

A Subexponential Algorithm for ARRIVAL

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Abstract

The ARRIVAL problem is to decide the fate of a train moving along the edges of a directed graph, according to a simple (deterministic) pseudorandom walk. The problem is in $\text{NP} \cap \text{coNP}$ but not known to be in P . The currently best algorithms have runtime $2^{\Theta(n)}$ where n is the number of vertices. This is not much better than just performing the pseudorandom walk. We develop a subexponential algorithm with runtime $2^{O(\sqrt{n} \log n)}$. We also give a polynomial-time algorithm if the graph is almost acyclic. Both results are derived from a new general approach to solve ARRIVAL instances.

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1 Introduction

Informally, the ARRIVAL problem is the following (we quote from Dohrau et al. [6]):

Suppose that a train is running along a railway network, starting from a designated origin, with the goal of reaching a designated destination. The network, however, is of a special nature: every time the train traverses a switch, the switch will change its position immediately afterwards. Hence, the next time the train traverses the same switch, the other direction will be taken, so that directions alternate with each traversal of the switch.

Given a network with origin and destination, what is the complexity of deciding whether the train, starting at the origin, will eventually reach the destination?

ARRIVAL is arguably the simplest problem in $\text{NP} \cap \text{coNP}$ that is not known to be in P . Due to its innocence and at the same time unresolved complexity status, ARRIVAL has attracted quite some attention recently. The train run can be interpreted as a deterministic simulation of a random walk that replaces random decisions at a switch by perfectly fair decisions. Such pseudorandom walks have been studied before under the names of *Eulerian walkers* [17], *rotor-router walks* [12], and *Propp machines* [4]. The reachability question as well as NP and coNP membership are due to Dohrau et al. [6].

Viewed somewhat differently, ARRIVAL is a *zero player game* (a process that runs without a controller); in contrast, three other well-known graph games in $\text{NP} \cap \text{coNP}$ that are not known to be in P are two-player (involving two controllers). These are *simple stochastic*



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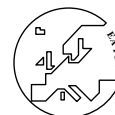
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games, mean-payoff games and parity games [3, 21, 13]. Moreover, it is stated in (or easily seen from) these papers that the one-player variants (the strategy of one controller is fixed) have polynomial-time algorithms. In light of this, one might expect a zero-player game such as ARRIVAL to be *really* simple. But so far, no polynomial-time algorithm could be found.

On the positive side, the $\text{NP} \cap \text{coNP}$ complexity upper bound could be strengthened in various ways. ARRIVAL is in $\text{UP} \cap \text{coUP}$, meaning that there are efficient verifiers that accept *unique* proofs [10]. A search version of ARRIVAL has been introduced by Karthik C. S. and shown to be in PLS [15], then in CLS [10], and finally in UniqueEOPL [10, 8]. The latter complexity class, established by Fearnley et al. [8], has an intriguing complete problem, but there is no evidence that ARRIVAL is complete for UniqueEOPL.

Concerning complexity lower bounds, there is one result: ARRIVAL is NL-hard [7]. This is not a very strong statement and means that every problem that can be solved by a nondeterministic log-space Turing machine reduces (in log-space) to ARRIVAL.

Much more interesting are the natural one- and two-player variants of ARRIVAL that have been introduced in the same paper by Fearnley et al. [7] and later expanded by Ani et al. [1]. These variants allow a better comparison with the previously mentioned graph games. It turns out that the one-player variants of ARRIVAL are NP-complete, and that the two-player variants are PSPACE-hard [7, 1]. This shows that the p -player variant of ARRIVAL is probably strictly harder than the p -player variants of the other graph games mentioned before, for $p = 1, 2$. This makes it a bit less surprising that ARRIVAL itself ($p = 0$) could so far not be shown to lie in P.

On the algorithmic side, the benchmark is the obvious algorithm for solving ARRIVAL on a graph with n vertices: simulate the train run. This is known to take at most $O(n2^n)$ steps (after this, we can conclude that the train runs forever) [6]. There is also an $\Omega(2^n)$ lower bound for the simulation [6]. The upper bound was improved to $O(p(n)2^{n/2})$ (in expectation) for some polynomial p , using a way to efficiently sample from the run [10]. The same bound was later achieved deterministically [11, 18], and the approach can be refined to yield a runtime of $O(p(n)2^{n/3})$, the currently best one for general ARRIVAL instances [18].

In this paper, we prove that ARRIVAL can be decided in subexponential time $2^{O(\sqrt{n} \log n)}$. While this is still far away from the desired polynomial-time algorithm, the new upper bound is making the first significant progress on the runtime. We also prove that polynomial runtime can be achieved if the graph is close to acyclic, meaning that it can be made acyclic by removing a constant number of vertices.

As the main technical tool from which we derive both results, we introduce a generalization of ARRIVAL. In this *multi-run* variant, there is a subset S of vertices where additional trains may start and also terminate. It turns out that if we start the right numbers of trains from the vertices in S , we also decide the original instance, so the problem is reduced to searching for these right numbers. We show that this search problem is well-behaved and can be solved by systematic guessing, where the number of guesses is exponential in $|S|$, not in n .

We are thus interested in cases where S is small but at the same time allows a sufficiently fast evaluation of a given guess. For the subexponential algorithm, we choose S as a set of size $O(\sqrt{n})$, with the property that a train can only take a subexponential number of steps until it terminates (in S or a destination). For almost acyclic graphs, we choose S as a minimum feedback vertex set, a set whose removal makes the graph acyclic. In this case, a train can visit any vertex only once before it terminates.

The multi-run variant itself is an interesting new approach to the ARRIVAL problem, and other applications of it might be found in the future.

