High-Girth Near-Ramanujan Graphs with Lossy Vertex Expansion

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─── **Abstract** ───

Kahale proved that linear sized sets in d-regular Ramanujan graphs have vertex expansion at least $\frac{d}{2}$ and complemented this with construction of near-Ramanujan graphs with vertex expansion no better than $\frac{d}{2}$. However, the construction of Kahale encounters highly local obstructions to better vertex expansion. In particular, the poorly expanding sets are associated with short cycles in the graph. Thus, it is natural to ask whether the vertex expansion of *high-girth* Ramanujan graphs breaks past the $\frac{d}{2}$ bound. Our results are two-fold:

1. For every $d = p + 1$ for prime $p \geq 3$ and infinitely many n, we exhibit an n-vertex d-regular graph with girth $\Omega(\log_{d-1} n)$ and vertex expansion of sublinear sized sets bounded by $\frac{d+1}{2}$ whose nontrivial eigenvalues are bounded in magnitude by $2\sqrt{d-1} + O\left(\frac{1}{\log_{d-1} n}\right)$.

2. In any Ramanujan graph with girth $C \log_{d-1} n$, all sets of size bounded by $n^{0.99C/4}$ have near-lossless vertex expansion $(1 - o_d(1))d$.

The tools in analyzing our construction include the nonbacktracking operator of an infinite graph, the Ihara–Bass formula, a trace moment method inspired by Bordenave's proof of Friedman's theorem [8], and a method of Kahale [16] to study dispersion of eigenvalues of perturbed graphs.

1 Introduction

This paper is concerned with expander graphs, which are ubiquitous in theoretical computer science. A natural and highly well-studied quantity associated with a d-regular graph is its *edge expansion* defined as

$$\min_{|S| \leq \epsilon n} E(S, \overline{S})/|S|,$$

for some constant ϵ. Namely it is the minimum ratio of edges leaving a set S to the size of S for all S of appropriately bounded size. While edge expansion is known to be intractable to compute, there are explicit constructions of good edge expanders, and it is closely related to the second largest magnitude eigenvalue of its adjacency matrix, also known as *spectral*

expansion of a graph, via the expander mixing lemma and Cheeger's inequality [1]. Spectral expansion is easily computable. In particular, an application of the expander mixing lemma proves that small enough sets in graphs with spectral expansion $o(d)$ have near-optimal edge expansion of $(1 - o_d(1))d$.

A natural analog to edge expansion is *vertex expansion*, defined as

$$\min_{|S| \leq \epsilon n} |\Gamma(S)|/|S|$$

for some constant ϵ, where $\Gamma(S)$ is the neighborhood of the set S (potentially containing vertices of S). However, as difficult as edge expansion is to ascertain, vertex expansion has proven far more challenging.

As witnessed by the neighborhood of a vertex, we cannot hope for vertex expansion greater than $d - 1$. Therefore we call a graph a *lossless vertex expander* if for every δ, there exists an ϵ such that there is vertex expansion $d - 1 - \delta$ for sets of size ϵn. Lossless vertex expanders exist since a random d-regular graph is one with high probability (see [15, Theorem 4.16] for a proof). However no deterministic construction of such graphs is known. In an effort to understand lossless vertex expansion better and give explicit constructions, a natural question to ask is: *what properties of random graphs leads to lossless vertex expansion?*

Since a random d-regular graph is near-Ramanujan with high probability [11], and since near-Ramanujan graphs have near-optimal edge expansion, it is natural to inquire if spectral expansion has any implications for vertex expansion as well. Kahale [16] showed that the spectral expansion gives a bound on the vertex expansion. Specifically, Ramanujan graphs (namely graphs with optimal spectral expansion) have vertex expansion at least $d/2$. While this is a nontrivial implication, it falls short of achieving the coveted *losslessness* property. Kahale also proved that the bound of $d/2$ is tight. In particular, he exhibited an infinite family of near-Ramanujan graphs with vertex expansion $d/2$, which means spectral expansion alone is not sufficient for lossless vertex expansion.

The occurrence of a copy of $K_{2,d}$[1] as a subgraph is the obstruction to lossless vertex expansion in Kahale's example. Kahale's example deviates from a random graph in that it is highly unlikely for a random graph to contain a copy of $K_{2,d}$ as a subgraph. More generally, random graphs have the property that with high probability any two "short" cycles are far apart, which Kahale's example doesn't satisfy. Thus, it is natural to ask if the "near-Ramanujan" property in conjunction with the "separatedness of cycles" property of random graphs break past the $d/2$ barrier of Kahale. The "separatedness of cycles" property is especially interesting to consider since it is a key property of random graphs exploited in proofs of Alon's conjecture [11, 8]. A concrete question we can ask is: *Do Ramanujan graphs with $\Omega(\log_{d-1} n)$ girth have lossless vertex expansion?*

An affirmative answer to the above question would prove that the Ramanujan graphs of Lubotzky, Phillips, and Sarnak [18] are lossless vertex expanders. Towards answering the above question, we prove the following negative result:

▶ **Theorem 1.** *For every $d = p + 1$ for prime $p \geq 3$, there is an infinite family of d-regular graphs G on n vertices of girth $\geq \left(\frac{2}{3} - o_n(1)\right) \log_{d-1} n$ where there is a set of vertices U such that $|\Gamma(U)| \leq (d+1)|U|/2$, $|U| \leq n^{1/3}$, and $\max\{\lambda_2(G), -\lambda_n(G)\} \leq 2\sqrt{d-1} + O(1/\log_{d-1} n)$.*

We also complement the above with a positive result which can be summarized as "small enough sets in Ramanujan graphs expand nearly losslessly":

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[1] complete bipartite graph with 2 vertices on one side and d vertices on the other

▶ **Theorem 2.** *Let G be a d-regular Ramanujan graph with girth $C \log_{d-1} n$, then every set of S of size $\leq n^\kappa$ for $\kappa < \frac{C}{4}$ has vertex expansion $(1 - o_d(1))d$.*

After posting our preprint, Amitay Kamber informed us that a theorem in an alternative version of [16] gives the same bound by a different argument. Moreover, his theorem does not depend on the spectral expansion of the graph. However, our proof may be of interest as an alternate method.

