Fault-Tolerant Syndrome Extraction and Cat State Preparation with Fewer Qubits

Prithviraj Prabhu

Department of Electrical and Computer Engineering, University of Southern California, Los Angeles, CA, USA

Ben W. Reichardt

Department of Electrical and Computer Engineering, University of Southern California, Los Angeles, CA, USA

Abstract

We reduce the extra qubits needed for two fault-tolerant quantum computing protocols: error correction, specifically syndrome bit measurement, and cat state preparation. For fault-tolerant syndrome extraction, we show an exponential reduction in qubit overhead over the previous best protocol. For a weight-w stabilizer, we demonstrate that stabilizer measurement tolerating one fault (distance-three) needs at most $\lceil \log_2 w \rceil + 1$ ancillas. If qubits reset quickly, four ancillas suffice. We also study the preparation of cat states, simple yet versatile entangled states. We prove that the overhead needed for distance-three fault tolerance is only logarithmic in the cat state size. These results could be useful both for near-term experiments with a few qubits, and for the general study of the asymptotic resource requirements of syndrome measurement and state preparation.

For a measured flag bits, there are 2^a possible flag patterns that can identify faults. Hence our results come from solving a combinatorial problem: the construction of maximal-length paths in the a-dimensional hypercube, corresponding to maximal-weight stabilizers or maximal-weight cat states.

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1 Introduction

A critical component of quantum error correction is syndrome measurement: a set of circuits used to pinpoint which qubits have errors. This process of error identification is itself susceptible to noise and may fail. To make this process robust, extra (ancilla) qubits can be used to identify damaging mid-circuit faults and mitigate the spread of errors. The objective of this paper is to reduce the overhead of ancilla qubits used in imparting this fault tolerance. In particular, we focus on optimizing the flag technique for distance-three fault tolerant stabilizer measurement. We also reduce qubit overhead in distance-three fault-tolerant cat state preparation. Cat states [11] have applications in many areas of quantum computing, including communication [9], information processing [12], and error correction [13, 14]. Besides practical applications, our results on cat state preparation are theoretically interesting since: *i*) we introduce the study of asymptotic estimates of qubit overhead for the fault-tolerant preparation of cat states of *arbitrary* size, and, *ii*) ideas developed for cat state preparation may provide clues for the fault-tolerant preparation of logical states of more complex codes.

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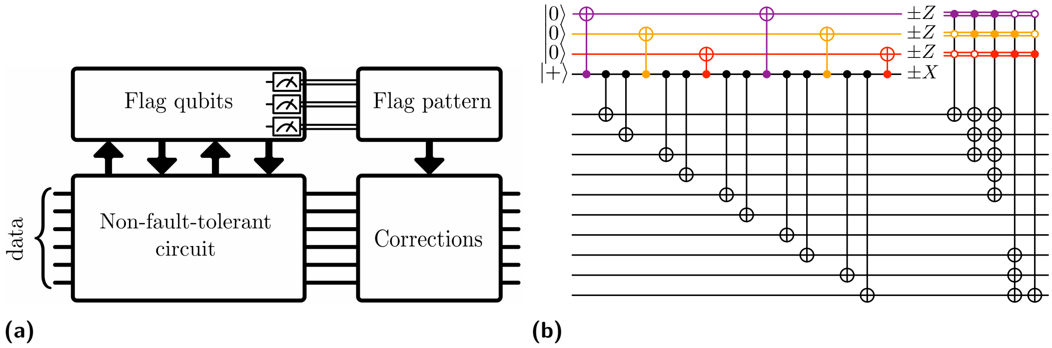


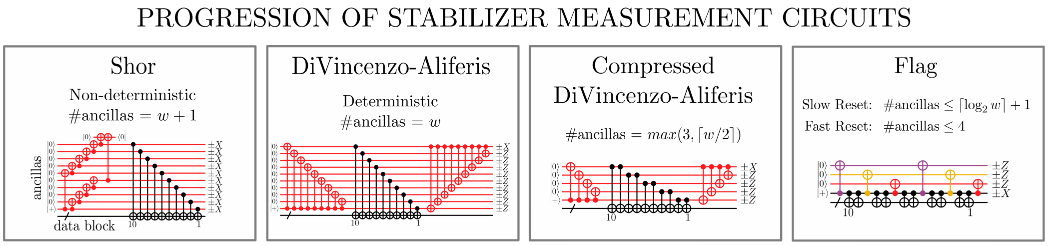
Figure 1 Functioning of a flag scheme. (a) Flag qubits interact with a non-fault-tolerant circuit to catch faults. Upon measurement, flag qubits yield a pattern of 1s and 0s. Based on the flag pattern, a correction is applied onto the data qubits. (b) Measurement of stabilizer $X^{\otimes 10}$ in the slow reset model, CSS fault-tolerant to distance three, using $a = 4$ ancilla qubits. Colored qubits and gates are used to impart distance-three fault tolerance.

We strive for low qubit overhead since quantum computers with limited qubits count resources preciously, and even minor improvements can free up extra qubits for other tasks. In topological codes where stabilizers are localized in space and low-weight, only a few flag qubits close to each stabilizer suffice to impart fault tolerance [16, 2, 3]. It has also been shown that with adaptive control and quickly resetting qubits, only four ancillas are needed for the universal fault-tolerant operation of some distance-three codes [4, 5]. In this paper, we present a general fault-tolerant protocol that works for a stabilizer of any size. If qubits are connected well enough, we show that only logarithmic overhead is required for fault-tolerant stabilizer measurement, an exponential space improvement over the previous linear overhead.