2 ARRIVAL

The ARRIVAL problem was introduced by Dohrau et al. [6] as the problem of deciding whether the train arrives at a given destination or runs forever. Here, we work in a different but equivalent setting (implicitly established by Dohrau et al. already) in which the train always arrives at one of two destinations, and we have to decide at which one. The definitions and results from Dohrau et al. [6] easily adapt to our setting. We still provide independent proofs, derived from the more general setting that we introduce in Section 3.

Given a finite set of vertices V , an origin $o \in V$, two destinations $d, \bar{d} \notin V$ and two functions $s_{\text{even}}, s_{\text{odd}} : V \rightarrow V \cup \{d, \bar{d}\}$, the 6-tuple $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ is an ARRIVAL instance. The vertices $s_{\text{even}}(v)$ and $s_{\text{odd}}(v)$ are called the even and the odd successor of v .

An ARRIVAL instance A defines a directed graph, connecting each vertex $v \in V$ to its even and its odd successor. We call this the *switch graph* of A and denote it by $G(A)$. To avoid special treatment of the origin later, we introduce an artificial vertex $Y \notin V \cup \{d, \bar{d}\}$ (think of it as the “train yard”) that only connects to the origin o . Formally, $G(A) = (V(A), E(A))$ where $V(A) = V \cup \{Y, d, \bar{d}\}$ and $E(A) = \{(Y, o)\} \cup \{(v, s_{\text{even}}(v)) : v \in V\} \cup \{(v, s_{\text{odd}}(v)) : v \in V\}$. We also refer to $E(A)$ simply as the edges of A . An edge $e \neq (Y, o)$ is called *proper*.

The *run procedure* is the following. For every vertex we maintain a current and a next successor, initially the even and the odd one. We put a token (usually referred to as the train) at o and move it along switch graph edges until it reaches either d or \bar{d} . Whenever the train is at a vertex v , we move it to v ’s current successor and then swap the current and the next successor; see Algorithm 1 for a formal description and Figure 1 for an example.

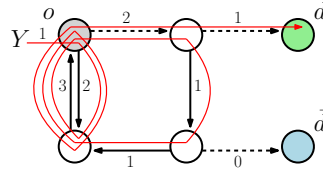
Algorithm 1 Run Procedure.

Input: ARRIVAL instance $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$
Output: destination of the train: either d or \bar{d}
 Let s_{curr} and s_{next} be arrays indexed by the vertices of V
for $v \in V$ **do**
 $s_{\text{curr}}[v] \leftarrow s_{\text{even}}(v)$
 $s_{\text{next}}[v] \leftarrow s_{\text{odd}}(v)$
 $v \leftarrow o$ /* traversal of edge (Y, o) */
while $v \neq d$ and $v \neq \bar{d}$ **do**
 $w \leftarrow s_{\text{curr}}[v]$
 $\text{swap}(s_{\text{curr}}[v], s_{\text{next}}[v])$
 $v \leftarrow w$ /* traversal of edge (v, w) */
return v

Algorithm 1 (Run procedure) may cycle, but we can avoid this by assuming that from every vertex $v \in V$, one of d and \bar{d} is reachable along a directed path in $G(A)$. We call such an ARRIVAL instance *terminating*, since it guarantees that either d or \bar{d} is eventually reached.

► **Lemma 1.** *Let $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ be a terminating ARRIVAL instance, $|V| = n$. Let $v \in V$ and suppose that the shortest path from v to a destination in $G(A)$ has length m . Then v is visited (the train is at v) at most 2^m times by Algorithm 1 (Run Procedure).*

Proof. Let $v = v_m, v_{m-1}, \dots, v_0 \in \{d, \bar{d}\}$ be the sequence of vertices on a shortest path from v to $\{d, \bar{d}\}$. Consider the first 2^m visits to v (if there are less, we are done). Once every two consecutive visits, the train moves on to v_{m-1} , so we can consider the first 2^{m-1} visits to



■ **Figure 1** A terminating ARRIVAL instance and the train run. Bold edges go to the even successors, dashed edges to the odd successors. The two successors may coincide (lower left vertex). The numbers indicate how often each edge is traversed by the train.

v_{m-1} and repeat the argument from there to show that v_i is visited at least 2^i times for all i , before v exceeds 2^m visits. In particular, $v_0 \in \{d, \bar{d}\}$ is visited, so the run indeed terminates within at most 2^m visits to v . ◀

► **Lemma 2.** Let $A = (V, o, d, \bar{d}, s_{even}, s_{odd})$ be a terminating ARRIVAL instance, $|V| = n$. Let ℓ be the maximum length of the shortest path from a vertex in V to a destination. Algorithm 1 (Run Procedure) traverses at most $(n - \ell + 2)2^\ell - 2$ proper edges.

Proof. By Lemma 1, the total number of visits to vertices $v \in V$ is bounded by $\sum_{i=1}^n n_i 2^i$, where n_i is the number of vertices with a shortest path of length i to a destination. We have $n_i > 0$ if and only if $i \leq \ell$, and hence the sum is maximized if $n_i = 1$ for all $i < \ell$, and $n_\ell = n - \ell + 1$. In this case, the sum is $(n - \ell + 2)2^\ell - 2$. The number of proper edges being traversed (one after every visit of $v \in V$) is the same. ◀

Given a terminating instance, ARRIVAL is the problem of deciding whether Algorithm 1 (Run Procedure) returns d (YES instance) or \bar{d} (NO instance). It is unknown whether $\text{ARRIVAL} \in \text{P}$, but it is in $\text{NP} \cap \text{coNP}$, due to the existence of *switching flows* that are certificates for the output of Algorithm 1 (Run Procedure).