1.1 Technical Overview

We give a brief description of how Theorem 1 and Theorem 2 are proved.

1.1.1 Overview of Proof of Theorem 1

Our proof is inspired by that of Kahale's. At a high level, Kahale embeds a copy of $K_{2,d}$ within a Ramanujan graph. We proceed similarly to Kahale, but instead of embedding a $K_{2,d}$, we embed a single subgraph H that is high girth but a lossy vertex expander and show that if H has size n^α for some $0 < \alpha \leq 1/3$, the overall graph is still near-Ramanujan.

Our proof involves two steps: the first step is in proving that the subgraph H being embedded has spectral radius bounded by $2\sqrt{d-1}$, and the second step is in proving that planting H within a Ramanujan graph results in a near-Ramanujan graph. For the first step, we describe an infinite graph containing H and bound its spectral radius via a trace moment method. The trace moment method involves bounding the number of closed walks satisfying certain properties within a graph, and is inspired by an encoding argument from Bordenave's proof of Friedman's theorem [8].

The second step is in proving that our method of embedding a copy of H within a Ramanujan graph does not perturb the eigenvalues by a large amount. Towards doing so, we use the fact that the spectral radius of H is bounded by $2\sqrt{d-1}$ in conjunction with Kahale's argument about dispersion of eigenvalues in high-girth graphs.

1.1.2 Overview of Proof of Theorem 2

We first prove that if a set S in a Ramanujan graph has "lossy" vertex expansion, then we can construct a graph H on vertex set S such that (i) the girth of H is at least half the girth of G, and (ii) the average degree of H is "high" (in particular, the worse the vertex expansion of S, the higher the average degree of H). We then employ the irregular Moore bound, which gives a quantitative tradeoff between the average degree of a graph and its girth. In particular, this would imply that a Ramanujan graph with "lossy" vertex expansion necessarily must have "low" girth.

1.2 Related Work

1.2.1 Applications of Vertex Expanders

There are many applications of expander graphs where having vertex expansion is particularly useful. For example, lossless expanders are particularly of interest in the field of error correcting codes [19, 26, 27]. Lossless vertex expanders give linear error correcting codes that are decodable in linear time [26]. Guruswami, Lee and Razborov [14] use bipartite vertex expanders to construct large subspaces of $\mathbb{R}^n$ where all vectors x in the subspace satisfy $(\log n)^{-O(\log \log \log n)}||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2$.

1.2.2   Explicit Constructions

Constructions of Ramanujan graphs of [18, 20, 22] of all degrees that are of the form $p^r + 1$ for p prime, as well as the construction of near-Ramanujan graphs of every degree of [21] have vertex expansion $\sim \frac{d}{2}$ just by virtue of being Ramanujan via Kahale's result. In fact no deterministic construction has improved upon the $d/2$ bound obtained from solely spectral information. In a remarkable work, Capalbo et al. [10] exhibited an explicit construction of a bipartite graph where subsets of one side of the bipartite graph expand losslessly to the other, using a zig-zag product so the the losslessness of a small, random-like graph boosts the expansion from a large, potentially lossy vertex expanding graph.

1.2.3   Quantum Ergodicity

Quantum ergodicity is another area where both local and global properties of random-like graphs are used. In particular, Anantharaman and Le Masson [6] proved that graphs that have few short cycles (and are therefore close to high girth) and spectral expansion are quantum ergodic, which in this context means the eigenvectors are equidistributed across vertices. Anantharaman, as well as Brooks, Le Masson, and Lindenstrauss exhibited alternative proofs [5, 9]. The proof from [9] shows that quantum ergodicity is equivalent to the mixing of a certain graphical operator. They then use high girth to show that this is equivalent to showing mixing on the infinite tree, then expansion to show the nonbacktracking operator mixes on the tree.

1.2.4   Eigenvector Delocalization

Ganguly and Srivastava, and later Alon, Ganguly and Srivastava [12, 3] give a perturbation of the LPS graph similar to Kahale's argument, but instead of individual vertices, two trees are added and connected to the graph. By assuming the tree is sufficiently deep and carefully connecting the tree to the rest of the graph, the authors create a graph that is high girth but contains eigenvectors that are localized. These graphs are also lossy vertex expanders. However, they show that these graphs cannot be Ramanujan, but rather have spectral radius at least $(2 + c)\sqrt{d - 1}$ where $c > 0$ is a constant. Alon [2] used eigenvector delocalization to create near-Ramanujan expanders of every degree by perturbing known constructions of Ramanujan or near-Ramanujan graphs. Paredes [23] used similar techniques to remove short cycles in a graph while preserving expansion and uses this to algorithmically create graphs that are near-Ramanujan and also have girth at least $\Omega(\sqrt{\log n})$.

1.2.5   Complexity of Constraint Satisfaction Problems

Proofs that it is hard for even linear degree Sum-of-Squares to refute random 3XOR and 3SAT instances on n variables [13, 25] rely on lossless vertex expansion of some sets in a graph underlying a random instance, which suggests a connection between deterministic algorithms for constructing lossless vertex expanders and algorithms for explicit hard instances for Sum-of-Squares.