The general model of flag-based fault-tolerance is displayed in Figure 1a. Here, a set of flag ancilla qubits monitor operations in a non-fault-tolerant circuit and when measured at the end, produce flag patterns which uniquely identify mid-circuit faults. Based on the observed flag pattern, a correction is applied to the data to minimize the spread of errors. As an example, Figure 1b measures a stabilizer on 10 data qubits while tolerating one fault. The three colored qubits are the flag ancillas and the measured flag patterns each imply different corrections. Also note that the sequence of flag patterns $(100, 110, 111, 011, 001)$ is a path on the hypercube and also corresponds to the order of the flag CNOTs (edge between 100 and 110 implies a CNOT is applied onto flag qubit 2).

In this paper, we restrict discussion to the measurement of individual stabilizers of a quantum code, as in Shor-style fault-tolerant stabilizer measurement [13]. We do not focus on other methods which measure multiple stabilizers in parallel. Figure 2 displays improvements made over the years to Shor's method. Note that Shor's method can tolerate any number of faults by increasing the fault tolerance of the ancillary cat state preparation. The subsequent schemes forgo this property and are only fault-tolerant to distance three. DiVincenzo and Aliferis first make the circuit deterministic by removing the need for cat state verification [7]. This ensures that a circuit designer need not wait for a fault-tolerantly prepared cat state before measuring the stabilizer. Subsequent improvements were made in [15], [16] and [5] to reduce ancilla count by coupling each ancilla qubit to two data qubits instead of one.

With our flag method, the ancilla cat state is prepared and unprepared while collecting the stabilizer. As in Figure 1b, an X fault occurring anywhere on the $|+\rangle$ qubit may spread into the data, but will also leave its imprint on the flags. This is then measured out as a flag pattern. Due to the particular chosen arrangement of the flag CNOTs, any fault that can



(a) [13]. **(b)** [7]. **(c)** [15, 16, 5]. **(d)**

Figure 2 Historical progression of stabilizer measurement circuits. A weight-10 X stabilizer measurement circuit is provided as an example. CNOTs in black have targets on the 10 data qubits, collectively represented by the black wire. In (b)(c)(d), fault-tolerance is only guaranteed to distance-three and Pauli corrections (or Pauli frame updates) are applied to the data based on the Z basis measurements. (a) Shor's method uses $w+1$ ancillas and requires a fault-tolerantly prepared cat state. (b)(c) The following two methods use unverified cat states with subsequent error decoding. Non-deterministic cat state verification is replaced with a deterministic circuit, allowing for uninterrupted circuit operation. (d) The flag method prepares and unprepares an ancilla cat state while collecting the stabilizer. Exponentially more flag configurations can thus be accessed for fault diagnosis.

Table 1 Distance-3 cat state preparation: Weight-w cat states can be prepared fault-tolerantly to distance-3 with m measurements of ancilla qubits. Slow reset requires m ancilla qubits whereas with fast reset, only one ancilla qubit is required.

Type	Bounds
Deterministic error correction Theorem 5	$w \leq 3\,(2^m - 2m + 2)$
Adaptive error correction Theorem 6	$w \leq 3\,(2^m - 2m + 3)$

spread to a high-weight data error triggers one of the five shown flag patterns. Each flag pattern then applies a unique correction that ensures that there is at most one data qubit in error. This satisfies the condition for fault tolerance, which states that k faults in a circuit should cause no more than k qubits to have errors.

For the distance-three fault-tolerant measurement of a weight-w stabilizer, we propose two methods based on the speed of qubit reset. With fast qubit reset, Theorem 3, only three flag ancillas are required in total, but each flag needs to be measured once per four data qubits. If more flags are used in parallel, the number of accessible flag patterns grows exponentially and the number of measurements per ancilla converges to one. This is the regime of slow qubit reset, Theorem 4, which uses at most $\lceil \log_2 w \rceil$ flag ancillas measured only at the end.

Table 1 contains bounds on the ancilla overhead for preparing weight-w cat states fault-tolerantly to distance-three. If the flag qubits can reset quickly, Theorem 5 states that only one flag qubit is required and it needs to be reset and measured m times. Since the flag qubits operate independently, it is also possible to use m flag qubits, with each one being measured once. We further show how to use an adaptive circuit in Theorem 6 to marginally increase the number of flag patterns in use.

The rest of this paper is divided into three sections. Section 2 details the construction of the two paths on the hypercube that we use as flag sequences. Section 3 describes how to use these sequences for distance-three fault-tolerant syndrome measurement, and Section 4 deals with cat state preparation.

2 Flag sequences

A flag pattern or flag configuration is a string of 1s and 0s that arises from measuring out the flags. If a flag ancillas are used, then the a-bit flag configuration labels a vertex of the a-dimensional hypercube. We show how to construct two paths on the hypercube to produce maximal-length sequences of flag configurations. Since they are paths, only one bit is flipped between subsequent flag configurations. This bit flip corresponds to the application of a flag CNOT from the syndrome ancilla to the flag qubit indexed by the flipped bit, thus providing a blueprint to construct the fault-tolerant circuit.

The first type of sequence just requires a maximal-length traversal of the a-dimensional hypercube. A simple choice for this is the Gray code [10, 8].