2.1 Switching flows

For a vertex v and a set of edges E , we will denote the set of outgoing edges of v by $E^+(v)$. Analogously, we will denote the set of incoming edges of v by $E^-(v)$. Furthermore, for a function $x : E \rightarrow \mathbb{N}_0$, we will also use the notation x_e instead of $x(e)$ to denote the value of x at some edge $e \in E$. Lastly, given some vertex v , edges E and a function $x : E \rightarrow \mathbb{N}_0$, we will use $x^+(v) := \sum_{e \in E^+(v)} x_e$ to denote the *outflow* of x at v and $x^-(v) := \sum_{e \in E^-(v)} x_e$ to denote the *inflow* of x at v . For two functions $x, x' : E \rightarrow \mathbb{N}_0$, we write $x \leq x'$ if this holds componentwise, i.e. $x_e \leq x'_e$ for all $e \in E$.

► **Definition 3** (Switching Flow [6]). Let $A = (V, o, d, \bar{d}, s_{even}, s_{odd})$ be a terminating ARRIVAL instance with edges E . A function $x : E \rightarrow \mathbb{N}_0$ is a switching flow for A if

$$\begin{aligned} x^+(Y) &= 1, \\ x^+(v) - x^-(v) &= 0, & v \in V & \text{ (flow conservation)} \\ x_{(v, s_{even}(v))} - x_{(v, s_{odd}(v))} &\in \{0, 1\}, & v \in V & \text{ (switching behavior)}. \end{aligned}$$

Moreover, x is called a switching flow to $t \in \{d, \bar{d}\}$ if $x^-(t) = 1$.

Note that due to flow conservation, a switching flow is a switching flow either to d or to \bar{d} : exactly one of the destinations must absorb the unit of flow emitted by Y . If we set x_e to the number of times the edge e is traversed in Algorithm 1 (Run Procedure), we obtain

a switching flow to the output; see Figure 1 for an example. Indeed, every time the train enters $v \in V$, it also leaves it; this yields flow conservation. The strict alternation between the successors (beginning with the even one) yields switching behavior.

Hence, the existence of a switching flow to the output is necessary for obtaining the output. Interestingly, it is also sufficient. For that, it remains to prove that we cannot have switching flows to both d and \bar{d} for the same instance.

► **Theorem 4** (Switching flows are certificates [6]). *Let $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ be a terminating ARRIVAL instance, $t \in \{d, \bar{d}\}$. Algorithm 1 (Run Procedure) outputs t if and only if there exists a switching flow to t .*

The switching flow corresponding to the actual train run can be characterized as follows.

► **Theorem 5** (The run profile is the minimal switching flow [6]). *Let $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ be a terminating ARRIVAL instance with edges E . Let \hat{x} be the run profile of A , meaning that \hat{x}_e counts the number of times edge e is traversed during Algorithm 1 (Run Procedure). Then $\hat{x} \leq x$ for all switching flows x . In particular, \hat{x} is the unique minimizer of the total flow $\sum_{e \in E} x_e$ over all switching flows.*

We note that this provides the missing direction of Theorem 4. Indeed, \hat{x} is a switching flow and hence either has $\hat{x}^-(d) = 1$ or $\hat{x}^-(\bar{d}) = 1$. By $\hat{x} \leq x$, every switching flow x is to the same destination. In general, there can be switching flows $x \neq \hat{x}$ [6].

We will derive Theorem 5 as a special case of Theorem 8 in the next section.

3 A general framework

In order to solve the ARRIVAL problem, we can simulate Algorithm 1 (Run Procedure) which takes exponential time in the worst case [6]; alternatively, we can try to get hold of a switching flow; via Theorem 4, this also allows us to decide ARRIVAL.

According to Definition 3, a switching flow can be obtained by finding a feasible solution to an integer linear program (ILP); this is a hard task in general, and it is unknown whether switching flow ILPs can be solved more efficiently than general ILPs.

In this section, we develop a framework that allows us to reduce the problem to that of solving a number of more constrained ILPs. At the same time, we provide direct methods for solving them that do not rely on using general purpose ILP solvers.

3.1 The idea

Given a terminating ARRIVAL instance, we consider the switching flow conditions in Definition 3. Given an arbitrary fixed subset $S = \{v_1, \dots, v_k\} \subseteq V$ of k vertices, we drop the flow conservation constraints at the vertices in S , but at the same time prescribe outflow values $x^+(v_1), \dots, x^+(v_k)$ that we can think of as guesses for their values in a switching flow.

If we minimize the total flow subject to these guesses, we obtain a unique solution (Theorem 8 (i) below) and hence unique inflow values $x^-(v_1), \dots, x^-(v_k)$ for the vertices in S . If we happen to stumble upon a fixed point of the mapping $x^+(v_1), \dots, x^+(v_k) \rightarrow x^-(v_1), \dots, x^-(v_k)$, we recover flow conservation also at S , which means that our guesses were correct and we have obtained a switching flow.

The crucial property is that the previously described mapping is *monotone* (Theorem 8 (ii) below), meaning that the theory of *Tarski fixed points* applies that guarantees the existence of a fixed point as well as efficient algorithms for finding it (Lemma 11 below).

Hence, we reduce the computation of a switching flow to a benign search problem (for a Tarski fixed point), where every search step requires us to solve a “guessing” ILP. We next present a “rail” way of solving the guessing ILP that turns out to be more efficient in the worst case (and also simpler) than general purpose ILP solvers. For suitable switch graphs and appropriate choices of the set S , it will be fast enough to yield the desired runtime results.