2   Preliminaries

2.1   Elementary Graph Theory

▶ **Definition 3.** *The girth $g(G)$ of a graph G is the length of the smallest cycle in G.*

▶ **Definition 4.** *For $G = (V, E)$, the valency of $a \in V$ to $B \subset V$ is $|\Gamma(a) \cap B|$, where $\Gamma(S)$ for $S \subset V$ is the set of neighbors of S in G.*

▶ **Definition 5.** *The ball of radius h around a set $U \subset V$, denoted $\mathrm{Ball}_h(U)$, is the set of vertices of distance at most h from U.*

▶ **Definition 6.** *The vertex expansion of a set $U \subset V$ is*

$$\Psi(U) := \frac{|\Gamma(U)|}{|U|}.$$

Similarly, the ϵ-vertex expansion of a graph G is:

$$\Psi_\epsilon(G) = \min_{|U| \leq \epsilon |V|} \Psi(U)$$

where U ranges over subsets of V, and ϵ is an arbitrary constant.

▶ **Definition 7.** *Given a graph G, we use A_G to denote its adjacency matrix. When G is a finite graph on n vertices, the eigenvalues of A_G can be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$.*

▶ **Definition 8.** *We use B_G to denote the* nonbacktracking matrix *of a graph G which is a matrix with rows and columns indexed by directed edges of G defined as follows:*

$$B[(u,v),(w,x)] = \begin{cases} 1 & \text{if } v = w \text{ and } u \neq x \\ 0 & \text{otherwise.} \end{cases}$$

▶ **Definition 9.** *The* spectral expansion *of a finite graph G, denoted $\lambda(G)$ is defined as $\max\{\lambda_2(G), -\lambda_n(G)\}$, which can equivalently be described as the "second largest absolute eigenvalue".*

We now state the following standard fact known as the *expander mixing lemma* (see [15, Lemma 2.5]).

▶ **Lemma 10** (Expander Mixing Lemma). *Let G be a d-regular graph on n vertices. For any two subsets of vertices, $S, T \subseteq V(G)$, let $e(S,T)$ be the number of pairs of vertices (x,y) such that $x \in S, y \in T$ and $\{x,y\}$ is an edge in G. Then:*

$$\left| e(S,T) - \frac{d}{n}|S| \cdot |T| \right| \leq \lambda(G)\sqrt{|S| \cdot |T|}.$$

And finally, we state the "irregular Moore bound" of [4] which articulates a tradeoff between the average degree of a graph and its girth.

▶ **Lemma 11.** *Let G be a n-vertex graph with average degree-d. Then*

$$g(G) \leq 2\log_{d-1} n + 2.$$

2.2 Operator Theory

In this section, let V be a countable set and $T : \ell_2(V) \to \ell_2(V)$ be a bounded linear operator.

▶ **Definition 12.** *The* spectrum *of T, which we denote $\mathsf{spec}(T)$, is the set of all $\lambda \in$ such that $\lambda I - T$ is not invertible.*

▶ **Definition 13.** *The* spectral radius *of T, which we denote $\rho(T)$ is defined as $\sup\{|\lambda| : \lambda \in \mathsf{spec}(T)\}$.*

▶ **Observation 14.** *The* operator norm *of T, which we write as $\|T\|$ is equal to $\sqrt{\rho(TT^*)}$ where T^* is the adjoint of T.*[2]

▶ **Observation 15.** $\rho(T) = \lim_{\ell \to \infty} \|T^\ell\|^{1/\ell}.$

▶ **Observation 16** (Consequence of [24, Theorem 6]). *Suppose T is a self-adjoint operator, and Φ is a basis of $\ell_2(V)$. Then:*

$$\rho(T) = \sup_{\phi \in \Phi} \limsup_{k \to \infty} |\langle \phi, T^k \phi \rangle|^{1/k}.$$

▶ **Observation 17.** *Let A be any principal submatrix of T. Then $\rho(A) \leq \rho(T)$.*

▶ **Corollary 18.** *If H is a subgraph of (possibly infinite) graph G, then $\rho(A_H) \leq \rho(A_G)$.*

3 Infinite Trees Hanging from a Biregular Graph

Let H be any $(2, d-1)$-biregular graph where the partition with degree-$(d-1)$ vertices is called U and the partition with degree-2 vertices is called V. Let X be the infinite graph constructed from H as follows:

At every vertex in U, the $(d-1)$-regular partition, glue an infinite tree where the root has degree-1 and the remaining vertices have degree-d. At every vertex in V, the 2-regular partition, glue an infinite tree where the root has degree-$(d-2)$ and every other vertex has degree-d.

Note that X is a d-regular infinite graph. The main result of this section is:

▶ **Lemma 19.** $\rho(A_X) \leq 2\sqrt{d-1}.$

To prove Lemma 19, we instead turn our attention to the nonbacktracking matrix of X, called B_X. In particular, we bound $\rho(B_X)$ and then employ the Ihara–Bass formula of [7] for infinite graphs to translate the bound on $\rho(B_X)$ into a bound on $\rho(A_X)$.

Thus, we first prove:

▶ **Lemma 20.** $\rho(B_X) \leq \sqrt{d-1}.$

We use the following version of the Ihara–Bass formula of [7] for infinite graphs.