▶ **Lemma 1.** *For $a \geq 1$, the Gray code creates a length-2^a Hamiltonian path in the a-dimensional hypercube.*

Proof. We provide a quick construction of the sequence. For $a = 1$, use the sequence $0, 1$. For $a > 1$, construct the sequence inductively. First, run the sequence for $a - 1$ with a 0 prepended, then run it backwards with a 1 prepended. ◀

For example, for $a = 2$, the sequence is $00, 01, 11, 10$. For $a = 3$, the sequence is $000, 001, 011, 010, 110, 111, 101, 100$.

In this paper we use a piece-wise definition of fault tolerance. Fault tolerance to distance-d implies that for all $k \leq t = \lfloor \frac{d-1}{2} \rfloor$, correlated errors of weight-k occur with k-th order probability. For distance-three CSS fault-tolerant syndrome bit measurement, any single fault should result in a data error with X and Z components having weight zero or one.

In order to ensure that the circuit is distance-three fault-tolerant, we need to ensure that a measurement fault does not trigger corrections of weight greater than one. Hence the second maximal-length sequence requires that there are no weight-one strings except at the start and end. As shown in Figure 1b, we may assign weight-one corrections to these two configurations, but for all others there exist multi-qubit corrections.

▶ **Lemma 2.** *For $a \geq 2$, in the a-dimensional hypercube $\{0, 1\}^a$ there exists a path $v_1 = 10^{a-1}, \dots, v_n = 0^{a-1}1$ such that all intermediate vertices $v_2, \dots, v_{n-1}$ have weight at least two, and each vertex appears at most once; with length $n = 2^a - 2a + 3$.*

Proof. Figure 3 illustrates the inductive construction of maximal-length flag sequences satisfying the above constraints. With $a = m - 1$ flag qubits, the sequence has length $2^a - 2a + 3$. The a-flag sequence is constructed by first running the previous flag sequence, on $a - 1$ flags, up to the second-to-last element (which for $a \geq 4$ is $\chi_{\{2,a-1\}}$[1]), and with 0 appended at the end. Then run the sequence backward, except with 1 appended at the end, and with the 2 and $a - 1$ coordinates swapped (the red and blue rows in the figure). Finally, finish the sequence from $\chi_{\{1,a\}}$ by walking through $\chi_{\{3,a\}}, \chi_{\{4,a\}}, \dots, \chi_{\{a-2,a\}}, \chi_{\{2,a\}}$, with the appropriate weight-three sequences (shown in gray) interposed.

To ensure that no vertex is visited more than once, one need only check that the last $2a-5$ sequences are distinct from those that came before. For this, one can track by induction the $2a - 3$ hypercube vertices that are not visited by each walk: 0^a, the $a - 2$ weight-one strings $\chi_2, \dots, \chi_{a-1}$, and the $a - 2$ weight-two strings $\chi_{\{1,3\}}, \chi_{\{3,4\}}, \chi_{\{4,5\}}, \dots, \chi_{\{a-1,a\}}$. ◀

[1] $\chi_{\{x,y\}}$ implies bits at positions x and y are set to 1.