3.2 The Multi-Run Procedure

Given $S = \{v_1, \dots, v_k\} \subseteq V$ and $w \in \mathbb{N}_0^k$ (guesses for the outflows from the vertices in S), we start one train from Y and w_i trains from v_i until they arrive back in S , or at a destination. In this way, we produce inflow values for the vertices in S .

By starting, we mean that we move each of the trains by one step: the one on Y moves to o , while $\lceil w_i/2 \rceil$ of the ones at v_i move to the even successor of v_i , and $\lfloor w_i/2 \rfloor$ to the odd successor. Trains that are now on vertices in $V \setminus S$ are called *waiting* (to move on).

For all $v \in V \setminus S$, we initialize current and next successors as before in Algorithm 1 (Run Procedure). Then we (nondeterministically) repeat the following until there are no more trains waiting.

We pick a vertex $v \in V \setminus S$ where some trains are waiting and call the number of waiting trains $t(v)$. We choose a number $\tau \in \{1, \dots, t(v)\}$ of trains to move on; we move $\lceil \tau/2 \rceil$ of them to the current successor and $\lfloor \tau/2 \rfloor$ to the next successor. If τ is odd, we afterwards swap the current and the next successor at v .

Algorithm 2 (Multi-Run Procedure) provides the details. For $S = \emptyset$, the procedure becomes deterministic and is equivalent to Algorithm 1 (Run Procedure).

■ Algorithm 2 Multi-Run Procedure.

Input: Terminating ARRIVAL instance $A = (V, o, d, \bar{d}, s_{even}, s_{odd})$ with edges E ; $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, $w = (w_1, w_2, \dots, w_k) \in \mathbb{N}_0^k$ (one train starts from Y , and w_i trains start from v_i).

Output: number of trains arriving at d, \bar{d} , and in S , respectively

Let t be a zero-initialized array indexed by the vertices of $V \cup \{d, \bar{d}\}$

$t[o] \leftarrow 1$ /* traversal of (Y, o) */

for $i = 1, 2, \dots, k$ **do**

$t[s_{even}(v_i)] \leftarrow t[s_{even}(v_i)] + \lceil w_i/2 \rceil$ /* $\lceil w_i/2 \rceil$ traversals of $(v_i, s_{even}(v_i))$ */
 $t[s_{odd}(v_i)] \leftarrow t[s_{odd}(v_i)] + \lfloor w_i/2 \rfloor$ /* $\lfloor w_i/2 \rfloor$ traversals of $(v_i, s_{odd}(v_i))$ */

Let s_{curr} and s_{next} be arrays indexed by the vertices of $V \setminus S$

for $v \in V \setminus S$ **do**

$s_{curr}[v] \leftarrow s_{even}(v)$
 $s_{next}[v] \leftarrow s_{odd}(v)$

while $\exists v \in V \setminus S : t[v] > 0$ **do**

pick $v \in V \setminus S$ such that $t[v] > 0$ and choose $\tau \in \{1, \dots, t[v]\}$
 $t[v] \leftarrow t[v] - \tau$
 $t[s_{curr}(v)] \leftarrow t[s_{curr}(v)] + \lceil \tau/2 \rceil$ /* $\lceil \tau/2 \rceil$ traversals of $(v, s_{curr}(v))$ */
 $t[s_{next}(v)] \leftarrow t[s_{next}(v)] + \lfloor \tau/2 \rfloor$ /* $\lfloor \tau/2 \rfloor$ traversals of $(v, s_{next}(v))$ */
if τ is odd **then**
 $\text{swap}(s_{curr}[v], s_{next}[v])$

return $(t[d], t[\bar{d}], t[v_1], t[v_2], \dots, t[v_k])$

► **Lemma 6.** *Algorithm 2 (Multi-Run Procedure) terminates.*

Proof. This is a qualitative version of the argument in Lemma 1. Let $x : E \rightarrow \mathbb{N}_0$ record how many times each edge $e \in E$ has been traversed in total, at any given time of Algorithm 2 (Multi-Run Procedure). For $v \in V \setminus S$, we always have $x^+(v) = x^-(v) - t(v)$, where $t(v)$ is the number of trains currently waiting at v . Suppose for a contradiction that the Multi-Run procedure cycles. Then $x^-(v)$ is unbounded for at least one $v \in V \setminus S$, which means that $x^+(v)$ is also unbounded, since $t(v)$ is bounded. This in turn means that $x^-(s_{\text{even}}(v))$ and $x^-(s_{\text{odd}}(v))$ are unbounded as well, since we distribute $x^+(v)$ evenly between the two successors. Repeating this argument, we see that $x^-(w)$ is unbounded for all vertices w reachable from v . But as $x^-(d)$ and $x^-(\bar{d})$ are bounded (by the number of trains that we started), neither d nor \bar{d} are reachable from v . This is a contradiction to A being terminating. ◀

3.3 Candidate switching flows

After Algorithm 2 (Multi-Run Procedure) has terminated, let \hat{x}_e be the number of times the edge e was traversed. We then have flow conservation at $v \in V \setminus S$, switching behavior at $v \in V$ and outflow w_i from v_i . Indeed, every train that enters $v \in V \setminus S$ eventually also leaves it; moreover, the procedure is designed such that it simulates moving trains out of $v \in V$ individually, strictly alternating between successors. Finally, as we start w_i trains from $v_i \in S$ and stop all trains once they arrive in S , we also have outflow w_i from v_i .

We remark that we do not have any control over how many trains end up at d or \bar{d} . Also, \hat{x} could in principle depend on the order in which we pick vertices, and on the chosen τ 's. We will show in Theorem 8 below that it does not. So far, we have only argued that \hat{x} is a *candidate switching flow* according to the following definition.