▶ **Theorem 21.** *Let G be a (possibly infinite) graph. Then*

$$\mathsf{spec}(B_G) = \{\pm 1\} \cup \{\lambda : (D_G - I) - \lambda A_G + \lambda^2 I \text{ is not invertible}\}.$$

An immediate corollary that we will use is:

▶ **Corollary 22.** *Let G be a d-regular graph. Then $\rho(B_G) \leq \sqrt{d-1}$ implies that $\rho(A_G) \leq 2\sqrt{d-1}$.*

Proof. If there is μ in $\mathsf{spec}(A_G)$ such that $|\mu| > 2\sqrt{d-1}$, then $\mu I - A_G$ is not invertible. Consequently, by Theorem 21 $\lambda = \frac{\mu + \sqrt{\mu^2 - 4(d-1)}}{2}$, which is greater than $\sqrt{d-1}$, is in $\mathsf{spec}(B_G)$. ◀

In light of Corollary 22, we see that Lemma 20 implies Lemma 19.

Towards proving Lemma 20, we first make a definition.

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[2] Since $\ell_2(V)$ comes equipped with the inner product $\langle f, g \rangle := \sum_{v \in V} f(v)g(v)$, T^* is simple the "transpose" of T.

▶ **Definition 23.** *We call a walk W a $(a \times b)$-linkage if it can be split into a segments, each of which is a length-b nonbacktracking walk.*

Proof of Lemma 20. By Observation 15

$$\rho(B_X) = \limsup_{\ell \to \infty} \|B_X^\ell\|^{1/\ell}.$$

Since $\|B_X^\ell\| = \sqrt{\rho(B_X^\ell (B_X^*)^\ell)}$ it suffices to bound $\rho(T)$ where $T := B_X^\ell (B_X^*)^\ell$ is a bounded self-adjoint operator, and hence by Observation 16:

$$\rho(T) = \max_{uv \in \vec{E}(X)} \limsup_{k \to \infty} |\langle 1_{uv}, T^k 1_{uv} \rangle|^{1/k}.$$

The quantity $\langle 1_{uv}, T^k 1_{uv} \rangle$ is bounded by the number of $(2k \times (\ell + 1))$-linkages that start and end at vertex u, which we can bound via an encoding argument. In particular, we will give an algorithm to uniquely encode such linkages and bound the total number of possible encodings.

3.1 Encoding Linkages

Each length-$(\ell + 1)$ nonbacktracking segment can be broken into 3 consecutive phases (of which some can possibly be empty): the phase where distance to H decreases on each step (Phase 1), the second phase where distance to H does not change on each step (Phase 2), and the third phase where distance to H increases on each step (Phase 3). We further break the third phase into two (possibly empty) subphases – the first subphase where the distance to u decreases on each step (Phase 3a), and the second subphase where the distance to u increases on each step (Phase 3b).

To encode the linkage, for each length-$(\ell + 1)$ nonbacktracking we specify four numbers denoting the lengths of Phases 1, 2, 3a, and 3b. Note that Phase 2 is nonempty only if it is contained in H. For each step ab in Phase 2 that goes from U (the $(d-1)$-regular partition) to V (the 2-regular partition) we specify a number i in $[d-1]$ such that b is the ith neighbor of a within H. If the first step ab in Phase 2 is from V to U we specify a number in $[2]$ denoting if b is the first or second neighbor of a. For each step ab in Phase 3b we specify a number i in $[d-1]$ such that b is the ith neighbor of a that does not lie in the path between between u and H.

3.2 Recovering Linkages from Encodings

We recover a linkage from its encoding "segment-by-segment". Suppose the first t segments have been recovered, we show how to recover the $(t+1)$-th segment. Let x be the vertex the walk is at after it has traversed the first t segments. The steps taken in Phase 1 can be recovered from the length of the Phase since there is a unique path from any vertex to H. The steps in Phase 2 alternate between stepping from V to U and from U to V. It is easy to recover the first step of Phase 2 as well as any step from U to V; a step ab from V to U that is not the first step of Phase 2 is uniquely determined by the previous step, since a has 2 neighbors in U and by the nonbacktracking nature of the walk there is only one choice for b. Note that Phase 3a is nonempty only if u is not in H and all the steps are contained in the same branch as u. Since there is a unique shortest path between the start vertex of Phase 3a and u, the steps taken in Phase 3a can be recovered from its length. Finally, it is easy to recover the steps taken in Phase 3b since they are explicitly given in the encoding.

3.3 Counting Encodings

Now we turn our attention to bounding the total number of encodings. For given $\alpha, \beta \geq 0$ such that $\alpha + \beta = 2k(\ell + 1)$ we first bound the number of walks such that α steps occur in Phase 2 (i.e. are within H) and β steps occur outside Phase 2 (i.e. are outside H). Let $v_1, v_2, \ldots, v_{2k(\ell+1)}$ be the sequence of vertices visited by the walk in order. By $d(x, y)$ we denote the graphical distance between vertices x and y, and for a set of vertices A, we write $d(x, A) := \min_{y \in A} d(x, y)$. Since $d(v_1, H) = d(v_{2k(\ell+1)}, H)$, $|d(v_i, H) - d(v_{i+1}, H)| \leq 1$ always and $|d(v_i, H) - d(v_{i+1}, H)| = 0$ for every step in Phase 2, the number of steps of the walk that occur in Phase 3 of their respective segments is at most $\frac{\beta}{2}$. In particular, the number of steps that occur in Phase 3b of their respective segments is bounded by $\frac{\beta}{2}$. The following bounds hold:

- The number of possible encodings of the lengths of phases is at most $(\ell + 1)^{8k}$.
- The number of possible encodings of the first step of Phase 2 of each segment is at most 2^{2k}.
- The number of possible encodings of the list of U-to-V steps in Phase 2 is at most $(d-1)^{\frac{\alpha+1}{2}}$ because the steps taken in Phase 2 alternate between going from V to U and from U to V.
- The number of possible encodings of the list of steps in Phase 3b is at most $(d-1)^{\frac{\beta}{2}}$.