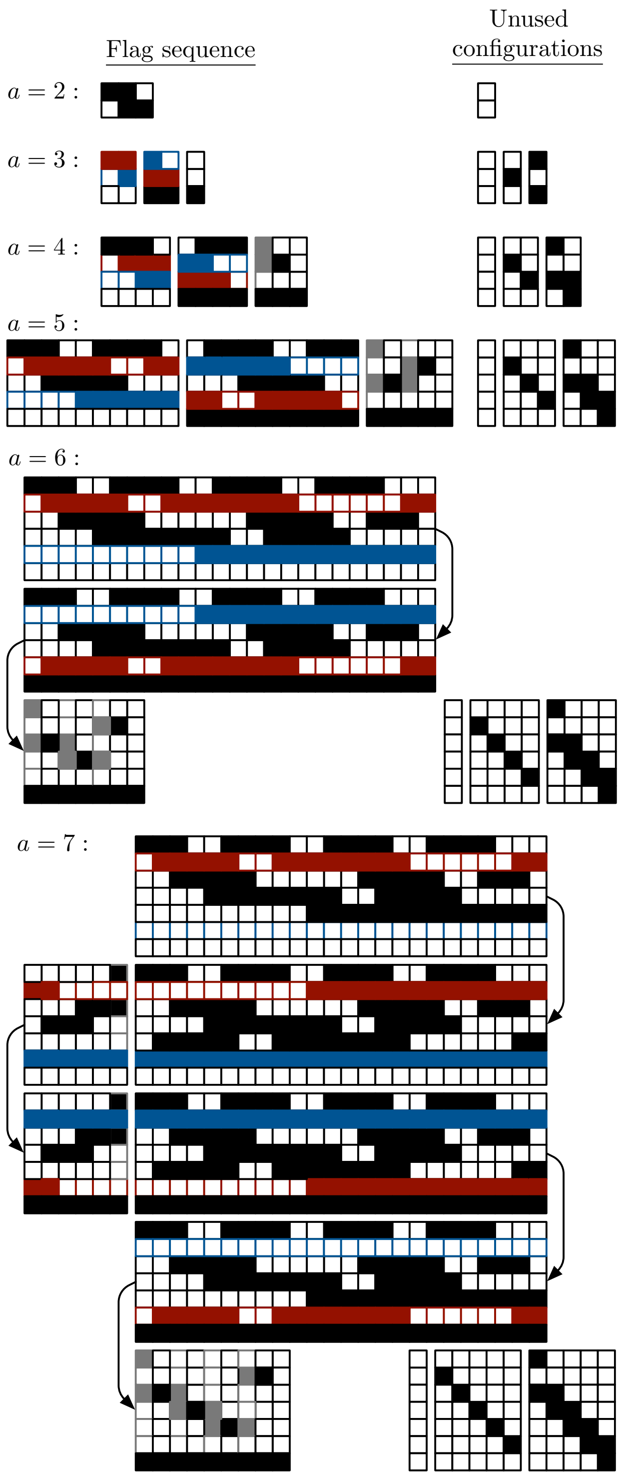


Figure 3 Flag sequences for distance-three fault-tolerant syndrome bit measurement, using a flag qubits, each measured once (the slow reset model). These sequences are walks through the a-dimensional hypercube, from 10^{a-1} to $0^{a-1}1$; passing through each vertex at most once and no other weight-one vertices. Flag configurations are stacked vertically and ordered initially left to right, with solid and empty squares representing 1 and 0, respectively, e.g., represents $10, 11, 01$.

3 Distance-three syndrome measurement

In this section, we outline two protocols for distance-three CSS fault-tolerant syndrome measurement. They differ based on the speed of qubit measurement and reset.

For $w \in \{4, 5, 6\}$, flag-fault-tolerant circuits are constructed the same way regardless of qubit reset speed. We show in Figure 4a that for $w = 6$, only two flag ancillas are required. Lower-weight stabilizers can be measured by removing data CNOTs and making appropriate changes to the Pauli corrections. For $7 \leq w \leq 10$, the different methods of construction yield the same circuits. It is only for $w > 10$ that the effects of qubit reset speed are pronounced.

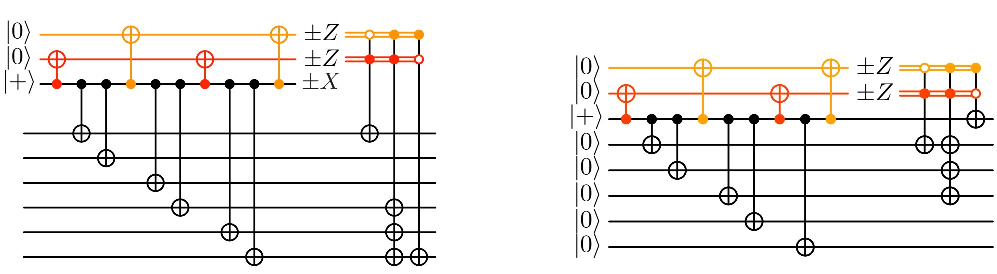
3.1 Fast reset

▶ **Theorem 3.** *If qubits can be measured and reset quickly, then for any w, four ancilla qubits are sufficient to measure the syndrome of $X^{\otimes w}$, CSS fault-tolerantly to distance three. Moreover, the number of measurements needed is $\lceil \frac{w+2}{4} \rceil + 1$.*

Proof. For $w \in \{4, 5, 6\}$, the circuit using two flag ancillas is shown in Figure 4a. It runs through a sequence of three flag configurations and a multi-qubit correction is only applied for the flag configuration 11. For $w > 6$, the general construction is shown in Figure 5. Each repetition of the highlighted region adds the X parity of four more data qubits, while measuring and quickly reinitializing one flag qubit. In terms of the number of measurements m, the construction achieves up to $w = 4\,(m - 1) - 2$. It is fault tolerant because X faults on the control wire cause flag configurations of alternating weights two or three, that localize the fault to three possible consecutive locations along the control wire: before, between or after two CNOT gates. The appropriate correction is for a fault between the CNOT gates. ◀

Theorem 3 may be optimal; it does not appear to be possible to use fewer than three flag qubits. With just one flag qubit, one can detect that an error has occurred, but not where. As illustrated in Figure 6, either the control wire is unprotected at some point or for $w \geq 4$ there is no consistent correction rule.

By a similar argument, two flag qubits are not enough. Any correction based on a single flag can have weight at most one, since the flag measurement itself could be faulty. However, if at some point in the middle the control wire is protected by just a single flag, a weight-one correction will not suffice. On the other hand, if both flags are used to protect the control wire across the entire sequence of CNOT gates, we are unable to locate faults well enough to correct them.



(a) **(b)**

Figure 4 (a) Circuit to measure an $X^{\otimes 6}$ stabilizer, CSS fault-tolerant to distance three. (b) Circuit to prepare a six-qubit cat state, fault-tolerant to distance three.

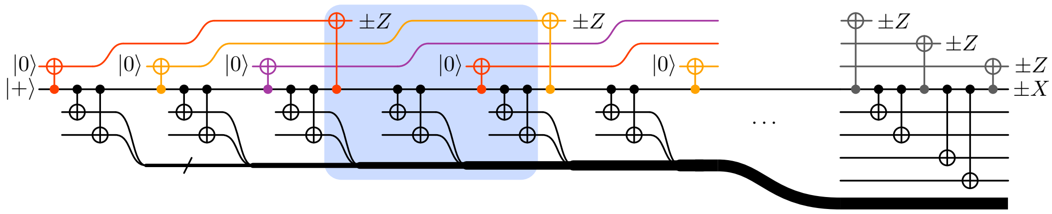


Figure 5 Distance-three fault-tolerant syndrome measurement only requires three flag qubits. The highlighted region can be repeated to fit the weight of the stabilizer being measured.