► **Definition 7** (Candidate Switching Flow). *Let $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ be a terminating ARRIVAL instance with edges E , $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, $w = (w_1, w_2, \dots, w_k) \in \mathbb{N}_0^k$.*

A function $x : E \rightarrow \mathbb{N}_0$ is a candidate switching flow for A (w.r.t. S and w) if

$$\begin{aligned} x^+(Y) &= 1, \\ x^+(v) - x^-(v) &= 0, & v \in V \setminus S & \quad (\text{flow conservation at } V \setminus S) \\ x^+(v_i) &= w_i, & i = 1, 2, \dots, k, & \quad (\text{outflow } w \text{ at } S) \\ x_{(v, s_{\text{even}}(v))} - x_{(v, s_{\text{odd}}(v))} &\in \{0, 1\}, & v \in V & \quad (\text{switching behavior}). \end{aligned} \tag{1}$$

► **Theorem 8** (Each Multi-Run profile is the minimal candidate switching flow). *Let A, E, S, w be as in Definition 7 and let \hat{x} be a Multi-Run profile of A , meaning that \hat{x}_e is the number of times edge $e \in E$ was traversed during some run of Algorithm 2 (Multi-Run Procedure). Then the following statements hold.*

- (i) $\hat{x} \leq x$ for all candidate switching flows x (w.r.t. S and w). In particular, \hat{x} is the unique minimizer of the total flow $\sum_{e \in E} x_e$ over all candidate switching flows.
- (ii) For fixed A, E, S , define $F(w) = (\hat{x}^-(v_1), \dots, \hat{x}^-(v_k)) \in \mathbb{N}_0^k$. Then the function $F : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^k$ is monotone, meaning that $w \leq w'$ implies that $F(w) \leq F(w')$.

Proof. We prove part (i) by the *pebble argument* [6]: Let x be any candidate switching flow w.r.t. S and w . For every edge e , we initially put x_e pebbles on e , and whenever a train traverses e in Algorithm 2 (Multi-Run Procedure), we let it collect a pebble. If we can show that we never run out of pebbles, $\hat{x} \leq x$ follows. By “running out of pebbles”, we concretely mean that we are for the first time trying to collect a pebble from an edge with no pebbles left.

Since x is a candidate switching flow, we cannot run out of pebbles while starting the trains. In fact, we exactly collect all the pebbles on the outgoing edges of $\{Y\} \cup S$. It remains to show that we cannot run out of pebbles while processing a picked vertex $v \in V \setminus S$. For this, we prove that we maintain the following additional invariants (which hold immediately after starting the trains). Let $p : E \rightarrow \mathbb{N}_0$ record for each edge e the remaining number of pebbles on e . Then for all $v \in V \setminus S$,

- (a) $p^+(v) = p^-(v) + t(v)$, where $t(v)$ is the number of trains waiting at v ;
- (b) $p((v, s_{curr}(v))) - p((v, s_{next}(v))) \in \{0, 1\}$.

Suppose that these invariants hold when picking a vertex $v \in V \setminus S$. As we have not run out of pebbles before, $p^-(v) \geq 0$ and (a) guarantees that we have $q \geq t(v)$ pebbles on the outgoing edges; by (b), $\lceil q/2 \rceil$ of them are on $(v, s_{curr}(v))$ and $\lfloor q/2 \rfloor$ on $(v, s_{next}(v))$. From the former, we collect $\lceil \tau/2 \rceil$, and from the latter $\lfloor \tau/2 \rfloor$ where $\tau \leq t(v) \leq q$, so we do not run out of pebbles. We maintain (a) at v where both p^+ and t are reduced by τ . We also maintain (a) at the successors; there, the gain in t exactly compensates the loss in p^- . Finally, we maintain (b) at v : If τ is even, both $p((v, s_{curr}(v)))$ and $p((v, s_{next}(v)))$ shrink by $\tau/2$. If τ is odd, we have $p((v, s_{curr}(v))) - p((v, s_{next}(v))) \in \{-1, 0\}$ after collecting one more pebble from $(v, s_{curr}(v))$ than from $(v, s_{next}(v))$, but then we reverse the sign by swapping s_{curr} and s_{next} .

For $S = \emptyset$, this proves Theorem 5, and for general S , we have now proved (i). In particular, the order in which we move trains in Algorithm 2 (Multi-Run Procedure) does not matter.

The proof of (ii) is now an easy consequence; recall that the inflow $F(w)_i$ is the number of trains that arrive at v_i . If $w \leq w'$, we run Algorithm 2 (Multi-Run Procedure) with input w' such that it first simulates a run with input w ; for this, we keep the extra trains corresponding to $w' - w$ waiting where they are after the start, until all other trains have terminated. At this point, we have inflow $f \geq F(w)$ at S , where $f - F(w)$ corresponds to the extra trains that have already reached S right after the start. We finally run the extra trains that are still waiting, and as this can only further increase the inflows at S , we get $F(w') \geq f \geq F(w)$. ◀

We remark that since the total inflow to the vertices in $\{d, \bar{d}\} \cup S$ equals the total outflow from Y and vertices in S , Theorem 8 (i) implies that the inflows to the vertices in $\{d, \bar{d}\} \cup S$ are the same in every candidate switching flow (w.r.t. S and w). This means that in Theorem 8 (ii), the Multi-Run profile \hat{x} can be replaced by an arbitrary candidate switching flow. This could in principle be easier to compute than the Multi-Run profile, but we currently do not know how.

3.4 Runtime

As we have proved in Theorem 8 (i), the Multi-Run procedure always generates the unique flow-minimal candidate switching flow. But the number of steps depends on the order in which vertices $v \in V \setminus S$ are picked, and on the chosen τ 's. We start with an upper bound on the number of edge traversals that generalizes Lemma 2.