The above bounds combine to give a bound of

$$(\ell+1)^{8k} 2^{2k} (d-1)^{\frac{\alpha+1}{2}} (d-1)^{\frac{\beta}{2}} \leq (\ell+1)^{8k} 2^{2k} \sqrt{d-1}^{2k(\ell+1)+1}.$$

As there are at most $2k\ell$ choices for (α, β) pairs, the number of $(2k \times (\ell+1))$-linkages is at most

$$2k\ell(\ell+1)^{8k} 2^{2k} \sqrt{d-1}^{2k(\ell+1)+1}.$$

Thus,

$$\rho(T) \leq \limsup_{k \to \infty} \left(2k\ell(\ell+1)^{8k} 2^{2k} \sqrt{d-1}^{2k(\ell+1)+1} \right)^{1/k} = 4(\ell+1)^8 \sqrt{d-1}^{2(\ell+1)}.$$

Consequently,

$$\rho(B_X) \leq \limsup_{\ell \to \infty} \rho(T)^{1/2\ell} \leq \limsup_{\ell \to \infty} \left(4(\ell+1)^8 \sqrt{d-1}^{2(\ell+1)} \right)^{1/2\ell} = \sqrt{d-1}. \qquad \blacktriangleleft$$

4   High-Girth Near-Ramanujan graphs with Lossy Vertex Expansion

We will plant a high girth graph with low spectral radius within a d-regular Ramanujan graph. We will show that such a construction is a spectral expander, but has low vertex expansion. By $u \sim_G v$, we mean that u and v are adjacent in the graph G. We will write $u \sim v$ when the graph is clear from context.

Consider a $(2, d-1)$ biregular bipartite graph $H = (U, V, E)$, with vertex components U and V. U is the degree-$(d-1)$ component and V the degree-2 component. Therefore if we define $\gamma := |U|$, requiring γ to be even, then $|V| = (d-1)\gamma/2$. Call the vertices of U and V $\{u_1, \ldots, u_\gamma\}$ and $\{v_1, \ldots, v_{\gamma(d-1)/2}\}$, respectively. We connect U and V in such a way to maximize the girth of H.

▶ **Lemma 24.**

$$g(H) \geq 2 \log_{d-1} \gamma.$$
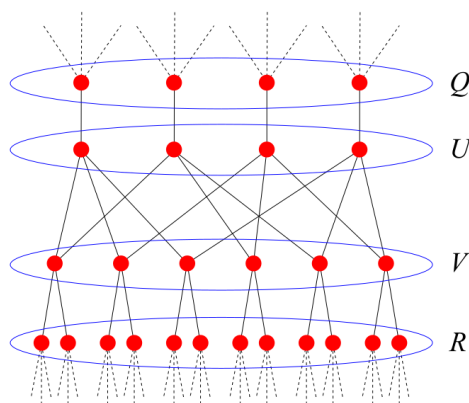
Figure 1 H', with labeled components for $d = 4$, $\gamma = 4$. Note that $\Psi(U) = (d+1)/2$. To create G', we connect Q and R to a well spaced matching in G.

Proof. Because of the valency conditions on H, there is a graph $\tilde{H}$ on γ vertices $\{\tilde{u}_1, \ldots, \tilde{u}_\gamma\}$, where $\tilde{u}_i \sim_{\tilde{H}} \tilde{u}_j$ if and only if $\exists v_k \in H$ such that $u_i \sim_H v_k$ and $u_j \sim_H v_k$. Namely, U corresponds to the vertex set of $\tilde{H}$, and V corresponds to the edge set. $\tilde{H}$ is $d-1$ regular, and, as paths in $\tilde{H}$ of length r correspond to paths of length $2r$ in H, $g(H) = 2g(\tilde{H})$.

By a result of Linial and Simkin [17], there exists a graph $\tilde{H}$ that has girth at least $c \log_{d-2} \gamma$, for any $c \in (0, 1)$, assuming γ is even. Therefore by setting $c = \log(d-2)/\log(d-1)$, we have that $g(\tilde{H}) \geq \log_{d-1} \gamma$ and $g(H) \geq 2 \log_{d-1} \gamma$. ◀

We add a new set of vertices $Q = \{q_1, \ldots, q_\gamma\}$ and add a matching between Q and U, adding the edge $q_i u_i$ for $1 \leq i \leq \gamma$. Similarly, we add another set of vertices $R = \{r_{i,j}\}, 1 \leq i \leq \gamma(d-1)/2, 1 \leq j \leq d-2$. For each $1 \leq i \leq \gamma(d-1)/2$, we then add an edge from v_i to each of $r_{i,j}$ for $1 \leq j \leq (d-2)$.

We call H' the graph on $U \cup V \cup Q \cup R$. At this point vertices of U and V have degree-d, and vertices of Q and R have degree-1. Also, note $\Psi(U) = (d+1)/2$. We wish to embed H' into a larger, high girth expander, and show that this new graph maintains high girth and expansion, even though the set U is a lossy vertex expander. Our argument follows that of [16, Section 5], but instead of embedding individual vertices, we will embed H'.

▶ **Theorem 25** (Theorem 1 in more detail). *For every $d = p + 1$ for prime $p \geq 3$, there is an infinite family of d-regular graphs $G_m = (V_m, E_m)$ on m vertices, such that $\exists U_m \subset V_m$ with $\Psi(U_m) = (d+1)/2$ for $|U_m| \leq m^{1/3}$, $g(G_m) = (\frac{2}{3} - o_m(1)) \log_{d-1} m$, and such that $\lambda(G_m) \leq 2\sqrt{d-1} + O(1/\log_{d-1} m)$.*

Proof. By the result of Lubotzky, Phillips and Sarnak [18], for such d, there exists an infinite family of d-regular graphs, where graphs of n vertices have girth $(\frac{4}{3} - o_n(1)) \log_{d-1} n$ and have spectral expansion $\leq 2\sqrt{d-1}$.