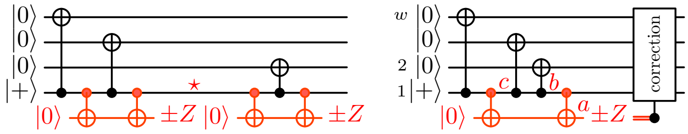


Figure 6 Distance-three error correction is not possible with only one flag qubit. Either (left) the control wire is unprotected at some point $\star$, from which an X fault can propagate to an error of weight at least two; or (right) faults at a, b, c, causing respective errors I, X_1, X_w have no consistent correction.

We would like to point out that this construction can also be used to prepare a w-qubit cat state fault tolerantly to distance three. The conversion to this circuit follows three steps: Remove one data qubit. Initialize the data qubits as $|0\rangle$. Remove the syndrome ancilla measurement, so as to retain the qubit in the support of the stabilizer. An example of this conversion is shown for $w = 6$ in Figure 4b. In Section 4, this method will be subsumed by a better protocol that uses just one ancilla qubit.

3.2 Slow reset

▶ **Theorem 4.** *The syndrome of $X^{\otimes w}$ can be measured CSS fault-tolerantly to distance three using $m \geq 3$ measurements, provided that*

$$w \leq 2\left(2^{m-1} - 2(m-1) + 3\right).$$

Proof. Two examples are shown in Figure 4a, for $w = 6$, and Figure 1b, for $w = 10$. As in these figures, in general we collect the syndrome two qubits at a time into a syndrome qubit that is initialized as $|+\rangle$. Between each of these pairs of CNOT gates, a CNOT is applied from the syndrome qubit into one of $m - 1$ flag qubits. This leads to a sequence of flag configurations, e.g., $100, 110, 111, 011, 001$ for the $w = 10$ example. Based on the observed flag configuration, a correction is applied as if an X fault had occurred between the corresponding pair of flag CNOT gates.

Observe that the flag sequence changes one bit at a time; it can be thought of as a path on the hypercube. It begins and ends with weight-one configurations, but otherwise the configurations all have weight at least two. This is important for distance-three fault tolerance because a fault could affect the flags, and only the first and last data corrections have weight one. Also, the flag configurations along the sequence are distinct, so each is associated with only one correction. The theorem then just follows from the flag sequence construction in Lemma 2. ◀

