambda x{:}\mathsf{D}_{\varphi,0}.\, X_4(X_5X_1)(X_8^1\, T\, (X_8^4 Z_1))),$$

where $T = X_8^3(\lambda w{:}\mathsf{P}_{\alpha,1}\lambda x_1{:}\mathsf{N}_{\varphi,1}\lambda y_1{:}\mathsf{N}_{p,1}.\, Y_3(\lambda y{:}\mathsf{D}_{p,1}.\, Y_4(Y_5Y_1y_1)\, S\, (Y_{11}^1 y)))$ has type $\mathsf{W}^1_{\varphi,0}$, and $S = Y_{11}^2(U_3(\lambda u{:}\mathsf{D}_{\varphi,1}.\, U_4(U_5X_1x_1)(U_8^1(U_8^2w)(U_8^5y))))$ has type $\mathsf{E}_{p,\varphi,1}$.

The above term represents the following refutation of φ. First check that φ is not assumed (this is the meaning of the subterm X_5X_1). Then check that φ cannot be obtained by introduction from $\alpha \vdash p$. Since α is not yet assumed $(X_8^4 Z_1)$ we now assume it (variable w) but not the other formulas (variables x_1, y_1). The goal p is now addressed for the first time and is marked as forbidden in this phase (variable y). It is easy to check that p is not an assumption $(Y_5Y_1y_1)$ and that it cannot be derived by elimination from α: indeed, the latter requires re-addressing the goal p in the same environment $(Y_{11}^1 y)$.

The subterm S refutes the possibility that p is obtained by elimination from φ, because an attempt to derive $\alpha \vdash \varphi$ will fail. Indeed, φ must be obtained by introduction from $\alpha \vdash p$ (the subterms $U_8^2 w$ and $U_5X_1x_1$ certify that α has already been assumed, but φ has not). But p is a forbidden goal $(U_8^5 y)$, hence using the introduction rule is illegal.

Conclusion

We have demonstrated a logarithmic space algorithm to construct a "dual" formula $\overline{\varphi}$ for any given propositional formula φ, so that $\overline{\varphi}$ is provable in IPC if φ is not. The construction is inspired by an automata-theoretic view of proof-search. This can be seen as alternative to introducing rules to derive refutability: just apply the old ones towards a different task.

The formula $\overline{\varphi}$ uses only implication and (as a simple type) is of order (depth) at most 3. Since $\overline{\overline{\varphi}}$ is provable iff so is φ, we conclude that IPC provability reduces to provability of formulas of particularly simple form.[2] The formula $\overline{\overline{\varphi}}$ is not equivalent to φ, but is computable in logarithmic space (note the analogy with CNF-SAT).

[2] Of course that can be done much simpler in a direct way [8].

Intuitionistic propositional logic can represent combinatorial problems as easily (or better) as classical propositional satisfiability, and it is far more expressive because it reaches beyond the class NP. Provability in IPC reduces to the case of order three. Those should be relatively easy to simplify and manipulate by various heuristics (like joining and deleting some formula components). It is about time for an intuitionistic analogue of Davis-Putnam algorithm. This issue is raised in a subsequent work [8].

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