For a given graph $G = (V, E)$ of this type of size n, we attach H' by removing a matching $M \subset E$, $M = \{(a_{1,1}, a_{1,2}), \ldots, (a_{k,1}, a_{k,2})\}$ for

$$k := \gamma(d-1)(2 + (d-1)(d-2))/4. \tag{1}$$

We take a matching such that the pairwise distance between edges in the matching is maximized in G.

▶ **Lemma 26.** *In a d-regular graph on n vertices, there exists a matching M of size k such that for every pair of edges $(a_{i_1,1}, a_{i_1,2}), (a_{i_2,1}, a_{i_2,2}) \in M$, $i_1 \neq i_2$,*

$$d((a_{i_1,1}, a_{i_1,2}), (a_{i_2,1}, a_{i_2,2})) \geq \log_{d-1} n - \log_{d-1} \gamma - O_n(1).$$

Proof. For a given pair of adjacent vertices $(a_{i,1}, a_{i,2})$, as our graph is d regular, there are at most $1 + d\frac{(d-1)^r - 1}{d-2}$ vertices at distance at most r from $a_{i,1}$, and at most $(d-1)^r$ vertices at distance r from $a_{i,2}$ and distance $r+1$ from $a_{i,1}$. Therefore for any $d \geq 4$, the number of edges at distance at most r from a given edge is less than $4(d-1)^r$. We then greedily add edges by choosing an arbitrary edge with vertices at distance at least r away from all already chosen edges. A kth such edge will exist as long as $4k(d-1)^r \leq n$. For our k given in (1) we can set $r = \log_{d-1} n - \log_{d-1} \gamma - O_n(1)$. ◀

To connect H' to G, we first delete the matching M. Then for every vertex of Q and R, we add $d-1$ edges to the set of vertices of M, connecting to each vertex of M exactly once. Namely, the induced subgraph on $(Q \cup R) \cup M$ is a $(d-1, 1)$ biregular bipartite graph. Call $G' = (V', E')$ the new graph formed from G and H'.

We wish to show that G' remains high girth and a good spectral expander. For the girth of G', cycles are either completely contained in H', completely contained in G, or a mix between the two. Cycles in H' have length at least $2 \log_{d-1} \gamma$ by Lemma 24. Cycles in G have length at least $(\frac{4}{3} - o_n(1)) \log_{d-1} n$ by the construction of [18]. For cycles that are a mix of H' and G, we must go from one vertex of H' to another vertex of H' through G. Therefore by Lemma 26, the length of such a cycle is at least $\log_{d-1} n - \log_{d-1} \gamma - O_n(1)$, giving

$$g(G') \geq \min\{2\log_{d-1} \gamma, \log_{d-1} n - \log_{d-1} \gamma - O_n(1)\}.$$

To show that the spectrum is not adversely affected, we follow the argument of [16, Theorem 5.2], with some adjustments. For our new graph, assume that there is an eigenvector $g \perp \mathbf{1}$ corresponding to an eigenvalue $|\mu| > 2\sqrt{d-1}$.

Call A the adjacency matrix of G', and A_G the adjacency matrix of G padded with zeros so it is of the same size as A. Then we have

$$g^* A g = g_G^* A_G g_G + g_{H'}^* A g_{H'} - 2 \sum_{i=1}^{k} g(a_{i,1}) g(a_{i,2}) + \sum_{\substack{u \in Q \cup R \\ a_{i,j} \in M \\ u \sim a_{i,j}}} g(u) g(a_{i,j})$$

where g_G and $g_{H'}$ are the projections of g onto G and H', respectively.

We know that

$$|g_G^* A_G g_G| \leq 2\sqrt{d-1} \|g_G\|^2 + \frac{d}{n} \left(\sum_{u \in G} g(u) \right)^2$$

by decomposing g into parts parallel and perpendicular to the all ones vector.

By a combination of Lemma 19 and Corollary 18, the spectral radius of H' is at most $2\sqrt{d-1}$, and therefore we have

$$|g_G^* A_G g_G| + |g_{H'}^* A g_{H'}| \leq 2\sqrt{d-1} \|g\|^2 + \frac{d}{n} \left(\sum_{u \in H'} g(u) \right)^2$$

as $\sum_G g(u) = -\sum_{H'} g(u)$, considering $g \perp \mathbf{1}$.

To show that $|\mu| = 2\sqrt{d-1} + O(1/\log n)$, we then need to show

$$\frac{1}{\|g\|^2} \left(\frac{d}{n} \left(\sum_{H'} g_{H'}(u) \right)^2 - 2\sum_{i=1}^k g(a_{i,1})g(a_{i,2}) + \sum_{\substack{u \in Q \cup R \\ a_{i,j} \in M \\ u \sim a_{i,j}}} g(u)g(a_{i,j}) \right) = O\left(\frac{1}{\log n} \right). \quad (2)$$

The first term of (2) can be bounded as

$$\frac{d}{n} \left(\sum_{H'} g_{H'}(u) \right)^2 \leq \frac{d}{n} |H'| \, \|g_{H'}\|^2 \leq \frac{\gamma(2 + (d-1)(d-2))d}{2n} \|g_{H'}\|^2. \quad (3)$$

The second term we can bound as

$$\left| 2\sum_{i=1}^k g(a_{i,1})g(a_{i,2}) \right| \leq \sum_{a_{i,j} \in M} g(a_{i,j})^2. \quad (4)$$

Now we will bound the last term of (2) using the Cauchy-Schwarz inequality.

$$\left| \sum_{\substack{u \in Q \cup R \\ a_{i,j} \in M \\ u \sim a_{i,j}}} g(u)g(a_{i,j}) \right| \leq \sqrt{(d-1) \sum_{u \in Q \cup R} g(u)^2} \sqrt{\sum_{a_{i,j} \in M} g(a_{i,j})^2}. \quad (5)$$

We use the following lemma to bound the right hand sides of (4) and (5). The lemma is a generalized version of [16, Lemma 5.1]. The result follows from the same proof, which we reproduce for completeness. Here, for two vectors $a, b \in \mathbb{R}^n$, $a \leq b$ if $\forall i \in [n], a(i) \leq b(i)$.