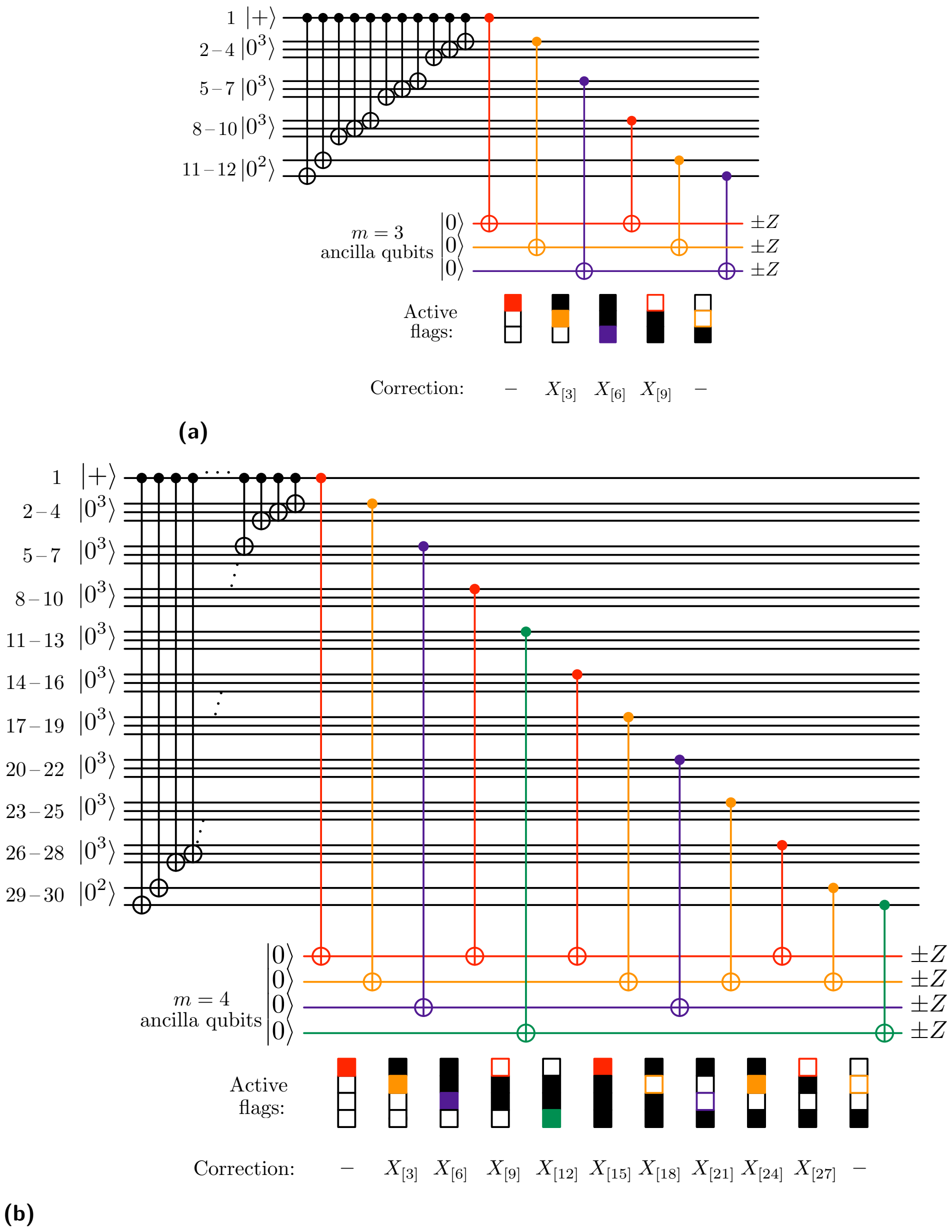


Figure 7 Distance-three fault-tolerant cat state preparation circuits. Note that, with fast reset, only one ancilla qubit is required.

The construction of Lemma 2 gives flag sequences of maximal length, $2^a - 2a + 3$. Indeed, this follows since the number of vertices with odd weight greater than one is $2^{a-1} - a$, and vertices must alternate between odd and even weight.

Note that the approach of Theorem 4, with slow reset, is different from the fast reset case of Theorem 3, in that a flag qubit is active and able to detect faults in more than one region of the circuit.

4 Distance-three cat state preparation

Next we turn to the question of distance-three fault-tolerant preparation of cat states. For preparing a two- or three-qubit cat state, any preparation circuit is automatically fault-tolerant, because every error has weight zero or one. For example, on three qubits $XXI \sim IIX$, since XXX is a stabilizer. Fault tolerance becomes interesting for preparing cat states on $w \geq 4$ qubits.

The ideas of Theorems 3 and 4 can also be applied to cat state preparation. For example, just as in Figure 4 a circuit for measuring $X^{\otimes 6}$ with three ancilla qubits corresponds to a circuit to prepare a six-qubit cat state with two ancillas, similarly adapting the construction of Theorem 4 allows preparing a $2(2^a - 2a + 3)$ qubit cat state with a ancilla qubits each measured once. However, we can do better.

▶ **Theorem 5.** *For $m \geq 2$, one ancilla qubit, measured m times, is sufficient to prepare a cat state on w qubits fault-tolerantly to distance three, for*

$$w \leq 3\left(2^m - 2m + 2\right).$$

Proof. Figure 7 illustrates our construction for the cases $m = 3$ and $m = 4$. In general, we prepare a w-qubit cat state using CNOT gates from the first qubit, so that the possible X errors from a single fault are $\mathbf{1}, X_1, X_{[2]}, X_{[3]}, \ldots$. (Here we are using the notation $[m] = \{1, 2, \ldots, m\}$ and $X_S = \prod_{j \in S} X_j$.) We then compute parities of subsets of the qubits into the ancillas, following the flag sequence from Lemma 2 and Figure 3. Although for clarity Figure 7 shows the m parity checks being made in parallel, they can also be made sequentially with just one ancilla qubit.

With the given correction rules, errors due to single faults are corrected up to possibly a weight-one remainder. (For example, in Figure 7a, errors $X_{[5]}$, $X_{[6]}$ and $X_{[7]}$ all result in the parity checks 111, for which the correction $X_{[6]}$ is applied.) The circuit also tolerates faults within the parity-check sub-circuit, because a single fault here can flip at most one parity, and no correction is applied for the weight-one configurations. ◀

By this method, the cat state is prepared in depth $w - 1$. The depth of the parity check circuit, however, increases exponentially as 2^{m-2} for $m \geq 3$ if we consider slow reset ($a = m$). This is evident from the flag sequences in Figure 3 as the maximum number of times any flag bit is switched. The total depth of the circuit is then $(w - 1) + 2^{m-2}$.

Note that the construction from Theorem 5 does not help for syndrome measurement, because the parity checks would in general become entangled with the data.

We can do slightly better if we allow an *adaptive* circuit, in which the parity checks are chosen based on the outcome of a flag qubit measurement. For example, Figure 8 gives a circuit to prepare a 15-qubit cat state using $m = 3$ measurements. Here, the result of measuring the red ancilla determines how the other two ancillas are used.

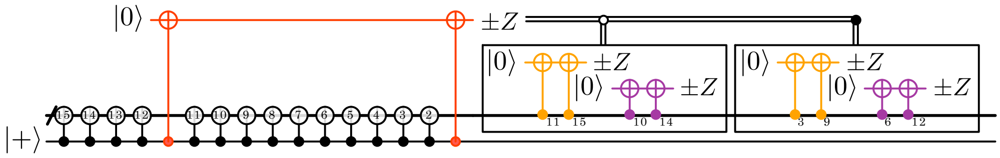


Figure 8 Circuit to prepare a 15-qubit cat state by adaptive error correction, fault-tolerant to distance three. Labels on the thick black wire indicate which data qubit in the block is being addressed as the control or target of the CNOT. If a fault occurs while preparing the cat state on the $|+\rangle$ qubit, it is partially localized by the red flag ancilla. The measurement result of this flag then determines a set of parity checks to completely localize a possible fault. After all the ancilla qubits have been measured, corrections are applied based on Table 2 and Table 3.