► **Lemma 9.** *Let $A = (V, o, d, \bar{d}, s_{even}, s_{odd})$ be a terminating ARRIVAL instance, $|V| = n$, $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, $w = (w_1, w_2, \dots, w_k) \in \mathbb{N}_0^k$. Let ℓ be the maximum length of the shortest path from a vertex in $V \setminus S$ to a vertex in $\{d, \bar{d}\} \cup S$. Further suppose that at the beginning of some iteration in Algorithm 2 (Multi-Run Procedure), R trains are still waiting. Then all subsequent iterations traverse at most $R((n - \ell + 2)2^\ell - 2)$ edges in total.*

Proof. We continue to run each of the R waiting trains individually and proceed with the next one only when the previous one has terminated. In Algorithm 2 (Multi-Run Procedure), this corresponds to always using $\tau = 1$ and the next vertex v as the head of the previously traversed edge, for each of the R trains. So we effectively perform Algorithm 1 (Run Procedure) for R trains.

As each train terminates once it reaches a vertex in $S \cup \{d, \bar{d}\}$, Lemmata 1 and 2 are easily seen to hold also here, after redefining “destination” as any vertex in $S \cup \{d, \bar{d}\}$. As a consequence, each train traverses at most $(n - \ell + 2)2^\ell - 2$ edges until it reaches a vertex in $\{d, \bar{d}\} \cup S$. This leads to at most $R((n - \ell + 2)2^\ell - 2)$ edge traversals overall. By Theorem 8 (i), this upper bound holds for all ways of continuing Algorithm 2 (Multi-Run Procedure). ◀

With $R = W := 1 + \sum_{i=1}^k w_i$, we obtain an upper bound for the total number of loop iterations since each iteration traverses at least one edge. But it turns out that we can be significantly faster (and polynomial in the encoding size of W) when we proceed in a greedy fashion, i.e. we always pick the next vertex as the one with the largest number of waiting trains, and move all these trains at once.

► **Lemma 10.** *Let A, n, S, w, ℓ as in Lemma 9, and suppose that in each iteration of Algorithm 2 (Multi-Run Procedure), we pick $v \in V \setminus S$ maximizing $t[v]$ and further choose $\tau = t[v]$. Then the number of iterations is at most $(\ln W + n)(n - k)((n - \ell + 2)2^\ell - 2)$, where $W = 1 + \sum_{i=1}^k w_i$.*

Proof. As in the proof of Theorem 8, we let each train collect a pebble as it traverses an edge, where we initially put \hat{x}_e pebbles on edge e , with \hat{x} being the unique Multi-Run profile. This means that we eventually collect all pebbles. Now consider an iteration and suppose that $R \leq W$ trains are still waiting. In the greedy algorithm, we move at least $R/(n - k)$ of them in this iteration and collect at least that many pebbles. On the other hand, with R trains still waiting, and with $T = (n - \ell + 2)2^\ell - 2$, there can be no more than RT pebbles left, as all of them will be collected in the remaining at most that many edge traversals, due to Lemma 9.

In summary, the number of pebbles is guaranteed to be reduced by a factor of

$$\left(1 - \frac{1}{(n - k)T}\right)$$

in each iteration, starting from at most WT pebbles before the first iteration. After $s = (\ln W + n)(n - k)T$ iterations, we therefore have at most

$$\left(1 - \frac{1}{(n - k)T}\right)^s WT \leq e^{-\ln W - n} WT < 1$$

pebbles left (using $T < e^n$). Hence, after at most s iterations, the greedy version of Algorithm 2 (Multi-Run Procedure) has indeed terminated. ◀

We remark that essentially the same runtime can be achieved by a round robin version that repeatedly cycles through $V \setminus S$ in some fixed order.

3.5 Tarski fixed points

Tarski fixed points arise in the study of order-preserving functions on complete lattices [19]. For our application, it suffices to consider finite sets of the form $L = \{0, 1, \dots, N\}^k$ for some $N, k \in \mathbb{N}^+$. For such a set, Tarski’s fixed point theorem [19] states that any monotone

function $D : L \rightarrow L$ has a fixed point, some $\hat{w} \in L$ such that $D(\hat{w}) = \hat{w}$. Moreover, the problem of finding such a fixed point has been studied: Dang, Qi and Ye [5] have shown that a fixed point can be found using $O(\log^k N)$ evaluations of D . Recently, Fearnley, Pálvölgyi and Savani [9] improved this to $O(\log^{2\lceil k/3 \rceil} N)$.

Via Theorem 8, we have reduced the problem of deciding a terminating ARRIVAL instance to the problem of finding a fixed point of a monotone function $F : \mathbb{N}_0^k \rightarrow \mathbb{N}_0^k$, assuming that we can efficiently evaluate F . Indeed, if we have such a fixed point, the corresponding (flow-minimal) candidate switching flow is an *actual* switching flow and hence decides the problem via Theorem 4.

The function F depends on a set $S \subseteq V$ of size k that we can choose freely (we will do so in the subsequent sections).

Here, we still need to argue that we can restrict F to a finite set $L = \{0, 1, \dots, N\}^k$ so that the Tarski fixed point theorem applies. We already know that outflow (and hence inflow) values never exceed $N = 2^n$ in *some* switching flow, namely the run profile (Lemma 1), so we simply restrict F to this range and at the same time cap the function values accordingly.