▶ **Lemma 27** (Lemma 5.1 of [16]). *Consider a graph on a vertex set W, a subset X of W, a positive integer h, and $s \in L^2(W)$. Let X_i be the set of nodes at distance i from X. Assume the following conditions hold:*
(1) *For $h-1 \leq i, j \leq h$, all nodes in X_i have the same number of neighbors in X_j.*
(2) *If $u \in X_{h-1}$ and $v \in X_h$ and $u \sim v$, then $s(u)/s(v)$ does not depend on the choices of u and v.*
(3) *s is nonnegative and $As \leq \mu s$ on $\text{Ball}_{h-1}(X)$, where μ is a positive real number.*

Proof. Let A be the adjacency matrix of W. Let P_{h-1} and $P_h(X)$ be the orthogonal projections onto X_{h-1} and onto X_h, respectively. Let $P_{\leq h-1}$ and $P_{\leq h}(X)$ be the orthogonal projections onto $\text{Ball}_{h-1}(X)$ and $\text{Ball}_h(X)$, respectively. We need to show that

$$\frac{\|P_h g\|^2}{\|P_h s\|^2} \geq \frac{\|P_{h-1}g\|^2}{\|P_{h-1}s\|^2}.$$

Call $A_h = P_{\leq h} A P_{\leq h}$ (so A_h performs the adjacency operator on $\text{Ball}_h(X)$). By the conditions of the lemma, we know that there are constants α, β and γ such that

$$P_h A_h s = \gamma P_h s \quad (6)$$

and

$$A_h P_h s = \alpha P_h s + \beta P_{h-1} s. \quad (7)$$

By assumption,

$$A_h s \leq \mu P_{\leq h-1} s + \gamma P_h s. \tag{8}$$

Therefore by applying $P_{\leq h-1}$ to both sides of (8),

$$P_{\leq h-1} A_h s \leq \mu P_{\leq h-1} s$$
$$\leq \mu P_{\leq h} s - \mu P_h s.$$

Now we apply A_h to both sides:

$$A_h P_{\leq h-1} A_h s \leq \mu A_h s - \mu A_h P_h s$$
$$\leq \mu A_h s - \mu(\alpha P_h s + \beta P_{h-1} s) \qquad \text{by (7)}$$
$$\leq \left(\mu^2 P_{\leq h-1} + \mu(\gamma - \alpha) P_h - \mu\beta P_{h-1} \right) s. \qquad \text{by (8)}$$

Define the matrix $B := \mu^2 P_{\leq h-1} + \mu(\gamma - \alpha) P_h - \mu\beta P_{h-1} - A_h P_{h-1} A_h$. B has no positive entries on the off-diagonal. Take any eigenvector ψ of B. Without loss of generality assume that ψ has a positive entry. Then take $i = \operatorname{argmax}_u \psi(u)/s(u)$. As $\psi \leq (\psi(i)/s(i))s$, $(B\psi)(i) \geq (B(\psi(i)/s(i))s)(i)$. The quantity on the right is nonnegative, meaning that the eigenvalue with eigenvector ψ is nonnegative. As ψ was arbitrary, B is positive semidefinite.

Because B is positive semidefinite,

$$g^* A_h P_{\leq h-1} A_h g \leq g^* \left(\mu^2 P_{\leq h-1} + \mu(\gamma - \alpha) P_h - \mu\beta P_{h-1} \right) g. \tag{9}$$

For any orthogonal projection P, $P^2 = P$. Therefore $g^* A_h P_{\leq h-1} A_h g = \|P_{\leq h-1} A_h g\|^2$. Moreover (9) becomes

$$\|P_{\leq h-1} A_h g\|^2 \leq \mu^2 \|P_{\leq h-1} g\|^2 + \mu(\gamma - \alpha)\|P_h g\|^2 - \mu\beta \|P_{h-1} g\|^2.$$

By assumption, $\|P_{\leq h} A_h g\| = \mu \|P_{\leq h} g\|$. Therefore

$$(\gamma - \alpha)\|P_h g\|^2 \geq \beta \|P_{h-1} g\|^2. \tag{10}$$

As A_h and P_h are self adjoint, $s^* A_h P_h s = s^* P_h A_h s$, so $\alpha \|P_h s\|^2 + \beta \|P_{h-1}\|^2 = \gamma \|P_h s\|^2$. Combining this with (10), we obtain (11). ◀

For any $g \in L^2(W)$ such that $|Ag(u)| = \mu|g(u)|$ for $u \in \operatorname{Ball}_{h-1}(X)$, we have

$$\frac{\sum_{v \in X_h} g(v)^2}{\sum_{v \in X_h} s(v)^2} \geq \frac{\sum_{v \in X_{h-1}} g(v)^2}{\sum_{v \in X_{h-1}} s(v)^2}. \tag{11}$$

To use the lemma, we set $X_0 = U \cup V$, and h will vary from $2 \leq h \leq \lfloor r/2 \rfloor$. Assuming that the girth of G' is at least r, the $\lfloor r/2 \rfloor$ neighborhoods of each vertex do not overlap.