Table 2 Parity checks and correction rules when the red flag ancilla in Figure 8 is measured as 1.

$3 \oplus 9$	$6 \oplus 12$	Possible errors	Correction
0	0	$\mathbf{1}, X_1, X_{[2]}$	X_1
1	0	$X_{[3]}, X_{[4]}, X_{[5]}$	$X_{[4]}$
1	1	$X_{[6]}, X_{[7]}, X_{[8]}$	$X_{[7]}$
0	1	$X_{[9]}, X_{[10]}, X_{[11]}$	$X_{[10]}$

▶ **Theorem 6.** *Using an adaptive circuit, for $m \geq 2$, one ancilla qubit, measured m times, can be used to prepare a cat state on w qubits fault-tolerantly to distance three, for*

$$w \leq 3\left(2^m - 2m + 3\right).$$

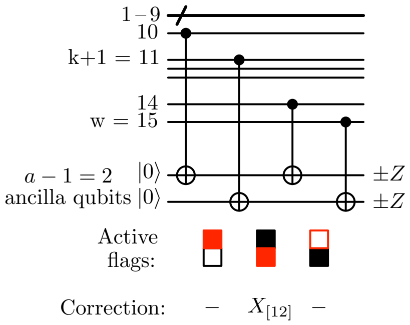
Proof. Our construction will follow the same basic structure as the circuit in Figure 8. Prepare the w data qubits as $|+0^{w-1}\rangle$, then apply $\mathrm{CNOT}_{1,w}, \mathrm{CNOT}_{1,w-1}, \ldots, \mathrm{CNOT}_{1,2}$ to get a cat state. Let $k = 3 \cdot 2^{m-1} - 2$. Just before $\mathrm{CNOT}_{1,k+1}$ and just after $\mathrm{CNOT}_{1,2}$, apply CNOTs into the first ancilla qubit, the red qubit in Figure 8, and measure it.

The remainder of the circuit depends on the measurement result. If it is 1, then a fault has been detected. The error on the cat state can be one of

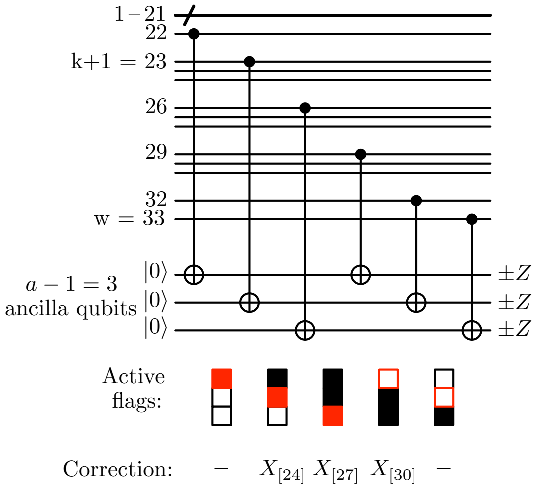
$$\mathbf{1}, X_1, X_{[2]}, \quad X_{[3]}, X_{[4]}, X_{[5]}, \quad \ldots, \quad X_{[k-1]}, X_{[k]}, X_{[k+1]}.$$

The correction procedure needs to determine in which of the above $1 + \frac{k-1}{3}$ groups of three the error lies; then for any error in $\{X_{[3j]}, X_{[3j+1]}, X_{[3j+2]}\}$ the correction $X_{[3j+1]}$ works. Perhaps the easiest way to locate the error is by binary search using the Gray code in Lemma 1, e.g., by computing parities between qubits $3j$ for $j \in \{1, 2, \ldots, 1 + \frac{k-1}{3}\}$. Since the measurement of the red ancilla could have been incorrect, it is important that the all-0s outcome of the binary search correspond to the $\mathbf{1}, X_1, X_{[2]}$ error triple, as in Table 2. Using $m-1$ measurements, we can search 2^{m-1} possibilities, which indeed is $1 + \frac{k-1}{3}$. (The search circuit can also be made nonadaptive, as in Figure 8.)