► **Lemma 11.** *Let $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ be a terminating ARRIVAL instance, $S = \{v_1, \dots, v_k\} \subseteq V$, $|V| = n$. Let F be the function defined in Theorem 8 (ii), let $N = 2^n$ and consider the function $D : \{0, 1, \dots, N\}^k \rightarrow \{0, 1, \dots, N\}^k$ defined by*

$$D(w) = \begin{pmatrix} \min(N, F(w)_1) \\ \min(N, F(w)_2) \\ \vdots \\ \min(N, F(w)_k) \end{pmatrix}, \quad w \in \{0, 1, \dots, N\}^k.$$

Then D is monotone and has a fixed point \hat{w} that can be found with $O(\log^{2\lceil k/3 \rceil} N)$ evaluations of D . Moreover, \hat{w} is also a fixed point of F , and when we apply Theorem 8 (i) with $w = \hat{w}$, the flow-minimal candidate switching flow resulting from the multi-run procedure is a switching flow for A .

We remark that the switching flow obtained in this way is not necessarily flow-minimal, so we cannot argue that we obtain the run profile of A as defined in Theorem 5. The function D may have several fixed points, each of them leading to a different switching flow; to obtain the run profile, we would have to find a particular fixed point, the one that leads to the unique switching flow of smallest total flow. The known Tarski fixed point algorithms cannot do this, and we do not know of any efficient method for computing the run profile from a given switching flow.

Proof. Monotonicity is clear: if $w \leq w'$, then $F(w) \leq F(w')$ by monotonicity of F ; see Theorem 8 (ii). But then also $D(w) \leq D(w')$ for the capped values. Hence, the Tarski fixed point theorem [19] yields a fixed point \hat{w} of D , and the algorithm of Fearnley, Pálvölgyi and Savani [9] finds it using $O(\log^{2\lceil k/3 \rceil} N) = O(n^{2\lceil k/3 \rceil})$ evaluations.

It remains to prove that \hat{w} is a fixed point of F . Suppose for a contradiction that it is not a fixed point. Then $F(\hat{w}) \neq D(\hat{w})$, i.e. some values were actually capped, and so $\hat{w}_j = D(\hat{w})_j = N < F(\hat{w})_j$ for at least one j . As we also have $\hat{w} = D(\hat{w}) \leq F(\hat{w})$, we get

$$\sum_{i=1}^k \hat{w}_i < \sum_{i=1}^k F(\hat{w})_i. \tag{2}$$

On the other hand, consider the candidate switching flow (1) with $w = \hat{w}$. At most the total flow emitted (at Y and the v_i 's) is absorbed at S , so we have

$$\sum_{i=1}^k F(\hat{w})_i \leq 1 + \sum_{i=1}^k \hat{w}_i. \quad (3)$$

Putting this together with (2), we get an equality in (3). In particular, v_j is the only vertex whose inflow value was capped (by one), all emitted flow is absorbed at S , and no flow arrives at d or \bar{d} .

But this is a contradiction to $\hat{w}_j = N = 2^n$: By the same arguments as in the proof of Lemma 1, based on flow conservation (at all $v \neq v_j$) and switching behavior, one of these 2^n outflow units is guaranteed to arrive at $\{d, \bar{d}\}$. ◀

4 Subexponential algorithm for ARRIVAL

In this section, we present our main application of the general framework developed in the previous section.

Given a terminating ARRIVAL instance A with $|V| = n$, the plan is to construct a set $S \subseteq V$ of size $O(\sqrt{n})$ such that from any vertex, the length of the shortest path in $G(A)$ to a vertex in $S \cup \{d, \bar{d}\}$ is also bounded by roughly $O(\sqrt{n})$. Since S is that small, we can find a Tarski fixed point with a subexponential number of F -evaluations; and since shortest paths are that short, each F -evaluation can also be done in subexponential time using the Multi-Run procedure. An overall subexponential algorithm ensues.

► **Lemma 12.** *Let $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ be a terminating ARRIVAL instance with $|V| = n$. Let $\phi \in (0, 1)$ be a real number. In $O(n)$ time, we can construct a ϕ -set S , meaning a set $S \subseteq V$ such that*

- (i) $|S| \leq \phi \cdot (n + 2)$;
- (ii) for all $v \in V$, the shortest path from v to $S \cup \{d, \bar{d}\}$ in $G(A)$ has length at most $\log_2(n + 2)/\phi$.

Proof. We adapt the ball-growing technique of Leighton and Rao [16], as explained by Trevisan [20].

We first decompose the switch graph $G(A)$ into layers based on the distance of the vertices to a destination [11]. More formally, for $v \in V \cup \{d, \bar{d}\}$, we denote by $\text{dist}(v)$ the length of the shortest path from v to $\{d, \bar{d}\}$ in $G(A)$. Then the layers are defined as $L_i := \{v \in V \cup \{d, \bar{d}\} : \text{dist}(v) = i\}$ for $i \geq 0$. Define $\ell := \max\{\text{dist}(v) : v \in V\}$. We can compute the layer decomposition (L_0, \dots, L_ℓ) using breadth-first search in $O(n)$ time.

Consider Algorithm 3 which computes a ϕ -set as a union of layers. It is clear that the procedure is done in $O(n)$ time. To prove (i), we observe that whenever we add a layer L_i to S , we have $|L_i| < \phi|U|$; moreover, the U 's considered in these inequalities are mutually disjoint subsets of $V \cup \{d, \bar{d}\}$. Hence, $|S| < \phi \cdot (n + 2)$.

For (ii), let $v \in V$. Then $v \in L_b$ for some $b \geq 1$. Let $0 \leq a \leq b$ be the largest index such that $L_a \subseteq S \cup \{d, \bar{d}\}$. Then the shortest path from v to a vertex in $S \cup \{d, \bar{d}\}$ has length at most $b - a$. It remains to bound $j := b - a$. The interesting case is $j > 0$.