Our test vector decays exponentially, with a small adjustment.

$$s(y) = \begin{cases} \dfrac{1}{(d-1)^{h/2}} & y \in X_{h,U} \\[2ex] \dfrac{2}{\sqrt{d-1}} - \dfrac{1}{(d-1)^{3/2}} & y \in X_{0,V} \\[2ex] \left(\dfrac{2}{d-2} - \dfrac{2}{(d-1)(d-2)} \right) \dfrac{1}{(d-1)^{(h-1)/2}} & y \in X_{h,V}, h \geq 1. \end{cases}$$

For this assignment of values we have $As \leq (2\sqrt{d-1})s$. In fact, this inequality is sharp at all coordinates except for $y \in X_{1,V}$.

For this s, we have that $\sum_{y \in X_h} s(y)^2$ is constant for $h = 1, \ldots, \lfloor r/2 \rfloor$. Also, recall $Q \cup R = X_1$ and $M = X_2$. By Lemma 27, as g corresponds to an eigenvalue $|\mu| > 2\sqrt{d-1}$, the mass on each of first 2 layers of X can only be at most $2/(r-2)$ of the total mass.

Combining (3), (4), and (5), we can bound (2) as

$$(2) \leq \frac{\gamma(2 + (d-1)(d-2))d}{2n} \|g_{H'}\|^2 + \sum_{a \in X_2} g(a)^2 + \sqrt{(d-1)\sum_{u \in X_1} g(u)^2} \sqrt{\sum_{a \in X_2} g(a)^2}$$

$$\leq \left(\frac{\gamma(2 + (d-1)(d-2))d}{2n} + (1 + \sqrt{d-1})\frac{2}{r-2} \right) \|g\|^2.$$

If we set $\gamma = n^{1/3}$ and $r = \frac{2}{3}\log_{d-1} n - O_n(1)$, for fixed d this becomes

$$O\left(\frac{1}{\log n} \right) \|g\|^2,$$

meaning that $\mu \leq 2\sqrt{d-1} + O(1/\log n)$. This also gives the desired bounds on vertex expansion and girth, by setting $U = U_m$. Because $|V'| = (1 + o_n(1))n$, the bounds on $\Psi(U_m)$, $g(G')$ and $\lambda(G')$ given in terms of n do not change when they are given in terms of m. ◄

5   Lossless Expansion of Small Sets

In this section, we prove that sufficiently small sets in a high-girth spectral expander expand losslessly.

▶ **Theorem 28** (Theorem 2 in detail). *Let G be a d-regular graph on n vertices with girth at least $2\alpha \log_{d-1} n + 4$. Then for any set S with n^κ vertices,*

$$\frac{|\Gamma(S)|}{|S|} \geq d - \lambda(G) - \frac{d^{2\kappa/\alpha}}{2} - \frac{d}{n^{1-\kappa}}.$$

Proof. Let S be a set of vertices of size n^κ in G. Let e_S denote the number of internal edges within S. Let n_i denote the number of vertices in $\Gamma(S)$ that have i edges from S incident to it. Then: $|\Gamma(S)| = n_1 + n_2 + \cdots + n_d$ and $|E(S, \Gamma(S))| = n_1 + 2n_2 + \cdots + dn_d$. Note that $|E(S, \Gamma(S))|$ is also equal to $d|S| - 2e_S$. Now consider the graph H_S on vertex set S and edge set given by induced edges on S along with new edges introduced by adding an arbitrary spanning tree for every set of i vertices that are neighbors of a vertex in $\Gamma(S)$ with exactly i neighbors in S. The number of edges in H_S is equal to

$$e_S + n_2 + 2n_3 + \cdots + (d-1)n_d = e_S + |E(S, \partial S)| - |\Gamma(S)| = d|S| - e_S - |\Gamma(S)|.$$

The edges in H_S that are not in G correspond to paths of length at most 2 in G. Therefore $g(H_S) \geq \frac{1}{2}g(G) \geq \alpha \log_{d-1} n + 2$. As a consequence of the expander mixing lemma (Lemma 10), $e_S \leq \left(\lambda(G) + \frac{d|S|}{n} \right) |S|$. Consequently,

$$|E(H_S)| \geq \left(d - \lambda(G) - \frac{d|S|}{n} \right) |S| - |\Gamma(S)|,$$

which means the average degree is lower bounded by

$$2\left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\Gamma(S)|}{|S|} \right).$$

Thus by the irregular Moore bound (Lemma 11),

$$g(H_S) \leq \frac{2 \log n^\kappa}{\log \left(2 \left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\Gamma(S)|}{|S|} \right) - 1 \right)} + 2$$

and hence

$$\frac{\alpha}{\log(d-1)} \leq \frac{2\kappa}{\log \left(2 \left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\Gamma(S)|}{|S|} \right) - 1 \right)}.$$

This implies

$$d - \lambda(G) - \frac{d|S|}{n} - \frac{|\Gamma(S)|}{|S|} - \frac{1}{2} \leq \frac{d^{2\kappa/\alpha}}{2},$$

and finally by rearranging the above and plugging in $|S| = n^\kappa$

$$\frac{|\Gamma(S)|}{|S|} \geq d - \lambda(G) - \frac{d^{2\kappa/\alpha} - 1}{2} - \frac{d}{n^{1-\kappa}}. \qquad \blacktriangleleft$$

▶ **Remark 29.** If G is a n-vertex d-regular Ramanujan graph with girth $\frac{4}{3} \log_{d-1} n$ (which is a condition satisfied by the Ramanujan graphs of [18]) then for every set S of size n^κ for $\kappa < 1/3$,

$$\frac{|\Gamma(S)|}{|S|} \geq d(1 - o_d(1)).$$

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