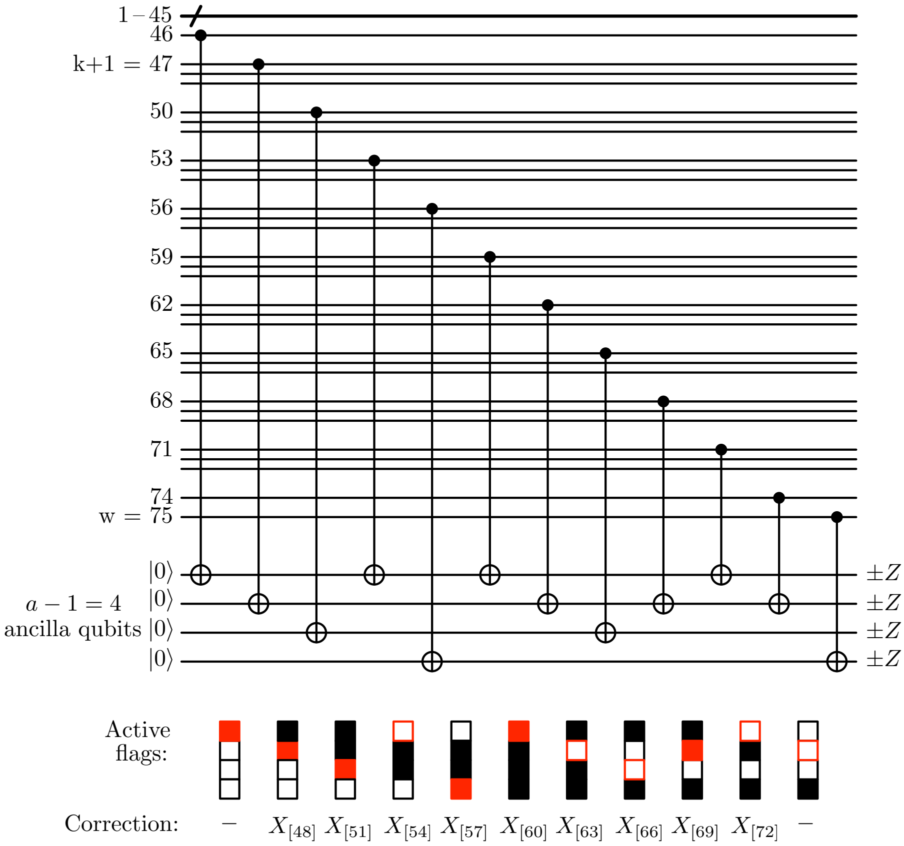
Next consider the case that the first measurement result is 0, so no fault has been detected. The error on the cat state can be one of $X_{[k+1]}, X_{[k+2]}, \ldots, X_{[w]} \sim \mathbf{1}$. We again use the remaining $m-1$ ancilla qubits to measure parities of subsets of cat state qubits. Since there is no guarantee of a fault having occurred yet, we use flag sequences from Lemma 2, where the length of the weight-at-least-two flag sequence is $J = (2^{m-1} - 2(m-1) + 1)$. The parity checks are now done between qubits $\{k, k+1+3j, k+2+3J\}$ for $j \in \{0, 1, \ldots, J\}$, as shown in Figure 9 and Table 3. We do not allow weight-one flag configurations to be able to correct any errors since they can be triggered by a measurement fault on any one of the data qubits involved in the parity check.



(a) $w = 15, a = 3$.



(b) $w = 33, a = 4$.



(c) $w = 75, a = 5$.

Figure 9 If the red ancilla flag in Figure 8 is not triggered, these circuits are used to find and correct a possible error. The flag sequences (from Figure 3) and corresponding corrections are listed at the bottom. Note that these sequences are nonadaptive, and can be used either with a ancilla qubits, in a slow reset model, or with just one ancilla qubit in a fast reset model (because many of the CNOT gates commute).

Table 3 Parity checks and correction rules when the red flag ancilla is measured as 0.

$11 \oplus 15$	$10 \oplus 14$	Possible errors	Correction
0	1	Flag/data qubit error	None
1	1	$X_{[11]}, X_{[12]}, X_{[13]}$	$X_{[12]}$
1	0	$X_{[14]}$ or flag/data qubit error	None
0	0	$\mathbf{1}$	None

Consolidating, we are allowed up to $3J + 1$ CNOTs before the red ancilla is initialized, and up to k CNOTs in the monitored region of the red ancilla. In total we can create a cat state on up to

$$w \leq 3J + k + 2 = 3\left(2^m - 2m + 3\right)$$

qubits, with m total measurements. ◀

We also tested protocols where multiple flags are used for the initial partial localization of a fault (in place of the red flag qubit). We found no improvement to our bounds on ancilla overhead. It appears that ancillas are better used in the parity checks than for partial fault localization.

5 Conclusion

In this paper, we optimize the overhead of distance-three fault tolerance for stabilizer measurement and cat state preparation. If the circuit on w qubits must tolerate one fault, we show that only $\sim \log w$ extra qubits are required. We detail the construction of a maximal-length path on the hypercube and show that it can be used to greatly increase the ability to catch and distinguish faults.

We describe two circuits for stabilizer measurement based on the speed of ancilla qubit reset. With slow reset, a weight-w stabilizer can be measured fault-tolerantly to distance-three using only $\lceil \log_2 w \rceil$ flag qubits for fault tolerance. With fast reset, only three flag qubits are required, but the number of times they are measured and reset grows as $\sim \frac{w}{4}$.

In our circuits for fault-tolerant cat state preparation we check for errors after the cat state is non-fault-tolerantly prepared. We show, using a deterministic and an adaptive circuit, that the overhead for fault tolerance can be as low as logarithmic in the size of the cat state. In fact, only one flag qubit suffices, as long as it can reset quickly.

There are numerous avenues for further improvements. The circuits detailed in this paper are only fault-tolerant to distance-three. Using more complex designs, flag-based fault tolerance can be used to effect fault tolerance to arbitrary distance [1, 6]. It may be interesting to try to develop higher-distance circuits for stabilizer measurement with logarithmic overhead.

From the perspective of stabilizer algebra, a cat state is a CSS ancilla state. A future avenue of research might look to extend these flag techniques to the fault-tolerant preparation of general CSS ancilla states.

In order to execute the circuits in this paper, one qubit needs to be connected to all the other qubits used. This does not bode well for architectures with limited connectivity. But by mixing flag and transversal gate concepts for fault tolerance, it is possible to construct stabilizer measurement circuits that can measure arbitrarily large stabilizers using only local interactions, fault-tolerantly. This can be especially useful in technologies such as superconducting qubits, where qubits only talk to neighbors on a 2-D lattice.

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