Consider Algorithm 3. After the a -th iteration, we have $|U| = |L_a| \geq 1$. Moreover, $|L_i| \geq \phi|U|$ for $i = a + 1, \dots, b$, meaning that for each iteration i in this range, the size of U has grown by a factor of at least $1 + \phi$. Hence, after the b -th iteration, $(1 + \phi)^j \leq |U| \leq n + 2$. This implies $j \leq \log_2(n + 2)/\log_2(1 + \phi) < \log_2(n + 2)/\phi$, where we use the inequality $\log_2(1 + \phi) > \phi$ for $\phi \in (0, 1)$. ◀

■ **Algorithm 3** Procedure to compute a ϕ -set.

Input: ARRIVAL instance with layer decomposition (L_0, \dots, L_ℓ) , $\phi \in (0, 1)$
Output: a ϕ -set S
 $S \leftarrow \emptyset$
 $U \leftarrow L_0$
for $i = 1, \dots, \ell$ **do**
 if $|L_i| < \phi|U|$ **then**
 $S \leftarrow S \cup L_i$
 $U \leftarrow \emptyset$
 $U \leftarrow U \cup L_i$
return S

► **Theorem 13.** Let $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ be a terminating ARRIVAL instance with $|V| = n$. A can be decided in time $O(p(n)n^{1.633\sqrt{n}})$, for some polynomial p .

Proof. By Lemma 12, we can find a ϕ -set S in $O(n)$ time, for any $\phi \in (0, 1)$. As $|S| \leq \phi \cdot (n+2)$, by Lemma 11, we can then decide A with $O(n^{2\lceil \phi \cdot (n+2)/3 \rceil})$ evaluations of the function D . Each evaluation in turn requires us to evaluate the function F in Theorem 8 (ii) for a given $w \in \{0, 1, \dots, 2^n\}^{|S|}$. We can do this by applying Algorithm 2 (Multi-Run Procedure). By Lemma 10 and the definition of a ϕ -set in Lemma 12, running this algorithm in a greedy fashion requires at most $(\ln W + n)(n - |S|)((n - \ell + 2)2^\ell - 2)$ iterations, where $W = 1 + \sum_{i=1}^{|S|} w_i$ and $\ell = \log_2(n+2)/\phi$. Further, from the choice of w , we have $W \leq 2^n \phi(n+2) + 1$. Therefore, the number of iterations is $O(q(n)n^{1/\phi})$ for some polynomial q . At each iteration, we need to find the vertex with the highest number of waiting trains, as stated in Lemma 10, and move the trains from the chosen vertex. All these operations take polynomial time.

In total, the runtime of the whole process is $O(n^{2\lceil \phi \cdot (n+2)/3 \rceil} \cdot p(n)n^{1/\phi})$ for some polynomial p . Choosing $\phi = \sqrt{3}/\sqrt{2n}$, the runtime becomes $O(p(n)n^{1.633\sqrt{n}})$. ◀

5 Feedback vertex sets

In the previous section, we used our framework to obtain an improved algorithm for ARRIVAL in general. In this section, we will instantiate the framework differently to obtain a polynomial-time algorithm for a certain subclass of ARRIVAL.

A subset $S \subseteq V$ of vertices in a directed graph $G = (V, E)$ is called a *feedback vertex set* if and only if the subgraph induced by $V \setminus S$ is acyclic (i.e. it contains no directed cycle). Karp [14] showed that the problem of finding a smallest feedback vertex set is NP-hard. However, there exists a parameterized algorithm by Chen et al. [2] which can find a feedback vertex set of size k in time $O(n^4 4^k k^3 k!)$ in a directed graph on n vertices, or report that no such set exists.

If we apply Theorem 8 with a feedback vertex set S , it turns out that we can compute the Multi-Run profile in polynomial time, meaning that we get a polynomial-time algorithm for ARRIVAL if there is a feedback vertex set of constant size k .

► **Theorem 14.** Let $A = (V, o, d, \bar{d}, s_{\text{even}}, s_{\text{odd}})$ be a terminating ARRIVAL instance with graph $G(A)$. If $G(A)$ has a feedback vertex set $S \subseteq V$ of size k (assumed to be fixed as $n = |V| \rightarrow \infty$), then A can be decided in time $O(n^{2(\lceil k/3 \rceil + 1)})$.

Proof. Using the algorithm by Chen et al. [2], we can find a feedback vertex set S in $O(n^4)$ time if it exists. According to Lemma 11, we can then decide A with $O(n^{2^{\lceil k/3 \rceil}})$ evaluations of the function D . Each evaluation in turn requires us to evaluate the function F in Theorem 8 (ii) for a given $w \in \{0, 1, \dots, 2^n\}^k$. To do this, we apply Algorithm 2 (Multi-Run Procedure) where we pick vertices $v \in V \setminus S$ in topological order and choose $\tau = t[v]$ always. As we never send any trains back to vertices that have previously been picked, we terminate within $n - k$ iterations, each of which can be performed in time $O(n)$ as it involves $O(n)$ -bit numbers. Hence, $F(w)$ can be computed in $O(n^2)$ time.

Overall, this gives a runtime of $O(n^4 + n^{2^{\lceil k/3 \rceil + 1}})$. For $k \geq 1$, the second term dominates the first one. For $k = 0$, we can check if a graph is acyclic, for example, via topological sorting in $O(n)$ time and do not need to employ the algorithm by Chen et al. [2]. The claimed runtime follows. \blacktriangleleft

We remark that even if k is not constant, we can still beat the subexponential algorithm in Section 4, as long as $k = O(n^\alpha)$ for some $\alpha < 1/2$.

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